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**Abstract** We study the numerical approximation by edge finite elements of fields whose divergence and curl, or divergence and curl-curl, are prescribed in a bounded set $\Omega$ of $\mathbb{R}^3$, together with a boundary condition. Special attention is paid to solutions with low-regularity, in terms of the Sobolev scale $(\boldsymbol{H}^{\mathtt{s}}(\Omega))_{\mathtt{s}>0}$. Among others, we consider an electromagnetic-like model including an interface between a classical medium and a metamaterial. In this setting the electric permittivity, and possibly the magnetic permeability, exhibit a sign-change at the interface. With the help of T-coercivity, we address the case of a model with one sign-changing coefficient, both for the model itself, and for its discrete version. Optimal error estimates are derived. Thanks to these results, we are also able to analyze the classical time-harmonic Maxwell equations, with one sign-changing coefficient.

## Introduction

We study the numerical approximation by finite elements of fields whose divergence and curl, or divergence and curl-curl, are prescribed in a bounded set of $\mathbb{R}^3$, together with a boundary condition on the tangential trace. Typically, one is looking for a field $\boldsymbol{u}$ such that $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f}$ or $\mathbf{curl}\,\zeta\,\mathbf{curl}\,\boldsymbol{u} = \underline{\boldsymbol{f}}$, together with $\operatorname{div}\xi\boldsymbol{u} = g$, in the bounded set, and $\boldsymbol{u}$ has a vanishing tangential trace on its boundary. This kind of model can be viewed as the fundamental building block for solving problems in electromagnetism, see for instance Chapter 6 in [3]. As a by-product, we provide a study of a well-known model in electromagnetism, ie. the time-harmonic Maxwell equations.

In a classical setting, that is if the medium is a classical dielectric, this model has been thoroughly explored, both from the theoretical an the numerical points of view. Among others, we refer to [38,1,2] for the well-posedness of

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the problem, to [53,54] for finite volume discretization, to [22,21,29] for finite element discretization, to [47,4] for the virtual element (previously known as mimetic finite difference) discretization, and to [17,5,46] for least-squares discretization, and to references therein.
On the other hand, there seems to be little knowledge regarding the case of a model including an interface between a classical medium and a metamaterial. In this setting the electric permittivity, and possibly the magnetic permeability, exhibit a sign-change at the interface. To the author's knowledge, the first attempt to address this situation theoretically is [13,12], see also [50]. However, little is known regarding the numerical approximation of the model. More precisely, in [12], the authors address the case of the time-harmonic Maxwell's equations. In the present paper, we propose the first step to solve this problem numerically, by studying problems with one sign-changing coefficient.
For the numerical part, we focus on (low-order) edge finite elements. We use some recent results [28] to interpolate low-regularity solutions that can occur both in a classical setting, and in the presence of an interface between a classical medium and a metamaterial.
The outline is as follows. We begin by introducing some notations, together with a precise definition of the mathematical framework considered hereafter to solve the div-curl problem. Before investigating the solution of this problem, we propose some comments in section 2 to help identify the difficulties to be addressed. For that, we rely on some well-known facts regarding the classical setting, that we shall apply to the new model. We introduce the companion scalar problem and tools, such as the T-coercivity to realize the inf-sup condition. Then in section 3, we solve the problem theoretically, recast as an equivalent variational formulation. Next, in section 4, we recall the numerical approximation via edge finite elements, and in particular how one can interpolate the solution of the div-curl problems, which can (possibly) be of low-regularity. To prove the results regarding convergence of the numerical method, we use some results regarding practical discrete T-coercivity for the companion scalar problem. These are recalled in the appendix A. As a matter of fact, these results allow us to prove the uniform discrete inf-sup condition for the div-curl problem: this is the object of sections 5-6. In section 7 we show how one can solve theoretically and numerically the div-curlcurl problem. Finally in section 8 we consider the case of the time-harmonic Maxwell equations (formulated in the electric field), before giving some concluding remarks in the last section.
The main novelties are contained in theorem 2 and its corollary (end of section 3) for the theoretical part, in propositions 8, 9 and 10 for the properties of the interpolation operator, and in sections 5 and 6 for the numerical analysis of the discrete problems. In passing, we note that we can also study the problem in the classical setting, in unusual configurations (see theorem 3).

## 1 Setting of the problem

As in [28], we denote constant fields by the symbol *cst*. Vector-valued (respectively tensor-valued) function spaces are written in boldface character (resp. blackboard bold characters). Unless otherwise specified, we consider spaces of real-valued functions. Given a non-empty open set $\mathcal{O}$ of $\mathbb{R}^3$, we use the notation $(\cdot|\cdot)_{0,\mathcal{O}}$ (respectively $\|\cdot\|_{0,\mathcal{O}}$) for the $L^2(\mathcal{O})$ and the $\boldsymbol{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3$ inner products (resp. norms). More generally, $(\cdot|\cdot)_{\mathtt{s},\mathcal{O}}$ and $\|\cdot\|_{\mathtt{s},\mathcal{O}}$ (respectively $|\cdot|_{\mathtt{s},\mathcal{O}}$) denote the inner product and the norm (resp. semi-norm) of the Sobolev spaces $H^{\mathtt{s}}(\mathcal{O})$ and $\boldsymbol{H}^{\mathtt{s}}(\mathcal{O}) := (H^{\mathtt{s}}(\mathcal{O}))^3$ for $\mathtt{s} \in \mathbb{R}$ (resp. for $\mathtt{s} > 0$). The index *zmv* indicates zero-mean-value fields. If moreover the boundary $\partial\mathcal{O}$ is Lipschitz, $\boldsymbol{n}$ denotes the unit outward normal vector field to $\partial\mathcal{O}$. It is assumed that the reader is familiar with function spaces related to Maxwell's equations, such as $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}_0(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}(\mathrm{div};\mathcal{O})$, $\boldsymbol{H}_0(\mathrm{div};\mathcal{O})$ etc. A priori, $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$ is endowed with the "natural" norm $\boldsymbol{v} \mapsto (\|\boldsymbol{v}\|_{0,\mathcal{O}}^2 + \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\mathcal{O}}^2)^{1/2}$, etc. We refer to the monographs of Monk [48] and Assous et al [3] for details.
The symbol $C$ is used to denote a generic positive constant which is independent of the meshsize, the mesh and the fields of interest ; $C$ may depend on the geometry, or on the coefficients defining the model. We use the notation $A \lesssim B$ for the inequality $A \leq CB$, where $A$ and $B$ are two scalar fields, and $C$ is a generic constant.

Let $\Omega$ be a domain in $\mathbb{R}^3$, ie. an open, connected and bounded subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. The domain $\Omega$ can be *topologically trivial or non-trivial* [39]. To simplify the computations (without restricting the scope of the study), we assume that the boundary $\partial\Omega$ is *connected*.

Given a domain $\Omega$, the definition of the div-curl model we choose to investigate, is to find the vector-valued field $\boldsymbol{u}$ governed by:

$$\begin{cases} \textit{Find } \boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \textit{ such that} \\ \mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \\ \mathrm{div}\,\xi\boldsymbol{u} = g \text{ in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

*A priori*, the real-valued (volume) source terms are $\boldsymbol{f}$ and $g$, respectively chosen in $\boldsymbol{L}^2(\Omega)$ and in $H^{-1}(\Omega)$.

*Remark 1* Note that one may also consider the div-curlcurl problem, with the curl equation $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f}$ in $\Omega$ replaced by $\mathbf{curl}\,\zeta\,\mathbf{curl}\,\boldsymbol{u} = \underline{\boldsymbol{f}}$ in $\Omega$. The results are completely similar. This is explained in Sect. 7 below.

Then, the real-valued coefficient $\xi$ fulfills one of the two sets of conditions below, which we refer to as the *classical case* and the *interface case* hereafter.
*Classical case:*

$$\begin{cases} \xi \text{ is a real-valued, symmetric, measurable tensor field on } \Omega, \\ \exists \xi_-, \xi_+ > 0, \forall \boldsymbol{z} \in \mathbb{R}^3,\ \xi_-|\boldsymbol{z}|^2 \leq \xi\boldsymbol{z}\cdot\boldsymbol{z} \leq \xi_+|\boldsymbol{z}|^2 \text{ a.e. in } \Omega. \end{cases} \tag{2}$$

*Interface case:* $\Omega$ is partitioned into the non-trivial partition $\mathcal{P} := (\Omega_p)_{p=+,-}$, where $\Omega_\pm$ are domains, and $\delta\xi$ fulfills (2), with $\delta = +1$ in $\Omega_+$ and $\delta = -1$ in $\Omega_-$.

*Remark 2* The study of the interface case is relevant for some problems in electromagnetism, such as an interface between a dielectric and a metamaterial, see [20] or §8.4 in [3]. See also section 8.

## 2 Some comments

To measure elements of $H_0^1(\Omega)$, we choose the norm $q \mapsto \|q\|_{H_0^1(\Omega)} := \|\nabla q\|_{0,\Omega}$. Note that if $\boldsymbol{f} = 0$, then the solution $\boldsymbol{u}$ may be written as $\boldsymbol{u} = \nabla p_{\boldsymbol{u}}$ for some $p_{\boldsymbol{u}} \in H_0^1(\Omega)$ (cf. Theorem 3.3.9 in [3], as $\partial\Omega$ is connected). Moreover, $p_{\boldsymbol{u}}$ is such that $\operatorname{div} \xi\nabla p_{\boldsymbol{u}} = g$ in $H^{-1}(\Omega)$. So to ensure well-posedness of the div-curl model, one must make an assumption on the companion scalar problem:

$$\begin{cases} \textit{Find } s \in H_0^1(\Omega) \textit{ such that} \\ (\xi\nabla s|\nabla q)_{0,\Omega} = \langle g, q\rangle_{H_0^1(\Omega)},\ \forall q \in H_0^1(\Omega), \end{cases} \tag{3}$$

namely, that this scalar problem is well-posed. In other words,

$$\exists C_\star > 0,\ \forall g \in H^{-1}(\Omega),\ \exists! s \text{ solution to (3), with } \|s\|_{H_0^1(\Omega)} \le C_\star \|g\|_{-1,\Omega}. \tag{4}$$

For the classical case, ie. under (2), the well-posedness of the scalar problem is automatically true. This is a straightforward consequence of the fact that $(q, q') \mapsto (\xi\nabla q|\nabla q')_{0,\Omega}$ defines an inner product on $H_0^1(\Omega)$, whose associated norm is equivalent to the $\|\cdot\|_{H_0^1(\Omega)}$-norm.

On the other hand, for the interface case, this is an *additional assumption*, which is addressed with the help of T-coercivity [15,11]. We recall the abstract framework below, see [27,25] for details. Let $V$ be a Hilbert space with norm $\|\cdot\|_V$, and $a(\cdot,\cdot)$ a *symmetric*, continuous bilinear form on $V \times V$. Then, the well-posedness of the problem

$$\textit{Find } u \in V \textit{ such that } a(u, v) = \langle f, v\rangle_V,\ \forall v \in V, \tag{5}$$

which reads

$$\exists C > 0,\ \forall f \in V',\ \exists! u \text{ solution to (5), with } \|u\|_V \le C\,\|f\|_{V'}, \tag{6}$$

can be addresssed as follows. One has to prove that the form $a$ is $T$-coercive, cf. Theorem 1 and Remark 2 of [25]:

$$\exists \alpha > 0,\ \exists T \in \mathcal{L}(V),\ \forall v \in V,\ |a(v, Tv)| \ge \alpha\,\|v\|_V^2. \tag{7}$$

In other words, the operator $T$ realizes the classical inf-sup condition (see eg. [8]) *explicitly.*

Hence, for the scalar problem (3), and because $\xi$ is a symmetric tensor field, well-posedness is equivalent to $(q, q') \mapsto (\xi\nabla q|\nabla q')_{0,\Omega}$ fulfilling an inf-sup condition:

$$\exists\gamma_0 > 0,\ \forall q \in H_0^1(\Omega), \qquad \sup_{q'\in H_0^1(\Omega)\setminus\{0\}} \frac{|(\xi\nabla q|\nabla q')_{0,\Omega}|}{\|q'\|_{H_0^1(\Omega)}} \geq \gamma_0 \|q\|_{H_0^1(\Omega)}. \tag{8}$$

Or, as noted above, this is equivalent to

$$\exists\alpha_0 > 0,\ \exists T_0 \in \mathcal{L}(H_0^1(\Omega)), \forall q \in H_0^1(\Omega),\ |(\xi\nabla q|\nabla(T_0 q))_{0,\Omega}| \geq \alpha_0 \|\nabla q\|_{0,\Omega}^2.$$

We observe that the absolute value can be removed. Indeed, the quadratic mapping $q \mapsto (\xi\nabla q|\nabla(T_0 q))_{0,\Omega}$ is continuous in $H_0^1(\Omega)$ and vanishes only for $q = 0$, so it takes either positive, or negative, values everywhere in $H_0^1(\Omega)$. Thus, (8) is also equivalent to

$$\begin{aligned}&\exists\alpha_0 > 0,\ \exists T_0 \in \mathcal{L}(H_0^1(\Omega)),\\ &\forall q \in H_0^1(\Omega),\ (\xi\nabla q|\nabla(T_0 q))_{0,\Omega} \geq \alpha_0 \|\nabla q\|_{0,\Omega}^2.\end{aligned} \tag{9}$$

To recapitulate, *we assume that:*

– *the coefficient $\xi$ is either as in the classical case, or as in the interface case ;*
– *assumption (8)-(9) holds.*

*Remark 3* Observe that it is possible to fit the classical case within the interface case, by choosing $T_0 = \mathbb{I}_{H_0^1(\Omega)}$ and $\alpha_0 = \xi_-$ in (9). In most instances, we will provide proofs below, that cover both the classical and the interface cases at once.

In the classical case, to characterize orthogonality, we introduce the weighted inner product on $\boldsymbol{H}_0(\mathbf{curl};\Omega)$:

$$(\boldsymbol{v}, \boldsymbol{v}')_{class} \mapsto (\xi\boldsymbol{v}|\boldsymbol{v}')_{0,\Omega} + (\mathbf{curl}\,\boldsymbol{v}|\,\mathbf{curl}\,\boldsymbol{v}')_{0,\Omega}.$$

Note that in the interface case, $(\cdot,\cdot)_{class}$ does not define an inner product.

When we study the *discrete div-curl problems* and in order to obtain explicit convergence rates between the exact and approximate solution to the div-curl problem, we shall make *two additional assumptions*:

– *the coefficient $\xi$ is piecewise smooth*: there exists a partition $\{\Omega_p\}_{p=1,\cdots,P}$ of $\Omega$, made of disjoint domains $(\Omega_p)_{p=1,\cdots,P}$, with $\overline{\Omega} = \cup_{p=1}^P \overline{\Omega_p}$, and such that $\xi_{|\Omega_p} \in W^{1,\infty}(\Omega_p)$ for $p = 1, \cdots, P$. In relation to the partition and for $\mathtt{s} \geq 0$, we define

$$PH^{\mathtt{s}}(\Omega) := \{v \in L^2(\Omega)\ :\ v_{|\Omega_p} \in H^{\mathtt{s}}(\Omega_p),\ 1 \leq p \leq P\}, \tag{10}$$

endowed with the "natural" norm $\|v\|_{PH^{\mathtt{s}}(\Omega)} := \Big(\sum_{1\leq p\leq P} \|v_p\|_{\mathtt{s},\Omega_p}^2\Big)^{1/2}$.

– *the data* $(\boldsymbol{f}, g)$ *has extra-regularity*, in the sense that

$$\boldsymbol{f} \in \boldsymbol{PH}^{\tau_1}(\Omega),\ g \in H^{-1+\tau_2}(\Omega),\ \text{with } \tau_1, \tau_2 > 0 \text{ given.} \tag{11}$$

For further analysis, let us introduce the scalar problem with *modified* right-hand side

$$\begin{cases} \textit{Find } s \in H^1_0(\Omega) \textit{ such that} \\ (\xi \nabla s | \nabla q)_{0,\Omega} = \langle g, q \rangle_{H^1_0(\Omega)} + (\xi \boldsymbol{g}, \nabla q)_{0,\Omega},\ \forall q \in H^1_0(\Omega). \end{cases} \tag{12}$$

In the classical case [30,51,34,44,40,9,33], one can prove a shift theorem for the problem (12) when the data $(g, \boldsymbol{g})$ has *extra-regularity* like

$$g \in H^{-1+\tau_2}(\Omega),\ \boldsymbol{g} \in \boldsymbol{H}^1(\Omega),\ \text{with } \tau_2 \in (0,1] \text{ given.}$$

In the interface case, there exist similar results in this direction. We refer to [31, 16,24,23,14] for a piecewise constant coefficient $\xi$. So we introduce $\tau_{Dir} \in (0,1]$ depending only on the geometry and on $\xi$ such that

$$\begin{aligned} &\forall \mathsf{s} \in [0, \tau_{Dir}) \setminus \{1/2\},\ \forall (g, \boldsymbol{g}) \in H^{-1+\mathsf{s}}(\Omega) \times \boldsymbol{H}^1(\Omega), \\ &\text{the solution } s \text{ to (12) is such that } s \in PH^{1+\mathsf{s}}(\Omega), \text{ and} \\ &\|s\|_{PH^{1+\mathsf{s}}(\Omega)} \lesssim (\|g\|_{-1+\mathsf{s},\Omega} + \|\boldsymbol{g}\|_{1,\Omega}). \end{aligned}$$

Above, the constant hidden in $\lesssim$ may depend on $s$, but not on $g$ nor on $\boldsymbol{g}$. By a slight abuse of vocabulary, we call this result the *shift theorem*, respectively $\tau_{Dir}$ the *limit regularity exponent*, and we assume that this result holds both in the classical and interface cases.

## 3 Basic mathematics

Recall that the domain $\Omega$ can be topologically trivial (tt) or non-trivial (tnt). This means that we assume that one of the two conditions below holds:

– (tt) *'for all curl-free vector field* $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega)$*, there exists* $p \in C^0(\Omega)$ *such that* $\boldsymbol{v} = \nabla p$ *in* $\Omega$*'*;
– (tnt) *'there exist* $I$ *non-intersecting, piecewise plane manifolds,* $(\Sigma_j)_{j=1,\cdots,I}$*, with boundaries* $\partial\Sigma_i \subset \partial\Omega$*, such that, if we let* $\dot\Omega = \Omega \setminus \bigcup_{i=1}^I \Sigma_i$*, for all curl-free vector field* $\boldsymbol{v}$*, there exists* $\dot p \in C^0(\dot\Omega)$ *such that* $\boldsymbol{v} = \nabla \dot p$ *in* $\dot\Omega$*'*.

If $\Omega$ is topologically trivial, we set $I = 0$.

