View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by INRIA a CCSD electronic archive server

Ground state solitary waves local controllability for the nonlinear focusing Schrödinger equation in the mass critical and slighlty mass subcritical regime

Ludovick Gagnon

To cite this version:

Ludovick Gagnon. Ground state solitary waves local controllability for the nonlinear focusing Schrödinger equation in the mass critical and slighlty mass subcritical regime. 2021. hal-03141045v2

HAL Id: hal-03141045

https://hal.archives-ouvertes.fr/hal-03141045v2

Preprint submitted on 1 Mar 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Ground state solitary waves local controllability for the nonlinear focusing Schrödinger equation in the mass critical and slightly mass subcritical regime

Ludovick Gagnon *

March 1, 2021

Abstract

In this paper, we prove the exact local controllability around different ground state solitary wave for the slightly subcritical mass and mass critical nonlinear Schrödinger equation. More precisely, if Q_{c_1} and Q_{c_2} denotes the ground states with two different scaling, we prove the exact local controllability from Q_{c_1} to Q_{c_2} in a minimal time depending on c_1 and c_2. The results presented relies on the blow-up profile in the mass slightly supercritical case and mass critical case.

Keywords : Nonlinear focusing Schrödinger equation, controllability, solitary waves

MSC2020 : 93B05 35Q51 35Q55 93C10

1 Introduction

We study in this article the controllability of the ground state solitary wave in the mass critical and slightly subcritical mass regime of the nonlinear focusing Schrödinger equation,

$$\begin{cases} i\psi_t + \Delta\psi + \psi|\psi|^{p-1} = 0, & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \psi(x,0) = \psi_0, & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

with $\psi_0 \in H^1(\mathbb{R}^d)$ and $p > 1$. Equation (1) has many physical applications for $p = 3$ and $d = 1, 2$, as it serves as a model for signal propagation in nonlinear optic for optic fibers and self-focusing laser beams in hollow core fibers ([15, 42]). The nonlinear focusing Schrödinger equation (1) is also completely integrable if $d = 1$ and $p = 3$ ([7]). The solutions to (1) conserve three quantities over time,

$$\text{Mass: } \|\psi(.,t)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)},$$

$$\text{Energy: } E(\psi(.,t)) := \frac{1}{2}\|\nabla\psi(.,t)\|_{L^2}^2 - \frac{1}{p+1}\|\psi(.,t)\|_{L^{p+1}}^{p+1} = E(\psi_0),$$

$$\text{Momentum: } \operatorname{Im}\left(\int \nabla\psi(x,t)\overline{\psi}(x,t)dx\right) = \operatorname{Im}\left(\int \nabla\psi_0(x)\overline{\psi}_0(x)dx\right).$$

[1]Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France. E-mail : `ludovick.gagnon@inria.fr`.

Moreover, solutions to (1) admits a number of symmetries in $H^1(\mathbb{R}^d)$ for $1 < p < (d+2)/(d-2)$ ([29]). If $\psi(x,t)$ solves (1), then so does

$$\text{Space, time and phase invariance: } e^{i\gamma}\psi(x+x_0,t+t_0),\ x_0,t_0,\gamma \in \mathbb{R},$$
$$\text{Galilean invariance: } \psi(x-\beta t,t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)},\beta \in \mathbb{R}.$$

Finally, solutions to (1) are invariant with respect to the scaling

$$\psi_\lambda(x,t) = \lambda^{\frac{2}{p-1}}\psi(\lambda x, \lambda^2 t), \quad \lambda > 0, \tag{2}$$

and the scaling is preserved for the homogeneous Sobolev norm $\|\psi_\lambda(t)\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|\psi(.,t)\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$ with

$$s_c := \frac{d}{2} - \frac{2}{p-1}.$$

We recall that equation (1) is said to be mass critical if $s_c = 0$, mass subcritical if $s_c < 0$ and mass supercritical if $s_c > 0$. Likewise, it is said to be energy subcritical if $s_c < 1$, energy critical if $s_c = 1$ and energy supercritical if $s_c > 1$.

1.1 Bound states

The bound states, or solitary waves, are stationary solutions to (1) of the form $\psi(x,t) = e^{i\lambda t}W_\lambda(x)$ where $W_\lambda(x) = \lambda^{\frac{1}{p-1}}W(\sqrt{\lambda}x)$ is solution to the nonlinear elliptic equation,

$$\Delta W_\lambda + |W_\lambda|^{p-1}W_\lambda = \lambda W_\lambda, \quad x \in \mathbb{R}^d. \tag{3}$$

Non-trivial solutions to (3) exist if the energy is subcritical and if and only if $\lambda > 0$. The solutions W_λ to (3) belongs to $W^{3,q}(\mathbb{R}^d)$ for any $2 \le q < \infty$ and numerous properties of bound states have been established. It is well-known that solutions to (3) are not unique, but unicity is recovered under the additional constraint of seeking a positive and radially symmetric solution to (3) [43, 44]. This unique solution denoted $Q_\lambda(x) = \lambda^{\frac{1}{p-1}}Q(\sqrt{\lambda}x)$, the ground state solitary wave, belongs to the Schwarz class $\mathcal{S}(\mathbb{R}^d)$ and has an explicit representation in dimension one,

$$Q(x) = \left(\frac{p+1}{2}\operatorname{sech}^2\left(\frac{p-1}{2}x\right)\right)^{\frac{1}{p-1}}.$$

In higher dimensions, the exponential decay of the ground state is characterized by the following estimate for $r \ge 1$ ([7, 42]),

$$\left|Q(r) - \kappa r^{-(d-1)/2}e^{-r}\right| + \left|Q'(r) - \kappa r^{-(d-1)/2}e^{-r}\right| \le Cr^{-(d+1)/2}e^{-r}, \tag{4}$$

where $r = |x|$ and where $\kappa, C > 0$ are two constants, depending on $d \ge 1$. Here and below, the constant κ will always refer to the constant appearing in the inequality (4). Finally, Q_λ satisfies $E(Q_\lambda) = \lambda E(Q) = 0$ in the mass critical case thanks to Pohozaev's identity [7]. We refer to [43, Appendix B] for additional properties of the ground state.

1.2 Well-posedness in the different regimes in $\mathbb{R}^d$

The ground state Q plays an important role in the description of the global well-posedness of (1). The local well-posedness of (1) with initial data $\psi_0 \in H^1(\mathbb{R}^d)$ is established by the Cauchy theory if the energy is subcritical for $d \ge 3$, and without any restriction on $p > 1$

for $d = 1, 2$ ([7, 13]). Such solutions belongs to $\psi \in C((-T_{\min}, T_{\max}); H^1(\mathbb{R}^d))$, with $T_{\min} = T_{\min}(\psi_0), T_{\max} = T_{\max}(\psi_0)$ and $T_{\min}, T_{\max} \in [0, \infty)$. Moreover, there is a blow-up alternative ([7]), that is to say, either $T_{\min} = T_{\max} = \infty$ and the solution is defined globally, or, if $T_{\max} < \infty$ (resp. $T_{\min} < \infty$), then $\lim_{t \to T_{\max}^-} \|\psi(t)\|_{H^1(\mathbb{R}^d)} = +\infty$ (resp. $\lim_{t \to T_{\min}^-} \|\psi(-t)\|_{H^1(\mathbb{R}^d)} = +\infty$). In the mass subcritical regime $s_c < 0$, the conserved quantities as well as the Gagliardo-Nirenberg inequality allows to extend the time existence, $T_{\min}$ and $T_{\max}$, of all solutions to (1), implying the global well-posedness in $H^1(\mathbb{R}^d)$ ([7]).

The mass critical regime $s_c = 0$ is the first regime to exhibit blow-up phenomenon. Weinstein proved in [44] that the estimate $\|\psi_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$ ensures the global existence of solutions to (1), thanks to the conservation of mass, energy and the Gagliardo-Nirenberg inequality

$$E(\psi) \geq \frac{1}{2}\|\psi\|_{\dot{H}^1(\mathbb{R}^d)}^2 \left(1 - \left(\frac{\|\psi\|_{L^2(\mathbb{R}^d)}^2}{\|Q\|_{L^2(\mathbb{R}^d)}^2}\right)^{4/d}\right), \quad \forall \psi \in H^1(\mathbb{R}^d).$$

This estimate was proved to be sharp by Merle [26] in the following sense. Assume $\|\psi_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)}$ and assume that the solution $\psi(.,t)$ of (1) blows up in finite time $T > 0$. Then the solution ψ is given (with an initial data at $t = -T$ for simplicity) by the pseudo conformal transformation profile

$$S(x,t) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}}. \tag{5}$$

This profile lies in the set $\Sigma = H^1(\mathbb{R}^d) \cap \{x\psi_0 \in L^2(\mathbb{R}^d)\}$, invariant by the flow, and blows up at the speed $\|\nabla S(.,t)\|_{L^2(\mathbb{R}^d)} \simeq 1/|t|$, $t \to 0^-$. A general blow-up phenomenon occurs in Σ in the range $0 \leq s_c < 1$ due to the so-called virial identity. Indeed, assume $\psi_0 \in \Sigma = H^1(\mathbb{R}^d) \cap \{x\psi_0 \in L^2(\mathbb{R}^d)\}$ and $E(\psi_0) < 0$. Then, the associated solution $\psi(.,t)$ of (1) belongs to Σ and

$$\frac{d^2}{dt^2}\|x\psi(.,t)\|_{L^2(\mathbb{R}^d)}^2 = 4d(p-1)E(\psi_0) < 0,$$

leading to a contradiction on the infinite time of existence of the solution.

The blow-up phenomenon in the mass critical regime was further investigated in a serie of works [27, 28, 29, 30, 31, 36]. The blow-up phenomenon was studied in the set

$$\mathcal{B}_{\alpha^*} = \left\{\psi_0 \in H^1(\mathbb{R}^d) \mid \|Q\|_{L^2}^2 < \|\psi_0\|_{L^2}^2 < \|Q\|_{L^2}^2 + \alpha^*\right\}$$

for $\alpha^* > 0$ sufficiently small. Assuming the *spectral property*[1], it was shown that if $E(\psi_0) > 0$, then blow-up occurs at rate $1/|t|$, the same rate as the pseudo conformal transformation (5) and if $E(\psi_0) < 0$, then blow-up occurs at the rate

$$\|\nabla\psi(.,t)\|_{L^2(\mathbb{R})} \simeq \sqrt{\frac{\log|\log(T-t)|}{T-t}}.$$

We shall denote here the concentration factor in the mass critical regime

$$\lambda^*(t) = \sqrt{\frac{T-t}{\log|\log(T-t)|}}, \quad t < T.$$

[1]Without going into details, the spectral property is related to the coercivity of linear form coming from the linearization around state Q along with H^1 orthogonality properties. It was proven in dimension $d = 1$ in [29] and in dimensions $d = 2, 3, 4$ using rigorously verified computing [11]. It is conjectured that the spectral property holds in any dimension. We refer to these works for a precise definition.

This rate is not self-similar as

$$\lim_{t\to T^-}\frac{\lambda^*(t)}{\sqrt{T-t}}=0.$$

In fact, more can be said about the blow-up profile in the case $E(\psi_0)<0$. Indeed, in [29], it is shown that the solution $\psi(.,t)$ coming from the initial data $\psi_0\in\mathcal{B}_{\alpha^*}$ such that $E(\psi_0)<0$ decomposes as

$$\psi(x,t)=\frac{e^{i\gamma(t)}}{\lambda(t)}(Q+\epsilon)\left(\frac{x-x(t)}{\lambda(t)},t\right),$$

with $\|\epsilon\|_{H^1(\mathbb{R}^d)}\to 0$ as $\alpha^*\to 0$. An important remark that we shall use in the present article is that, if the initial data ψ_0 is radial, then translation parameter $x(t)$ is equal to 0 ([29]).

Let us underline here that the mass subcritical case $s_c<0$ is in sharp contrast with the mass critical and supercritical regime. Not only the solutions are always globally defined in $H^1(\mathbb{R}^d)$, but the ground state is in fact orbitally stable with respect to $H^1(\mathbb{R}^d)$ pertubations, that is, stable up to translation in space and phase shift [8]. More precisely, for any $\epsilon>0$, there exists $\delta>0$ such that if,

$$\inf_{(\gamma,x_0)\in[0,2\pi]\times\mathbb{R}}\|\psi_0(.-x_0)-Q(.)e^{i\gamma}\|_{H^1(\mathbb{R}^d)}<\delta,$$

then

$$\inf_{(\gamma,x_0)\in[0,2\pi]\times\mathbb{R}}\|\psi(.-x_0,t)-e^{i(t+\gamma)}Q(.)\|_{H^1(\mathbb{R}^d)}<\epsilon.$$

Finally, the mass intercritical case $0<s_c<1$ is conjectured ([41, 45]) to exhibit self-similar blow-up in finite time for initial data close to the ground state Q in $\dot{H}^1(\mathbb{R}^d)$. The existence and stability of the self-similar blow-up was proved in [32] for $0<s_c\ll 1$ and a precise description of the blow-up profile was obtained in a recent work of Bahri, Martel and Raphaël [1].

