How to Take a Function Apart with SboxU

Léo Perrin

How to Take a Function Apart with SboxU

(Also Featuring some New Results on Ortho-Derivatives)

Anne Canteaut[1], Léo Perrin[1]

[1]Inria, France

leo.perrin@inria.fr

@lpp_crypto

Boolean Functions and their Applications 2020

A wild vectorial Boolean function appears!

A wild vectorial Boolean function appears!

What do you do?

Outline

Plan of this Section

Plan of this Section

How to

- You need to have SAGE installed
- Then head to `https://github.com/lpp-crypto/sboxU`

`Demo`

Sbox from SAGE vs. sboxU

There are already many functions for investigating vectorial boolean functions in SAGE:

- Class `SBox` from `sage.crypto.sbox` (or `sage.crypto.mq.sbox` in older versions)
- Module `boolean_function` from `sage.crypto`

Sbox from SAGE vs. sboxU

There are already many functions for investigating vectorial boolean functions in SAGE:

- Class SBox from `sage.crypto.sbox` (or `sage.crypto.mq.sbox` in older versions)
- Module `boolean_function` from `sage.crypto`

SAGE SBox

- Supports output size $\neq$ input size
- Sub-routines written in Python or Cython
- Built-in SAGE

`sboxU`

- **Assumes** output size $=$ input size
- Sub-routines written in Python or multi-threaded C++
- Cutting edge functionalities

Plan of this Section

Some Tools

1. DDT/LAT (+ Pollock representation thereof)

Demo

Some Tools

1. DDT/LAT (+ Pollock representation thereof)

   `Demo`

2. ANF, algebraic degree

   `Demo`

Some Tools

1. DDT/LAT (+ Pollock representation thereof)

   Demo

2. ANF, algebraic degree

   Demo

3. Finite field arithmetic

   Demo

Some Tools

1. DDT/LAT (+ Pollock representation thereof)

   Demo

2. ANF, algebraic degree

   Demo

3. Finite field arithmetic

   Demo

4. Linear mappings

   Demo

Plan of this Section

Plan of this Section

CCZ- and EA-equivalence

Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \to \mathbb{F}_2^m$ are *C(arlet)-C(harpin)-Z(inoviev) equivalent* if

$$\Gamma_G = \{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = L\left(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}\right) = L(\Gamma_F),$$

where $L : \mathbb{F}_2^{n+m} \to \mathbb{F}_2^{n+m}$ is an affine permutation.

CCZ- and EA-equivalence

Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are *C(arlet)-C(harpin)-Z(inoviev) equivalent* if

$$\Gamma_G = \{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = L\left(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}\right) = L(\Gamma_F),$$

where $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ is an affine permutation.

Definition (EA-Equivalence; EA-mapping)

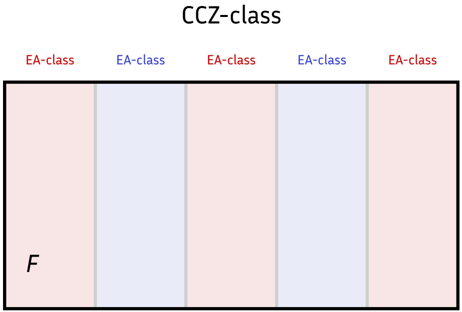
F and G are *E(xtented) A(ffine) equivalent* if $G(x) = (B \circ F \circ A)(x) + C(x)$, where A, B, C are affine and A, B are permutations; so that

$$\{(x, G(x)), \forall x \in \mathbb{F}_2^n\} = \begin{bmatrix} A^{-1} & 0 \\ CA^{-1} & B \end{bmatrix} \left(\{(x, F(x)), \forall x \in \mathbb{F}_2^n\}\right).$$

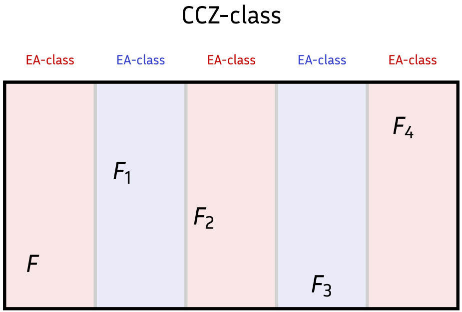
Some Algorithmic Problems with CCZ-Equivalence

CCZ-class

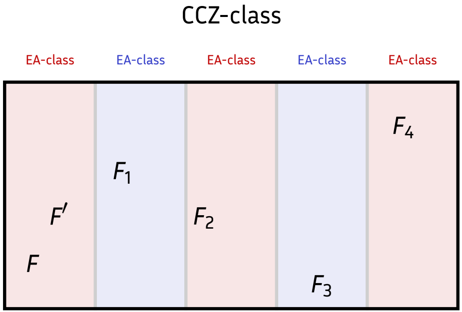
Some Algorithmic Problems with CCZ-Equivalence