We note that, thanks to the a priori regularity of $\boldsymbol{f}$ and the boundary condition, the solution to (1) is such that $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$. Let us introduce

$$\boldsymbol{H}^\Sigma_0(\operatorname{div} 0; \Omega) := \{\boldsymbol{f}' \in \boldsymbol{H}_0(\operatorname{div}; \Omega)\ :\ \operatorname{div} \boldsymbol{f}' = 0 \text{ in } \Omega,\ \langle \boldsymbol{f}' \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_i} = 0,\ 1 \le i \le I\}.$$

In the classical case, according to Theorem 6.1.4 in [3] (recall that $\partial\Omega$ is connected):

$$\boldsymbol{v} \mapsto (\mathbf{curl}\, \boldsymbol{v}, \operatorname{div} \xi \boldsymbol{v})$$

is a bijective mapping from $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ to $\boldsymbol{H}_0^{\Sigma}(\mathrm{div}\,0;\Omega)\times H^{-1}(\Omega)$.
Hence, to ensure well-posedness of the div-curl model in the classical case, the source terms must be chosen such that

$$\boldsymbol{f}\in\boldsymbol{H}_0^{\Sigma}(\mathrm{div}\,0;\Omega),\quad g\in H^{-1}(\Omega). \tag{13}$$

This is the choice we make from now on. We will check that this choice is also valid for the div-curl model in the interface case under assumption (9). To that aim, we use an equivalent variational formulation to the div-curl problem. Let us introduce

$$\begin{cases} \textit{Find } (\boldsymbol{u},p)\in\boldsymbol{H}_0(\mathbf{curl};\Omega)\times H_0^1(\Omega) \textit{ such that} \\ (\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega}+(\xi\boldsymbol{u}|\nabla q)_{0,\Omega}+(\xi\boldsymbol{v}|\nabla p)_{0,\Omega} \\ \quad=(\boldsymbol{f}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega}-\langle g,q\rangle_{H_0^1(\Omega)},\ \forall(\boldsymbol{v},q)\in\boldsymbol{H}_0(\mathbf{curl};\Omega)\times H_0^1(\Omega). \end{cases} \tag{14}$$

In (14), the left-hand side defines a continuous bilinear form on $\boldsymbol{H}_0(\mathbf{curl};\Omega)\times H_0^1(\Omega)$, and the right-hand side defines a continuous linear form on the same function space. The norm of the right-hand side is bounded from above by $\|\boldsymbol{f}\|_{0,\Omega}+\|g\|_{-1,\Omega}$.
We begin by an elementary, yet fundamental, result.

**Lemma 1** *Let $\boldsymbol{f}\in\boldsymbol{H}_0^{\Sigma}(\mathrm{div}\,0;\Omega)$ and $g\in H^{-1}(\Omega)$ be given. Then if $(\boldsymbol{u},p)$ is a solution to the variational formulation (14), it holds that $p=0$.*

*Proof* Choose the test function $(\nabla(T_0p),0)$ in (14). This yields:

$$(\xi\nabla(T_0p)|\nabla p)_{0,\Omega}=0.$$

Recall that $\xi$ is a symmetric tensor field, so one has $\alpha_0\|\nabla p\|_{0,\Omega}^2=0$ according to (9), and it follows that $p=0$. □

**Proposition 1** *Let $\boldsymbol{f}\in\boldsymbol{H}_0^{\Sigma}(\mathrm{div}\,0;\Omega)$ and $g\in H^{-1}(\Omega)$ be given. Then it holds that $\boldsymbol{u}$ is a solution to the div-curl problem (1) if, and only if, $(\boldsymbol{u},0)$ is a solution to the variational formulation (14).*

*Proof* If $\boldsymbol{u}$ solves (1), then by definition $\boldsymbol{u}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$.
Also it is obvious by integration on $\Omega$ together with the definition of duality $\langle\mathrm{div}\,\xi\boldsymbol{u},q\rangle_{H_0^1(\Omega)}=-(\xi\boldsymbol{u},|\nabla q)_{0,\Omega}$ that $(\boldsymbol{u},0)$ is a solution to the variational formulation (14). Indeed, given $(\boldsymbol{v},q)\in\boldsymbol{H}_0(\mathbf{curl};\Omega)\times H_0^1(\Omega)$, one has

$$(\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega}+(\xi\boldsymbol{u}|\nabla q)_{0,\Omega}=(\boldsymbol{f}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega}-\langle g,q\rangle_{H_0^1(\Omega)}.$$

On the other hand, let $(\boldsymbol{u},p)$ be a solution to (14). We already know from lemma 1 that $p=0$, and that by definition $\boldsymbol{u}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Next, choose $(0,q)$ as a test function in (14):

$$(\xi\boldsymbol{u}|\nabla q)_{0,\Omega}=-\langle g,q\rangle_{H_0^1(\Omega)},\ \forall q\in H_0^1(\Omega),$$

so that $\mathrm{div}\,\xi\boldsymbol{u}=g$ in $H^{-1}(\Omega)$.
Since $\boldsymbol{f}\in\boldsymbol{H}_0^{\Sigma}(\mathrm{div}\,0;\Omega)$, we know from Theorem 3.5.1 in [3] that there exists

$\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{w}$ in $\Omega$. So, using $(\boldsymbol{u}-\boldsymbol{w},0)$ as a test function in (14) now yields:

$$(\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}(\boldsymbol{u}-\boldsymbol{w}))_{0,\Omega} = (\mathbf{curl}\,\boldsymbol{w}|\,\mathbf{curl}(\boldsymbol{u}-\boldsymbol{w}))_{0,\Omega},$$

or $\|\,\mathbf{curl}(\boldsymbol{u}-\boldsymbol{w})\|^2_{0,\Omega} = 0$, ie. $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f}$ in $\boldsymbol{L}^2(\Omega)$.
In other words, $\boldsymbol{u}$ is a solution to the div-curl problem (1). □

We next recall a result on the splitting of fields in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. To that end, define

$$\boldsymbol{K}_N(\Omega,\xi) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ :\ \operatorname{div}\xi\boldsymbol{v} = 0\}.$$

An equivalent (variational) definition is

$$\boldsymbol{K}_N(\Omega,\xi) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ :\ (\xi\boldsymbol{v}|\nabla q)_{0,\Omega} = 0,\ \forall q \in H^1_0(\Omega)\}.$$

**Proposition 2** *One has the continuous, direct sum*

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) = \nabla[H^1_0(\Omega)] \oplus \boldsymbol{K}_N(\Omega,\xi). \tag{15}$$

*In the classical case, the sum is orthogonal with respect to the inner product* $(\cdot,\cdot)_{class}$.

*Proof* Obviously, $\nabla[H^1_0(\Omega)] + \boldsymbol{K}_N(\Omega,\xi)$ is a subset of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. Let $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$. According to (4), there exists $p_{\boldsymbol{v}} \in H^1_0(\Omega)$ such that

$$(\xi\nabla p_{\boldsymbol{v}}|\nabla q)_{0,\Omega} = (\xi\boldsymbol{v}|\nabla q)_{0,\Omega},\ \forall q \in H^1_0(\Omega). \tag{16}$$

Now, let $\boldsymbol{k}_{\boldsymbol{v}} = \boldsymbol{v} - \nabla p_{\boldsymbol{v}}$, one has $\boldsymbol{k}_{\boldsymbol{v}} \in \boldsymbol{K}_N(\Omega,\xi)$ by construction. It follows that $\boldsymbol{H}_0(\mathbf{curl};\Omega) = \nabla[H^1_0(\Omega)] + \boldsymbol{K}_N(\Omega,\xi)$.
Next, let $\boldsymbol{z} \in \nabla[H^1_0(\Omega)] \cap \boldsymbol{K}_N(\Omega,\xi)$ be given. There exists $s \in H^1_0(\Omega)$ such that $\boldsymbol{z} = \nabla s$ and, by definition of $\boldsymbol{K}_N(\Omega,\xi)$, $s$ is governed by (3) with zero right-hand side. By uniqueness of the solution, one has $s = 0$ and so $\boldsymbol{z} = 0$: the sum is direct.
Finally, by definition (16) of $p_{\boldsymbol{v}}$ and according to (9), one has $\alpha_0\,\|\nabla p_{\boldsymbol{v}}\|^2_{0,\Omega} \le (\xi\nabla p_{\boldsymbol{v}}|\nabla(T_0 p_{\boldsymbol{v}}))_{0,\Omega} = (\xi\boldsymbol{v}|\nabla(T_0 p_{\boldsymbol{v}}))_{0,\Omega} \le \|\xi\boldsymbol{v}\|_{0,\Omega}\|\nabla(T_0 p_{\boldsymbol{v}})\|_{0,\Omega}$, so that

$$\begin{aligned} \|\nabla p_{\boldsymbol{v}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} &= \|\nabla p_{\boldsymbol{v}}\|_{0,\Omega} \le \alpha_0^{-1}\|\xi\|_{\infty,\Omega}\|T_0\|_{\mathcal{L}(H^1_0(\Omega))}\|\boldsymbol{v}\|_{0,\Omega}, \\ \text{and}\ \ \|\boldsymbol{k}_{\boldsymbol{v}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} &\le (1+\alpha_0^{-1}\|\xi\|_{\infty,\Omega}\|T_0\|_{\mathcal{L}(H^1_0(\Omega))})\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \end{aligned}$$

So the sum is continuous.
Additionally, in the classical case, let $q \in H^1_0(\Omega)$ and $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\xi)$:

$$(\nabla q,\boldsymbol{k})_{class} = (\xi\nabla q|\boldsymbol{k})_{0,\Omega} = (\xi\boldsymbol{k}|\nabla q)_{0,\Omega} = 0,$$

where we use successively the properties: $\nabla q$ is curl-free; $\xi$ is a symmetric tensor field; $\xi\boldsymbol{k}$ is divergence-free. □

In other words, we may introduce the operators of $\mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega), H^1_0(\Omega))$, resp. of $\mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega))$

$$\pi_1 \;:\; \begin{cases} \boldsymbol{H}_0(\mathbf{curl};\Omega) \to H^1_0(\Omega) \\ \boldsymbol{v} \mapsto p_{\boldsymbol{v}} \end{cases}, \quad \boldsymbol{\pi}_2 \;:\; \begin{cases} \boldsymbol{H}_0(\mathbf{curl};\Omega) \to \boldsymbol{K}_N(\Omega,\xi) \\ \boldsymbol{v} \mapsto \boldsymbol{k}_{\boldsymbol{v}} \end{cases}$$

and write, for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, $\boldsymbol{v} = \nabla(\pi_1\boldsymbol{v}) + \boldsymbol{\pi}_2\boldsymbol{v}$. Note that $(\boldsymbol{\pi}_2)^2 = \boldsymbol{\pi}_2$.

We then recall an important result on the measure of elements of $\boldsymbol{K}_N(\Omega,\xi)$. For its proof, we refer the reader to Corollary 5.2 of [12].

**Theorem 1** *Elements of $\boldsymbol{K}_N(\Omega,\xi)$ can be measured with the $\|\mathbf{curl}\cdot\|_{0,\Omega}$-norm:*

$$\exists C_W > 0,\ \forall \boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\xi), \quad \|\boldsymbol{k}\|_{0,\Omega} \le C_W \,\|\mathbf{curl}\,\boldsymbol{k}\|_{0,\Omega}, \tag{17}$$

$$\exists C'_W > 1,\ \forall \boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\xi), \quad \|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C'_W \,\|\mathbf{curl}\,\boldsymbol{k}\|_{0,\Omega}. \tag{18}$$

Before proving the main result of this section, let us introduce the notations:

$$\mathbb{V} := \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H^1_0(\Omega) \text{ endowed with } \|(\boldsymbol{v},q)\|_{\mathbb{V}} := (\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|q\|^2_{H^1_0(\Omega)})^{1/2};$$

$$a((\boldsymbol{u},p),(\boldsymbol{v},q)) := (\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + (\xi\boldsymbol{u}|\nabla q)_{0,\Omega} + (\xi\boldsymbol{v}|\nabla p)_{0,\Omega},\ \forall(\boldsymbol{u},p),(\boldsymbol{v},q) \in \mathbb{V}.$$

Observe that the form $a$ is bilinear and symmetric on $\mathbb{V}\times\mathbb{V}$.
The variational formulation (14) writes

$$\begin{cases} \textit{Find } (\boldsymbol{u},p) \in \mathbb{V} \textit{ such that} \\ a((\boldsymbol{u},p),(\boldsymbol{v},q)) = (\boldsymbol{f}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \langle g,q\rangle_{H^1_0(\Omega)},\ \forall(\boldsymbol{v},q) \in \mathbb{V}. \end{cases} \tag{19}$$

We show next that it is well-posed. We already noted that, for all $\boldsymbol{f} \in \boldsymbol{H}^{\Sigma}_0(\mathrm{div}\,0;\Omega)$ and $g \in H^{-1}(\Omega)$, the right-hand side of (19) defines a continuous form on $\mathbb{V}$, with norm bounded from above by $\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{-1,\Omega}$.

**Theorem 2** *The form $a$ is T-coercive.*

*Remark 4* In the spirit of the T-coercivity theory [27,25], we provide a constructive proof. Namely, given $(\boldsymbol{u},p)$, we build some *ad hoc* test-field $(\boldsymbol{v}^\star,q^\star)$ to realize the condition (7) via the design of an operator $\mathbb{T} \in \mathcal{L}(\mathbb{V})$, by setting $\mathbb{T}((\boldsymbol{u},p)) = (\boldsymbol{v}^\star,q^\star)$. Regarding the companion scalar problem, we cover both the classical and the interface cases at once by choosing $T_0 = \mathbb{I}_{H^1_0(\Omega)}$ and $\alpha_0 = \xi_-$ in the classical case.

*Proof* Let $(\boldsymbol{u},p) \in \mathbb{V}$ be given. Let us decompose $\boldsymbol{u}$ using (15): $\boldsymbol{u} = \nabla p_{\boldsymbol{u}} + \boldsymbol{k}_{\boldsymbol{u}}$ with $(p_{\boldsymbol{u}},\boldsymbol{k}_{\boldsymbol{u}}) := (\pi_1\boldsymbol{u},\boldsymbol{\pi}_2\boldsymbol{u}) \in H^1_0(\Omega)\times\boldsymbol{K}_N(\Omega,\xi)$.
(i) Assume first that $\boldsymbol{u} = 0$. Then $a((0,p),(\boldsymbol{v},q)) = (\xi\boldsymbol{v}|\nabla p)_{0,\Omega}$. One chooses in this case $(\boldsymbol{v}^\star,q^\star) = (\nabla(T_0p),0)$. Indeed, because $\xi$ is a symmetric tensor that fulfills (9), it holds that

$$a((0,p),(\boldsymbol{v}^\star,q^\star)) = (\xi\nabla(T_0p)|\nabla p)_{0,\Omega} \ge \alpha_0\,\|\nabla p\|^2_{0,\Omega} = \alpha_0\,\|(0,p)\|^2_{\mathbb{V}}.$$

(ii) Consider next that $p = 0$. Then $a((\boldsymbol{u},0),(\boldsymbol{v},q)) = (\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + (\xi\boldsymbol{u}|\nabla q)_{0,\Omega}$. Because $\boldsymbol{k_u} \in \boldsymbol{K}_N(\Omega,\xi)$ with $\mathbf{curl}\,\boldsymbol{k_u} = \mathbf{curl}\,\boldsymbol{u}$, one has now

$$a((\boldsymbol{u},0),(\boldsymbol{v},q)) = (\mathbf{curl}\,\boldsymbol{k_u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + (\xi\nabla p_{\boldsymbol{u}}|\nabla q)_{0,\Omega}.$$

One chooses in this case $(\boldsymbol{v}^\star,q^\star) = (\boldsymbol{k_u}, T_0 p_{\boldsymbol{u}})$. Indeed with the help of (9) and (18)

$$\begin{aligned} a((\boldsymbol{u},0),(\boldsymbol{v}^\star,q^\star)) &= \|\,\mathbf{curl}\,\boldsymbol{k_u}\|_{0,\Omega}^2 + (\xi\nabla p_{\boldsymbol{u}}|\nabla(T_0 p_{\boldsymbol{u}}))_{0,\Omega} \\ &\geq (C'_W)^{-2}\|\boldsymbol{k_u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}^2 + \alpha_0\,\|\nabla p_{\boldsymbol{u}}\|_{0,\Omega}^2 \\ &\geq \min((C'_W)^{-2},\alpha_0)\left(\|\boldsymbol{k_u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}^2 + \|\nabla p_{\boldsymbol{u}}\|_{0,\Omega}^2\right) \\ &\geq \gamma\,\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}^2 = \gamma\,\|(\boldsymbol{u},0)\|_{\mathbb{V}}^2, \end{aligned}$$

where $\gamma := \frac{1}{2}\min((C'_W)^{-2},\alpha_0) > 0$.
(iii) In the general case, let us consider a linear combination of the above, ie. $(\boldsymbol{v}^\star,q^\star) = (\beta_1\nabla(T_0 p)+\beta_2\boldsymbol{k_u}, \beta_2 T_0 p_{\boldsymbol{u}})$, for $\beta_1,\beta_2 \in \mathbb{R}$. Then using the property $\boldsymbol{k_u} \in \boldsymbol{K}_N(\Omega,\xi)$ with $\mathbf{curl}\,\boldsymbol{k_u} = \mathbf{curl}\,\boldsymbol{u}$, one finds

$$a((\boldsymbol{u},p),(\boldsymbol{v}^\star,q^\star)) = \beta_2\|\,\mathbf{curl}\,\boldsymbol{k_u}\|_{0,\Omega}^2 + \beta_2(\xi\nabla p_{\boldsymbol{u}}|\nabla(T_0 p_{\boldsymbol{u}}))_{0,\Omega} + \beta_1(\xi\nabla(T_0 p)|\nabla p)_{0,\Omega}.$$

Let $\gamma$ be as above. Choosing $\beta_1 = \beta_2 = \beta > 0$ leads to

$$\begin{aligned} a((\boldsymbol{u},p),(\boldsymbol{v}^\star,q^\star)) &\geq \beta\left((C'_W)^{-2}\|\boldsymbol{k_u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}^2 + \alpha_0\,\|\nabla p_{\boldsymbol{u}}\|_{0,\Omega}^2 + \alpha_0\,\|\nabla p\|_{0,\Omega}^2\right) \\ &\geq \beta\left(\gamma\,\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}^2 + \alpha_0\,\|\nabla p\|_{0,\Omega}^2\right) \\ &\geq \beta\,\gamma\,\|(\boldsymbol{u},p)\|_{\mathbb{V}}^2, \end{aligned}$$

because $\gamma < \alpha_0$. To conclude the proof, remark that the operator

$$\mathbb{T}\ :\ (\boldsymbol{u},p) \mapsto (\nabla(T_0 p) + \boldsymbol{\pi}_2\boldsymbol{u}, T_0(\pi_1\boldsymbol{u}))$$

belongs to $\mathcal{L}(\mathbb{V})$. □

The conclusions are summarized below.