Before stating our main results, we present a short overview of the controllability of solitary waves as well as the controllability of the Schrödinger equation, as it will ease the presentation and allow us to state more precisely the nature of our results.

1.3 Literature overview

Despite the extensive literature on solitary waves, it seems that little is known so far on its controllability properties. If the solitary waves are defined on the whole space, only one result exists on the controllability of the ground state for the generalized KdV equation (where the ground state is also solution to (3)) due to Muñoz. In [34], considering the ground state on the whole line and using a moving distributed control, Muñoz proved the approximate controllability in large time from Q_{c_1} to Q_{c_2} for $c_1,c_2\in\mathbb{R}^+$. More precisely, the nonlinear interaction between the control and the solitary wave allows Muñoz to add (or remove, depending on the sign of c_2-c_1) an important mass to the ground state with a slow varying distributed control. It is important to highlight that the control strategy is not destructive, meaning that the solution over time remains close to the ground state for $t\in[0,T]$. We also note a second result on linking the controllability and the solitary waves defined on the real line. In [12], the N-solitons solution was used by the author as a trajectory to achieve small-time Lagrangian controllability for the Korteweg-de Vries equation.

Other results are found in the literature on the controllability of the ground state when defined on bounded domains. Lange and Teismann [19], considered the 1-D nonlinear focusing Schrödinger equation on a bounded domain with homogeneous Dirichlet boundary conditions and with a distributed control. In [19], the ground state is defined as the unique positive solution in $x \in (0,1)$ of

$$\begin{cases} \psi''(x) + \psi^3(x) = \psi(x), \quad x \in (0,1), \\ \psi(0) = \psi(1) = 0. \end{cases} \tag{6}$$

The local controllability around the ground state was obtained using the HUM method [24] and a fine spectral analysis of the underlying linearized equation. A similar question was addressed by Castelli and Teismann in [6], but this time with the size $L(t)$ of the 1-D domain having the role of the control.

We also find few results on the stabilization of the solitary waves. In [33], the approximate stability of the bilinear Schrödinger equation in $\mathbb{R}^d$, modelling the action of a laser on a quantum particle, around ground state solutions to

$$(\Delta + V(x))\psi(x) = 0, \quad x \in \mathbb{R}^d,$$

was proven by Mirrahimi under spectral hypothesis of the operator $\Delta + V$. In the particular case of a dipolar function $\mu(x) = x$ and with a domain equal to the interval $(-1/2, 1/2)$, a global practical stabilisation was obtained by Beauchard and Mirrahimi through a Lyapunov analysis in [3].

We now turn to general controllability properties of the nonlinear Schrödinger equation. We first emphasize on the result of Rosier and Zhang [37], as the extension of their results shall be needed in the present work. They have proved the exact local controllability around smooth trajectories to (1) of

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^2\psi = 0, & (x,t) \in \Omega \times (0,T), \\ \psi(x,0) = \psi_0(x), & x \in \Omega, \end{cases} \tag{7}$$

for $\Omega \subset \mathbb{R}^d, d \geq 1$ an open, connected and bounded domain with a boundary $\partial\Omega$ of class C^2. The boundary control v is imposed either on the Dirichlet or Neumann boundary condition,

$$\psi(x,t) = v(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \quad \text{or} \quad \frac{\partial \psi}{\partial \nu}(x,t) = v(x,t), \quad (x,t) \in \partial\Omega \times (0,T), \tag{8}$$

where ν denotes the outward normal vector to Ω. More precisely,

Theorem 1 (Theorem 1.1, [37]). *Let $T > 0$ be given and let $w \in C^\infty([-\epsilon, T+\epsilon]; S(\mathbb{R}))$ be a solution of the nonlinear Schrödinger equation, with $\lambda \in \mathbb{C}^*$,*

$$iw_t + \Delta w + \lambda |w|^2 w = 0,$$

for any $(x,t) \in \Omega_1 \times (-\epsilon, T+\epsilon)$ where $\epsilon > 0$ and Ω_1 is a bounded domain in $\mathbb{R}^d$ with $\overline{\Omega} \subset \Omega_1$. Assume

$$s > \frac{d}{2}, \quad \text{or} \quad 0 \leq s < \frac{d}{2} \text{ with } 1 \leq d \leq 2 + 2s, \quad \text{or} \quad s = 0, 1 \text{ with } d = 2.$$

Then there exists a $\delta > 0$ such that for any $\psi_0, \psi_1 \in H^s(\Omega)$ satisfying

$$\|\psi_0 - w(.,0)\|_{H^s(\Omega)} \leq \delta, \quad \|\psi_1 - w(.,T)\|_{H^s(\Omega)} \leq \delta,$$

one can find an appropriate boundary control function $v(x,t)$ such that (7)-(8) admits a solution $\psi \in C([0,T]; H^s(\Omega))$ such that

$$\psi(x,0) = \psi_0(x), \quad \psi(x,T) = \psi_1(x), \text{ in } \Omega.$$

The control strategy deployed in [37] consists to extend the initial data ψ_0 to $\mathbb{R}^d$ and to consider the control problem in the whole space

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^2\psi = \varphi(x)h(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{9}$$

where $\varphi \in C^\infty(\mathbb{R}^d; [0,1])$ is a cut-off function such that

$$\varphi(x) = \begin{cases} 1, & \text{if } |x| \geq R+1 \\ 0, & \text{if } |x| \leq R \end{cases} \tag{10}$$

with $R > 0$ sufficiently large so that $\Omega \subset B_R(0)$. This strategy of extending the solution to the whole space to deduce boundary controllability property is attributed to Russell who first used it for the wave equation in [38, 39]. Littman and Taylor later on gave a general principle for the boundary controllability of a linear partial differential equation, that is reversibility, smoothing properties and uniqueness usually leads to controllability [25]. In [37], the smoothing properties come from the Strichartz estimates available in $\mathbb{R}^d$. The controllability of the linearized equation associated to (9) around the smooth trajectory w is obtained in [37] via Carleman estimates for $C^\infty([-\epsilon, T+\epsilon]; \mathcal{S}(\mathbb{R}))$ potentials. The local exact controllability of (9) around smooth trajectories is deduced by a fixed point argument. Theorem 1 is obtained by constructing the control v in (8) as the trace of (9), and the controlled solution of (7) is obtained as the restriction of the solution of (9) to Ω. Such solutions may be defined in a weak sense (see Section 2.4).

Other controllability results were obtained for the nonlinear Schrödinger equation. The controllability of the nonlinear defocusing Schrödinger equation on a compact manifold without boundary was proved by Dehman, Gérard and Lebeau [10] using the exponential decay of the nonlinear problem and local controllability, assuming that the support of the internal control satisfies the Geometric Control Condition [2], where the time of controllability depends on the size of the initial data. This argument of first stabilizing the solution and then using local controllability around zero was also used by Laurent in [20, 21]. The Geometric Control Condition (GCC) was shown by Lebeau to be sufficient for the controllability of the linear Schrödinger equation [23], but unlike the linear wave equation where GCC is necessary and sufficient (see [4] for a precise statement), GCC is not always necessary, as illustred by Jaffard [16] and Burq and Zworski [5]. It was more recently proved by Jin that a control supported in any nonempty open set yields the controllability on hyperbolic surfaces (see [17] and reference therein). Extensive study was also done on the controllability of the linear and bilinear Schrödinger equation and with boundary or internal control. We refer the reader to the surveys [22, 46] on the subject

1.4 Main results

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, with $d \geq 1$ be an open, connected and bounded domain with a boundary $\partial\Omega$ of class C^2. In the spirit of [37], we consider the boundary controllability of the focusing nonlinear Schrödinger equation

$$\begin{cases} i\psi_t + \Delta\psi + \psi|\psi|^{p-1} = 0, & (x,t) \in \Omega \times (0,T), \\ \psi(x,t) = v(x,t), & (x,t) \in \partial\Omega \times (0,T), \\ \psi(x,0) = \psi_0(x), & x \in \Omega, \end{cases} \tag{11}$$

where $v \in L^2((0,T); L^2(\partial\Omega))$ is the boundary control. We first state the exact local controllability around ground state with different scaling in the mass critical regime $p-1 = 4/d$. This result covers the physically relevant case $p = 3$ in dimension $d = 2$.

Theorem 2. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be an open, connected and bounded with a boundary $\partial\Omega$ of class C^2. Let $c_1, c_2 \in \mathbb{R}^+$ and $c_1 < c_2$. For $d \geq 5$, assume that the spectral property ([29]) holds. Then there exists $\epsilon > 0$ such that, for every $\psi_0, \psi_1 \in H^1(\Omega)$ such that*

$$\inf_{\gamma\in[0,2\pi]} \|\psi_0 - e^{i\gamma}Q_{c_1}\|_{H^1(\Omega)} < \epsilon, \quad \inf_{\gamma\in[0,2\pi]} \|\psi_1 - e^{i\gamma}Q_{c_2}\|_{H^1(\Omega)} < \epsilon,$$

there exists a control $v \in C([0,T]; H^{1/2^-}(\partial\Omega))$ such that (11) *admits a solution $\psi \in C([0,T]; H^1(\Omega))$ satisfying*

$$\psi(x,0) = \psi_0(x), \quad \psi(x,T) = \psi_1(x), \ \textit{in } \Omega,$$

for $T \gtrsim T_1^* - T_2^* > 0$, where

$$\frac{\log|\log(T_1^*)|}{T_1^*} = c_1, \quad \frac{\log|\log(T_1^* - T_2^*)|}{T_1^* - T_2^*} = c_2.$$

Remark. The minimal time of controllability $T_1^* - T_2^*$ given by Theorem 2 is approximately the minimal time of controllability with the technique employed in this article. A precise time of controllability is given in the proof of Theorem 5, and is obtained by taking into account the phase shift of the blow-up profile as well as the initial and final data. We highlight that $T_1^* - T_2^*$ is the minimal time of controllability if the phase of ψ_0 and ψ_T are those of the blow-up profile, hence the statement of Theorem 2.

The proof of Theorem 2 relies on two main arguments. One is to exploit the nonlinear nature of the equation to use the blow-up profile ψ_b in the set $\mathcal{B}_{\alpha^*}$, and such that $E(\psi_0) < 0$, as a trajectory to go from the vicinity of Q_{c_1} to the vicinity of Q_{c_2}. The second is to use the $C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ regularity of the ground state to adapt the exact local controllability around smooth trajectories [37, Theorem 1.1] for (11) around the ground state.

Theorem 3. *Let $T > 0$, $p \in [1, p^*]$ and $\lambda \geq 0$. Then there exists a $\delta > 0$ such that for any $\psi_0, \psi_T \in H^1(\Omega)$ satisfying*

$$\|\psi_0 - Q_\lambda(.,0)\|_{H^1(\Omega)} \leq \delta, \quad \|\psi_T - Q_\lambda(.,T)\|_{H^1(\Omega)} \leq \delta, \tag{12}$$

one can find an appropriate boundary control function $v(x,t)$ such that (11) *admits a solution $\psi \in C([0,T]; H^1(\Omega))$ such that*

$$\psi(x,0) = \psi_0(x), \quad \psi(x,T) = \psi_T(x), \ \textit{in } \Omega.$$

The control strategy therefore consists to drive the initial data to the blow-up profile $\psi_b(.,\epsilon)$, for $\epsilon > 0$ arbitrarily small, and for α^* sufficiently small. Then, the blow-up profile is used for $(\epsilon, T-\epsilon)$ to reach the vicinity of Q_{c_2} where the local exact controllability around Q_{c_2} is used again to reach ψ_T at time $t = T$. The minimal time is therefore dependent on the finite speed of blow-up. This strategy of proof is similar to the result of Muñoz [34], as the control strategy is non destructive and as the controlled solution ψ remains close to the ground state $\forall t \in [0,T]$. Note that the assumption $c_1 < c_2$ is not restrictive as the case $c_2 < c_1$ is easily obtained by the time reversibility of (1) through the change of variables $\psi(x,t) \mapsto \overline{\psi}(x,-t)$ (the case $c_1 = c_2$ is trivially deduced from Theorem 3). In particular, we deduce the null-controllability in large time in the neighborhood of the ground state.