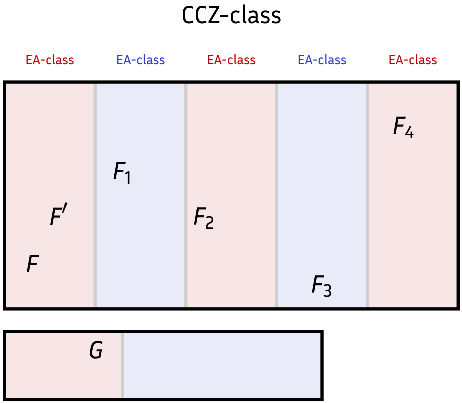
Some Algorithmic Problems with CCZ-Equivalence



Some Algorithmic Problems with CCZ-Equivalence



Some Algorithmic Problems with CCZ-Equivalence



Plan of this Section

Exploring a CCZ-class

Demo

Class Invariants

Definition (Differential spectrum)

Recall that $DDT_F[a, b] = \#\{x, F(x + a) + F(x) = b\}$. The differential spectrum is the number of occurrences of each number in the DDT.

Class Invariants

Definition (Differential spectrum)

Recall that $DDT_F[a, b] = \#\{x, F(x + a) + F(x) = b\}$. The differential spectrum is the number of occurrences of each number in the DDT.

Definition (Walsh spectrum)

Recall that $\mathcal{W}_F[a, b] = \sum_x (-1)^{a \cdot x + b \cdot F(x)}$. The Walsh spectrum is the number of occurrences of each number in the LAT. The extended Walsh spectrum considers only absolute values.

Class Invariants

Definition (Differential spectrum)

Recall that $DDT_F[a, b] = \#\{x, F(x + a) + F(x) = b\}$. The differential spectrum is the number of occurrences of each number in the DDT.

Definition (Walsh spectrum)

Recall that $\mathcal{W}_F[a, b] = \sum_x (-1)^{a \cdot x + b \cdot F(x)}$. The Walsh spectrum is the number of occurrences of each number in the LAT. The extended Walsh spectrum considers only absolute values.

- Differential and extended Walsh spectra are constant in a **CCZ**-class.
- The algebraic degree and the **thickness spectrum** are constant in an **EA**-class.

Demo

Plan of this Section

Plan of this Section

Definition

Definition

Ortho-Derivative Let F be a quadratic function of $\mathbb{F}_2^n$. The ortho-derivatives of F are the functions of $\mathbb{F}_2^n$ such that

$$\forall x \in \mathbb{F}_2^n,\ \pi_F(a) \cdot \Big(\underbrace{F(x+a) + F(x)}_{\Delta_a F(x)} + F(a) + F(0) \Big) = 0 \ .$$

Definition

Definition

Ortho-Derivative Let F be a quadratic function of $\mathbb{F}_2^n$. The ortho-derivatives of F are the functions of $\mathbb{F}_2^n$ such that

$$\forall x \in \mathbb{F}_2^n,\ \pi_F(a) \cdot \Big(\underbrace{F(x+a) + F(x)}_{\Delta_a F(x)} + F(a) + F(0) \Big) = 0\ .$$

- $\pi_F(a)$ is orthogonal to the linear part of the hyperplane $\mathrm{Im}(\Delta_a F)$
- π_F can take any value in 0.

Basic Properties

Lemma (Ortho-derivatives of APN functions)

F is APN if and only if $\pi_F(a)$ is uniquely defined for all $a \in (\mathbb{F}_2^n)^*$.

[1] See also *A note on the properties of associated Boolean functions of quadratic APN functions* by Anastasiya Gorodilova on ArXiv.

Basic Properties

Lemma (Ortho-derivatives of APN functions)

F is APN if and only if $\pi_F(a)$ is uniquely defined for all $a \in (\mathbb{F}_2^n)^*$.

Lemma (Interaction with EA-equivalence)

If $G = B \circ F \circ A$ where A and B are linear permutations, then

$$\pi_F = (B^T)^{-1} \circ F \circ A$$

[1]See also *A note on the properties of associated Boolean functions of quadratic APN functions* by Anastasiya Gorodilova on ArXiv.

Basic Properties

Lemma (Ortho-derivatives of APN functions)

F is APN if and only if $\pi_F(a)$ is uniquely defined for all $a \in (\mathbb{F}_2^n)^*$.

Lemma (Interaction with EA-equivalence)

If $G = B \circ F \circ A$ where A and B are linear permutations, then

$$\pi_F = (B^T)^{-1} \circ F \circ A$$

It seems like[1] the algebraic degree of the ortho-derivative of an APN function is always $n - 2$.

[1]See also *A note on the properties of associated Boolean functions of quadratic APN functions* by Anastasiya Gorodilova on ArXiv.

Preimages of the Ortho-Derivative

Theorem

Linear Structures (APN case) If

$$T_F(b) = \left\{x \in \mathbb{F}_2^n : \pi_F(x) = b\right\} ,$$

then $T_F(b) = \mathrm{LS}(x \mapsto b \cdot F(x))$.