**Corollary 1** *Let $\boldsymbol{f} \in \boldsymbol{H}_0^\Sigma(\mathrm{div}\,0;\Omega)$ and $g \in H^{-1}(\Omega)$ be given. Then there exists one, and only one, solution to $(\boldsymbol{u},p)$ to (14). In addition, it holds that $p = 0$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{-1,\Omega}$.*

## 4 Approximation by Nédélec's finite elements

For the ease of exposition[1], we assume that $\Omega$ and $\{\Omega_p\}_{p=1,\cdots,P}$ are Lipschitz polyhedra. We consider a family of simplicial meshes of $\Omega$, and we choose

[1] The results obtained in this paper carry over to curved polyhedra, that is domains with piecewise smooth boundaries (see eg. p. 81 in [3] for a precise definition). When dealing with the discretization by first-order edge finite elements in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one may use [32]. Respectively, when dealing with the discretization by Lagrange's first-order finite elements in $H_0^1(\Omega)$, one may use [37]. In particular, it is proven there that optimal interpolation properties hold, ie. one may recover up to $O(h)$ accuracy, provided the field to be interpolated is sufficiently smooth.

the Nédélec's first family of edge finite elements [49,48] to define finite dimensional subspaces $(\boldsymbol{V}_h)_h$ of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. So $\overline{\Omega}$ is triangulated by a shape regular family of meshes $(\mathcal{T}_h)_h$, made up of (closed) simplices, generically denoted by $K$. Each mesh is indexed by $h := \max_K h_K$ (the meshsize), where $h_K$ is the diameter of $K$. And meshes are conforming with respect to the partition $\{\Omega_p\}_{p=1,\cdots,P}$ induced by the coefficient $\xi$: namely, for all $h$ and all $K \in \mathcal{T}_h$, there exists $p \in \{1,\cdots,P\}$ such that $K \subset \overline{\Omega_p}$. Nédélec's $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming (first family, first-order) finite element spaces are then defined by

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \ : \ \boldsymbol{v}_{h|K} \in \mathcal{R}_1(K),\ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{R}_1(K)$ is the vector space of polynomials on $K$ defined by

$$\mathcal{R}_1(K) := \{\boldsymbol{v} \in \boldsymbol{P}_1(K) \ : \ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x},\ \boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^3\}.$$

To approximate the div-curl problem, we need to define a suitable approximation of elements of $H^1_0(\Omega)$. So we introduce finite dimensional subspaces $(M_h)_h$ of $H^1_0(\Omega)$. Lagrange's first-order finite element spaces are defined by

$$M_h := \{q_h \in H^1_0(\Omega) \ : \ q_{h|K} \in P_1(K),\ \forall K \in \mathcal{T}_h\}.$$

The discrete companion scalar problems are

$$\begin{cases} \textit{Find } s_h \in M_h \textit{ such that} \\ (\xi \nabla s_h | \nabla q_h)_{0,\Omega} = \langle g, q_h \rangle_{H^1_0(\Omega)},\ \forall q_h \in M_h. \end{cases} \tag{20}$$

For approximation purposes, one can use the Lagrange interpolation operator $\Pi_h^L$, or the Scott-Zhang interpolation operator $\Pi_h^{SZ}$. The latter allows one to interpolate any element of $H^1_0(\Omega)$, with values in $M_h$, at the expense of local interpolation operators that are not localized to each tetraedron, but are localized to the union of the tetrahedron and its neighbouring tetrahedra. We refer to [35] for details. Unless otherwise specified, we choose $\Pi_h^{grad} = \Pi_h^{SZ}$. Then, for all $h$, we introduce the finite dimensional subspaces $\mathbb{V}_h := \boldsymbol{V}_h \times M_h$ of $\mathbb{V}$. For $h$ given, the discrete variational formulation of the div-curl problem is

$$\begin{cases} \textit{Find } (\boldsymbol{u}_h, p_h) \in \mathbb{V}_h \textit{ such that} \\ a((\boldsymbol{u}_h, p_h),(\boldsymbol{v}_h, q_h)) = (\boldsymbol{f} | \,\mathbf{curl}\, \boldsymbol{v}_h)_{0,\Omega} - \langle g, q_h \rangle_{H^1_0(\Omega)},\ \forall (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h. \end{cases} \tag{21}$$

To obtain explicit error estimates for the div-curl problem, we shall use the interpolation of its solution $\boldsymbol{u}$. Let $\Pi_h^{curl}$ be the classical global Raviart-Thomas-Nédélec interpolant in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ with values in $\boldsymbol{V}_h$ [49]. We then denote by $\Pi_h^{div}$ the classical global Raviart-Thomas-Nédélec interpolation operator in $\boldsymbol{H}_0(\mathrm{div};\Omega)$ with values in $\boldsymbol{W}_h$ [55,49], where $(\boldsymbol{W}_h)_h$ are designed with the help of $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming, first-order finite element spaces:

$$\boldsymbol{W}_h := \{\boldsymbol{w}_h \in \boldsymbol{H}_0(\mathrm{div};\Omega) \ : \ \boldsymbol{w}_{h|K} \in \mathcal{D}_1(K),\ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{D}_1(K)$ is the vector space of polynomials on $K$ defined by

$$\mathcal{D}_1(K) := \{\boldsymbol{v} \in \boldsymbol{P}_1(K) \ : \ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x},\ \boldsymbol{a} \in \mathbb{R}^3,\ b \in \mathbb{R}\}.$$

Let us recall a few useful properties (see Chapter 5 in [48]). To start with,

**Proposition 3** *For all $h$, it holds that*

$$\nabla[M_h] \subset \boldsymbol{V}_h\,; \tag{22}$$
$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad \varPi_h^{curl}\boldsymbol{v}_h = \boldsymbol{v}_h\,; \tag{23}$$
$$\mathbf{curl}[\boldsymbol{V}_h] \subset \boldsymbol{W}_h\,; \tag{24}$$
$$\forall \boldsymbol{w}_h \in \boldsymbol{W}_h, \quad \varPi_h^{div}\boldsymbol{w}_h = \boldsymbol{w}_h\,; \tag{25}$$
$$\forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ s.t. } \varPi_h^{curl}\boldsymbol{v} \text{ exists, } \varPi_h^{div}(\mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}(\varPi_h^{curl}\boldsymbol{v}). \tag{26}$$

There are useful additional properties regarding $\varPi_h^{curl}$ listed below. Below, when we refer to piecewise-$H^s$ fields, the partition is understood as in (10).

**Proposition 4 (discrete exact sequence [49])** *Let $h$ be given, and let $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ that can be written as $\boldsymbol{v} = \nabla q$ in $\Omega$, for some $q \in H^1_0(\Omega)$. Then if $\varPi_h^{curl}\boldsymbol{v}$ is well-defined, there exists $q_h \in M_h$ such that $\varPi_h^{curl}\boldsymbol{v} = \nabla q_h$ in $\Omega$.*

**Proposition 5 (classical interpolation results)** *Assume that $\boldsymbol{v} \in \boldsymbol{PH}^{\mathtt{s}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{PH}^{\mathtt{s}'}(\Omega)$ for some $\mathtt{s} > 1/2$, $\mathtt{s}' > 0$. Then one can define $\varPi_h^{curl}\boldsymbol{v}$ and, in addition, one has the approximation result [7]:*

$$\|\boldsymbol{v} - \varPi_h^{curl}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},\mathtt{s}',1)}\{\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}'}(\Omega)}\}. \tag{27}$$

*Furthermore, if $\mathbf{curl}\,\boldsymbol{v}$ is piecewise constant on $\mathcal{T}_h$, one has the improved approximation result (cf. Theorem 5.41 in [48]):*

$$\|\boldsymbol{v} - \varPi_h^{curl}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},1)}\{\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}\}.$$

*Remark 5* When $\Omega_2$ is a domain of $\mathbb{R}^2$, note that one can define the Raviart-Thomas-Nédélec interpolant of a field $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl};\Omega_2)$ as soon as $\boldsymbol{v} \in \boldsymbol{PH}^{\mathtt{s}}(\Omega_2)$ for some $\mathtt{s} > 0$ (there is no requirement on the regularity of $\mathrm{curl}\,\boldsymbol{v}$). This result is proven in [6] for fields in $\boldsymbol{H}(\mathrm{div};\Omega_2)$, and it obviously carries over to fields in $\boldsymbol{H}(\mathrm{curl};\Omega_2)$ by appropriate coordinates transform. Further, one has the approximation result:

$$\|\boldsymbol{v} - \varPi_h^{curl}\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{curl};\Omega_2)} \lesssim h^{\min(\mathtt{s},1)}\{\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega_2)} + \|\,\mathrm{curl}\,\boldsymbol{v}\|_{0,\Omega_2}\}.$$

We recall that, according to (11), $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{PH}^{\tau_1}(\Omega)$ for $\tau_1 > 0$. Hence, to guarantee that $\varPi_h^{curl}$ can be applied to the solution $\boldsymbol{u}$ of the div-curl problem, one must have $\boldsymbol{u} \in \boldsymbol{PH}^{\mathtt{s}}(\Omega)$ for some $\mathtt{s} > 1/2$. To evaluate the exponent $\mathtt{s}$ *a priori*, we use the following decomposition (see Lemma 2.4 of [41]).

**Proposition 6** *There exist operators*

$$\boldsymbol{P} \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega), \boldsymbol{H}^1(\Omega)), \quad Q \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega), H^1_0(\Omega)),$$

*such that*

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \quad \boldsymbol{v} = \boldsymbol{P}\boldsymbol{v} + \nabla(Q\boldsymbol{v}). \tag{28}$$

This yields some useful results for elements of $\boldsymbol{K}_N(\Omega,\xi)$.

**Corollary 2** *The a priori regularity of elements of $\boldsymbol{K}_N(\Omega,\xi)$ is governed by the imbedding:*

$$\boldsymbol{K}_N(\Omega,\xi) \subset \cap_{\mathsf{s}\in[0,\tau_{Dir})} \boldsymbol{PH}^{\mathsf{s}}(\Omega).$$

*Moreover, for all $\mathsf{s} \in [0, \tau_{Dir})$,*

$$\forall \boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\xi), \quad \|\boldsymbol{k}\|_{\boldsymbol{PH}^{\mathsf{s}}(\Omega)} \lesssim \|\,\mathbf{curl}\,\boldsymbol{k}\|_{0,\Omega}. \tag{29}$$

*Proof* Let $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\xi)$. According to proposition 6, one can write $\boldsymbol{k} = \boldsymbol{k}^\star + \nabla s_{\boldsymbol{k}}$ with $\boldsymbol{k}^\star \in \boldsymbol{H}^1(\Omega)$, resp. $s_{\boldsymbol{k}} \in H^1_0(\Omega)$, and it holds that $\|\boldsymbol{k}^\star\|_{1,\Omega} + \|s_{\boldsymbol{k}}\|_{H^1_0(\Omega)} \lesssim \|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. In particular, $\mathrm{div}(\xi\nabla s_{\boldsymbol{k}}) = -\,\mathrm{div}\,\xi\boldsymbol{k}^\star$ in $\Omega$, so $s_{\boldsymbol{k}}$ solves the modified scalar problem (12) with data $(0,\boldsymbol{k}^\star)$. Thanks to the shift theorem, we know that, for all $\mathsf{s} \in [0,\tau_{Dir})$, $s_{\boldsymbol{k}}$ belongs to $PH^{1+\mathsf{s}}(\Omega)$, with the bound $\|s_{\boldsymbol{k}}\|_{PH^{1+\mathsf{s}}(\Omega)} \lesssim \|\boldsymbol{k}^\star\|_{1,\Omega}$. Using the triangle inequality, we conclude that

$$\forall \mathsf{s} \in [0,\tau_{Dir}),\ \boldsymbol{k} \in \boldsymbol{PH}^{\mathsf{s}}(\Omega),\ \text{and } \|\boldsymbol{k}\|_{\boldsymbol{PH}^{\mathsf{s}}(\Omega)} \lesssim \|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

This proves the first part of the corollary. Using finally theorem 1 on the equivalence of norms in $\boldsymbol{K}_N(\Omega,\xi)$, we conclude that (29) holds. □

One has the first interpolation result below.

**Proposition 7 (classical interpolation of $\boldsymbol{u}$)** *Let $\boldsymbol{u}$ be the solution to the div-curl problem. Let the extra-regularity of the data $(\boldsymbol{f}, g)$ be as in (11) with $\tau_1, \tau_2 > 0$ given. If $\tau_2 > 1/2$, and if the limit regularity exponent $\tau_{Dir} > 1/2$, one can define $\Pi_h^{curl}\boldsymbol{u}$, and moreover one has the approximation result, for all $\mathsf{s} \in (1/2, \min(\tau_2, \tau_{Dir}))$,*

$$\|\boldsymbol{u} - \Pi_h^{curl}\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathsf{s},\tau_1)}\{\|g\|_{-1+\mathsf{s},\Omega} + \|\boldsymbol{f}\|_{\boldsymbol{PH}^{\tau_1}(\Omega)}\}.$$

*Proof* Let $\mathsf{s} \in (1/2, \min(\tau_2, \tau_{Dir}))$. Let us write $\boldsymbol{u} = \boldsymbol{u}^\star + \nabla s_{\boldsymbol{u}}$ with $\boldsymbol{u}^\star = \boldsymbol{P}\boldsymbol{u}$, resp. $s_{\boldsymbol{u}} = Q\boldsymbol{u}$. It holds that $\|\boldsymbol{u}^\star\|_{1,\Omega} + \|s_{\boldsymbol{u}}\|_{H^1_0(\Omega)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
By definition, $s_{\boldsymbol{u}}$ solves the modified scalar problem (12) with data $(g, \boldsymbol{u}^\star)$. Thanks to the shift theorem, $s_{\boldsymbol{u}} \in PH^{1+\mathsf{s}}(\Omega)$, with the bound

$$\|s_{\boldsymbol{u}}\|_{PH^{1+\mathsf{s}}(\Omega)} \lesssim \|g\|_{-1+\mathsf{s},\Omega} + \|\boldsymbol{u}^\star\|_{1,\Omega}.$$

Using the triangle inequality, the bound on $\|\boldsymbol{u}^\star\|_{1,\Omega}$ and finally corollary 1 leads to

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{PH}^{\mathsf{s}}(\Omega)} &\le \|\boldsymbol{u}^\star\|_{\boldsymbol{PH}^{\mathsf{s}}(\Omega)} + \|s_{\boldsymbol{u}}\|_{PH^{1+\mathsf{s}}(\Omega)} \lesssim \|\boldsymbol{u}^\star\|_{1,\Omega} + \|g\|_{-1+\mathsf{s},\Omega} \\ &\lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|g\|_{-1+\mathsf{s},\Omega} \lesssim \|g\|_{-1+\mathsf{s},\Omega} + \|\boldsymbol{f}\|_{0,\Omega}.\end{aligned}$$

By definition, $\mathsf{s} < 1$, and so the claim follows with the help of (27). □

In practice however, it may happen that the field to be interpolated, eg. the solution $\boldsymbol{u}$, does not belong to $\cup_{\mathtt{s}>1/2}\boldsymbol{PH}^{\mathtt{s}}(\Omega)$. In the classical case, the occurence of such a situation is explained in Section 7 of [30]. In the interface case, this can be inferred from the results obtained in [11,14].
On the other hand, to interpolate such a *low regularity field*, one may still choose the quasi-interpolation operator of [36], or the combined interpolation operator of [28]. We choose the latter, and follow Section 4.2 in [28]. To get a definition for the combined interpolation operator, denoted by $\Pi_h^{comb}$, one needs to be able to split low regularity fields defined on $\Omega$. To that end, we apply proposition 6.

**Definition 1 (combined interpolation operator)** Let $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, with $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^{\mathtt{s}'}(\Omega)$ for some $\mathtt{s}' > 0$. We define

$$\Pi_h^{comb}\boldsymbol{v} := \Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v}) + \nabla(\Pi_h^{grad}(Q\boldsymbol{v})).$$

*Remark 6* To interpolate low regularity fields (of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$) via the combined interpolation approach, one has to provide a decomposition like in proposition 6. In proposition 6, the decomposition is independent of $\xi$. In this sense, it is less involved than the $\xi$-dependent decomposition that is used in Proposition 4 of [28]. However, both decompositions yield identical approximation results.

Then, the approximation results for the combined interpolation are a straightforward consequence of the available results for $\Pi_h^{curl}$ and $\Pi_h^{grad}$.

**Proposition 8 (combined interpolation results)** *Let $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, with $Q\boldsymbol{v} \in PH^{1+\mathtt{s}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{PH}^{\mathtt{s}'}(\Omega)$ for some $\mathtt{s} \geq 0$, $\mathtt{s}' > 0$. One has the approximation result:*

$$\|\boldsymbol{v} - \Pi_h^{comb}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},\mathtt{s}',1)}\{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|Q\boldsymbol{v}\|_{PH^{1+\mathtt{s}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}'}(\Omega)}\}. \quad (30)$$

*Furthermore, if $\mathbf{curl}\,\boldsymbol{v}$ is piecewise constant on $\mathcal{T}_h$, one has the improved approximation result:*

$$\|\boldsymbol{v} - \Pi_h^{comb}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},1)}\{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|Q\boldsymbol{v}\|_{PH^{1+\mathtt{s}}(\Omega)}\}.$$

Together with this definition of the combined interpolation operator, we have the results below, to be compared with the well-known results (23) and (26), and to proposition 7, for the classical interpolation operator.