Theorem 4. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be an open, connected and bounded domain with a boundary $\partial\Omega$ of class C^2. Let $c \in \mathbb{R}^+$. For $d \geq 5$, assume that the spectral property ([29]) holds. Then there exists $\epsilon > 0$ such that, for every $\psi_0 \in H^1(\Omega)$ such that*

$$\inf_{\gamma\in[0,2\pi]} \|\psi_0 - e^{i\gamma}Q_c\|_{H^1(\Omega)} < \epsilon$$

there exists a control $v \in C([0,T]; H^{1/2^-}(\partial\Omega))$ *such that* (11) *admits a solution* $\psi \in C([0,T]; H^1(\Omega))$ *satisfying*

$$\psi(x,0) = \psi_0(x), \quad \psi(x,T) = 0, \ \text{in } \Omega,$$

for $T \simeq 1/\delta > 0$*, with* $\delta > 0$ *given by the smallness assumption* (12) *of Theorem 3.*

Proof. Theorem 2 implies Theorem 4. Indeed, choose $c_2 = c$ and, for any given $\delta > 0$, choose $0 < c_1$ sufficiently small so that $\|Q_{c_1}\|_{H^1(\Omega)} < \delta$. Such c_1 always exist since Ω is bounded, and therefore $\|.\|_{L^2(\Omega)}$ is not invariant with respect to the scaling $\psi_\lambda(x,t) = \lambda^{\frac{2}{p-1}}\psi(\lambda x, \lambda^2 t)$. For instance, if $\Omega \subset B_1(0)$, the ball of radius 1 and centered at the origin, then $\|Q_{c_1}\|_{L^2(\Omega)} \leq \|Q_{c_1}\|_{L^2(B_1(0))} = \|Q\|_{L^2(B_{\sqrt{c_1}}(0))}$. Since Q is bounded on $B_{\sqrt{c_1}}(0)$, we have $\|Q\|_{B_{\sqrt{c_1}}(0)} \to 0$ as $c_1 \to 0$. The same argument is used for $\|Q_{c_1}\|_{\dot{H}^1(\Omega)}$. Then, the time reversibility of (11) yields a trajectory starting from ψ_0 to Q_{c_1}. We use Theorem 3 to drive the solution near Q_{c_1} to 0. □

A natural question following Theorem 2 and Theorem 4 is to understand whether this type of controllability results hold for the ground state in the mass subcritical regime. Indeed, the ground state is known to be orbitally stable in this regime, and one could assert that this stability is sufficient to disrupt the control strategy employed for Theorem 2. By proving that the blow-up profile of Bahri, Martel and Raphaël [1] holds in the mass slightly subcritical regime, we prove the exact controllability between the vicinity of two different ground states for the mass slightly subcritical case.

Theorem 5. *Let* $d \geq 1$ *and* $\Omega \subset \mathbb{R}^d$ *be an open, connected and bounded domain with a boundary* $\partial\Omega$ *of class* C^2*. Let* $c_1, c_2 \in \mathbb{R}^+$ *with* $c_2 > c_1$*. Then, there exists* $1 < \overline{p} < p^*$ *such that for every* $p \in (\overline{p}, p^*)$*, there exists* $\epsilon > 0$ *such that, for every* $\psi_0, \psi_1 \in H^1(\Omega)$ *such that*

$$\inf_{\gamma\in[0,2\pi]} \|\psi_0 - e^{i\gamma}Q_{c_1}\|_{H^1(\Omega)} < \epsilon, \qquad \inf_{\gamma\in[0,2\pi]} \|\psi_1 - e^{i\gamma}Q_{c_2}\|_{H^1(\Omega)} < \epsilon,$$

there exists a control $v \in C([0,T]; H^{1/2^-}(\partial\Omega))$ *such that* (11) *admits a solution* $\psi \in C([0,T]; H^1(\Omega))$ *satisfying*

$$\psi(x,0) = \psi_0(x), \quad \psi(x,T) = \psi_1(x), \ \text{in } \Omega,$$

in time $T \gtrsim T_{c_1,c_2} > 0$*, where*

$$T_{c_1,c_2} := C\frac{c_2 - c_1}{c_1 c_2},$$

with $C(p) > 0$ *independant of* c_1, c_2*.*

As for Theorem 2, we deduce that Theorem 5 holds for $c_2 < c_1$ as well as the null controllability.

Theorem 6. *Let* $d \geq 1$ *and* $\Omega \subset \mathbb{R}^d$ *be an open, connected and bounded domain with a boundary* $\partial\Omega$ *of class* C^2*. Let* $c \in \mathbb{R}^+$*. Then, there exists* $1 < \overline{p} < p^*$ *such that for every* $p \in (\overline{p}, p^*)$*, there exists* $\epsilon > 0$ *such that, for every* $\psi_0 \in H^1(\Omega)$ *such that*

$$\inf_{\gamma\in[0,2\pi]} \|\psi_0 - e^{i\gamma}Q_c\|_{H^1(\Omega)} < \epsilon,$$

there exists a control $v \in C([0,T]; H^{1/2^-}(\partial\Omega))$ *such that* (11) *admits a solution* $\psi \in C([0,T]; H^1(\Omega))$ *satisfying*

$$\psi(x,0) = \psi_0(x), \quad \psi(x,T) = 0, \ \text{in } \Omega,$$

in time $T \simeq 1/\delta > 0$*, with* $\delta > 0$ *given by the smallness assumption* (12) *of Theorem 3.*

We underline that Theorem 2, Theorem 4, Theorem 5 and Theorem 6 are closely related to the open question of global exact controllability of (11). Not only the solitary waves are special solutions to (1), but they are in fact conjectured to be generic in the decomposition of the solutions to (1). Indeed, the *soliton resolution*, conjectured to hold in the energy subcritical regime [40], states that the solutions of (1) evolve as a finite number of solitons, and a radiation behaving as a solution to the linear Schrödinger equation. Therefore, any control strategy of (11) relying on the extension on the Euclidean space either needs to rule out altogether the existence of solitary waves, or to understand, in some extent, their global controllability properties. In this sense, we view the results above as a step in this direction.

1.5 Structure of the article

In Section 2, we recall the well-posedness results for the focusing nonlinear Schrödinger equation. In Section 3, we extend the results of Rosier and Zhang [37] to deduce the exact local controllability around the ground state for the mass critical and subcritical regime. Section 4 is dedicated to the various results on the blow-up profiles that we shall need for the proof of the main results. Finally, Section 5 is reserved for the proof of Theorem 2 and Theorem 5.

2 Well-posedness

2.1 Functional framework

We consider, for Ω an open subset of $\mathbb{R}^d$, the space $L^p(\Omega)$ of measurable complex-valued functions $u : \Omega \to \mathbb{C}$ such that $\|u\|_{L^p(\Omega)} < \infty$ with the norm

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{1/p}, & \text{if } p < \infty, \\ \text{ess sup}_{x\in\Omega}|u(x)|, & \text{if } p = \infty. \end{cases}$$

The space $L^2(\Omega)$ is a real Hilbert space equipped with the scalar product

$$(u,v)_{L^2(\Omega)} = \mathrm{Re}\left(\int_\Omega u(x)\overline{v(x)}dx\right), \quad \forall u,v \in L^2(\Omega).$$

Likewise, $H^m(\Omega), m \in \mathbb{N}$ is a real Hilbert space with the scalar product

$$(u,v)_{H^m(\Omega)} = \sum_{0\le|\alpha|\le m} \mathrm{Re}\left(\int_\Omega D^\alpha u(x)\overline{D^\alpha v(x)}dx\right), \quad \forall u,v \in H^m(\Omega),$$

where $\alpha \in \mathbb{N}^d$ is a multi-index and $D^\alpha = D^{(\alpha_1,\dots,\alpha_d)} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$.

2.2 Strichartz estimates

Let us now introduce the definition of admissible pair for the Strichartz estimates. Let $1 \le q \le \infty$ and $r \in \mathbb{R}^+$. We say that the pair (q,r) is admissible if and only if

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \tag{13}$$

and if $2 \le r \le \infty$ for $d = 1$, $2 \le r < \infty$ for $d = 2$ and

$$2 \le r \le \frac{2d}{d-2},$$

if $d \geq 3$. We denote q' the conjugate of q such that $\frac{1}{q} + \frac{1}{q'} = 1$. Let us recall the Strichartz estimates that we shall use in this paper.

Lemma 1 (Strichartz's estimates, [7]). *For any $s \in \mathbb{R}$, the following holds*

- *If (q,r) is an admissible pair, then there exists a constant $C > 0$ such that, for every $\psi \in H^s(\mathbb{R}^d)$,*
$$\|S(t)\psi\|_{L^q(\mathbb{R};W^{s,r}(\mathbb{R}^d))} \leq C\|\psi\|_{H^s(R^d)},$$
- *Let $I \subset \mathbb{R}$, be an interval, bounded or not , $J = \overline{I}$ satisfying $0 \in J$. If (γ,ρ) and (q,r) are two admissible pairs, then there exists a constant $C > 0$ such that for every $f \in L^{\gamma'}(I, W^{s,\rho'}(\mathbb{R}^d))$*
$$\left\| \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L^q(I;W^{s,r}(\mathbb{R}^d))} \leq C\|f\|_{L^{\gamma'}(I,W^{s,\rho'}(\mathbb{R}^d))}.$$

2.3 Linear Schrödinger equation

We begin with the well-posedness of the linear Schrödinger equation. For any $s \in \mathbb{R}$, denote $A\psi = i\Delta\psi, \forall \psi \in D(A)$ with $A : D(A) = H^{s+2}(\mathbb{R}) \to H^s(\mathbb{R})$ the infinitesimal generator of the group of isometry $S(t)$ on $H^s(\mathbb{R})$. The solution of

$$\begin{cases} i\psi_t + \Delta\psi = 0, & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases}$$

for $\psi_0 \in H^s(\mathbb{R})$, is given by $\psi(.,t) = S(t)\psi_0$ and satisfies $\psi(.,t) \in C(\mathbb{R}; H^s(\mathbb{R}^d))$. We now turn to the inhomogeneous problem. Consider $I \subset \mathbb{R}$ an interval such that $0 \in I$, and consider

$$\begin{cases} i\psi_t + \Delta\psi = f, & (x,t) \in \mathbb{R}^d \times I, \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{14}$$

We follow [7] for the definition of the well-posedness. We say that $\psi \in C(I; H^1(\mathbb{R}^d))$ is a H^1 strong solution to (14) if and only if ψ satisfies

$$\psi(t) = S(t)\psi_0 - i\int_0^t S(t-\tau)f(\tau)d\tau, \quad \forall t \in I,$$

and the well-posedness is defined as follow.

Definition 1. *We say that* (14) *is locally well-posed in H^1 if :*

- *There is uniqueness in H^1 of the solution ψ;*
- *For every $\psi_0 \in H^1(\mathbb{R}^d)$, there exists a strong $H^1(\mathbb{R}^d)$ solution which is defined on a maximal interval $(-T_{min}, T_{max})$ with $T_{min} = T_{min}(\psi_0)$ and $T_{max} = T_{max}(\psi_0)$ such that $T_{min}, T_{max} \in (0,\infty]$.*
- *There is a blow-up alternative : if $T_{max} < \infty$, then $\lim_{t\to T_{max}^-} \|\psi(.,t)\|_{H^1(\mathbb{R}^d)} = \infty$ (respectively $T_{min} < \infty$, then $\lim_{t\to -T_{max}^+} \|\psi(.,t)\|_{H^1(\mathbb{R}^d)} = \infty$).*
- *The solution depends continuously on the initial data : if $\psi_{0,n} \to \psi_0$ in $H^1(\mathbb{R}^d)$, and if $I \subset (-T_{min}(\psi_0), T_{max}(\psi_0))$ is a closed interval, then the maximal solution ψ_n of* (14) *with the initial data $\psi_n(0) = \psi_{0,n}$ is defined on I for n large enough and satisfies $\psi_n \to \psi, n \to \infty$ in $C(I; H^1(\mathbb{R}^d))$.*

Then, the Strichartz estimates and a classical semigroup result [7, 35] yields the local well-posedness of (14).