Corollary

For any b, $T_F(b)$ *is a linear subspace of* $\mathbb{F}_2^n$ *whose dimension has the same parity as* n. *Furthermore,*

$$\left(\mathcal{W}_F[a, b]\right)^2 \in \left\{0,\ 2^{n+\dim T_F(b)}\right\}$$

Plan of this Section

Identifying EA- and CCZ-classes

Corollary (Ortho-derivatives of APN functions)

The differential and ***extended Walsh spectra*** *of the ortho-derivative of an APN function is the same within an EA-class.*

Identifying EA- and CCZ-classes

Corollary (Ortho-derivatives of APN functions)

*The differential and **extended Walsh spectra** of the ortho-derivative of an APN function is the same within an EA-class.*

Observation

In practice, these spectra differ from one EA-class to the next!

Identifying EA- and CCZ-classes

Corollary (Ortho-derivatives of APN functions)

*The differential and **extended Walsh spectra** of the ortho-derivative of an APN function is the same within an EA-class.*

Observation

In practice, these spectra differ from one EA-class to the next!

We can use this to very efficiently sort large numbers of quadratic functions into distinct EA-classes.

Demo

Plan of this Section

Principle

Is it possible to recover F given π_F?

Principle

Is it possible to recover F given π_F? Yes!

The Key Observation

We can write the scalar product $x \cdot y$ as $(\vec{x})^T \times \vec{y}$, where $\times$ is a matrix operation.

Principle

Is it possible to recover F given π_F? Yes!

The Key Observation

We can write the scalar product $x \cdot y$ as $(\vec{x})^T \times \vec{y}$, where $\times$ is a matrix operation.

We represent F as a vector of $\mathbb{F}_2^{n2^n}$ by concatenating the n-bit representation of each of the 2^n values $F(x)$:

$$\text{vec}(F) = \begin{bmatrix} F_0(0) \\ F_1(0) \\ \dots \\ F_{n-1}(0) \\ F_0(1) \\ \dots \\ F_{n-1}(2^n - 1) \end{bmatrix}.$$

Re-Defining Ortho-Derivatives

Let G be a function and $\zeta_a(G)$ be a matrix defined by

1. $\zeta_G(a)[x, x] = \vec{G(a)}^T$, $\zeta_G(a)[x, x + a] = \vec{G(a)}^T$,
2. $\zeta_G(a)[x, 0] = \vec{G(a)}^T$, $\zeta_G(a)[x, a] = \vec{G(a)}^T$,

Re-Defining Ortho-Derivatives

Let G be a function and $\zeta_a(G)$ be a matrix defined by

1. $\zeta_G(a)[x, x] = \vec{G(a)}^T$, $\quad \zeta_G(a)[x, x + a] = \vec{G(a)}^T$,
2. $\zeta_G(a)[x, 0] = \vec{G(a)}^T$, $\quad \zeta_G(a)[x, a] = \vec{G(a)}^T$,

so that

$$\zeta_G(a) \times \text{vec}(F) = \begin{bmatrix} G(a) \cdot \big(F(0) + F(0 + a) + F(a) + F(0)\big) \\ G(a) \cdot \big(F(1) + F(1 + a) + F(a) + F(1)\big) \\ \dots \\ G(a) \cdot \big(F(2^n - 1) + F(2^n - 1 + a) + F(a) + F(2^n - 1)\big) \end{bmatrix},$$

Re-Defining Ortho-Derivatives

Let G be a function and $\zeta_a(G)$ be a matrix defined by

1. $\zeta_G(a)[x,x] = \vec{G(a)}^T$, $\quad \zeta_G(a)[x,x+a] = \vec{G(a)}^T$,
2. $\zeta_G(a)[x,0] = \vec{G(a)}^T$, $\quad \zeta_G(a)[x,a] = \vec{G(a)}^T$,

so that

$$\zeta_G(a) \times \text{vec}(F) = \begin{bmatrix} G(a) \cdot (F(0)+F(0+a)+F(a)+F(0)) \\ G(a) \cdot (F(1)+F(1+a)+F(a)+F(1)) \\ \dots \\ G(a) \cdot (F(2^n-1)+F(2^n-1+a)+F(a)+F(2^n-1)) \end{bmatrix},$$

from which we deduce that if π_F is an ortho-derivative of F then

$$\text{vec}(F) \in \ker\left(\zeta(\pi_F)\right) \text{ where } \zeta(\pi_F) = \begin{bmatrix} \zeta_0(\pi_F) \\ \dots \\ \zeta_{2^n-1}(\pi_F) \end{bmatrix}.$$

Inverting the DDT of a Quadratic Function

1. Find a DDT,
2. deduce the corresponding π,
3. build $\zeta(\pi)$,
4. find $\ker(\zeta(\pi))$,
5. obtain vec(F)!

[2] Tricks are used to get rid of redundancies in ζ, and trivial solutions.

Inverting the DDT of a Quadratic Function

1. Find a DDT,
2. deduce the corresponding π,
3. build $\zeta(\pi)$,
4. find $\ker(\zeta(\pi))$,
5. obtain vec(F)!

In practice, starting from "cleverly" built functions π yields $\zeta(\pi)$ with empty[2] kernels...

[2] Tricks are used to get rid of redundancies in ζ, and trivial solutions.

Plan of this Section

Conclusion

Go an use `sboxU`!

Conclusion

Go an use `sboxU`!

Thank you!