**Proposition 9** *For all $h$, it holds that*

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \exists q_h \in M_h, \quad \Pi_h^{comb}\boldsymbol{v}_h = \boldsymbol{v}_h + \nabla q_h\,; \quad (31)$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ s.t. } \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^{\mathtt{s}'}(\Omega) \text{ for some } \mathtt{s}' > 0, \quad \Pi_h^{div}(\mathbf{curl}\,\boldsymbol{v}) = \mathbf{curl}(\Pi_h^{comb}\boldsymbol{v}). \quad (32)$$

*Proof* Let $\boldsymbol{v}_h \in \boldsymbol{V}_h$. We note that because $\boldsymbol{v}_h$ is piecewise smooth on $\mathcal{T}_h$, one has $\boldsymbol{v}_h, \mathbf{curl}\,\boldsymbol{v}_h \in \boldsymbol{PH}^{\mathtt{t}}(\Omega)$ for all $\mathtt{t} \in [0, 1/2)$. Hence $\Pi_h^{comb}\boldsymbol{v}_h$ is well-defined according to definition 1. If we write $\boldsymbol{v}_h = (\boldsymbol{v}_h)^\star + \nabla s_{\boldsymbol{v}_h}$, with $(\boldsymbol{v}_h)^\star = \boldsymbol{P}\boldsymbol{v}_h$, resp. $s_{\boldsymbol{v}_h} = Q\boldsymbol{v}_h$, we have $\Pi_h^{comb}\boldsymbol{v}_h := \Pi_h^{curl}(\boldsymbol{v}_h)^\star + \nabla(\Pi_h^{grad} s_{\boldsymbol{v}_h})$.
On the other hand, $\nabla s_{\boldsymbol{v}_h} = \boldsymbol{v}_h - (\boldsymbol{v}_h)^\star$. Since $\Pi_h^{curl}(\boldsymbol{v}_h - (\boldsymbol{v}_h)^\star)$ is well-defined, so is $\Pi_h^{curl}(\nabla s_{\boldsymbol{v}_h})$ and, according to proposition 4, there exists $q_h' \in M_h$ such that $\Pi_h^{curl}(\nabla s_{\boldsymbol{v}_h}) = \nabla q_h'$. Applying now $\Pi_h^{curl}$ to $(\boldsymbol{v}_h)^\star = \boldsymbol{v}_h - \nabla s_{\boldsymbol{v}_h}$, it follows that

$$\Pi_h^{curl}(\boldsymbol{v}_h)^\star = \Pi_h^{curl}\boldsymbol{v}_h - \nabla q_h' = \boldsymbol{v}_h - \nabla q_h',$$

where the second equality now follows from (23). One concludes that

$$\Pi_h^{comb}\boldsymbol{v}_h := \boldsymbol{v}_h + \nabla(\Pi_h^{grad} s_{\boldsymbol{v}_h} - q_h'),$$

which is precisely (31) with $q_h = \Pi_h^{grad} s_{\boldsymbol{v}_h} - q_h'$.
To check (32), let $\boldsymbol{v}$ be split as $\boldsymbol{v} = \boldsymbol{P}\boldsymbol{v} + \nabla(Q\boldsymbol{v})$. Since $\mathbf{curl}(\boldsymbol{P}\boldsymbol{v}) \in \boldsymbol{H}^{\mathtt{s}'}(\Omega)$, according to proposition 5 one may apply (26) to $\boldsymbol{P}\boldsymbol{v}$, leading to $\Pi_h^{div}(\mathbf{curl}(\boldsymbol{P}\boldsymbol{v})) = \mathbf{curl}(\Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v}))$. On the other hand, because of the definition 1 of $\Pi_h^{comb}\boldsymbol{v} = \Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v}) + \nabla(\Pi_h^{grad}(Q\boldsymbol{v}))$ one has

$$\mathbf{curl}(\Pi_h^{curl}(\boldsymbol{P}\boldsymbol{v})) = \mathbf{curl}(\Pi_h^{comb}\boldsymbol{v} - \nabla(\Pi_h^{grad}(Q\boldsymbol{v}))) = \mathbf{curl}(\Pi_h^{comb}\boldsymbol{v}).$$

Using finally the equality $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}(\boldsymbol{P}\boldsymbol{v})$ leads to the claim. □

**Proposition 10 (combined interpolation of $\boldsymbol{u}$)** *Let $\boldsymbol{u}$ be the solution to the div-curl problem. Let the extra-regularity of the data $(\boldsymbol{f}, g)$ be as in (11) with $\tau_1, \tau_2 > 0$ given. One can define $\Pi_h^{comb}\boldsymbol{u}$, and moreover one has the approximation result, for all $\mathtt{s} \in [0, \min(\tau_2, \tau_{Dir}))$,*

$$\|\boldsymbol{u} - \Pi_h^{comb}\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},\tau_1)}\{\|g\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{f}\|_{\boldsymbol{PH}^{\tau_1}(\Omega)}\}.$$

*Proof* Let $\mathtt{s} \in [0, \min(\tau_2, \tau_{Dir}))$ ; because $\tau_{Dir} \le 1$, one has $\mathtt{s} < 1$.
According to proposition 6, we may write $\boldsymbol{u} = \boldsymbol{u}^\star + \nabla s_{\boldsymbol{u}}$ with $\boldsymbol{u}^\star \in \boldsymbol{H}^1(\Omega)$, $s_{\boldsymbol{u}} \in H_0^1(\Omega)$, and $\|\boldsymbol{u}^\star\|_{1,\Omega} + \|s_{\boldsymbol{u}}\|_{H_0^1(\Omega)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. By construction, $s_{\boldsymbol{u}}$ solves the modified scalar problem (12) with data $(g, \boldsymbol{u}^\star)$. But $\mathtt{s} < \min(\tau_2, \tau_{Dir})$ so, thanks to the shift theorem, $s_{\boldsymbol{u}} \in PH^{1+\mathtt{s}}(\Omega)$, with the bound $\|s_{\boldsymbol{u}}\|_{PH^{1+\mathtt{s}}(\Omega)} \lesssim \|g\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{u}^\star\|_{1,\Omega}$. As before, using corollary 1 for the last inequality below, we find

$$\|s_{\boldsymbol{u}}\|_{PH^{1+\mathtt{s}}(\Omega)} \lesssim \|g\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|g\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{f}\|_{0,\Omega}.$$

Since $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f} \in \boldsymbol{PH}^{\tau_1}(\Omega)$, one can define $\Pi_h^{comb}\boldsymbol{u}$. Using (30) and corollary 1 once more, one finds now

$$\begin{aligned}\|\boldsymbol{u} - \Pi_h^{comb}\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} &\lesssim h^{\min(\mathtt{s},\tau_1)}\{\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|s_{\boldsymbol{u}}\|_{PH^{1+\mathtt{s}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{PH}^{\tau_1}(\Omega)}\}\\ &\lesssim h^{\min(\mathtt{s},\tau_1)}\{\|g\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{f}\|_{\boldsymbol{PH}^{\tau_1}(\Omega)}\},\end{aligned}$$

which is the desired estimate. □

In particular, we conclude that even when the solution $\boldsymbol{u}$ to the div-curl problem does not belong to $\cup_{\mathtt{s}>1/2}\boldsymbol{PH}^{\mathtt{s}}(\Omega)$, one may use the combined interpolation operator and still obtain "best" interpolation error.

## 5 Case of a "full" T-coercivity operator

We assume in this section that we have at hand a "full" T-coercivity involution operator $T_0$ to solve the companion scalar problem (3) (see section A.1), and that the meshes are T-conform, such that (70)-(71) are fulfilled, with consequences listed in section A.2.

*Remark 7* Properties (70) are always true in the classical case ($T_0 = \mathbb{I}_{H_0^1(\Omega)}$).

Define, for any $h$,

$$\boldsymbol{K}_h(\xi) := \{\boldsymbol{v}_h \in \boldsymbol{V}_h \ : \ (\xi\boldsymbol{v}_h|\nabla q_h)_{0,\Omega} = 0, \ \forall q_h \in M_h\}. \tag{33}$$

**Proposition 11** *Assume that (70) holds. For all $h$, one has the direct sum*

$$\boldsymbol{V}_h = \nabla[M_h] \oplus \boldsymbol{K}_h(\xi). \tag{34}$$

*In the classical case, the sum is orthogonal with respect to the inner product $(\cdot,\cdot)_{class}$.*

*Proof* Let $h$ be given. Thanks to (22), we know that $\nabla[M_h] + \boldsymbol{K}_h(\xi)$ is a subset of $\boldsymbol{V}_h$. Then for $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and because the discrete scalar problem (20) is well-posed, there exists one, and only one, $p_{\boldsymbol{v}_h} \in M_h$ such that

$$(\xi\nabla p_{\boldsymbol{v}_h}|\nabla q_h)_{0,\Omega} = (\xi\boldsymbol{v}_h|\nabla q_h)_{0,\Omega}, \ \forall q_h \in M_h. \tag{35}$$

And one has

$$\boldsymbol{k}_{\boldsymbol{v}_h} = \boldsymbol{v}_h - \nabla p_{\boldsymbol{v}_h} \in \boldsymbol{K}_h(\xi), \tag{36}$$

so $\boldsymbol{V}_h = \nabla[M_h] + \boldsymbol{K}_h(\xi)$. Using (70), the fact that the sum is direct is derived exactly as in the continuous case (see the proof of proposition 2).
In the classical case, let $q_h \in M_h$, $\boldsymbol{k}_h \in \boldsymbol{K}_h(\xi)$: $(\nabla q_h, \boldsymbol{k}_h)_{class} = (\xi\nabla q_h|\boldsymbol{k}_h)_{0,\Omega} = 0$, because $\nabla q_h$ is curl-free, and $\xi$ is a symmetric tensor field. □

For all $h$, we can use the splitting (34) and the explicit definitions (35)-(36) to introduce the operators

$$\pi_{1h} \ : \ \begin{cases} \boldsymbol{V}_h \to M_h \\ \boldsymbol{v}_h \mapsto p_{\boldsymbol{v}_h} \end{cases}, \quad \boldsymbol{\pi}_{2h} \ : \ \begin{cases} \boldsymbol{V}_h \to \boldsymbol{K}_h(\xi) \\ \boldsymbol{v}_h \mapsto \boldsymbol{k}_{\boldsymbol{v}_h} \end{cases}. \tag{37}$$

In other words, one may write, for all $h$, for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\boldsymbol{v}_h = \nabla(\pi_{1h}\boldsymbol{v}_h) + \boldsymbol{\pi}_{2h}\boldsymbol{v}_h$. Also, one has for all $h$, $(\boldsymbol{\pi}_{2h})^2 = \boldsymbol{\pi}_{2h}$.

**Proposition 12** *Assume that (70) holds. The continuity moduli of the operators $(\pi_{1h})_h$, $(\boldsymbol{\pi}_{2h})_h$ are bounded independently of $h$.*

*Proof* This is obvious in the classical case, because the sum is orthogonal with respect to the inner product $(\cdot,\cdot)_{class}$.
In the interface case, given $h$ and $\boldsymbol{v}_h \in \boldsymbol{V}_h$, one has according to (70) and (35)

$$\begin{aligned}\alpha_0' \|\nabla(\pi_{1h}\boldsymbol{u}_h)\|_{0,\Omega}^2 &\le (\xi\nabla(\pi_{1h}\boldsymbol{u}_h)|\nabla(T_0(\pi_{1h}\boldsymbol{u}_h)))_{0,\Omega} = (\xi\boldsymbol{u}_h|\nabla(T_0(\pi_{1h}\boldsymbol{u}_h)))_{0,\Omega}\\ &\le \|\xi\boldsymbol{u}_h\|_{0,\Omega}\|\nabla(T_0(\pi_{1h}\boldsymbol{u}_h))\|_{0,\Omega}\\ &\le \|\xi\boldsymbol{u}_h\|_{0,\Omega}\|T_0\|_{\mathcal{L}(H_0^1(\Omega))}\|\nabla(\pi_{1h}\boldsymbol{u}_h)\|_{0,\Omega},\end{aligned}$$

so that

$$\|\nabla(\pi_{1h}\boldsymbol{u}_h)\|_{0,\Omega} \le (\alpha_0')^{-1}\|\xi\|_{\infty,\Omega}\|T_0\|_{\mathcal{L}(H_0^1(\Omega))}\|\boldsymbol{u}_h\|_{0,\Omega}.$$

And then

$$\|\boldsymbol{\pi}_{2h}\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le (1+(\alpha_0')^{-1}\|\xi\|_{\infty,\Omega}\|T_0\|_{\mathcal{L}(H_0^1(\Omega))})\|\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},$$

so the claim follows. □

Let us now establish a uniform discrete inf-sup condition, for the discrete div-curl problems, by applying the T-coercivity approach. Namely, we design explicit (discrete) operators $\mathbb{T}_h$ that yield T-coercivity for the discrete problems (21). The first step is as follows.

**Proposition 13** *Assume that (70) holds. For all $h$, let $\mathbb{T}_h \in \mathcal{L}(\mathbb{V}_h)$ be defined as*

$$\mathbb{T}_h \;:\; (\boldsymbol{u}_h,p_h) \mapsto (\nabla(T_0p_h)+\boldsymbol{\pi}_{2h}\boldsymbol{u}_h, T_0(\pi_{1h}\boldsymbol{u}_h)).$$

*Then it holds*

$$\begin{aligned}&\forall h,\ \forall(\boldsymbol{u}_h,p_h)\in\mathbb{V}_h,\quad a((\boldsymbol{u}_h,p_h),\mathbb{T}_h(\boldsymbol{u}_h,p_h)) \ge\\ &\min(1,\alpha_0')\left(\|\mathbf{curl}(\boldsymbol{\pi}_{2h}\boldsymbol{u}_h)\|_{0,\Omega}^2 + \|\nabla(\pi_{1h}\boldsymbol{u}_h)\|_{0,\Omega}^2 + \|\nabla p_h\|_{0,\Omega}^2\right). \quad (38)\end{aligned}$$

*Moreover, the continuity modulus of $(\mathbb{T}_h)_h$ is bounded independently of $h$.*

*Proof* Using the definition of $\boldsymbol{K}_h(\xi)$ and the fact that $\xi$ is a symmetric tensor field, one computes

$$\begin{aligned}&a((\boldsymbol{u}_h,p_h),\mathbb{T}_h(\boldsymbol{u}_h,p_h))\\ &= a((\nabla(\pi_{1h}\boldsymbol{u}_h)+\boldsymbol{\pi}_{2h}\boldsymbol{u}_h,p_h),(\nabla(T_0p_h)+\boldsymbol{\pi}_{2h}\boldsymbol{u}_h,T_0(\pi_{1h}\boldsymbol{u}_h)))\\ &= \|\mathbf{curl}(\boldsymbol{\pi}_{2h}\boldsymbol{u}_h)\|_{0,\Omega}^2 + (\xi\nabla(\pi_{1h}\boldsymbol{u}_h)|\nabla(T_0(\pi_{1h}\boldsymbol{u}_h)))_{0,\Omega} + (\xi\nabla(T_0p_h)|\nabla p_h)_{0,\Omega}\\ &\ge \|\mathbf{curl}(\boldsymbol{\pi}_{2h}\boldsymbol{u}_h)\|_{0,\Omega}^2 + \alpha_0'\left(\|\nabla(\pi_{1h}\boldsymbol{u}_h)\|_{0,\Omega}^2 + \|\nabla p_h\|_{0,\Omega}^2\right),\end{aligned}$$

which proves (38).
The uniform bound on the continuity modulus of $(\mathbb{T}_h)_h$ is a straightforward consequence of proposition 12. □

To obtain a lower bound with $\|(\boldsymbol{u}_h,p_h)\|_{\mathbb{V}}^2$ in the right-hand side of (38), one has to check that $\boldsymbol{k}_h \mapsto \|\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$ defines a norm on $\boldsymbol{K}_h(\xi)$. And, if the answer is positive, whether this norm of uniformly equivalent in $h$ (ie. with constants that are independent of $h$) to the $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$-norm on $\boldsymbol{K}_h(\xi)$.
So the second step is the ...

**Proposition 14** *Assume that (70) holds. For all $h$, $\boldsymbol{k}_h \mapsto \|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$ defines a norm on $\boldsymbol{K}_h(\xi)$.*

*Proof* Let $\boldsymbol{k}_h \in \boldsymbol{K}_h(\xi)$ be such that $\mathbf{curl}\,\boldsymbol{k}_h = 0$ in $\Omega$. Since the boundary $\partial\Omega$ is connected, we get from Theorem 3.3.9 of [3] that there exists $q \in H^1_0(\Omega)$ such that $\boldsymbol{k}_h = \nabla q$ in $\Omega$. Since $\Pi_h^{curl}\boldsymbol{k}_h$ is well-defined (and equal to $\boldsymbol{k}_h$), we know from proposition 4 that there exists $q_h \in M_h$ such that $\Pi_h^{curl}\boldsymbol{k}_h = \nabla q_h$ in $\Omega$. In other words, $\boldsymbol{k}_h = \Pi_h^{curl}\boldsymbol{k}_h = \nabla q_h \in \nabla[M_h]$. So one has $\boldsymbol{k}_h \in \nabla[M_h] \cap \boldsymbol{K}_h(\xi)$ which reduces to $\{0\}$ according to proposition 11: this proves the result. □

*Remark 8* Propositions 13 and 14 already yield incomplete, yet promising results. As a matter of fact, it follows from the above that, for all $h$, the discrete problem (21) is well-posed. Also, its solution $(\boldsymbol{u}_h, p_h)$ is such that $p_h = 0$: indeed, using $(\nabla(T_0 p_h), 0)$ as a test function in (21), one finds that $(\xi\nabla(T_0 p_h)|\nabla p_h)_{0,\Omega} = 0$, so $p_h = 0$ (cf. (70) plus symmetry of $\xi$).

**Theorem 3** *Assume that (70) holds. Then*

$$\exists C_W^\star > 0,\ \forall h,\ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\xi), \quad \|\boldsymbol{k}_h\|_{0,\Omega} \le C_W^\star \,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}. \tag{39}$$

*Remark 9* The proof is split into three parts. The first part is well-known, see eg. chapter 7 in [48]. To the author's knowledge, the other two parts are new.

*Proof* We consider three cases:

(i) classical case with use of $\Pi_h^{curl}$;
(ii) classical case with use of $\Pi_h^{comb}$;
(iii) interface case.

Let $\boldsymbol{k}_h \in \boldsymbol{K}_h(\xi)$ be given. We know from corollary 1 that there exists one, and only one, solution to the div-curl problem with data $(\boldsymbol{f}, g) = (\mathbf{curl}\,\boldsymbol{k}_h, 0)$, as $\mathbf{curl}\,\boldsymbol{k}_h \in \boldsymbol{H}_0^\Sigma(\mathrm{div}\,0;\Omega)$ (cf. Theorem 6.1.4 in [3]). We denote its solution by $\boldsymbol{k}$. By definition, $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\xi)$. We observe that

$$\begin{aligned}\|\boldsymbol{k}_h\|_{0,\Omega} &\le \|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega} + \|\boldsymbol{k}\|_{0,\Omega}\\ &\le \|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega} + C_W\,\|\,\mathbf{curl}\,\boldsymbol{k}\|_{0,\Omega}\\ &= \|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega} + C_W\,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}\end{aligned} \tag{40}$$

thanks to the triangle inequality, (17) and the definition of $\boldsymbol{k}$. To obtain (39), one has now to bound $\|\boldsymbol{k}_h - \boldsymbol{k}\|_{0,\Omega}$ by $\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$, uniformly with respect to $h$.