Proposition 1. *Let $I \subset \mathbb{R}$ an interval such that $0 \in I$. Then, for $\psi_0 \in H^1(\mathbb{R}^d)$ and $f \in L^1(I; H^s(\mathbb{R}^d))$, (14) is locally well-posedness.*

The above Strichartz estimates allows us to deduce the local well-posedness of the linearized equation of (9) around smooth trajectories of (1). Indeed, consider

$$\begin{cases} i\psi_t + \Delta\psi + a(x,t)\psi + b(x,t)\overline{\psi} = f(x,t), & (x,t) \in \mathbb{R}^d \times I, \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{15}$$

with $\psi_0 \in H^1(\mathbb{R}^d)$, $f \in L^1(I; H^s(\mathbb{R}^d))$ and a, b complex-valued functions belonging to $C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$. The solution to (15) is given by the Duhamel formula,

$$\psi(t) = S(t)\psi_0 - i\int_0^t S(t-\tau)(a\psi + b\overline{\psi} + f)(\tau)d\tau, \quad \text{ in } H^s(\mathbb{R}^d).$$

The local well-posedness of (15) in $C(I; H^s(\mathbb{R}^d))$ was proved [37] by proving that the Strichartz estimates of Lemma 1 also holds for the flow map $S_L(t)$ associated to

$$L\psi := i\psi_t + \Delta\psi + a(x,t)\psi + b(x,t)\overline{\psi}.$$

Proposition 2 (Proposition 2.4, [37])**.** *For any $s \in \mathbb{R}$ and any an admissible pair (q,r) such that $q > 2$, then the following holds:*

- *There exists a constant $C > 0$ such that, for every $\psi \in H^s(\mathbb{R}^d)$,*

$$\|S_L(t)\psi\|_{L^q(\mathbb{R}; W^{s,r}(\mathbb{R}^d))} \le C\|\psi\|_{H^s(R^d)},$$

- *Let $I \subset \mathbb{R}$, be an interval, bounded or not, $J = \overline{I}$ satisfying $0 \in J$. Let (γ, ρ) be another admissible pair, then there exists a constant $C > 0$ such that for every $f \in L^{\gamma'}(I, W^{s,\rho'}(\mathbb{R}^d))$,*

$$\left\| \int_0^t S_L(t-\tau)f(\tau)d\tau \right\|_{L^q(I; W^{s,r}(\mathbb{R}^d))} \le C\|f\|_{L^{\gamma'}(I, W^{s,\rho'}(\mathbb{R}^d))}.$$

These estimates together with standard semigroup arguments [35] allows to prove the well-posedness of

$$\begin{cases} i\psi_t + \Delta\psi + a(x,t)\psi + b(x,t)\overline{\psi} = f(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{16}$$

Proposition 3. *Let $I \subset \mathbb{R}$ such that $0 \in I$. Let $\psi_0 \in H^1(\mathbb{R}^d)$, $f \in L^1(I; H^1(\mathbb{R}^d))$. Then equation (16) is locally well-posed with $\psi \in C(I; H^1(\mathbb{R}^d))$.*

2.4 Well-posedness of the nonlinear equation on $\mathbb{R}^d$

We now turn to the well-posedness of the nonlinear equation

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0, & (x,t) \in \mathbb{R}^d \times I, \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{17}$$

where $\psi_0 \in H^1(\mathbb{R}^d)$. The local well-posedness holds using Kato's argument [18] (see also [7, Section 4.4]).

Theorem 7 ([18]). *Let $j \in C(\mathbb{C};\mathbb{C})$ such that $j(0)=0$,*

$$|j(u)-j(v)| \le L(K)|u-v|,$$

for all $u,v \in \mathbb{C}$ such that $|u|,|v| \le K$ for $K>0$ with

$$\begin{cases} L(t) \in C([0,\infty)), & d=1,\\ L(t) \le C(1+t^\alpha), & \text{with } 0 \le \alpha < \dfrac{4}{d-2}, \text{ if } d \ge 2. \end{cases}$$

Set $g(u)(x) = j(u(x))$ for all measurable $u : \mathbb{R}^d \to \mathbb{C}$ almost everywhere in $\mathbb{R}^d$. Assume j, considered as a function of $\mathbb{R}^2 \to \mathbb{R}^2$, is of class C^1. Then, (17) is locally well-posed in $H^1(\mathbb{R}^d)$.

In turn,

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = f(x,t), & (x,t) \in \mathbb{R}^d \times (0,T),\\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{18}$$

is also locally well-posed with $f \in L^1(I;H^1(\mathbb{R}^d))$. In fact, one deduces the local well-posedness of (18) around any smooth trajectory w of (1) from [7, Theorem 4.4.6],

$$\begin{cases} i\psi_t + \Delta\psi + |w+\psi|^{p-1}(w+\psi) - |w|^{p-1}w = f(x,t), & (x,t) \in \mathbb{R}^d \times (0,T),\\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{19}$$

Proposition 4. *Let $\psi_0 \in H^1(\mathbb{R}^d)$, $f \in L^1(I;H^1(\mathbb{R}^d))$, $w \in C^\infty(I;\mathcal{S}(\mathbb{R}^d))$ solution to (1). Then, (19) is locally well-posed in $H^1(\mathbb{R}^d)$.*

Proof. Indeed, [7, Theorem 4.4.6] is stated as follow for general nonlinear equation of the form

$$\begin{cases} i\psi_t + \Delta\psi + g(\psi) = 0, & (x,t) \in \mathbb{R}^d \times (0,T),\\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{20}$$

with the following assumption on the non-linearity $g \in C(H^1(\mathbb{R}^d);H^{-1}(\mathbb{R}^d))$: suppose there exist $2 \le r,\rho < \frac{2d}{d-2}$ ($2 \le r,\rho < \infty$ if $d=1,2$) such that

$$\|g(u)-g(v)\|_{L^{\rho'}(\mathbb{R}^d)} \le C(\delta)\|u-v\|_{L^r(\mathbb{R}^d)}, \tag{21}$$

for all $u,v \in H^1(\mathbb{R}^d)$ such that $\|u\|_{H^1(\mathbb{R}^d)} < \delta$, $\|v\|_{H^1(\mathbb{R}^d)} < \delta$ and

$$\|g(u)\|_{W^{1,\rho'}(\mathbb{R}^d)} \le C(\delta)(1+\|u\|_{W^{1,r}(\mathbb{R}^d)}), \tag{22}$$

for all $u \in H^1(\mathbb{R}^d) \cap W^{1,r}(\mathbb{R}^d)$, such that $\|u\|_{H^1(\mathbb{R}^d)} < \delta$. Then, (we refer to [7] for a more precise statement)

Theorem 8 (Theorem 4.4.6 [7]). *Let $g = g_1 + g_2 + \ldots + g_k$ such that each $g_j, j=1,\ldots k$ satisfies (21) and (22) for some exponent r_j,ρ_j. Then, for every $\psi_0 \in H^1(\mathbb{R}^d)$, there exists a unique strong H^1 solution of (20) defined on a maximal time interval $(0,T_{\psi_0})$.*

Then, Proposition 4 follows easily. Indeed, define

$$g(\psi) = |w+\psi|^{p-1}(w+\psi) - |w|^{p-1}w,$$

$\theta(z) = 1$ if $|z| \le 1$ and

$$g_1(\psi) = \theta(\psi)g(\psi), \quad g_2(\psi) = (1-\theta(\psi))g(\psi).$$

Then g_1, g_2 satisfies (21) and (22) (see the proof of Proposition 6). Hence the local well-posedness follows for $f=0$, and the case with $f \ne 0$ with $\|f\|_{L^1((0,T);H^1(\mathbb{R}^d))} < \delta$ is dealt classically by including f in the Duhamel formula and using the Strichartz estimates. □

Let us finally give a meaning to a solution of (11), following for instance [24]. We underline that this definition is weaker than Definition 1, as nothing is stated about the uniqueness, the blow-up alternative or the continuous dependence on the initial data.

Definition 2. *We say that ψ is a solution to* (11) *if* $\forall \tau \in [0,T)$,

$$i\int_\Omega \psi(x,\tau)\overline{\phi}(x,\tau) - \psi_0(x)\overline{\phi}(x,0)dx - \int_0^\tau\int_\Omega \nabla\psi(x,t)\nabla\overline{\phi}(x,t)dxdt \\ + \int_0^\tau\int_\Omega \psi|\psi|^{p-1}\phi dxdt = \int_0^\tau\int_{\partial\Omega} v(x,t)\partial_\nu\phi(x,t)dxdt, \quad (23)$$

for any $\phi \in C([0,T];H^1(\Omega)) \cap \{\partial_\nu\phi \in C([0,T];L^2(\partial\Omega))\}$.

We underline that (23) make sense since, as $p-1 < 4/(d-2)$, $|\psi|^{p-1}\psi \in C([0,T);H^{-1}(\mathbb{R}^d))$. Therefore, denoting $\tilde{\phi}$ the extension of ϕ by 0 outside Ω,

$$\left|\int_0^\tau\int_\Omega \psi|\psi|^{p-1}\phi dxdt\right| = \left|\int_0^\tau\int_{\mathbb{R}^d} \psi|\psi|^{p-1}\tilde{\phi} dxdt\right| \leq C\tau \||\psi|^{p-1}\psi\|_{L^\infty((0,\tau);H^{-1}(\mathbb{R}^d))}\|\phi\|_{L^\infty((0,\tau);H^1(\Omega))}.$$

3 Local exact controllability around the ground state

The goal of this section is to extend the local exact controllability around smooth trajectories of (1) obtained by Rosier and Zhang [37] for $p=3$ and $d \geq 1$ to the local exact controllability around the ground state for the mass critical and mass subcritical regime $0 < p-1 \leq 4/d$ with $d \geq 1$, as given by Theorem 3. We begin by recalling their exact controllability result for the linear equation.

3.1 Controllability of the linearized equation

Let $T > 0$ and consider

$$\begin{cases} i\psi_t + \Delta\psi + a(x,t)\psi + b(x,t)\overline{\psi} = \varphi(x)h(x,t), & (x,t) \in \mathbb{R}^d \times (0,T), \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (24)$$

with $\psi_0 \in H^s(\mathbb{R}^d)$, $a,b \in C^\infty((0,T);\mathcal{S}(\mathbb{R}^d))$, $h \in L^2((0,T);H^s(\mathbb{R}^d))$ and $\varphi \in C^\infty(\mathbb{R}^d;[0,1])$ defined by (10).

Theorem 9 ([37, Theorem 3.1])**.** *Let $T > 0$ and $s \geq 0$ be given and assume $a,b \in C^\infty((0,T);\mathcal{S}(\mathbb{R}^d))$. There exists a bounded linear operator*

$$G : H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \to L^2((0,T);H^s(\mathbb{R}^d)),$$

such that for any $\psi_0, \psi_T \in H^s(\mathbb{R}^d)$, if one chooses $h = G(\psi_0,\psi_T)$ as a control input, then the system (24) *admits a solution $\psi \in C([0,T];H^s(\mathbb{R}^d))$ satisfying*

$$\psi(.,T) = \psi_T, \text{ in } H^s(\mathbb{R}^d).$$

3.2 Local exact controllability around smooth trajectories

We now turn to the local exact controllability of,

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = \varphi(x)h(x,t), & (x,t) \in \mathbb{R}^d \times (0,T), \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (25)$$

around ground state solitary wave $\psi_\lambda(x,t) = e^{i\lambda t}Q_\lambda(x)$. We recall that, in this case, $\psi_\lambda \in C^\infty(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$ and that Q_λ is positive. We first write solution to (25) under the form $\psi = \psi_\lambda + y$, where y solves,

$$\begin{cases} iy_t + \Delta y + \dfrac{p+1}{2}Q_\lambda^{p-1}y + \dfrac{p-1}{2}e^{i2\lambda t}Q_\lambda^{p-1}\overline{y} + g(y) = \varphi(x)h(x,t), & (x,t) \in \mathbb{R}^d \times (0,T), \\ y(x,0) = y_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{26}$$

where,

$$g(y) = |e^{i\lambda t}Q_\lambda + y|^{p-1}(e^{i\lambda t}Q_\lambda + y) - e^{i\lambda t}Q_\lambda^p - \frac{p+1}{2}Q_\lambda^{p-1}y - \frac{p-1}{2}e^{i2\lambda t}Q_\lambda^{p-1}\overline{y}. \tag{27}$$

Denote $S_L(t)$ the flow map associated to (24) with

$$a(x,t) = \frac{p+1}{2}Q_\lambda^{p-1}(x), \quad b(x,t) = \frac{p-1}{2}e^{2i\lambda t}Q_\lambda^{p-1}(x). \tag{28}$$

The following result give the sufficient functional framework to obtain the local exact controllability of (26).