(i) In the classical case, and assuming one can apply the classical interpolation operator $\Pi_h^{curl}$ to $\boldsymbol{k}$; that is, if one assumes that $\tau_{Dir} \in (1/2, 1]$ (see corollary 2 and proposition 5). The classic proof (see eg. chapter 7 in [48]) allows to bound uniformly $\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega}$ by $\|\boldsymbol{k} - \Pi_h^{curl}\boldsymbol{k}\|_{0,\Omega}$ as follows. Thanks to the assumption (2) on $\xi$,

$$\begin{aligned}\xi_-\,\|\boldsymbol{k} - \boldsymbol{k}_h\|^2_{0,\Omega} &\le (\xi(\boldsymbol{k} - \boldsymbol{k}_h)|\boldsymbol{k} - \boldsymbol{k}_h)_{0,\Omega}\\ &= (\xi(\boldsymbol{k} - \boldsymbol{k}_h)|\boldsymbol{k} - \Pi_h^{curl}\boldsymbol{k})_{0,\Omega}\\ &\quad + (\xi(\boldsymbol{k} - \boldsymbol{k}_h)|\Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h)_{0,\Omega}.\end{aligned} \tag{41}$$

Let us study $\Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h$.
First, we remark that $\mathbf{curl}(\Pi_h^{curl}\boldsymbol{k}) = \Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k})$ according to (26). Next, we express $\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k})$ in terms of $\mathbf{curl}\,\boldsymbol{k}_h$. By definition of $\boldsymbol{k}$, it holds that $\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k}) = \Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k}_h)$, so using (24)-(25), we get that $\Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k}) = \mathbf{curl}\,\boldsymbol{k}_h$. In other words, $\mathbf{curl}(\Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h) = 0$ in $\Omega$. According to Theorem 3.3.9. in [3], there exists $q \in H_0^1(\Omega)$ such that $\Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h = \nabla q$ in $\Omega$. Moreover, $\Pi_h^{curl}(\Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h)$ is well-defined (and equal to $\Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h$), so we conclude from proposition 4 that there exists $q_h \in M_h$ such that $\nabla q_h\big( = \Pi_h^{curl}(\Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h)\big) = \Pi_h^{curl}\boldsymbol{k} - \boldsymbol{k}_h$.
Hence the last term in (41) is equal to $(\xi(\boldsymbol{k}-\boldsymbol{k}_h)|\nabla q_h)_{0,\Omega}$: it vanishes, because $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\xi)$, resp. $\boldsymbol{k}_h \in \boldsymbol{K}_h(\xi)$. Using Cauchy-Schwarz inequality, (41) now yields

$$\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega} \le \frac{\xi_+}{\xi_-}\,\|\boldsymbol{k} - \Pi_h^{curl}\boldsymbol{k}\|_{0,\Omega}.$$

Finally, using proposition 5, the bound (29), (18) and the definition of $\boldsymbol{k}$, we find that, for any $\mathtt{s} \in (1/2, \tau_{Dir})$ it holds $\|\boldsymbol{k} - \Pi_h^{curl}\boldsymbol{k}\|_{0,\Omega} \lesssim h^{\mathtt{s}}\,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$. Since $h \lesssim \mathrm{diam}(\Omega)$, using (40) we conclude that the uniform bound (39) holds.

(ii) In the classical case, and if $\tau_{Dir} \in (0, 1/2]$, one can still use the combined interpolation operator $\Pi_h^{comb}$ on $\boldsymbol{k}$ (see proposition 8). One finds now

$$\xi_-\,\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega}^2 \le (\xi(\boldsymbol{k} - \boldsymbol{k}_h)|\boldsymbol{k} - \Pi_h^{comb}\boldsymbol{k})_{0,\Omega} + (\xi(\boldsymbol{k} - \boldsymbol{k}_h)|\Pi_h^{comb}\boldsymbol{k} - \boldsymbol{k}_h)_{0,\Omega}. \quad (42)$$

According to property (32) for the combined interpolation operator, we know that $\mathbf{curl}(\Pi_h^{comb}\boldsymbol{k}) = \Pi_h^{div}(\mathbf{curl}\,\boldsymbol{k})$. Proceeding as before (cf. the proof for case (i)), we find now that $\Pi_h^{comb}\boldsymbol{k} - \boldsymbol{k}_h$ belongs to $\nabla[M_h]$, so the last term in (42) vanishes too. Using Cauchy-Schwarz inequality, it yields

$$\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega} \le \frac{\xi_+}{\xi_-}\,\|\boldsymbol{k} - \Pi_h^{comb}\boldsymbol{k}\|_{0,\Omega}.$$

Thanks to proposition 8, for any $\mathtt{s} \in (0, \tau_{Dir})$ it holds that

$$\|\boldsymbol{k} - \Pi_h^{comb}\boldsymbol{k}\|_{0,\Omega} \lesssim h^{\mathtt{s}}\{\|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|Q\boldsymbol{k}\|_{PH^{1+\mathtt{s}}(\Omega)}\}.$$

On the other hand, we know that $\|Q\boldsymbol{k}\|_{PH^{1+\mathtt{s}}(\Omega)} \lesssim \|\boldsymbol{k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$ (see the proof of corollary 2), so using (18) and the definition of $\boldsymbol{k}$, for any $\mathtt{s} \in (0, \tau_{Dir})$, it actually holds that $\|\boldsymbol{k} - \Pi_h^{comb}\boldsymbol{k}\|_{0,\Omega} \lesssim h^{\mathtt{s}}\,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$. We conclude as in case (i).

(iii) In the interface case, we cannot proceed like in the classical case, indeed the mapping $(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\xi\boldsymbol{v}|\boldsymbol{w})_{0,\Omega}$ is not an inner product on $\boldsymbol{L}^2(\Omega)$ anymore. On the other hand, we know that $\mathbf{curl}(\boldsymbol{k} - \boldsymbol{k}_h) = 0$ in $\Omega$ by definition of $\boldsymbol{k}$, so according to Theorem 3.3.9. in [3], there exists $q \in H_0^1(\Omega)$ such that $\boldsymbol{k} - \boldsymbol{k}_h = \nabla q$ in $\Omega$. Thus, using (9), we have the bound

$$\alpha_0\,\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega}^2 = \alpha_0\,\|\nabla q\|_{0,\Omega}^2 \le (\xi\nabla q|\nabla(T_0 q))_{0,\Omega} = (\xi(\boldsymbol{k} - \boldsymbol{k}_h)|\nabla(T_0 q))_{0,\Omega}.$$

As in cases (i)-(ii), we note that $(\xi(\boldsymbol{k}-\boldsymbol{k}_h)|\nabla q'_h)_{0,\Omega}=0$, for all $q'_h\in M_h$. Or equivalently, if we recall (70) and its consequence $T_0[M_h]=M_h$: $(\xi(\boldsymbol{k}-\boldsymbol{k}_h)|\nabla(T_0 q_h))_{0,\Omega}=0$, for all $q_h\in M_h$. Hence, it holds that, for all $q_h\in M_h$:

$$\begin{aligned}\alpha_0\,\|\boldsymbol{k}-\boldsymbol{k}_h\|^2_{0,\Omega} &\le (\xi(\boldsymbol{k}-\boldsymbol{k}_h)|\nabla(T_0(q-q_h)))_{0,\Omega}\\ &\le \xi_+\,\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\,\|\nabla(T_0(q-q_h))\|_{0,\Omega}\\ &\le \xi_+\,\|T_0\|_{\mathcal{L}(H^1_0(\Omega))}\,\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\,\|\nabla(q-q_h)\|_{0,\Omega}.\end{aligned}$$

This implies that

$$\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\le\frac{\xi_+}{\alpha_0}\,\|T_0\|_{\mathcal{L}(H^1_0(\Omega))}\,\inf_{q_h\in M_h}\|\nabla(q-q_h)\|_{0,\Omega}.$$

There remains to choose some *ad hoc* $q_h\in M_h$. We know that $\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h$ belongs to $\nabla[M_h]$ (same as in case (ii)) or, in other words, that there exists $q_h^0\in M_h$ such that $\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h=\nabla q_h^0$. But

$$\nabla(q-q_h^0)=(\boldsymbol{k}-\boldsymbol{k}_h)-(\Pi_h^{comb}\boldsymbol{k}-\boldsymbol{k}_h)=\boldsymbol{k}-\Pi_h^{comb}\boldsymbol{k},$$

so choosing $q_h=q_h^0$ yields

$$\|\boldsymbol{k}-\boldsymbol{k}_h\|_{0,\Omega}\le\frac{\xi_+}{\alpha_0}\,\|T_0\|_{\mathcal{L}(H^1_0(\Omega))}\,\|\boldsymbol{k}-\Pi_h^{comb}\boldsymbol{k}\|_{0,\Omega}.$$

We can conclude as in case (ii). □

We can now state the error estimates for the div-curl problem.

**Corollary 3** *Assume that (70) holds. Then, for all $h$, the discrete problem (21) is well-posed. In addition, its solution $(\boldsymbol{u}_h,p_h)$ is such that $p_h=0$. Without further assumption on the regularity of the data $(\boldsymbol{f},g)$, one has*

$$\lim_{h\to0}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}=0. \tag{43}$$

*Let the extra-regularity of the data $(\boldsymbol{f},g)$ be as in (11) with $\tau_1,\tau_2>0$ given, then one has the error estimate, for all $\mathtt{s}\in[0,\min(\tau_2,\tau_{Dir}))$,*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\lesssim h^{\min(\mathtt{s},\tau_1)}\{\|g\|_{-1+\mathtt{s},\Omega}+\|\boldsymbol{f}\|_{\boldsymbol{PH}^{\tau_1}(\Omega)}\}. \tag{44}$$

*Proof* We observed in remark 8 that the discrete problems are well-posed, with $p_h=0$. In addition, we can now prove that the form $a$ fulfills a uniform discrete inf-sup condition. This result is an obvious consequence of (38), because (39) allows one to bound uniformly $\|\,\mathbf{curl}(\boldsymbol{\pi}_{2h}\boldsymbol{u}_h)\|^2_{0,\Omega}$ from below by $\|\boldsymbol{\pi}_{2h}\boldsymbol{u}_h\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. Indeed, we get

$$\begin{aligned}&\forall h,\ \forall(\boldsymbol{u}_h,p_h)\in\mathbb{V}_h,\\ &a((\boldsymbol{u}_h,p_h),\mathbb{T}_h(\boldsymbol{u}_h,p_h))\gtrsim\left(\|\,\mathbf{curl}(\boldsymbol{\pi}_{2h}\boldsymbol{u}_h)\|^2_{0,\Omega}+\|\nabla(\pi_{1h}\boldsymbol{u}_h)\|^2_{0,\Omega}+\|\nabla p_h\|^2_{0,\Omega}\right)\\ &\qquad\gtrsim\left(\|\boldsymbol{\pi}_{2h}\boldsymbol{u}_h\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\nabla(\pi_{1h}\boldsymbol{u}_h)\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\nabla p_h\|^2_{0,\Omega}\right),\\ &\qquad\gtrsim\left(\|\boldsymbol{u}_h\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\nabla p_h\|^2_{0,\Omega}\right),\end{aligned}$$

where we used successively (38), (39) and finally the triangle inequality

$$\|\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq \|\nabla(\pi_{1h}\boldsymbol{u}_h)\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\pi}_{2h}\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

Hence it holds that

$$\exists\underline{\alpha} > 0,\ \forall h,\ \forall(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h, \quad a((\boldsymbol{u}_h, p_h), \mathbb{T}_h(\boldsymbol{u}_h, p_h)) \geq \underline{\alpha}\, \|(\boldsymbol{u}_h, p_h)\|_{\mathbb{V}}^2. \quad (45)$$

We recall that the continuity modulus of $(\mathbb{T}_h)_h$ is bounded independently of $h$ (see proposition 13). Using Theorem 2 in [25], we conclude that the form $a$ fulfills a uniform discrete inf-sup condition, and so that the classical error estimate holds (recall that $p = p_h = 0$)

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.$$

In the absence of extra-regularity of the data, according to the basic approximability property of $(\boldsymbol{V}_h)_h$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one finds (43). On the other hand, in the case of extra-regularity of the data, we then recover (44) by choosing $\boldsymbol{v}_h = \Pi_h^{comb}\boldsymbol{u}$ (see proposition 10). □

## 6 Case of a ”weak” T-coercivity operator

As usual we assume in this section that the companion scalar problem (3) is well-posed. But that we only have at hand a ”weak” explicit T-coercivity involution operator $T$, cf. (69) in section A.1. This situation may occur only in the interface case. At the discrete level, one can build ”weak” discrete T-coercivity operators provided the meshes are locally T-conform (see section A.2). This yields uniformly bounded discrete operators $(T_h)_{h\leq h_0}$, where $h_0 > 0$ is a threshold value, such that (72)-(73) are fulfilled, with consequences listed in section A.2. Then, introducing $\boldsymbol{K}_h(\xi)$ as before (see (33)), one has the...

**Proposition 15** *In the ”weak” T-coercivity framework, for all $h \leq h_0$, one has the direct sum*

$$\boldsymbol{V}_h = \nabla[M_h] \oplus \boldsymbol{K}_h(\xi). \quad (46)$$

*In addition, $\boldsymbol{k}_h \mapsto \|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}$ defines a norm on $\boldsymbol{K}_h(\xi)$.*
*Finally, the operators $(\pi_{1h})_{h\leq h_0}$ and $(\boldsymbol{\pi}_{2h})_{h\leq h_0}$ introduced in (37) are well-defined, and their continuity moduli are bounded independently of $h \leq h_0$.*

As a consequence,

$$|||\cdot|||_{\mathbb{V}}\ :\ (\boldsymbol{u}_h, p_h) \mapsto \left(\|\,\mathbf{curl}(\boldsymbol{\pi}_{2h}\boldsymbol{u}_h)\|_{0,\Omega}^2 + \|\nabla(\pi_{1h}\boldsymbol{u}_h)\|_{0,\Omega}^2 + \|\nabla p_h\|_{0,\Omega}^2\right)^{1/2}$$

defines a norm on $\mathbb{V}$ for all $h \leq h_0$. We now prove a uniform discrete inf-sup condition in $|||\cdot|||_{\mathbb{V}}$-norm.

**Proposition 16** *In the "weak" T-coercivity framework, for h small enough, the form a fulfills a uniform discrete inf-sup condition in $|||\cdot|||_{\mathbb{V}}$-norm:*

$$\begin{cases} \exists C, h_1 > 0,\ \forall h \le h_1,\ \forall(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h, \\ \sup_{(\boldsymbol{v}_h,q_h)\in\mathbb{V}_h\setminus\{0\}} \dfrac{|a((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h))|}{|||(\boldsymbol{v}_h, q_h)|||_{\mathbb{V}}} \ge C\,|||(\boldsymbol{u}_h, p_h)|||_{\mathbb{V}}. \end{cases} \tag{47}$$

*Proof* We argue by contradiction. Namely, we assume that

$$\begin{cases} \forall k \in \mathbb{N}\setminus\{0\},\ \exists h_k \le k^{-1},\ \exists(\boldsymbol{u}_{h_k}, p_{h_k}) \in \mathbb{V}_{h_k}, \\ |||(\boldsymbol{u}_{h_k}, p_{h_k})|||_{\mathbb{V}} = 1, \quad \text{and} \\ \sup_{(\boldsymbol{v}_{h_k},q_{h_k})\in\mathbb{V}_{h_k}\setminus\{0\}} \dfrac{|a((\boldsymbol{u}_{h_k}, p_{h_k}), (\boldsymbol{v}_{h_k}, q_{h_k}))|}{|||(\boldsymbol{v}_{h_k}, q_{h_k})|||_{\mathbb{V}}} \le k^{-1}. \end{cases} \tag{48}$$

In particular, $\lim_{k\to\infty} h_k = 0$, so it holds that $h_k < h_0$ for $k$ large enough, where $h_0$ is defined by (73). So from now on, we consider that $h_k < h_0$. Compute

$$\begin{aligned} &a((\boldsymbol{u}_{h_k}, p_{h_k}), (\nabla(T_{h_k}p_{h_k}) + \boldsymbol{\pi}_{2h_k}\boldsymbol{u}_{h_k}, T_{h_k}(\pi_{1h_k}\boldsymbol{u}_{h_k}))) \\ &\quad= \|\operatorname{\mathbf{curl}}(\boldsymbol{\pi}_{2h_k}\boldsymbol{u}_{h_k})\|_{0,\Omega}^2 + (\xi\nabla(\pi_{1h_k}\boldsymbol{u}_{h_k})|\nabla(T_{h_k}(\pi_{1h_k}\boldsymbol{u}_{h_k})))_{0,\Omega} \\ &\qquad\qquad + (\xi\nabla(T_{h_k}p_{h_k})|\nabla p_{h_k})_{0,\Omega} \\ &\quad\ge \|\operatorname{\mathbf{curl}}(\boldsymbol{\pi}_{2h_k}\boldsymbol{u}_{h_k})\|_{0,\Omega}^2 + \underline{\alpha}\,\|\nabla(\pi_{1h_k}\boldsymbol{u}_{h_k})\|_{0,\Omega}^2 + \underline{\alpha}\,\|\nabla p_{h_k}\|_{0,\Omega}^2 \\ &\qquad\qquad - \underline{\beta}\,\|\pi_{1h_k}\boldsymbol{u}_{h_k}\|_{0,\Omega}^2 - \underline{\beta}\,\|p_{h_k}\|_{0,\Omega}^2 \\ &\quad\ge \min(1,\underline{\alpha})\,|||(\boldsymbol{u}_{h_k}, p_{h_k})|||_{\mathbb{V}}^2 - \underline{\beta}\,\|\pi_{1h_k}\boldsymbol{u}_{h_k}\|_{0,\Omega}^2 - \underline{\beta}\,\|p_{h_k}\|_{0,\Omega}^2. \end{aligned}$$

Above, we used the definition of $\boldsymbol{K}_{h_k}(\xi)$, the fact that $\xi$ is a symmetric tensor field, the "weak" discrete T-coercivity property (72), and finally the definition of $|||(\boldsymbol{u}_{h_k}, p_{h_k})|||_{\mathbb{V}}$.
By assumption (48), $|||(\boldsymbol{u}_{h_k}, p_{h_k})|||_{\mathbb{V}} = 1$, so $(\pi_{1h_k}\boldsymbol{u}_{h_k})_k$ and $(p_{h_k})_k$ are bounded in $H_0^1(\Omega)$, and moreover it holds that

$$\begin{aligned} &a((\boldsymbol{u}_{h_k}, p_{h_k}), (\nabla(T_{h_k}p_{h_k}) + \boldsymbol{\pi}_{2h_k}\boldsymbol{u}_{h_k}, T_{h_k}(\pi_{1h_k}\boldsymbol{u}_{h_k}))) \\ &\qquad\qquad \ge \min(1,\underline{\alpha}) - \underline{\beta}\,\|\pi_{1h_k}\boldsymbol{u}_{h_k}\|_{0,\Omega}^2 - \underline{\beta}\,\|p_{h_k}\|_{0,\Omega}^2. \end{aligned} \tag{49}$$