Proposition 5. *Let $s \geq 0$, $T > 0$ and g defined by (27). If the following holds in a Banach space $X_{s,T} \subset C([0,T];H^s(\mathbb{R}^d))$: there exists $C > 0$ such that $\phi \in H^s(\mathbb{R}^d)$,*

$$\|S_L(t)\phi\|_{X_{s,T}} \leq C\|\phi\|_{H^s(\mathbb{R}^d)},$$

for every $f \in L^2((0,T);H^s(\mathbb{R}^d))$,

$$\left\| \int_0^t S_L(t-\tau)f(.,\tau)\,d\tau \right\|_{X_{s,T}} \leq C\|f\|_{L^2((0,T);H^s(\mathbb{R}^d))},$$

for every $z_1, z_2 \in X_{s,T}$,

$$\left\| \int_0^t S_L(t-\tau)g(z_1)(\tau)\,d\tau \right\|_{X_{s,T}} \leq C(1 + \|z_1\|_{X_{s,T}})\|z_1\|_{X_{s,T}}^{p-1},$$

and

$$\left\| \int_0^t S_L(t-\tau)(g(z_1) - g(z_2))(\tau)\,d\tau \right\|_{X_{s,T}} \leq C(\|z_1\|_{X_{s,T}} + \|z_2\|_{X_{s,T}} + \|z_1\|_{X_{s,T}}^{p-1} + \|z_2\|_{X_{s,T}}^{p-1})\|z_1 - z_2\|_{X_{s,T}},$$

then (26) is locally exactly controllable in $H^s(\mathbb{R}^d)$, that is, there exists $\delta > 0$ such that for every $y_0, y_1 \in H^s(\mathbb{R}^d)$ such that,

$$\|y_0\|_{H^s(\mathbb{R}^d)} < \delta, \quad \|y_1\|_{H^s(\mathbb{R}^d)} < \delta,$$

then one can find a control $h \in L^2((0,T);H^s(\mathbb{R}^d))$ such that the solution $y \in C([0,T];H^s(\mathbb{R}^d))$ of (26) satisfies,

$$y(.,T) = y_1, \text{ in } H^s(\mathbb{R}^d).$$

Proof. The proof is very similar to [37, Theorem 4.1]. First, notice that (26) linearized around zero is given by (24), with a and b are defined by (28). Since a, b defined by (28) belong to $C^\infty(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$, the exact controllability given by Theorem 9 holds.

The solution to (26) is given by the Duhamel formula,

$$y(t) = S_L(t)y_0 - \int_0^t S_L(t-\tau)g(y)(\tau)d\tau + \int_0^t S_L(t-\tau)\varphi(x)h(x,\tau)d\tau.$$

Using the time-reversibility, we can assume $y_1 = 0$. The proof of Proposition 5 is obtained by a classical Fixed Point Theorem. Indeed, let

$$h = G\left(y_0, \int_0^t S_L(t-\tau)g(u)(\tau)d\tau\right)$$

for every $u \in C([0,T]; H^s(\mathbb{R}))$. Then, let

$$\Gamma(u)(t) = WS_L(t)y_0 - \int_0^t S_L(t-\tau)g(y)(\tau)d\tau + \int_0^t S_L(t-\tau)\varphi(x)h(x,\tau))d\tau.$$

We easily see that

$$\Gamma(u)(0) = y_0, \quad \Gamma(u)(T) = 0.$$

It therefore suffices to prove that $\Gamma(u)(t)$ is a contraction in $X_{s,T}$ thanks to the various estimates in hypothesis of Proposition 5. From now on, the proof is the same as in [37, Theorem 4.1] and we therefore omit it here.

□

It remains to define the proper space $X_{s,T}$ and to prove the estimates of Proposition 5, which yields Theorem 3.

Proposition 6. *Let $T > 0$, $d \geq 1$, $s = 1$ and $r = p+1$ for $p \in (1, p^*]$. Then the estimates of Proposition 5 hold in $X_T := C([0,T]; H^1(\mathbb{R}^d)) \cap L^q((0,T); W^{1,r}(\mathbb{R}^d))$ endowed with the norm $\|.\|_{X_{s,T}} := \|.\|_{L^\infty((0,T);H^1(\mathbb{R}^d))} + \|.\|_{L^q((0,T);W^{1,r}(\mathbb{R}^d))}$.*

Proof. Let $r := p+1$ and q defined by (13). The pair (q,r) is admissible pair since $r \geq 2$, r is always finite and satisfies

$$2 \leq r \leq \frac{2d}{d-2},$$

if $d \geq 3$. Following [37], let $\xi \in C_0^\infty(\mathbb{C})$ be such that $\xi(z) = 1$ for $|z| \leq 1$ and set

$$g_1(z) = \xi(z)g(z), \quad g_2(v) = (1-\xi(z))g(z)$$

Since $p > 1$, we have the following for $C_i > 0$,

$$\begin{aligned}
\|g_1(z)\|_{L^2(\mathbb{R}^d)} &\leq C\|z\|_{L^2(\mathbb{R}^d)} + \left\|\xi(z)\left((1+|\psi_c|^{p-1})(z+\psi_c) - e^{ict}Q_c^p\right)\right\|_{L^2(\mathbb{R}^d)} \\
&\leq C_1(C_2 + \|z\|_{L^2(\mathbb{R}^d)}) \\
&\leq C_3\|z\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

and

$$\begin{aligned}
\|g_2(z)\|_{L^{r'}(\mathbb{R}^d)} &\leq C\|z\|_{L^{r'}(\mathbb{R}^d)} + \left\|\xi(z)\left((1+|\psi_c|^{p-1})(z+\psi_c) - e^{ict}Q_c^p\right)\right\|_{L^{r'}(\mathbb{R}^d)} \\
&\leq C_1(C_2 + \|z\|^{p-1}_{L^r(\mathbb{R}^d)}) \\
&\leq C_3\|z\|^{p-1}_{L^r(\mathbb{R}^d)}
\end{aligned}$$

A straightforward estimation yields

$$\|g_1(z_1) - g_1(z_2)\|_{L^2(\mathbb{R}^d)} \leq C\|z_1 - z_2\|_{L^2(\mathbb{R}^d)}$$

and using the inequality $||u|^{p-1}u - |v|^{p-1}v| \leq C(|u|^{p-1} + |v|^{p-1})|u-v|$ for $p > 1$ ([43]), Hölder and Minkowski inequalities, we deduce

$$\begin{aligned}
\|g_2(z_1) - g_2(z_2)\|_{L^{r'}} &\leq C\|Q_c^{p-1}(z_1 - z_2)\|_{L^{r'}} + C\|(|z_1+\psi_c|^{p-1} + |z_2+\psi_c|^{p-1})|z_1 - z_2|\|_{L^{r'}} \\
&\leq C\|z_1 - z_2\|_{L^r} + C\left(\|z_1+\psi_c\|_{L^r}^{p-1} + \|z_2+\psi_c\|_{L^r}^{p-1}\right)\|z_1 - z_2\|_{L^r} \\
&\leq C\left(\|z_1\|_{L^r}^{p-1} + \|z_2\|_{L^r}^{p-1}\right)\|z_1 - z_2\|_{L^r}.
\end{aligned}$$

where r' denotes the conjugate of r. Moreover, using the diamagnetic inequality ([43]) : for all $f \in H^1(\mathbb{R}^d)$

$$\nabla|f| \leq |\nabla f|,$$

in the sense of distribution, we obtain

$$\|\nabla g_1(z)\|_{L^2(\mathbb{R}^d)} \leq C\|\nabla z\|_{L^2(\mathbb{R}^d)}$$

and

$$\|\nabla g_2(z)\|_{L^{r'}(\mathbb{R}^d)} \leq C\|z\|_{L^r(\mathbb{R}^d)}^{p-1}\|\nabla z\|_{L^r(\mathbb{R}^d)}$$

Since $p \in (1, p^*]$, we have $q > 2$ (recall (13)) and therefore from [37, Proposition 2.4] (or see also [7]), we have that for every admissible pair (q, r), the following holds

$$\begin{aligned}
\|S_L(t)\psi_0\|_{X_T} &\leq C\|\psi_0\|_{H^1(\mathbb{R}^d)}, \\
\left\|\int_0^t S_L(t-\tau)\varphi(.)h(.,\tau)\mathrm{d}\tau\right\|_{X_{s,T}} &\leq C\|\varphi h\|_{L^1((0,T);H^1(\mathbb{R}^d))} \leq C\|\varphi h\|_{L^2((0,T);H^1(\mathbb{R}^d))}, \\
\left\|\int_0^t S_L(t-\tau)\left(g(z)\right)(\tau)\mathrm{d}\tau\right\|_{X_T} &\leq C\left(\|g_1(z)\|_{L^1((0,T);H^1(\mathbb{R}^d))} + \|g_2(z)\|_{L^{q'}((0,T);W^{1,r'}(\mathbb{R}^d))}\right), \\
\left\|\int_0^t S_L(t-\tau)\left(g(v_1) - g(v_2)\right)(\tau)\mathrm{d}\tau\right\|_{X_T} &\leq C\left(\|g_1(z_1) - g_1(z_2)\|_{L^1((0,T);H^1(\mathbb{R}^d))}\right) \\
&\quad + \|g_2(z_1) - g_2(z_2)\|_{L^{q'}((0,T);W^{1,r'}(\mathbb{R}^d))}\Big).
\end{aligned}$$

In a similar fashion as above, one deduces that for $z_1, z_2 \in X_T$, $g_1(z_1) \in L^\infty((0,T); H^1(\mathbb{R}^d))$ and $g_2(z_1) \in L^q((0,T); W^{1,r'}(\mathbb{R}^d))$ with the following estimates

$$\begin{aligned}
\|g_1(z_1)\|_{L^\infty((0,T);H^1(\mathbb{R}^d))} &\leq C\|z_1\|_{L^\infty((0,T);H^1(\mathbb{R}^d))} \\
\|g_2(z_1)\|_{L^{q'}((0,T);W^{1,r'}(\mathbb{R}^d))} &\leq C\|z_1\|_{L^\infty((0,T);L^{1,r}(\mathbb{R}^d))}^{p-1}\|z_1\|_{L^q((0,T);W^{1,r}(\mathbb{R}^d))} \\
\|g_1(z_1) - g_1(z_2)\|_{L^\infty(H^1)} &\leq C\left(\|z_1\|_{L^\infty(H^1)} + \|z_2\|_{L^\infty(H^1)}\right)\|z_1 - z_2\|_{L^\infty(H^1)} \\
\|g_2(z_1) - g_2(z_2)\|_{L^{q'}(W^{1,r'})} &\leq C\left(\|z_1\|_{L^\infty(L^{1,r})}^{p-1} + \|z_2\|_{L^\infty(L^{1,r})}^{p-1}\right)\|z_1 - z_2\|_{L^\infty(W^{1,r})}
\end{aligned}$$

Consequently,

$$\|g_1(z)\|_{L^1((0,T);H^1(\mathbb{R}^d))} + \|g_2(z)\|_{L^{q'}((0,T);W^{1,r'}(\mathbb{R}^d))} \leq C(1 + \|z\|_{X_T})\|z\|_{X_T}^{p-1}$$

and

$$\begin{aligned}
&\|g_1(z_1) - g_1(z_2)\|_{L^1((0,T);H^1(\mathbb{R}^d))} + \|g_2(z_1) - g_2(z_2)\|_{L^{q'}((0,T);W^{1,r'}(\mathbb{R}^d))} \\
&\quad \leq C\left(\|z_1\|_{X_T} + \|z_2\|_{X_T} + \|z_1\|_{X_T}^{p-1} + \|z_2\|_{X_T}^{p-1}\right)\|z_1 - z_2\|_{X_T},
\end{aligned}$$

that is, the desired estimates.

□

We finally obtain the local exact controllability of (11) around smooth trajectories of (1) stated in Theorem 3 by taking the trace of the solution of (25) as the control. The well-posedness and the properties of the control operator G ensures that the solution ψ defined this way belongs to $C([0,T];H^1(\mathbb{R}^d))$.

4 Properties of the blow-up trajectory

We present here the properties of the blow-up trajectory in the mass critical and mass slightly subcritical regime that shall be needed for the proof of Theorem 5. We begin by recalling the blow-up profile close in $\dot{H}^1(\mathbb{R}^d)$ to the ground state in the mass slightly supercritical regime obtained in [1]. Using this construction, we prove first in Theorem 11 that this construction yields a blow-up profile close in $\dot{H}^1(\mathbb{R}^d)$ to the ground state in the slightly subcritical regime. The second part of the proof of Theorem 11 lies in the proof that the blow-up profile is close to the ground state in $H^1_{loc}(\mathbb{R}^d)$. This closeness to the ground state shall be used in the proof of Theorem 5 to drive the initial or final data to the blow-up trajectory using the local controllability around the ground state. We recall that one cannot expect the blow-up profile to be close to the ground state in $H^1(\mathbb{R}^d)$ as the blow-up profile does not belong to $L^2(\mathbb{R}^d)$ [1].