Let $\rightharpoonup$ denote weak convergence. Thanks to Rellich's Theorem (cf. Theorem 9.6 in [18]), there exists $\tilde{p}_{\boldsymbol{u}}, \tilde{p} \in H_0^1(\Omega)$ such that

$$\begin{aligned} &\lim_{k\to\infty} \|\pi_{1h_k}\boldsymbol{u}_{h_k} - \tilde{p}_{\boldsymbol{u}}\|_{0,\Omega} = 0, && \pi_{1h_k}\boldsymbol{u}_{h_k} \rightharpoonup \tilde{p}_{\boldsymbol{u}} \text{ in } H_0^1(\Omega)\,; \\ &\lim_{k\to\infty} \|p_{h_k} - \tilde{p}\|_{0,\Omega} = 0, && p_{h_k} \rightharpoonup \tilde{p} \text{ in } H_0^1(\Omega). \end{aligned}$$

We prove that $\tilde{p}_{\boldsymbol{u}} = \tilde{p} = 0$.
We start with $\tilde{p}$: let $q \in H_0^1(\Omega)$, it holds that (cf. Proposition 3.5 in [18])

$$(\xi\nabla q|\nabla p_{h_k})_{0,\Omega} \to (\xi\nabla q|\nabla\tilde{p})_{0,\Omega}.$$

Now, according to the basic approximability property of $(M_h)_h$ in $H_0^1(\Omega)$, there exists $(q_{h_k})_k$ such that, for all $k \ge 1$, $q_{h_k} \in M_{h_k}$, and moreover $\lim_{k\to\infty} \|q_{h_k} -$

$q\|_{H_0^1(\Omega)} = 0$. Recall that $a((\boldsymbol{u}_{h_k}, p_{h_k}), (\nabla q_{h_k}, 0)) = (\xi \nabla q_{h_k} | \nabla p_{h_k})_{0,\Omega}$, so one has

$$\begin{aligned}|(\xi \nabla q | \nabla p_{h_k})_{0,\Omega}| &\leq |(\xi \nabla (q - q_{h_k}) | \nabla p_{h_k})_{0,\Omega}| + |(\xi \nabla q_{h_k} | \nabla p_{h_k})_{0,\Omega}| \\ &\leq \xi_+ \|\nabla (q - q_{h_k})\|_{0,\Omega} \|\nabla p_{h_k}\|_{0,\Omega} + |a((\boldsymbol{u}_{h_k}, p_{h_k}), (\nabla q_{h_k}, 0))| \\ &\leq \xi_+ \|\nabla (q - q_{h_k})\|_{0,\Omega} \|\nabla p_{h_k}\|_{0,\Omega} + k^{-1} \|\nabla q_{h_k}\|_{0,\Omega}.\end{aligned}$$

Above, the last inequality is a consequence of assumption (48). Passing to the limit, one concludes that $(\xi \nabla q | \nabla \tilde{p})_{0,\Omega} = 0$. This result holds holds for all $q \in H_0^1(\Omega)$. Because $\xi$ is a symmetric tensor field and the scalar problem (3) is well-posed, it follows that $\tilde{p} = 0$.
Let us carry on with $\tilde{p}_{\boldsymbol{u}}$: let again $q \in H_0^1(\Omega)$, it now holds that

$$(\xi \nabla q | \nabla (\pi_{1h_k} \boldsymbol{u}_{h_k}))_{0,\Omega} \to (\xi \nabla q | \nabla \tilde{p}_{\boldsymbol{u}})_{0,\Omega}.$$

Take $(q_{h_k})_k$ as before and recall that $a((\boldsymbol{u}_{h_k}, p_{h_k}), (0, q_{h_k})) = (\xi \nabla (\pi_{1h_k} \boldsymbol{u}_{h_k}) | \nabla q_{h_k})_{0,\Omega}$, so one has

$$\begin{aligned}|(\xi \nabla q | \nabla (\pi_{1h_k} \boldsymbol{u}_{h_k}))_{0,\Omega}| &\leq |(\xi \nabla (q - q_{h_k}) | \nabla (\pi_{1h_k} \boldsymbol{u}_{h_k}))_{0,\Omega}| + |(\xi \nabla q_{h_k} | \nabla (\pi_{1h_k} \boldsymbol{u}_{h_k}))_{0,\Omega}| \\ &\leq \xi_+ \|\nabla (q - q_{h_k})\|_{0,\Omega} \|\nabla (\pi_{1h_k} \boldsymbol{u}_{h_k})\|_{0,\Omega} + |a((\boldsymbol{u}_{h_k}, p_{h_k}), (0, q_{h_k}))| \\ &\leq \xi_+ \|\nabla (q - q_{h_k})\|_{0,\Omega} \|\nabla (\pi_{1h_k} \boldsymbol{u}_{h_k})\|_{0,\Omega} + k^{-1} \|\nabla q_{h_k}\|_{0,\Omega}.\end{aligned}$$

Above, the last inequality is again a consequence of assumption (48). Proceeding as before, we find now that $\tilde{p}_{\boldsymbol{u}} = 0$.
Hence, going back to (49), we have that the lower bound goes to

$$\lim_{k \to \infty} \left( \min(1, \underline{\alpha}) - \underline{\beta} \, \|\pi_{1h_k} \boldsymbol{u}_{h_k}\|_{0,\Omega}^2 - \underline{\beta} \, \|p_{h_k}\|_{0,\Omega}^2 \right) = \min(1, \underline{\alpha}) > 0.$$

On the other hand, according to the assumption (48)

$$\begin{aligned}&|a((\boldsymbol{u}_{h_k}, p_{h_k}), (\nabla (T_{h_k} p_{h_k}) + \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}, T_{h_k}(\pi_{1h_k} \boldsymbol{u}_{h_k}))| \\ &\qquad \leq k^{-1} \, \|(\nabla (T_{h_k} p_{h_k}) + \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}, T_{h_k}(\pi_{1h_k} \boldsymbol{u}_{h_k}))\|_{\mathbb{V}}.\end{aligned}$$

Then, $\boldsymbol{\pi}_{2h_k}(\nabla (T_{h_k} p_{h_k}) + \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}) = \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}$ and $\pi_{1h_k}(\nabla (T_{h_k} p_{h_k}) + \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}) = T_{h_k} p_{h_k}$, so:

$$\begin{aligned}&\|(\nabla (T_{h_k} p_{h_k}) + \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}, T_{h_k}(\pi_{1h_k} \boldsymbol{u}_{h_k}))\|_{\mathbb{V}}^2 = \\ &\|\,\mathbf{curl}(\boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k})\|_{0,\Omega}^2 + \|\nabla (T_{h_k} p_{h_k})\|_{0,\Omega}^2 + \|\nabla (T_{h_k}(\pi_{1h_k} \boldsymbol{u}_{h_k}))\|_{0,\Omega}^2.\end{aligned}$$

It follows that

$$\|(\nabla (T_{h_k} p_{h_k}) + \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}, T_{h_k}(\pi_{1h_k} \boldsymbol{u}_{h_k}))\|_{\mathbb{V}} \lesssim \max \Big( 1, \sup_k \|T_{h_k}\|_{\mathcal{L}(M_{h_k})} \Big),$$

and one finds that the upper bounds goes to

$$\lim_{k \to \infty} \left( k^{-1} \, \|(\nabla (T_{h_k} p_{h_k}) + \boldsymbol{\pi}_{2h_k} \boldsymbol{u}_{h_k}, T_{h_k}(\pi_{1h_k} \boldsymbol{u}_{h_k}))\|_{\mathbb{V}} \right) = 0,$$

ie. a contradiction. □

*Remark 10* As already observed in remark 8, it follows from the above that, for $h$ small enough, the discrete problem (21) is well-posed. And its solution $(\boldsymbol{u}_h, p_h)$ is such that $p_h = 0$.

To conclude the study, we now prove the results below.

**Theorem 4** *In the "weak" T-coercivity framework, for h small enough, the* $\|\,\mathbf{curl}\cdot\|_{0,\Omega}$*-norm defines a norm that is uniformly equivalent to the* $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$*-norm on* $\boldsymbol{K}_h(\xi)$*:*

$$\exists C_W^\star, h_2 > 0,\ \forall h \le h_2,\ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\xi), \quad \|\boldsymbol{k}_h\|_{0,\Omega} \le C_W^\star\,\|\,\mathbf{curl}\,\boldsymbol{k}_h\|_{0,\Omega}. \tag{50}$$

*Proof* It follows closely the proof of theorem 3, case (iii). Let $\boldsymbol{k}_h \in \boldsymbol{K}_h(\xi)$ be given, and let $\boldsymbol{k}$ be the solution to the div-curl problem with data $(\boldsymbol{f}, g) = (\mathbf{curl}\,\boldsymbol{k}_h, 0)$. As before, we find that there exists $q \in H_0^1(\Omega)$ such that $\boldsymbol{k}-\boldsymbol{k}_h = \nabla q$ in $\Omega$.
Let $h \le h_0$, where $h_0$ appears in the uniform discrete inf-sup condition (73). Then, for any $\bar{q}_h \in M_h$, we write the triangle inequality

$$\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega} = \|\nabla q\|_{0,\Omega} \le \|\nabla(q - \bar{q}_h)\|_{0,\Omega} + \|\nabla \bar{q}_h\|_{0,\Omega}.$$

According to (73), there exists $q'_h \in M_h \setminus \{0\}$ such that

$$\|\nabla \bar{q}_h\|_{0,\Omega} \le (\underline{\gamma}_0)^{-1} \frac{|(\xi \nabla \bar{q}_h | \nabla q'_h)_{0,\Omega}|}{\|\nabla q'_h\|_{0,\Omega}}.$$

Since $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega, \xi)$ and $\boldsymbol{k}_h \in \boldsymbol{K}_h(\xi)$, one has $(\xi(\boldsymbol{k} - \boldsymbol{k}_h)|\nabla q'_h)_{0,\Omega} = 0$ or, in other words, $(\xi \nabla q | \nabla q'_h)_{0,\Omega} = 0$. Hence,

$$\|\nabla \bar{q}_h\|_{0,\Omega} \le (\underline{\gamma}_0)^{-1} \frac{|(\xi \nabla(\bar{q}_h - q) | \nabla q'_h)_{0,\Omega}|}{\|\nabla q'_h\|_{0,\Omega}} \le \frac{\xi_+}{\underline{\gamma}_0} \|\nabla(q - \bar{q}_h)\|_{0,\Omega}.$$

We find that $\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega} \le (1 + \xi_+/\underline{\gamma}_0)\|\nabla(q - \bar{q}_h)\|_{0,\Omega}$. Since the result holds for any $\bar{q}_h \in M_h$, we have actually proved that

$$\|\boldsymbol{k} - \boldsymbol{k}_h\|_{0,\Omega} \lesssim \inf_{q_h \in M_h} \|\nabla(q - \bar{q}_h)\|_{0,\Omega}.$$

We conclude the proof as before (theorem 3, case (iii)), by choosing $q_h \in M_h$ such that $\Pi_h^{comb}\boldsymbol{k} - \boldsymbol{k}_h = \nabla q_h$. □

We can finally state the error estimates for the div-curl problem in the "weak" T-coercivity framework. The proof is similar to the proof of corollary 3.

**Corollary 4** *In the "weak" T-coercivity framework, for h small enough, the discrete problem (21) is well-posed. In addition, its solution* $(\boldsymbol{u}_h, p_h)$ *is such that* $p_h = 0$*. Without further assumption on the regularity of the data* $(\boldsymbol{f}, g)$*, one has*

$$\lim_{h\to 0} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0. \tag{51}$$

*Let the extra-regularity of the data* $(\boldsymbol{f}, g)$ *be as in (11) with* $\tau_1, \tau_2 > 0$ *given, then one has the error estimate, for all* $\mathtt{s} \in [0, \min(\tau_2, \tau_{Dir}))$*,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\min(\mathtt{s},\tau_1)}\{\|g\|_{-1+\mathtt{s},\Omega} + \|\boldsymbol{f}\|_{\boldsymbol{PH}^{\tau_1}(\Omega)}\}. \tag{52}$$

## 7 The div-curlcurl problem

The div-curlcurl problem writes:

$$\begin{cases} \textit{Find } \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ \mathbf{curl}\,\zeta\,\mathbf{curl}\,\boldsymbol{u} = \underline{\boldsymbol{f}} \text{ in } \Omega, \\ \operatorname{div}\xi\boldsymbol{u} = g \text{ in } \Omega. \end{cases} \tag{53}$$

Above, we assume that the new coefficient[2] $\zeta$ fulfills (2) (classical case), whereas $\xi$ is as before (classical case or interface case). Introducing $\boldsymbol{H}(\operatorname{div}0;\Omega) := \{\boldsymbol{f}' \in \boldsymbol{H}(\operatorname{div};\Omega) \ : \ \operatorname{div}\boldsymbol{f}' = 0 \text{ in } \Omega\}$, the assumptions on the source terms are as follows

$$\underline{\boldsymbol{f}} \in \boldsymbol{H}(\operatorname{div}0;\Omega), \quad g \in H^{-1}(\Omega). \tag{54}$$

Indeed, let us prove existence and uniqueness of the solution $\boldsymbol{u}$ to the div-curlcurl problem (see also proposition 17 below).
*Uniqueness.* Assume first that $(\underline{\boldsymbol{f}}, g) = (0,0)$. Then $\boldsymbol{f} = \zeta\,\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl};\Omega)$ is such that $\mathbf{curl}\,\boldsymbol{f} = 0$. Because $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, one has $\zeta^{-1}\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}_0(\operatorname{div};\Omega)$, with $\operatorname{div}\zeta^{-1}\boldsymbol{f} = 0$ in $\Omega$. Finally, $\langle\zeta^{-1}\boldsymbol{f}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_i} = 0$ for $1 \le i \le I$ (cf. Remark 3.5.2 in [3]). Since $\zeta^{-1}$ is as in the classical case, we thus conclude that $\boldsymbol{f} = 0$ according to Proposition 6.2.1 in [3]. Hence, $\boldsymbol{u}$ solves the div-curl problem (coefficient $\xi$) with vanishing data: $\boldsymbol{u} = 0$.
*Existence.* Let $\underline{\boldsymbol{f}} \in \boldsymbol{H}(\operatorname{div}0;\Omega)$. As $\partial\Omega$ is connected, according now to Theorem 6.2.5 in [3], there exists $\boldsymbol{w}$ in

$$\boldsymbol{X}_T(\Omega,\zeta^{-1}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega) \ : \ \zeta^{-1}\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{div};\Omega)\}$$

such that $\mathbf{curl}\,\boldsymbol{w} = \underline{\boldsymbol{f}}$ and $\operatorname{div}\zeta^{-1}\boldsymbol{w} = 0$ in $\Omega$, and $\langle\zeta^{-1}\boldsymbol{w}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_i} = 0$ for $1 \le i \le I$. In particular, $\zeta^{-1}\boldsymbol{w}$ belongs to $\boldsymbol{H}_0^{\Sigma}(\operatorname{div}0;\Omega)$ (see definition of this function space at the beginning of section 3). Hence there exists $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ such that $\mathbf{curl}\,\boldsymbol{u} = \zeta^{-1}\boldsymbol{w}$ and $\operatorname{div}\xi\boldsymbol{u} = g$ in $\Omega$ because the div-curl problem (coefficient $\xi$) is well-posed. Obviously, $\boldsymbol{u}$ solves the div-curlcurl problem (53) with data $(\underline{\boldsymbol{f}}, g)$.
Next, define the continuous bilinear form on $\mathbb{V}$:

$$a_\zeta((\boldsymbol{u},p),(\boldsymbol{v},q)) := (\zeta\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + (\xi\boldsymbol{u}|\nabla q)_{0,\Omega} + (\xi\boldsymbol{v}|\nabla p)_{0,\Omega}.$$

For such source terms as in (54), we introduce the variational formulation

$$\begin{cases} \textit{Find } (\boldsymbol{u},p) \in \mathbb{V} \textit{ such that} \\ a_\zeta((\boldsymbol{u},p),(\boldsymbol{v},q)) = (\underline{\boldsymbol{f}}|\boldsymbol{v})_{0,\Omega} - \langle g, q\rangle_{H_0^1(\Omega)}, \ \forall(\boldsymbol{v},q) \in \mathbb{V}. \end{cases} \tag{55}$$

The right-hand side defines a continuous linear form on $\mathbb{V}$, with norm bounded from above by $\|\underline{\boldsymbol{f}}\|_{0,\Omega} + \|g\|_{-1,\Omega}$. Observe that, since $\zeta$ is as in the classical case, the bilinear form $a_\zeta$ is similar to the form $a$ for the div-curl problem, so one recovers easily results that correspond to lemma 1 and proposition 1.

[2] Note that $\zeta^{-1}$ fulfills (2) automatically.

**Lemma 2** *Let* $\underline{\boldsymbol{f}} \in \boldsymbol{H}(\mathrm{div}\,0;\Omega)$ *and* $g \in H^{-1}(\Omega)$ *be given. Then if* $(\boldsymbol{u},p)$ *is a solution to the variational formulation (55), it holds that* $p=0$.

*Proof* Choose $(\boldsymbol{v},q) = (\nabla(T_0 p), 0)$ in (55). This yields:

$$(\xi\nabla(T_0 p)|\nabla p)_{0,\Omega} = 0.$$

Since $\xi$ is symmetric, one has $\alpha_0\|\nabla p\|^2_{0,\Omega} = 0$ according to (9), so $p=0$. □

**Proposition 17** *Let* $\underline{\boldsymbol{f}} \in \boldsymbol{H}(\mathrm{div}\,0;\Omega)$ *and* $g \in H^{-1}(\Omega)$ *be given. Then it holds that* $\boldsymbol{u}$ *is a solution to the div-curlcurl problem (53) if, and only if,* $(\boldsymbol{u},0)$ *is a solution to the variational formulation (55).*

*Proof* Let $\boldsymbol{u}$ solve (53), then $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$.
Also, for $(\boldsymbol{v},q) \in \mathbb{V}$, one has by integration on $\Omega$, and integration by parts (resp. definition of duality):

$$\begin{aligned}&(\zeta\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} = (\underline{\boldsymbol{f}}|\boldsymbol{v})_{0,\Omega}, \text{ resp.}\\ &(\xi\boldsymbol{u}|\nabla q)_{0,\Omega} = -\langle g,q\rangle_{H^1_0(\Omega)}.\end{aligned}$$

So it follows that

$$(\zeta\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + (\xi\boldsymbol{u}|\nabla q)_{0,\Omega} = (\underline{\boldsymbol{f}}|\boldsymbol{v})_{0,\Omega} - \langle g,q\rangle_{H^1_0(\Omega)}.$$

Hence, $(\boldsymbol{u},0)$ solves (55).
Conversely, let $(\boldsymbol{u},p)$ solve (55). By definition $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and, according to lemma 2, $p=0$. Choosing $(0,q)$ as a test function in (55), one finds next that $\mathrm{div}\,\xi\boldsymbol{u} = g$ in $H^{-1}(\Omega)$. Then, one has that

$$(\zeta\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} = (\underline{\boldsymbol{f}}|\boldsymbol{v})_{0,\Omega}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega).$$

We thus conclude that $\mathbf{curl}\,\zeta\,\mathbf{curl}\,\boldsymbol{u} = \underline{\boldsymbol{f}}$ in $(\boldsymbol{H}_0(\mathbf{curl};\Omega))'$. Hence $\boldsymbol{u}$ solves the div-curlcurl problem (53). □

And the well-posedness of (55) is obtained as in theorem 2 and corollary 1, so the div-curlcurl problem is well-posed. Indeed, observe that one can recover T-coercivity of the form $a_\zeta$ with the *same operator* $\mathbb{T}$ as the one provided in the proof of theorem 2: in this sense, the forms $a_\zeta$ and $a$ exhibit the same properties. Well-posedness then follows.