4.1 Blow-up profile in the slightly mass supercritical regime

The blow-up profile constructed in [1] is based on the ansatz

$$\psi(x,t) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} e^{i\theta(t)}\tilde{\psi}\left(\frac{x}{\lambda(t)}\right), \tag{29}$$

where $\tilde{\psi}$ is assumed to be radially symmetric, $\tilde{\psi}(x) = \tilde{\psi}(|x|) = \tilde{\psi}(r), r := |x|$ and where, for a given $T \in \mathbb{R}$ and $b > 0$, $\lambda(t)$ and $\theta(t)$ are defined for $t \in [0,T)$ by

$$\lambda(t) := \sqrt{2b(T-t)}, \quad \theta(t) := -\frac{\log(T-t)}{2b}. \tag{30}$$

The parameter b, that shall be small, is used as a bifurcation parameter. Indeed, for ψ defined by (29) to be a solution to (1), $\tilde{\psi}$ has to satisfy

$$\Delta\tilde{\psi} - \tilde{\psi} + ib\left(\frac{2}{p-1}\psi + x.\nabla\psi\right) + |\tilde{\psi}|^{p-1}\tilde{\psi} = 0, \quad x \in \mathbb{R}^d. \tag{31}$$

Notice that taking $b = 0$ in (31) is equivalent to (3), which makes the bifurcation parameter b apparent. Bahri, Martel and Raphaël are able to give a precise description of the finite energy self-similar blow-up according to the ansatz (29).

Theorem 10. *[1, Theorem 1] Let $d \geq 1$, $0 < s_c \ll 1$ and $p^* = 1 + 4/d$ be the mass critical exponent. There exists $\epsilon > 0$ such that for any p satisfying*

$$0 < p - p^* < \epsilon,$$

there exists $b(p) > 0$ and a non-zero radially symmetric solution $\tilde{\psi}$ to (31) such that

$$\tilde{\psi} \in \dot{H}^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d), \quad E(\tilde{\psi}) = 0.$$

Moreover, as $p \to p^{*+}$, the following hold

- *Law for the nonlinear eigenvalue :* $b = b_{s_c}(1 + o(1))$ *where* $0 < b_{s_c} \ll 1$ *is defined by*

$$s_c = \frac{\kappa^2}{N_c b_{s_c}} \exp\left(-\frac{\pi}{b_{s_c}}\right), \quad N_c = \int_0^\infty Q^2(r) r^{d-1} dr, \tag{32}$$

  and κ *is defined as in* (4)*;*

- *Bifurcation from the soliton profile :* $\|\tilde{\psi} - Q\|_{\dot{H}^1(\mathbb{R}^d)} = o(1)$*;*
- *Non-oscillatory behaviour for the outgoing wave:*

$$\lim_{r\to\infty} r^{\frac{2}{p-1}} |\tilde{\psi}(r)| = \rho_{s_c}(1 + o(1)), \quad \limsup_{r\to\infty} r^{\frac{p+1}{p-1}} |\tilde{\psi}'(r)| < \infty,$$

  where

$$\rho_{s_c} = \sqrt{2N_c s_c}.$$

Based on the construction of [1], we shall prove that there exists a blow-up profile in the mass slightly subcritical case satisfying the following.

Theorem 11. *Let* $d \geq 1$*,* $-1 \ll s_c < 0$ *and* $p^* = 1 + 4/d$ *be the mass critical exponent. Then there exists* $\epsilon > 0$ *such that for any* p *satisfying*

$$-\epsilon < p - p^* < 0$$

there exists $b(p) > 0$ *and a non-zero radially symmetric solution* $\tilde{\psi}$ *to* (31) *such that*

$$\tilde{\psi} \in \dot{H}^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d), \quad E(\tilde{\psi}) = 0.$$

Moreover, as $p \to p^{*-}$*, the following hold*

- *Law for the nonlinear eigenvalue :* $b = b_{s_c}(1 + o(1))$ *where* $0 < b_{s_c} \ll 1$ *is defined by*

$$|s_c| = \frac{\kappa^2}{N_c b_{s_c}} \exp\left(-\frac{\pi}{b_{s_c}}\right), \quad N_c = \int_0^\infty Q^2(r) r^{d-1} dr;$$

- *Bifurcation from the soliton profile :* $\|\tilde{\psi} - Q\|_{H^1_{loc}(\mathbb{R}^d)} = o(1)$*;*
- *Non oscillatory behaviour for the outgoing wave:*

$$\lim_{r\to\infty} r^{\frac{2}{p-1}} |\tilde{\psi}(r)| = \rho_{s_c}(1 + o(1)), \quad \limsup_{r\to\infty} r^{\frac{p+1}{p-1}} |\tilde{\psi}'(r)| < \infty,$$

  where

$$\rho_{s_c} = \sqrt{2N_c |s_c|}.$$

For the controllability purposes of Theorem 5, we shall prove that the blow-up profile satisfies $\|\tilde{\psi} - Q\|_{H^1_{loc}(\mathbb{R}^d)} = o(1)$. Compared to Theorem 10, this requires to prove the additional convergence $\|\tilde{\psi} - Q\|_{L^2_{loc}(\mathbb{R}^d)} = o(1)$. We therefore recall below the main results behind Theorem 10 to collect the estimates of the blow-up profile in the different regions.

Let $|x| = r \in [0, \infty)$ and consider the change of variables

$$\tilde{\psi}(x) = \exp\left(-i\frac{br^2}{4}\right) P(r). \tag{33}$$

Then $\tilde{\psi}$ is a radial symmetric solution to (31) if we are able to find $P : [0, \infty) \to \mathbb{C}$ solution to

$$\begin{cases} P'' + \frac{d-1}{r}P' + \left(\frac{b^2r^2}{4} - 1 - ibs_c\right)P + |P|^{p-1}P = 0, \quad r > 0, \\ P'(0) = 0. \end{cases} \tag{34}$$

The dependence between b and s_c in (34) is relaxed by looking for a solution P_σ of

$$\begin{cases} P'' + \frac{d-1}{r}P' + \left(\frac{b^2r^2}{4} - 1 - ib\sigma\right)P + |P|^{p-1}P = 0, \quad r > 0, \\ P'(0) = 0, \end{cases} \tag{35}$$

with $\sigma > 0$ small. Clearly, if P_σ is solution to (35) for any $\sigma > 0$ small, then taking $\sigma = s_c$ yields P_{s_c} solution to (34) and therefore a radially symmetric profile $\tilde{\psi}$ satisfying (31). Additional parameters $(\sigma, \rho, \gamma, \theta) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ are introduced in order to construct the profile P_σ, with an a priori control with respect to σ:

$$b \in \left[b_\sigma - \frac{1}{2}b_\sigma^{\frac{13}{6}}, b_\sigma + \frac{1}{2}b_\sigma^{\frac{13}{6}}\right], \tag{36}$$

where $b_\sigma > 0$ is defined by

$$\sigma = \frac{\kappa^2}{N_c b_\sigma}\exp\left(-\frac{\pi}{b_\sigma}\right), \quad N_c = \int_0^\infty Q^2(r)r^{d-1}dr,$$

and

$$\rho \in \left[\frac{1}{2}\rho_\sigma, \frac{3}{2}\rho_\sigma\right], \quad \gamma \in \left[-\frac{1}{2}\gamma_\sigma, \frac{1}{2}\gamma_\sigma\right], \quad \theta \in \left[-\frac{1}{2}\theta_\sigma, \frac{1}{2}\theta_\sigma\right], \tag{37}$$

where

$$\rho_\sigma = \sqrt{2}N_c^{\frac{1}{2}}\sqrt{\sigma}, \quad \gamma_\sigma = b_\sigma^{\frac{1}{6}}\exp\left(-\frac{2}{\sqrt{b_\sigma}}\right), \quad \theta_\sigma = b_\sigma^{\frac{1}{6}}\exp\left(-\frac{\pi}{b_\sigma}\right)\exp\left(\frac{2}{\sqrt{b_\sigma}}\right).$$

In particular, there exists $C > 0$,

$$\left|\exp\left(\frac{\pi}{b} - \frac{\pi}{b_\sigma}\right) - 1\right| \leq C\left|\frac{\pi}{b} - \frac{\pi}{b_\sigma}\right| \leq Cb_\sigma^{\frac{1}{6}}, \quad \sigma \leq Cb^{-1}\exp\left(-\frac{\pi}{b}\right). \tag{38}$$

These parameters allows are used to define the solutions of (35) over three intervals, $K := [0, r_K]$, $J = [r_K, r_I]$ and $I = [r_I, \infty)$, where $r_K = b^{-1/2}$ and $r_I = b^{-2}$. The three intervals take into account three different dynamic of the solution near the ground state : in K the non-linearity plays an important role, while in I, the ground state is exponentially small and therefore the equation is essentially linear. The interval J includes the so-called turning point, where the real part of the zeroth order operator vanishes[2]. In the proof of Theorem 10, the solution in I is extended to J, and is denoted P_{ext}. It is then matched to the solution P_{int} in K at $r = r_K$ to yield the profile P_σ.

4.1.1 Solution P_{int} of (35) in K

In K, the solution to P_{int} of (35) on K satisfies the following

[2]it is in fact on (44) that this turning point is seen, and it depends on the dimension d (see [1, 14])

Proposition 7 (Proposition 4.1, [1]). *For $\sigma > 0$ small enough and for any b, γ satisfying* (36) *and* (37)*, there exists a solution $P_{int} = P_{int}[\sigma, b, \gamma]$ of* (35) *on K satisfying*

$$
\begin{aligned}
\mathrm{Re}(P_{int}(r_K)) &= \kappa b^{\frac{d-1}{4}} \exp\left(-\frac{1}{\sqrt{b}}\right)(1 + O(b^{\frac{1}{3}})) + \kappa_A \gamma b^{\frac{d-1}{4}} \exp\left(\frac{1}{\sqrt{b}}\right), \\
\mathrm{Re}(P'_{int}(r_K)) &= -\kappa b^{\frac{d-1}{4}} \exp\left(-\frac{1}{\sqrt{b}}\right)(1 + O(b^{\frac{1}{3}})) + \kappa_A \gamma b^{\frac{d-1}{4}} \exp\left(\frac{1}{\sqrt{b}}\right), \\
\mathrm{Im}(P_{int}(r_K)) &= \kappa_B \sigma b^{\frac{d+3}{4}} \exp\left(\frac{1}{\sqrt{b}}\right)(1 + O(b^{\frac{1}{4}})), \\
\mathrm{Im}(P'_{int}(r_K)) &= \kappa_B \sigma b^{\frac{d+3}{4}} \exp\left(\frac{1}{\sqrt{b}}\right)(1 + O(b^{\frac{1}{4}})),
\end{aligned}
$$

and there exists $C > 0$ such that

$$\|P_{int} - Q\|_{\dot{H}^1(K)} \leq C b^{\frac{1}{12}}.$$

Moreover, the map $(\sigma, b, \gamma) \mapsto (P_{int}[\sigma, b, \gamma](r_K), P'_{int}[\sigma, b, \gamma](r_K))$ is continuous.

The solution P_{int} is obtained from the decomposition

$$P = (Q + \gamma A + \phi_+) + i(b\sigma B + \phi_-) \tag{39}$$

by a fixed point argument on $(\phi_+, \phi_-) \in E_K$ where E_K is the complete metric space

$$E_K := \{(\phi_+, \phi_-) : K \to \mathbb{R}^2 \text{ is continuous and satisfies} \|(\phi_+, \phi_-)\|_K \leq 1\} \tag{40}$$

endowed with the norm

$$\|(\phi_+, \phi_-)\|_K := \max(b^{-1/3}\mathcal{N}_+(\phi_+), b^{-5/4}\sigma^{-1}\mathcal{N}_-(\phi_-)),$$

where

$$\mathcal{N}_+(\phi_+) = \|\phi_+/Q\|_{L^\infty(K)}, \quad \mathcal{N}_-(\phi_-) = \|H\phi_-\|_{L^\infty(K)}, \quad H(r) = (1+r)^{d-1}Q(r), \quad r \geq 0.$$

Moreover, the functions A and B are associated with the linearized operators around the ground state, defined by

$$L_+ = -\partial_{rr} - \frac{d-1}{r}\partial_r + 1 - pQ^{p-1}, \tag{41}$$

$$L_- = -\partial_{rr} - \frac{d-1}{r}\partial_r + 1 - Q^{p-1}, \tag{42}$$

and satisfies, by collecting the results from Lemma 4.1 and Lemma 4.4 of [1],

Lemma 2 ([1]). *There exist C^2 functions $A : [0, \infty) \to \mathbb{R}$ and $B : [0, \infty) \to \mathbb{R}$ solutions respectively of $L_+A = 0$ and $L_-B = -Q$ on $(0, \infty)$, such that $A(0) = 1$, $A'(0) = B(0) = B'(0) = 0$ and*

$$
\begin{aligned}
A(r) &= \kappa_A r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), \quad r \in [1, \infty), \\
A'(r) &= \kappa_A r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), \quad r \in [1, \infty), \\
B(r) &= \kappa_B r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), \quad r \in [1, \infty), \\
B'(r) &= \kappa_B r^{-\frac{d-1}{2}} e^r (1 + O(r^{-1})), \quad r \in [1, \infty),
\end{aligned}
$$

for a constant $\kappa_A \neq 0$ and $\kappa_B = N_c/(2\kappa) > 0$.

Proposition 7 is obtained in part with the following estimates on the decomposition of P_{int} (39) which relates to the closeness to the ground state with respect to the parameter b (see for instance eq (4.15) in [1]). We shall use these estimates for the closeness of the ground state to the blow-up profile in $H^1_{loc}(\mathbb{R}^d)$.

Lemma 3 ([1]). *There exists $C > 0$ such that*

$$|\gamma A| \le C b^{\frac{1}{6}} Q, \quad |\phi_+| \le C b^{\frac{1}{3}} Q, \quad b|\sigma B| + |\phi_-| \le C \exp\left(-\frac{\pi}{b} + \frac{2}{\sqrt{b}}\right) Q.$$

Proof. These estimates come from the a priori bounds on the parameters σ, b, γ, (4), (36), (37) and (38) that, A and B given by Lemma 2 and for every $(\phi_+, \phi_-) \in E_K$, using the definition of the norm $\mathcal{N}_\pm$. Indeed, one has for $r \in K$, since Q is positive,

$$\left|\frac{\phi_+}{Q}\right| \le \left\|\frac{\phi_+}{Q}\right\|_{L^\infty(K)} \le b^{1/3},$$

from the definition of $\mathcal{N}_+$. Moreover, from the definition A and the a priori bound on γ, we have

$$|\gamma A| \le \frac{1}{2}\gamma_\sigma |A| \le C b_\sigma^{1/6} \exp\left(-\frac{2}{\sqrt{b_\sigma}}\right) r^{-\frac{d-1}{2}} e^r,$$

Using (4) we have $\kappa r^{-(d-1)/2} e^{-r} \le Q(r) + C r^{-(d+1)/2} e^{-r}$,

$$\begin{aligned}
|\gamma A| &\le C b_\sigma^{1/6} \exp\left(-\frac{2}{\sqrt{b_\sigma}}\right) e^{2r} \left(Q(r) + C r^{-(d+1)/2} e^{-r}\right) \\
&\le C b_\sigma^{1/6} \exp\left(\frac{2}{\sqrt{b}} - \frac{2}{\sqrt{b_\sigma}}\right) (Q(r) + C) \\
&\le C b_\sigma^{1/6} \left(Q(r) + \left|\frac{2}{\sqrt{b}} - \frac{2}{\sqrt{b_\sigma}}\right| Q(r)\right) \\
&\le C b_\sigma^{1/6} Q(r).
\end{aligned}$$

The bound on $b\sigma B$ and ϕ_- are obtained similarly.

□

4.1.2 Solution of (35) in $I \cup J$

The profile in P_{ext} in $I \cup J$ is constructed in two main steps. First, the solution is defined in $I = [b^{-2}, \infty)$ by letting

$$P(r) = r^{-\frac{d-1}{2}} U(r), \tag{43}$$

where

$$U'' + \left(\frac{b^2 r^2}{4} - 1 - \frac{(d-1)(d-3)}{4r^2} - ib\sigma\right) U + r^{-\frac{1}{2}(d-1)(p-1)} |U|^{p-1} U = 0 \tag{44}$$

The equation (44) is essentially linear near the profile Q in the region I as profile Q is exponentially decreasing.

Proposition 8 (Proposition 2.1, [1]). *For $\sigma > 0$ small enough and for any b, ρ satisfying* (36), (37), *there exists a C^2 solution U of* (44) *on I satisfying*

$$U(r) = \rho r^{-\frac{1}{2}+\sigma} \exp\left(ib\frac{r^2}{4}\right) \exp\left(-i\frac{\ln r}{b}\right) (1 + O(b^{-3} r^{-2})) \tag{45}$$

$$U'(r) = i\frac{b}{2}\rho r^{\frac{1}{2}+\sigma} \exp\left(ib\frac{r^2}{4}\right) \exp\left(-i\frac{\ln r}{b}\right) (1 + O(b^{-3} r^{-2})), \tag{46}$$

and

$$U' - i\frac{br}{2}U = O(b^{-1}r^{-1}|U|) = O(\rho b^{-1}r^{-\frac{3}{2}+\sigma}).$$

Moreover, the map $(\sigma, b, \rho) \mapsto (U[\sigma, b, \rho](b^{-2}), U'[\sigma, b, \rho](b^{-2}))$ *is continuous.*

As pointed out in [1, Remark 2.3], the profile P given by (43) in I belongs to $\dot{H}^1(I)$ but not in $L^2(I)$ due to the asymptotic given by (45), (46). However, (45) is sufficient to recover $L^2_{loc}(I)$ for the profile P given by (43), which shall be used to deduce $\|P - Q\|_{L^2_{loc}(I)} = o(1)$.

The solution U on I is then extended to $I \cup J$, the region including the turning point, which complicates significantly the analysis.

Proposition 9 (Proposition 3.1, [1])**.** *For* $\sigma > 0$ *small enough and for any* b, ρ*, satisfying* (36)*,* (37)*, the solution* U *of* (44) *on* I *constructed in Proposition 8 extends to a solution of* (44) *on* $J \cup I$*. Moreover, there exists a real* $\theta_{ext} \in [0, 2\pi)$ *such that the function* P_{ext} *defined by*

$$P_{ext}(r) = e^{i\theta_{ext}} r^{-\frac{d-1}{2}} U(r), r \in J \cup I,$$

is a solution of (35) *on* $J \cup I$ *and satisfies*

$$\begin{aligned}
\mathrm{Re}(P_{ext}(r_K)) &= \frac{\rho b^{\frac{d+1}{4}}}{\sqrt{2}} \exp\left(\frac{\pi}{2b} - \frac{1}{\sqrt{b}}\right)(1 + O(b^{\frac{1}{4}}))\\
\mathrm{Re}(P'_{ext}(r_K)) &= -\frac{\rho b^{\frac{d+1}{4}}}{\sqrt{2}} \exp\left(\frac{\pi}{2b} - \frac{1}{\sqrt{b}}\right)(1 + O(b^{\frac{1}{4}}))\\
\mathrm{Im}(P_{ext}(r_K)) &= \frac{\rho b^{\frac{d+1}{4}}}{2\sqrt{2}} \exp\left(\frac{1}{\sqrt{b}} - \frac{\pi}{2b}\right)(1 + O(b^{\frac{1}{4}}))\\
\mathrm{Im}(P'_{ext}(r_K)) &= \frac{\rho b^{\frac{d+1}{4}}}{2\sqrt{2}} \exp\left(\frac{1}{\sqrt{b}} - \frac{\pi}{2b}\right)(1 + O(b^{\frac{1}{4}}))
\end{aligned}$$

Moreover, the map $(\sigma, b, \rho) \mapsto (P_{ext}[\sigma, b, \rho](r_K), P'_{ext}[\sigma, b, \rho](r_K))$ *is continuous*

4.1.3 Matching asymptotic

The solution P_{int}, defined over K, and P_{ext}, defined over $J \cup I$, are matched at r_K using a fixed-point argument on the parameters.

Theorem 12 (Theorem 2, [1])**.** *Let* $d \geq 1$ *and* $p_* < \overline{p}$*. There exists* $\sigma_0 > 0$ *such that for any* $\sigma \in (0, \sigma_0)$*, and* $p \in [p_*, \overline{p}]$*, there exists* b, ρ, γ, θ *satisfying* (36)*,* (37) *such that the solution* $P_{ext}[\sigma, b, \rho]$ *of* (35) *on* $I \cup J$ *given by Proposition 3.1 and* $P_{int}[\sigma, b, \gamma]$ *of* (35) *on* K *given by Proposition 4.1 satisfy the following matching conditions*

$$P_{int}(r_K) = P_{ext}(r_K), \quad P'_{int}(r_K) = P'_{ext}(r_K).$$

In particular, the function P *defined on* $[0, \infty)$ *by*

$$P(r) = \begin{cases} P_{int}(r), & r \in K,\\ P_{int}(r), & r \in I \cup J, \end{cases}$$

is a C^2 *solution of* (35) *on* $[0, \infty)$ *satisfying the asymptotic: for* r *large,*

$$\begin{aligned}
|P(r)| &= |\rho| r^{-\frac{d}{2}+\sigma}(1 + O(b^{-3}r^{-2}))\\
P'(r) - i\frac{br}{2}P(r) &= O(\rho b^{-1} r^{-\frac{d}{2}-1+\sigma})
\end{aligned}$$

Theorem 10 is obtained from Theorem 12 by choosing $\sigma = s_c = \frac{d}{2} - \frac{2}{p-1}$.

4.2 Blow-up profile for the mass slightly subcritical case

We deduce Theorem 11 in the mass slightly subcritical regime $-1 \ll s_c < 0$.

Proof. Consider ψ according to the ansatz

$$\psi(x,t) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)} e^{i\theta(t)} \tilde{\psi}\left(\frac{x}{\lambda(t)}\right), \tag{47}$$

where $\tilde{\psi}$ is assumed radially symmetric and $\lambda(t), \theta(t)$ defined by (30). Then, if $\tilde{\psi}$ satisfies

$$\Delta\tilde{\psi} - \tilde{\psi} + ib\left(\frac{2}{p-1}\psi + x.\nabla\psi\right) + |\tilde{\psi}|^{p-1}\tilde{\psi} = 0, \quad x \in \mathbb{R}^d. \tag{48}$$

Then ψ defined by (47) satisfies (1). Now, let

$$\tilde{\psi}(x) = \exp\left(-i\frac{br^2}{4}\right) P(r).$$

If P is a solution to

$$\begin{cases} P'' + \dfrac{d-1}{r}P' + \left(\dfrac{b^2r^2}{4} - 1 - ibs_c\right) P + |P|^{p-1}P = 0, \quad r > 0, \\ P'(0) = 0. \end{cases} \tag{49}$$

then $\tilde{\psi}$ is a solution to (48). The proof of Theorem 10 is obtained from Proposition 7, Proposition 8 and Proposition 9 by relaxing the dependence of (49) with respect to $s_c > 0$ by considering the family of equations,

$$\begin{cases} P'' + \dfrac{d-1}{r}P' + \left(\dfrac{b^2r^2}{4} - 1 - ib\sigma\right) P + |P|^{p-1}P = 0, \quad r > 0, \\ P'(0) = 0. \end{cases} \tag{50}$$

with respect to the parameter $\sigma > 0$. To prove that the conclusion of Theorem 10 holds in the mass slightly subcritical case $-1 \ll s_c < 0$, we seek a profile P solution to

$$\begin{cases} P'' + \dfrac{d-1}{r}P' + \left(\dfrac{b^2r^2}{4} - 1 + ib\sigma\right) P + |P|^{p-1}P = 0, \quad r > 0, \\ P'(0) = 0. \end{cases} \tag{51}$$

where the sign of $ib\sigma P$ is chosen so that $b > 0$ and $\sigma > 0$ (recall that in the mass subcritical case, $\text{Im}(-ibs_c) > 0$). Therefore, if P denotes the solution of (50) in the mass supercritical case, then $\overline{P}$ is solution to (51). Indeed, taking the complex conjugate of (51) yields,

$$\begin{cases} \overline{P}'' + \dfrac{d-1}{r}\overline{P}' + \left(\dfrac{b^2r^2}{4} - 1 + ib\sigma\right) \overline{P} + |\overline{P}|^{p-1}\overline{P} = 0, \quad r > 0, \\ \overline{P}'(0) = 0. \end{cases}$$

with $b > 0$ and $\sigma > 0$. Hence, Proposition 7, Proposition 8 and Proposition 9 hold, and taking $\sigma = s_c = \frac{d}{2} - \frac{2}{p-1}$, the conclusion of Theorem 10 holds in the mass slightly subcritical case.