When we study the discrete div-curlcurl problems and to obtain explicit convergence rates, we make again *two additional assumptions*. First, that *the coefficients* $\zeta^{-1}$ *and* $\xi$ *are piecewise smooth* on a common partition. Second, regarding the *extra-regularity* of the data $(\underline{\boldsymbol{f}},g)$, we only ask that

$$g \in H^{-1+\tau_2}(\Omega), \text{ with } \tau_2 \in (0,1] \text{ given.}$$

Indeed, when $\boldsymbol{u}$ solves the div-curlcurl problem, we already noticed that $\boldsymbol{c} = \zeta\,\mathbf{curl}\,\boldsymbol{u}$ belongs to $\boldsymbol{X}_T(\Omega,\zeta^{-1})$. Using a shift theorem for the companion scalar problem with Neumann boundary condition

$$\begin{cases}\text{Find } s \in H^1_{zmv}(\Omega) \text{ such that}\\ (\zeta^{-1}\nabla s|\nabla q)_{0,\Omega} = \langle g',q\rangle_{H^1_{zmv}(\Omega)}, \ \forall q \in H^1_{zmv}(\Omega),\end{cases} \tag{56}$$

and a regular plus gradient splitting (see eg. [30,28]), we introduce $\tau_{Neu} \in (0,1]$ depending only on the geometry and on $\zeta^{-1}$ such that

$$\boldsymbol{X}_T(\Omega,\zeta^{-1}) \subset \cap_{\mathtt{s}\in[0,\tau_{Neu})} \boldsymbol{PH}^{\mathtt{s}}(\Omega),$$

with continuous imbedding for all $\mathtt{s} \in [0,\tau_{Neu})$. Furthermore, using a Weber inequality (cf. Theorem 6.2.5 in [3]), one has that for all $\mathtt{s} \in [0,\tau_{Neu})$,

$$\forall \boldsymbol{v} \in \boldsymbol{X}_T(\Omega,\zeta^{-1}),$$
$$\|\boldsymbol{v}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega)} \lesssim \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega} + \|\operatorname{div}\zeta^{-1}\boldsymbol{v}\|_{0,\Omega} + \textstyle\sum_{1\le i\le I} |\langle \zeta^{-1}\boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Sigma_i}|.$$

As a consequence, we note that since $\zeta^{-1}$ is piecewise smooth, it also holds that $\mathbf{curl}\,\boldsymbol{u} \in \cap_{\mathtt{s}\in[0,\tau_{Neu})} \boldsymbol{PH}^{\mathtt{s}}(\Omega)$ with continuous dependence. And, because $\boldsymbol{c} = \zeta\,\mathbf{curl}\,\boldsymbol{u}$ is such that $\operatorname{div}\zeta^{-1}\boldsymbol{c} = 0$ in $\Omega$ and $\langle \zeta^{-1}\boldsymbol{c}\cdot\boldsymbol{n},1\rangle_{\Sigma_i} = 0$ for $1 \le i \le I$ (cf. Remark 3.5.2 in [3]), one has the estimate, for all $\mathtt{s} \in [0,\tau_{Neu})$,

$$\|\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{PH}^{\mathtt{s}}(\Omega)} \lesssim \|\,\mathbf{curl}\,\zeta\,\mathbf{curl}\,\boldsymbol{u}\|_{0,\Omega}.$$

*Remark 11* In other words, for the div-curlcurl problem (compared to the div-curl problem), $\tau_{Neu}$ plays the role of $\tau_1$. Observe that choosing $\underline{\boldsymbol{f}} \in \boldsymbol{L}^2(\Omega)$ is already *sufficient* to have extra regularity on the curl of $\boldsymbol{u}$.

So one may find interpolation results that are completely similar to the ones previously obtained. In particular regarding the classical, or combined, interpolation of the solution $\boldsymbol{u}$ to the div-curlcurl problem, with $\tau_{Neu}$ now replacing $\tau_1$ in propositions 7 and 10.
Then, because the forms $a_\zeta$ and $a$ exhibit the same properties, one concludes that error estimates similar to those of corollaries 3 and 4 hold, that is when a "full" T-coercivity operator, resp. a "weak" T-coercivity operator, is available for the companion scalar problem (3), again with $\tau_{Neu}$ replacing $\tau_1$.

## 8 The time-harmonic Maxwell equations in the electric field

We let the medium in $\Omega$ be surrounded by a perfect conductor. Classically, for a given pulsation $\omega > 0$, the time-harmonic Maxwell's equations set in $\Omega$ can be expressed in terms of the complex-valued electric field $\boldsymbol{e}$ only. They write

$$\begin{cases} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) - \omega^2\varepsilon\boldsymbol{e} = \imath\omega\boldsymbol{j} \text{ in } \Omega \\ \operatorname{div}\varepsilon\boldsymbol{e} = \varrho \text{ in } \Omega. \end{cases} \tag{57}$$

Above, the real-valued coefficient $\varepsilon$ is the electric permittivity tensor and the real-valued coefficient $\mu$ is the magnetic permeability tensor. The complex-valued source terms $\boldsymbol{j}$ and $\varrho$ are respectively the current density and the charge density. They are related by the charge conservation equation

$$-\imath\omega\varrho + \operatorname{div}\boldsymbol{j} = 0 \text{ in } \Omega. \tag{58}$$

The *a priori* regularity of the source terms is $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$, and $\varrho \in H^{-1}(\Omega)$.
Note that in (57), the equation $\operatorname{div} \varepsilon \boldsymbol{e} = \varrho$ is implied by the second-order equation $\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}) - \omega^2 \varepsilon \boldsymbol{e} = \imath\omega \boldsymbol{j}$, together with the charge conservation equation (58), so it can be omitted.
Finally, note that one can split the problem into two parts, where $\Re(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ is related to $-\Im(\boldsymbol{j}) \in \boldsymbol{L}^2(\Omega)$, resp. $\Im(\boldsymbol{e}) \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ is related to $\Re(\boldsymbol{j}) \in \boldsymbol{L}^2(\Omega)$. So, we carry on with $\boldsymbol{e}$ standing either for $\Re(\boldsymbol{e})$ or $\Im(\boldsymbol{e})$, resp. $\boldsymbol{j}$ standing for $-\Im(\boldsymbol{j})$ or $\Re(\boldsymbol{j})$, that is with real-valued fields. One can check that the equivalent variational formulation in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ writes

$$\begin{cases} \textit{Find } \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \textit{ such that} \\ a_\omega(\boldsymbol{e},\boldsymbol{v}) = \omega(\boldsymbol{j}|\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \end{cases} \tag{59}$$

where

$$a_\omega(\boldsymbol{u},\boldsymbol{v}) := (\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2(\varepsilon\boldsymbol{u}|\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega).$$

Next, we assume that $\mu$ is as in the classical case (cf. (2)), resp. $\varepsilon$ is as in the interface case, and assumption (8)-(9) holds for $\varepsilon$. Obviously, $\zeta := \mu^{-1}$ fulfills (2). Using operators $\pi_1$ and $\boldsymbol{\pi}_2$, one can provide a reformulation of the variational formuation (59). Its solution $\boldsymbol{e}$ may be be split as

$$\boldsymbol{e} = \boldsymbol{e}_0 + \nabla\phi, \text{ with } \boldsymbol{e}_0 = \boldsymbol{\pi}_2\boldsymbol{e} \text{ and } \phi = \pi_1\boldsymbol{e}. \tag{60}$$

By using the (variational) definition of $\boldsymbol{K}_N(\Omega,\varepsilon)$ (recall that $\varepsilon$ is a symmetric tensor field), we notice that $\boldsymbol{e}_0$ and $\phi$ are respectively governed by

$$\begin{cases} \textit{Find } \boldsymbol{e}_0 \in \boldsymbol{K}_N(\Omega,\varepsilon) \textit{ such that} \\ (\zeta\,\mathbf{curl}\,\boldsymbol{e}_0|\,\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2(\varepsilon\boldsymbol{e}_0|\boldsymbol{v})_{0,\Omega} = \omega(\boldsymbol{j}|\boldsymbol{v})_{0,\Omega},\ \forall \boldsymbol{v} \in \boldsymbol{K}_N(\Omega,\varepsilon). \end{cases} \tag{61}$$

$$\begin{cases} \textit{Find } \phi \in H^1_0(\Omega) \textit{ such that} \\ (\varepsilon\nabla\phi|\nabla q)_{0,\Omega} = -\omega^{-1}(\boldsymbol{j}|\nabla q)_{0,\Omega},\ \forall q \in H^1_0(\Omega). \end{cases} \tag{62}$$

Actually, there is an equivalence result (the proof is left to the reader).

**Proposition 18** *A field $\boldsymbol{e}$ is a solution to (59) if, and only if, $\boldsymbol{\pi}_2\boldsymbol{e}$ is a solution to (61) and $\pi_1\boldsymbol{e}$ is a solution to (62).*

According to the assumption on $\varepsilon$, we already know that problem (62) is well-posed. Hence proving the well-posedness of (59) amounts to proving the well-posedness of (61). We recall Theorem 8.15 of [12].

**Theorem 5** *The imbedding of $\boldsymbol{K}_N(\Omega,\varepsilon)$ in $\boldsymbol{L}^2(\Omega)$ is compact.*

As a consequence (cf. Theorem 8.16 of [12]), one has the

**Corollary 5** *The formulation (61) with unknown $\boldsymbol{e}_0$ enters the Fredholm alternative:*

– *either the problem (61) is well-posed, ie. it admits a unique solution $\boldsymbol{e}_0$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, which depends continuously on the data $\boldsymbol{j}$:*

$$\|\boldsymbol{e}_0\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{j}\|_{0,\Omega}\,;$$

– *or, the problem (61) has solutions if, and only if,* $\boldsymbol{j}$ *satisfy a finite number of compatibility conditions; in this case, the space of solutions is an affine space of finite dimension, and the component of the solution which is orthogonal (in the sense of the* $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ *inner product) to the corresponding linear vector space, depends continuously on the data* $\boldsymbol{j}$*.*

*Finally, each alternative occurs simultaneously for formulation (61) and formulation (59) with unknown* $\boldsymbol{e}$*.*

From now on, we assume that formulation (59) is well-posed. After discretization, our aim is to obtain the well-posedness result for the discretized problems, and convergence to the exact solution $\boldsymbol{e}$, by deriving a uniform discrete inf-sup condition. As before, we assume that there exists a limit regularity exponent $\tau_{Dir} = \tau_{Dir}(\varepsilon) \in (0,1]$. For $h$ given, the discrete variational formulation of the time-harmonic problem (59) is

$$\begin{cases} \textit{Find } \boldsymbol{e}_h \in \boldsymbol{V}_h \textit{ such that} \\ a_\omega(\boldsymbol{e}_h, \boldsymbol{v}_h) = \omega(\boldsymbol{j}|\boldsymbol{v}_h)_{0,\Omega}, \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{63}$$

In the "weak" T-coercivity framework that is, assuming (72)-(73) holds for $\xi = \varepsilon$, we remark that $(\boldsymbol{k}_h, \boldsymbol{k}'_h) \mapsto (\zeta\, \mathbf{curl}\, \boldsymbol{k}_h | \,\mathbf{curl}\, \boldsymbol{k}'_h)_{0,\Omega}$ fulfills a uniform discrete inf-sup condition on $\boldsymbol{K}_h(\varepsilon) \times \boldsymbol{K}_h(\varepsilon)$, for $h$ small enough. Indeed, according to theorem 4, we have

$$\begin{cases} \exists \tilde{\gamma}, h_2 > 0, \ \forall h \le h_2, \ \forall \boldsymbol{k}_h \in \boldsymbol{K}_h(\varepsilon), \\ \displaystyle\sup_{\boldsymbol{k}'_h \in \boldsymbol{K}_h(\varepsilon)\setminus\{0\}} \frac{|(\zeta\, \mathbf{curl}\, \boldsymbol{k}_h |\, \mathbf{curl}\, \boldsymbol{k}'_h)_{0,\Omega}|}{\|\boldsymbol{k}'_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \ge \tilde{\gamma}\, \|\boldsymbol{k}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \end{cases} \tag{64}$$

Next, we introduce $A_\omega \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega))$ defined by

$$(A_\omega \boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{H}_0(\mathbf{curl};\Omega)} := a_\omega(\boldsymbol{v}, \boldsymbol{w}), \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),$$

and

$$|||a_\omega||| := \sup_{\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v}, \boldsymbol{w})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} < \infty.$$

**Theorem 6** *Assume that the formulation (59) is well-posed. In the "weak" T-coercivity framework for* $\xi = \varepsilon$*, the form* $a_\omega$ *fulfills a uniform discrete inf-sup condition on* $\boldsymbol{V}_h \times \boldsymbol{V}_h$ *for* $h$ *small enough, ie.*

$$\begin{cases} \exists C_\omega, h_\omega > 0, \ \forall h \le h_\omega, \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \displaystyle\sup_{\boldsymbol{v}'_h \in \boldsymbol{V}_h\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v}_h, \boldsymbol{v}'_h)|}{\|\boldsymbol{v}'_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \ge C_\omega\, \|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \end{cases} \tag{65}$$

*Remark 12* Next, we proceed in the spirit of the proof of Theorem 2.2 in [15].

*Proof* We argue by contradiction. Namely, we assume that

$$\begin{cases} \forall k \in \mathbb{N} \setminus \{0\}, \ \exists h_k \le k^{-1}, \ \exists \boldsymbol{v}_{h_k} \in \boldsymbol{V}_{h_k}, \\ \|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 1 \quad \text{and} \quad \displaystyle\sup_{\boldsymbol{v}'_{h_k} \in \boldsymbol{V}_{h_k}\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{v}'_{h_k})|}{\|\boldsymbol{v}'_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \le k^{-1}. \end{cases} \tag{66}$$

In particular, $\lim_{k\to\infty} h_k = 0$, so it holds that $h_k < h_2$ for $k$ large enough, where $h_2$ is defined by (64). So from now on, we consider that $h_k < h_2$. We write $\boldsymbol{v}_{h_k} = \nabla q_{h_k} + \boldsymbol{k}_{h_k}$, where $q_{h_k} = \pi_{1h_k}\boldsymbol{v}_{h_k}$ and $\boldsymbol{k}_{h_k} = \boldsymbol{\pi}_{2h_k}\boldsymbol{v}_{h_k}$. Note that $(\nabla q_{h_k})_k$ and $(\boldsymbol{k}_{h_k})_k$ are bounded sequences in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, because the continuity moduli of $(\pi_{1h_k})_k$ and $(\boldsymbol{\pi}_{2h_k})_k$ are bounded uniformly with respect to $k$ (cf. Proposition 15).

Step 1. Let us show that $\lim_{k\to\infty} \|\nabla q_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$. This is a simple consequence of (73). According to (66):

$$\forall k \in \mathbb{N}\setminus\{0\}, \sup_{q'_{h_k}\in M_{h_k}\setminus\{0\}} \frac{|a_\omega(\boldsymbol{v}_{h_k}, \nabla q'_{h_k})|}{\|\nabla q'_{h_k}\|_{0,\Omega}} \le k^{-1}.$$

But $a_\omega(\boldsymbol{v}_{h_k}, \nabla q'_{h_k}) = -\omega^2(\varepsilon \boldsymbol{v}_{h_k}|\nabla q'_{h_k})_{0,\Omega} = -\omega^2(\varepsilon\nabla q_{h_k}|\nabla q'_{h_k})_{0,\Omega}$. From (73), we infer that

$$\underline{\gamma}_0\, \omega^2 \|\nabla q_{h_k}\|_{0,\Omega} \le k^{-1} \to 0 \text{ as } k\to\infty.$$

Step 2. Let us show that $\lim_{k\to\infty} \|\boldsymbol{\pi}_2 \boldsymbol{v}_{h_k}\|_{0,\Omega} = 0$. This is a consequence of (64), and of step 1. Let $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, and $\boldsymbol{w}_{h_k} \in \boldsymbol{V}_{h_k}$:

$$\begin{aligned}|a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w})| &\le |a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w}-\boldsymbol{w}_{h_k})| + |a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w}_{h_k})|\\ &\le |||a_\omega|||\, \|\boldsymbol{w}-\boldsymbol{w}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + k^{-1}\|\boldsymbol{w}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}.\end{aligned}$$

According to the basic approximability property of $(\boldsymbol{V}_{h_k})_k$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, one can choose $(\boldsymbol{w}_{h_k})_k$ such that $\lim_{k\to\infty}\|\boldsymbol{w}-\boldsymbol{w}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$. In particular, $(\boldsymbol{w}_{h_k})_k$ is a bounded sequence in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, and one finds that

$$\lim_{k\to\infty} |a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{w})| = 0.$$

This result holds for all $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, so we have proved that $A_\omega \boldsymbol{v}_{h_k} \rightharpoonup 0$ (weakly) in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. On the other hand, the formulation (59) is well-posed, so $A_\omega^{-1}$ exists and $A_\omega^{-1} \in \mathcal{L}(\boldsymbol{H}_0(\mathbf{curl};\Omega))$. Hence $\boldsymbol{v}_{h_k} \rightharpoonup 0$ (weakly) in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. This implies that $\boldsymbol{\pi}_2\boldsymbol{v}_{h_k} \rightharpoonup 0$ (weakly) in $\boldsymbol{K}_N(\Omega,\varepsilon)$. And because the imbedding of $\boldsymbol{K}_N(\Omega,\varepsilon)$ in $\boldsymbol{L}^2(\Omega)$ is compact, one finds that $\lim_{k\to\infty}\|\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}\|_{0,\Omega} = 0$.

Step 3. Let us show that $\lim_{k\to\infty}\|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$. According to (64),

$$\|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le \tilde{\gamma}^{-1} \sup_{\boldsymbol{k}'_{h_k}\in \boldsymbol{K}_{h_k}(\varepsilon)\setminus\{0\}} \frac{|(\zeta\,\mathbf{curl}\,\boldsymbol{k}_{h_k}|\,\mathbf{curl}\,\boldsymbol{k}'_{h_k})_{0,\Omega}|}{\|\boldsymbol{k}'_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}}.$$

Let $\boldsymbol{k}'_{h_k} \in \boldsymbol{K}_{h_k}(\varepsilon)$. By definition of $\boldsymbol{k}_{h_k}$, one finds that

$$\begin{aligned}(\zeta\,\mathbf{curl}\,\boldsymbol{k}_{h_k}|\,\mathbf{curl}\,\boldsymbol{k}'_{h_k})_{0,\Omega} &= (\zeta\,\mathbf{curl}\,\boldsymbol{v}_{h_k}|\,\mathbf{curl}\,\boldsymbol{k}'_{h_k})_{0,\Omega}\\ &= a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{k}'_{h_k}) + \omega^2(\varepsilon\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}.\end{aligned}$$

According to (66), one has $|a_\omega(\boldsymbol{v}_{h_k}, \boldsymbol{k}'_{h_k})| \le k^{-1}\|\boldsymbol{k}'_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
On the other hand, by definition of $\boldsymbol{K}_{h_k}(\varepsilon)$,

$$\begin{aligned}|(\varepsilon\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}| &= |(\varepsilon\boldsymbol{\pi}_{2h_k}\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}|\\ &\le |(\varepsilon(\boldsymbol{\pi}_{2h_k}-\boldsymbol{\pi}_2)\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}| + |(\varepsilon\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}|.\end{aligned}$$

The last term is bounded by the Cauchy-Schwarz inequality

$$|(\varepsilon\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}| \leq \|\varepsilon\|_{L^\infty(\Omega)}\,\|\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}\|_{0,\Omega}\|\boldsymbol{k}'_{h_k}\|_{0,\Omega}.$$

There remains to evaluate the first term. Noting that $(\boldsymbol{\pi}_{2h_k} - \boldsymbol{\pi}_2)\boldsymbol{v}_{h_k} = -\nabla(\pi_{1h_k}\boldsymbol{v}_{h_k}) + \nabla(\pi_1\boldsymbol{v}_{h_k})$, one finds that, by definition of $\boldsymbol{K}_{h_k}(\varepsilon)$,

$$\begin{aligned}
|(\varepsilon(\boldsymbol{\pi}_{2h_k} - \boldsymbol{\pi}_2)\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}| &= |(\varepsilon(\nabla(\pi_1\boldsymbol{v}_{h_k}) - \nabla(\pi_{1h_k}\boldsymbol{v}_{h_k}))|\boldsymbol{k}'_{h_k})_{0,\Omega}| \\
&= |(\varepsilon\nabla(\pi_1\boldsymbol{v}_{h_k})|\boldsymbol{k}'_{h_k})_{0,\Omega}| \\
&= |(\varepsilon\nabla(\pi_1\boldsymbol{v}_{h_k})|\boldsymbol{k}'_{h_k} - \boldsymbol{k})_{0,\Omega}| \\
&\leq \|\varepsilon\|_{L^\infty(\Omega)}\,\|\nabla(\pi_1\boldsymbol{v}_{h_k})\|_{0,\Omega}\|\boldsymbol{k}'_{h_k} - \boldsymbol{k}\|_{0,\Omega},
\end{aligned}$$

where $\boldsymbol{k} \in \boldsymbol{K}_N(\Omega,\varepsilon)$ is chosen as in theorem 4. Thanks to the fact that $\|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 1$ (cf. (66)), one gets the bound

$$|(\varepsilon(\boldsymbol{\pi}_{2h_k} - \boldsymbol{\pi}_2)\boldsymbol{v}_{h_k}|\boldsymbol{k}'_{h_k})_{0,\Omega}| \lesssim {h_k}^{\mathsf{s}}\,\|\,\mathbf{curl}\,\boldsymbol{k}'_{h_k}\|_{0,\Omega}.$$

Aggregating the above estimates, one finds that

$$\|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \left(k^{-1} + \|\boldsymbol{\pi}_2\boldsymbol{v}_{h_k}\|_{0,\Omega} + {h_k}^{\mathsf{s}}\right),$$

thus leading to $\lim_{k\to\infty}\|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$ according to Step 2.
Step 4. For all $k$, one has $\|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq \|\nabla q_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{k}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$ by the triangle inequality, so one concludes that $\lim_{k\to\infty}\|\boldsymbol{v}_{h_k}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = 0$, which contradicts (66). □

One can finally derive the (classical) error estimate.

**Corollary 6** *Let the assumptions of theorem 6 be fulfilled, and let $h_\omega > 0$ be the threshold value defined there. Then, for all $h \leq h_\omega$, the discrete variational formulation (63) is well-posed. Moreover, one has the error estimate:*

$$\forall h \leq h_\omega, \quad \|\boldsymbol{e} - \boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{e} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \tag{67}$$

As before, one can find the a priori regularity of the electric field and of its curl, to derive explicit convergence rates. The analysis is very similar to the one of the div-curlcurl problem, and is left to the reader.

## 9 Conclusions and extensions

We have studied the div-curl and div-curlcurl problems, and the time-harmonic Maxwell equations, for classical and interface models. For the latter model, a sign-change of the coefficient appearing in the divergence condition is possible. We have also proved optimal convergence rates on the error, when the numerical approximation is computed with the help of the Nédélec's first family of edge finite elements. For low-regularity solutions, those results are achieved with the help of the combined interpolation operator introduced in [28]. All those results have been obtained with the help of explicit T-coercivity operators for the derivation of the inf-sup conditions. There are several possible extensions.

The first obvious extension is to deal with a non-vanishing tangential trace, namely replacing $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega$ by $\boldsymbol{u} \times \boldsymbol{n} = \mathrm{u}_\Gamma$ on $\partial\Omega$ in (1), (53) or (57), where the data $\mathrm{u}_\Gamma$ defined on $\partial\Omega$ is actually equal to the tangential trace of some field $\boldsymbol{u}^\star \in \boldsymbol{H}(\mathbf{curl};\Omega)$. Introducing $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{u}^\star \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, one finds that $\boldsymbol{u}_0$ solves the div-curl problem (1), the div-curlcurl problem (53), or the time-harmonic Maxwell equations (57), with modified right-hand sides. Hence one may study these problems as before. In order to determine explicit convergence rates, one needs to have some *ad hoc* extra-regularity assumptions on $\boldsymbol{u}^\star$.

Among other possible extensions, one could prove similar results both from the theoretical and numerical viewpoints for the div-curl model with vanishing normal trace, namely

$$\begin{cases} \textit{Find } \boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \textit{ such that} \\ \mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \\ \operatorname{div} \xi\boldsymbol{u} = g \text{ in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{68}$$

under appropriate assumptions on the data. We refer to §6.2 in [3] for pointers on how to address the classical case theoretically, from which one may iterate as in the present paper.

And for a div-curl problem with mixed boundary conditions, we refer to Fernandes and Gilardi [38] and Jochmann [43] to initiate the studies.

Last, another interesting extension is to investigate the div-curlcurl problem, or the time-harmonic Maxwell equations, with *two* sign-changing coefficients.

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# A Practical T-coercivity for the companion scalar problem

## A.1 Explicit T-coercivity operators

In practice, how to realize *explicitly* the T-coercivity for a well-posed companion scalar problem (3) in the interface case? The concept was originally introduced in [15] (see Theorem 2.1).

We provide a list *à la Prévert* to describe a number of situations where explicit T-coercivity operators are available, taking into account the geometry of the domain $\Omega$, and the shape of the interface induced by the partition $\mathcal{P} = (\Omega_p)_{p=+,-}$. In some cases the results are known for domains in $\mathbb{R}^2$ (we use the notations $\Omega_2$, resp. $(\Omega_{2p})_{p=+,-}$). We rely on Refs. [52, 11, 25, 12, 19, 10] for the precise results:

- the geometry is symmetric with respect to the interface, cf. §5.1 in [52] or §3.1 in [11]; this implies that the interface is a subset of a hyperplane;
- the geometry is tubular with respect to the interface, with a smooth interface, cf. §3.4 in [11];
- the domain $\Omega_2$ is a disk or an angular sector in $\mathbb{R}^2$, and $\Omega_{2+}$ and $\Omega_{2-}$ are angular subsectors, cf. §3.2 in [11], or the domain $\Omega_2$ is the union of self-replicating triangles in $\mathbb{R}^2$, and $\Omega_{2+}$ and $\Omega_{2-}$ are union of contiguous triangles, cf. §3 in [10]; this implies that the interface has exactly one corner inside $\Omega_2$. This can be generalized to a geometry in $\mathbb{R}^3$, by taking $\Omega := \Omega_2 \times (a, b)$, resp. $\Omega_\pm := \Omega_{2\pm} \times (a, b)$, cf. §7.2 in [12], for some $a < b$; this implies that the interface has exactly one edge, and no vertex, inside $\Omega$.
- $\Omega$ is the cube $(-a, a)^3$, $\Omega_+$ or $\Omega_-$ is the sub-cube $(0, a)^3$, cf. §7.3 in [12], for some $a > 0$; or §5.2 in [52] for the same setting in a square domain $\Omega_2$ in $\mathbb{R}^2$.

Then one can build explicitly an operator $T_0$ that fulfills (9). We say that there is a "full" T-coercivity operator $T_0$ available. In all of the above, the operator $T_0$ is derived from elementary geometrical transforms, such as symmetries, rotations and angle dilation. Except for the latter, all those transforms can be used after discretization, provided the underlying discrete geometrical structures (in our case, the meshes, see section 4) are *conforming* with respect to the transforms.

One can check that, thanks to the generic definition of the operators $T_0$ that is used (cf. p. 1915 in [15] or p. 4274 in [52]), in all instances, one has $(T_0)^2 = \mathbb{I}_{H^1_0(\Omega)}$. This leads to a similar property for the operator $\mathbb{T} \in \mathcal{L}(\mathbb{V})$ provided in the proof of theorem 2. As a matter of fact, one has $T_0 p \in H^1_0(\Omega)$ and $\boldsymbol{\pi_2 u} \in \boldsymbol{K}_N(\Omega, \xi)$, so that

$$\begin{aligned}\mathbb{T}^2((\boldsymbol{u}, p)) &= \mathbb{T}((\nabla(T_0 p) + \boldsymbol{\pi_2 u}, T_0(\pi_1 \boldsymbol{u}))) \\ &= ((\nabla(T_0(T_0(\pi_1 \boldsymbol{u}))) + \boldsymbol{\pi}_2[\nabla(T_0 p) + \boldsymbol{\pi_2 u}], T_0(T_0 p))) \\ &= (\nabla((T_0)^2(\pi_1 \boldsymbol{u})) + (\boldsymbol{\pi}_2)^2 \boldsymbol{u}, (T_0)^2 p) \\ &= (\nabla(\pi_1 \boldsymbol{u}) + \boldsymbol{\pi_2 u}, p) = (\boldsymbol{u}, p).\end{aligned}$$

On the other hand, in many other configurations, and even though the scalar problem (3) is well-posed, only a "weak" T-coercivity operator $T$, defined in the following sense (see Lemma 2 in [10]), can be built explicitly:

$$\begin{cases} \exists \alpha, \beta > 0,\ \exists T \in \mathcal{L}(H^1_0(\Omega)) \text{ bijective}, \\ \forall q \in H^1_0(\Omega),\ (\xi \nabla q | \nabla(Tq))_{0,\Omega} \geq \alpha \, \|\nabla q\|^2_{0,\Omega} - \beta \, \|q\|^2_{0,\Omega}. \end{cases} \tag{69}$$

The main idea (see §4.3 in [11]) to build those operators is to use localization arguments. For that, the mathematical tool of choice is an *ad hoc* partition of unity function. First, one can focus on a neighborhood of the interface. Second, one separates corners and edges (in $\mathbb{R}^2$), or one splits a smooth interface, etc., into elementary blocks that fit locally the situations described above. We provide another list *à la Prévert* in which such a "weak" T-coercivity operator $T$ can be built. The geometry of the domain $\Omega$, and the partition $\mathcal{P} = (\Omega_p)_{p=+,-}$ are such that:

- the geometry is locally symmetric with respect to the interface, cf. §4 in [25] or §7.4 in [12];

- the interface is smooth, cf. §2.B.1 in [19].
- the partition of the domain $\Omega_2$ is such that the interface separating $\Omega_{2+}$ and $\Omega_{2-}$ is polygonal, cf. §4 in [10]; this can be generalized to a geometry in $\mathbb{R}^3$, by taking $\Omega := \Omega_2 \times (a,b)$, resp. $\Omega_\pm := \Omega_{2\pm} \times (a,b)$, for some $a < b$; in principle, in $\mathbb{R}^3$, it could be generalized to a polyhedral interface.

Again in all instances above, one has $T^2 = \mathbb{I}_{H_0^1(\Omega)}$, see Lemma 2 in [10].

*Remark 13* Notice that (69) also fits the original concept of T-coercivity, cf. §2 in [15].

## A.2 Discrete T-coercivity for the companion scalar problem

We assume below that the companion scalar problem (3) is well-posed.
With the help of "full" or "weak" T-coercivity operators for this problem, one may define discrete T-coercivity operators that help prove well-posedness of the discrete scalar problems (20). As a matter of fact, this is made possible thanks to the use, in the definition of the exact operators $T_0$ ("full" T-coercivity operator) and $T$ ("weak" T-coercivity operator), of elementary geometrical transforms, such as symmetries and rotations. This happens when the interface is part of a hyperplane, polygonal (in $\mathbb{R}^2$) or polyhedral (in $\mathbb{R}^3$). Also, one needs to interpolate the partition of unity function for the "weak" T-coercivity operator. Then, one can implement the discrete operators: this amounts to using (locally for the "weak" T-coercivity operator) T-conform meshes. Namely, the mesh is first built in $\Omega_-$, and then mapped to $\Omega_+$ via the same geometrical transforms as the ones that were chosen to design $T_0$ or $T$, in order to define the mesh there. Or the other way around, from $\Omega_+$ to $\Omega_-$. For the "weak" T-coercivity operator, the process is localized to a neighborhood of the interface. We refer to [25, 10] for details.
Consequently, when one has at hand a "full" T-coercivity operator $T_0$, it can also be used to establish the uniform discrete T-coercivity of the discrete scalar problems (20). Namely, $T_0$ is such that

$$\begin{cases} \forall h, \quad T_0[M_h] \subset M_h, \text{ and} \\ \exists \alpha_0' > 0, \ \forall h, \ \forall q_h \in M_h, \ (\xi \nabla q_h | \nabla (T_0 q_h))_{0,\Omega} \geq \alpha_0' \, \|\nabla q_h\|^2_{0,\Omega}. \end{cases} \tag{70}$$

As a first consequence of (70), we note that since $(T_0)^2 = \mathbb{I}_{H_0^1(\Omega)}$, one has actually $T_0[M_h] = M_h$ for all $h$. Another by-product of (70) is that $(q_h, q_h') \mapsto (\xi \nabla q_h | \nabla q_h')_{0,\Omega}$ fulfills a *uniform discrete inf-sup condition*, ie.

$$\exists \underline{\gamma}_0 > 0, \ \forall h, \ \forall q_h \in M_h, \qquad \sup_{q_h' \in M_h \setminus \{0\}} \frac{|(\xi \nabla q_h | \nabla q_h')_{0,\Omega}|}{\|q_h'\|_{H_0^1(\Omega)}} \geq \underline{\gamma}_0 \, \|q_h\|_{H_0^1(\Omega)}. \tag{71}$$

So, the discrete scalar problems (20) are well-posed, and the classical error estimate holds: $\|s - s_h\|_{H_0^1(\Omega)} \lesssim \inf_{q_h \in M_h} \|s - q_h\|_{H_0^1(\Omega)}$. See Theorem 2 in [25] for details.
On the other hand, when one has at hand a "weak" T-coercivity operator $T$, because of the presence of the partition of unity function, one builds "weak" discrete T-coercivity operators (see Lemma 3 in [10]), that is discrete operators $(T_h)_h$ such that

$$\exists C, h_0 > 0, \ \forall h \leq h_0, \ \exists T_h \in \mathcal{L}(M_h), \qquad \sup_{q \in M_h \setminus \{0\}} \frac{\|\nabla (T - T_h) q_h\|_{0,\Omega}}{\|\nabla q_h\|_{0,\Omega}} \leq C\, h.$$

Obviously, $\sup_h \|T_h\|_{\mathcal{L}(M_h)} < \infty$. We call this situation the *"weak" T-coercivity framework*. It follows that one has a *"weak"* discrete T-coercivity property (pp. 820-821 in [10]):

$$\exists \underline{\alpha}, \underline{\beta}, h_0 > 0, \ \forall h \leq h_0, \quad \forall q_h \in M_h, \ (\xi \nabla q_h | \nabla (T_h q_h))_{0,\Omega} \geq \underline{\alpha} \, \|\nabla q_h\|^2_{0,\Omega} - \underline{\beta} \, \|q_h\|^2_{0,\Omega}. \tag{72}$$

Then, thanks to Proposition 3 in [25] where one argues by contradiction[3], one can prove that $(q_h, q'_h) \mapsto (\xi\nabla q_h|\nabla q'_h)_{0,\Omega}$ fulfills a *uniform discrete inf-sup condition, for h small enough*, ie.

$$\exists \underline{\gamma}_0, h_0 > 0,\ \forall h \le h_0,\ \forall q_h \in M_h, \qquad \sup_{q'_h \in M_h\setminus\{0\}} \frac{|(\xi\nabla q_h|\nabla q'_h)_{0,\Omega}|}{\|q'_h\|_{H^1_0(\Omega)}} \ge \underline{\gamma}_0 \, \|q_h\|_{H^1_0(\Omega)}. \quad (73)$$

So, one can derive results for the discrete scalar problems (20) that are similar to those that where obtained when a "full" T-coercivity operator was available, now for $h$ small enough, that is when $h \le h_0$.
Finally, when the interface is smooth, the same guidelines apply, see §2.B.1 in [19]. In this case, one needs to have at hand some curvilinear finite elements, such as isoparametric finite elements (cf. §4.3 in [26]), near the interface. It is known that optimal interpolation properties hold, ie. one may recover up to $O(h)$ accuracy using Lagrange's first-order finite elements for a sufficiently smooth scalar field. Or, one can choose the approach of [45] to achieve again optimal convergence rate: for that one needs a family of simplicial meshes which resolve the smooth interface sufficiently well. Observe that for first-order edge finite elements, the latter approach can also be used, to yield $O(h)$ interpolation accuracy for a sufficiently smooth vector field of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ (see [42]).

[3] In proposition 16 in section 6, we proceed similarly to derive a uniform discrete inf-sup condition for the form $a$. A proof is given there. Note that because we argue by contradiction, bounds are not explicit anymore.