To conclude the proof of Theorem 11, it remains to prove $\|\tilde{\psi} - Q\|_{L^2_{loc}(\mathbb{R}^d)} = o(1)$. Let C be a compact subset of $\mathbb{R}^d$. Denote first $C_K = B_{r_K}(0) \cap C$, where $B_{r_K}(0)$ is the ball of $\mathbb{R}^d$ centered at the origin and of radius $r_K = b^{-1/2}$. We have,

$$\begin{aligned}
\|\tilde{\psi} - Q\|^2_{L^2(C_K)} \leq & \|\tilde{\psi} - Q\|^2_{L^2(B_{r_K}(0))} \\
= & \int_{0\leq r\leq r_K} \left| \exp\left(-i\frac{br^2}{4}\right) P(r) - Q(r)\right|^2 r^{d-1}dr \\
= & \int_{0\leq r\leq r_K} \left| \left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1}dr + \int_{0\leq r\leq r_K} |P(r)-Q(r)|^2 r^{d-1}dr.
\end{aligned}$$

We divide the region of integration of the first integral,

$$\begin{aligned}
\int_{r\leq r_K} \left| \left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1}dr = & \int_{0\leq r\leq b^{-1/4}} \left| \left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1}dr \\
& + \int_{b^{-1/4}\leq r\leq r_K} \left| \left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1}dr.
\end{aligned}$$

Then, on one hand

$$\begin{aligned}
\int_{0\leq r\leq b^{-1/4}} \left| \left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1}dr \leq & \sup_{0\leq r\leq b^{-1/4}} \left|1-\exp\left(-i\frac{br^2}{4}\right)\right| \int_{0\leq r<\infty} Q^2(r) r^{d-1}dr \\
\leq & N_c \sup_{0\leq r\leq b^{-1/4}} \left|1-\exp\left(-i\frac{br^2}{4}\right)\right| \\
\leq & Cb^{1/2}.
\end{aligned}$$

On the other hand, using the exponential decay of the ground state (4)

$$\int_{b^{-1/4}\leq r\leq r_K} \left| \left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1}dr \leq 2\int_{b^{-1/4}\leq r\leq r_K} |Q(r)|^2 r^{d-1}dr \leq Ce^{-b^{-1/4}} \tag{52}$$

To deal with the profile P, we use the profile decomposition of P in K, the bounds of Lemma 3. Indeed, there exists $(\phi_+, \phi_-) \in E_K$ (recall (40)) such that

$$P = (Q + \gamma A + \phi_+) + i(b\sigma B + \phi_-)$$

and such that Lemma 3 holds. Then,

$$\begin{aligned}
\int_{0\leq r\leq r_K} |P(r)-Q(r)|^2 r^{d-1}dr = & \int_{0\leq r\leq r_K} |P(r)-Q(r)|^2 r^{d-1}dr \\
= & \int_{0\leq r\leq r_K} |\gamma A + \phi_+ + i(b\sigma B + \phi_-)|^2 r^{d-1}dr \\
\leq & Cb^{1/6} \int_{0\leq r\leq r_K} |Q(r)|^2 r^{d-1}dr \\
\leq & Cb^{1/6} \int_{0\leq r<\infty} |Q(r)|^2 r^{d-1}dr \\
\leq & Cb^{1/6}.
\end{aligned}$$

Hence, we deduce $\|\tilde{\psi} - Q\|_{L^2(C_K)} \to 0$ as $b \to 0$. Second, we deal with $C_{I\cup J} := C \setminus C_K$. Denote $r_C > 0$, the radius of $B_{r_C}(0)$ such that $C \subset B_{r_C}(0)$. Since the profile P_{int} on $I \cup J$ satisfies (45),

(46) of Proposition 8, we have, using again (4)

$$
\begin{aligned}
\|\tilde{\psi}-Q\|^2_{L^2(C_{I\cup J})} \leq & \|\tilde{\psi}-Q\|^2_{L^2(B_{r_C}(0)\setminus B_{r_K}(0))} \\
\leq & \int_{r_K\leq r\leq r_C} |P(r)-Q(r)|^2 r^{d-1} dr \\
& + \int_{r_K\leq r\leq r_C} \left|\left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1} dr
\end{aligned}
$$

Dealing with the second integral as for (52), we deduce

$$
\int_{r_K\leq r\leq r_C} \left|\left(1-\exp\left(-i\frac{br^2}{4}\right)\right) Q(r)\right|^2 r^{d-1} dr \leq Ce^{-b^{-1/2}}.
$$

We are left with the first integral. We use the exponential decay of the ground state given by (45) to obtain

$$
\begin{aligned}
\int_{r_K\leq r\leq r_C} |P(r)-Q(r)|^2 r^{d-1} dr \leq & \int_{r_K\leq r\leq r_C} \left|r^{-(d-1)/2}U(r)-\kappa r^{-(d-1)/2}e^{-r}\right|^2 r^{d-1} dr \\
& + C\int_{r_K\leq r\leq r_C} e^{-2r} r^{-1} dr \\
\leq & \int_{r_K\leq r\leq r_C} |U(r)|^2 dr + \kappa \int_{r_K\leq r\leq r_C} e^{-2r} dr \\
& + C\int_{r_K\leq r\leq r_C} e^{-2r} r^{-1} dr.
\end{aligned}
$$

The last two integrals are bounded by $Ce^{-2b^{-1/2}}$. It remains to bound the first integral. In order to do so, we use (52), and the bound on ρ with respect to b given by (37) and (38) to deduce

$$
\begin{aligned}
\int_{r_K\leq r\leq r_C} |U(r)|^2 dr \leq & (r_C-b^{-1/2})\|U\|^2_{L^\infty(r_K\leq r\leq r_C)} \\
\leq & |r_C-b^{-1/2}| b^{-5} e^{-\pi/b}
\end{aligned}
$$

which allows to finally prove that $\|\tilde{\psi}-Q\|_{L^2(C_{I\cup J})} \to 0$ as $b\to 0$. Combining this fact with $\|\tilde{\psi}-Q\|_{\dot{H}^1(\mathbb{R}^d)} \to 0$ as $b \to 0$, we finally deduce $\|\tilde{\psi}-Q\|_{H^1_{loc}(\mathbb{R}^d)} \to 0$ as $b\to 0$, which ends the proof.

□

5 Controllability to the blow-up trajectory

We are now in position to prove Theorem 2 and Theorem 5. We begin by the proof of Theorem 5.

Proof. Let $d \geq 1$ and $\epsilon > 0$. Consider for now $p \in [1, p^*)$. Denote ψ_b the blow-up profile

$$
\psi_b(x,t) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} e^{i\theta(t)} \tilde{\psi}\left(\frac{x}{\lambda(t)}\right).
$$

where $\tilde{\psi}$ is given by Theorem 11, where

$$
\lambda(t) = \sqrt{2b(T-t)}, \quad \theta(t) = -\frac{\log(T-t)}{2b}, \quad t < T,
$$

and where $T > 0$ is fixed the following way,

$$T - \epsilon = \frac{1}{2bc_1}, \quad T - T_{c_1,c_2} + \epsilon = \frac{1}{2bc_2}. \tag{53}$$

Hence,

$$\psi_b(x,\epsilon) = e^{i\theta(\epsilon)} c_1^{\frac{1}{p-1}} \tilde{\psi}(\sqrt{c_1}x), \quad \psi_b(x, T_{c_1,c_2} - \epsilon) = e^{i\theta(T_{c_1,c_2}-\epsilon)} c_2^{\frac{1}{p-1}} \tilde{\psi}(\sqrt{c_1}x). \tag{54}$$

For $p \in (1,p^*]$, denote $\delta_1(p), \delta_2(p) \in \mathbb{R}^+$ such that Theorem 3 apply in time $T = \epsilon$ for the smooth trajectory $\psi_{c_1}(x,t) = e^{itc_1}Q_{c_1}(x)$ and $\psi_{c_2}(x,t) = e^{itc_2}Q_{c_2}(x)$ respectively. Define $\tilde{\delta} := \min_{p\in[1,p^*]}\{\delta_1(p), \delta_2(p)\}$. We easily see that $\tilde{\delta} > 0$ since Theorem 3 holds for $p \in [1,p^*]$. We fix $\underline{p} \in (1,p^*)$ such that, $\forall p \in (\underline{p}, p^*)$,

$$\|Q_{c_1} - c_1^{\frac{1}{p-1}} \tilde{\psi}(\sqrt{c_1}x)\|_{H^1(\Omega)} < \tilde{\delta}, \quad \|Q_{c_2} - c_2^{\frac{1}{p-1}} \tilde{\psi}(\sqrt{c_2}x)\|_{H^1(\Omega)} < \tilde{\delta}. \tag{55}$$

This is always possible since $\|Q - \tilde{\psi}\|_{H^1_{loc}(\mathbb{R}^d)} \to 0$ as $b \to 0$. Now consider $\psi_0, \psi_T \in H^1(\Omega)$ such that

$$\min_{\gamma\in[0,2\pi]} \|\psi_0 - e^{i\gamma}Q_{c_1}\|_{H^1(\Omega)} < \tilde{\delta}, \quad \min_{\gamma\in[0,2\pi]} \|\psi_T - e^{i\gamma}Q_{c_2}\|_{H^1(\Omega)} < \tilde{\delta}, \tag{56}$$

and denote

$$\gamma_1 = \arg \min_{\gamma\in[0,2\pi]} \|\psi_0 - e^{i\gamma}Q_{c_1}\|_{H^1(\Omega)}, \quad \gamma_2 = \arg \min_{\gamma\in[0,2\pi]} \|\psi_T - e^{i\gamma}Q_{c_2}\|_{H^1(\Omega)}.$$

We are now in position to prove Theorem 2. First, we need to adjust the phase of the blow-up profile as well as the initial and final data. Using the phase invariance of the solutions to (1), we define the trajectory

$$\psi_{traj}(x,t) = e^{i(c_1\epsilon+\gamma_1-\theta(\epsilon))}\psi_b(x,t). \tag{57}$$

By construction, using (54), (55) and (57),

$$\|e^{i(c_1\epsilon+\gamma_1)}Q_{c_1} - \psi_{traj}(.,\epsilon)\|_{H^1(\Omega)} < \tilde{\delta}.$$

Therefore, we can apply Theorem 3 in time $T = \epsilon$ around the smooth trajectory $e^{i(c_1t+\gamma_1)}Q_{c_1}$. This implies that the solution to (11), denoted ψ^1, satisfies $\psi^1(.,0) = \psi_0$ and $\psi^1(.,\epsilon) = \psi_{traj}(.,\epsilon)$. We denote the associated control v_1.

During the time interval $(\epsilon, T_{c_1,c_2} - \epsilon)$ we define $\psi^2(.,t) = \psi_{traj}(.,t)$. Then, using again (54), (55) and (57), we apply Theorem 3 to drive the solution to (11), denoted ψ^3, from $\psi_3(.,T_{c_1,c_2} - \epsilon) = \psi_{traj}(.,T_{c_1,c_2} - \epsilon)$ to $\psi_3(.,T_{c_1,c_2}) = e^{i(c_1\epsilon+\gamma_1+\theta(T_{c_1,c_2}-\epsilon)-\theta(\epsilon))}Q_{c_2}$. Now, define the smallest $T_{\gamma_2} \geq 0$ such that

$$e^{ic_2T_{\gamma_2}}e^{i(c_1\epsilon+\gamma_1+\theta(T_{c_1,c_2}-\epsilon)-\theta(\epsilon))} = e^{i\gamma_2},$$

and define $\psi^4(.,t) = e^{i(c_1\epsilon+\gamma_1+\theta(T_{c_1,c_2}-\epsilon)-\theta(\epsilon))}e^{ic_2(t-T_{c_1,c_2})}Q_{c_2}$. We therefore have $\psi^3(.,T_{c_1,c_2}) = \psi^4(.,T_{c_1,c_2})$ and $\psi^4(.,T_{c_1,c_2} + T_{\gamma_2}) = e^{i\gamma_2}Q_{c_2}$. We can finally apply Theorem 3 one last time to deduce that there exists a solution ψ^5 to (11) such that $\psi^5(.,T_{c_1,c_2} + T_{\gamma_2}) = \psi^4(.,T_{c_1,c_2} + T_{\gamma_2}) = e^{i\gamma_2}Q_{c_2}$ and $\psi^5(.,T_{c_1,c_2} + T_{\gamma_2} + \epsilon) = \psi_T$.

Hence, the solution

$$\psi(x,t) = \begin{cases} \psi^1(x,t), & t \in (0,\epsilon), \\ \psi^2(x,t), & t \in (\epsilon, T_{c_1,c_2} - \epsilon), \\ \psi^3(x,t), & t \in (T_{c_1,c_2} - \epsilon, T_{c_1,c_2}), \\ \psi^4(x,t), & t \in (T_{c_1,c_2}, T_{c_1,c_2} + T_{\gamma_2}), \\ \psi^5(x,t), & t \in (T_{c_1,c_2} + T_{\gamma_2}, T_{c_1,c_2} + T_{\gamma_2} + \epsilon), \end{cases}$$

belongs to $C([0,T_{c_1,c_2}+T_{\gamma_2}+\epsilon];H^1(\Omega))$. By a classical trace theorem, the trace of ψ provides the definition of the control $v \in C([0,T_{c_1,c_2}+T_{\gamma_2}+\epsilon];H^{1/2^-}(\partial\Omega))$.

□

We now turn to the proof of Theorem 2.

Proof. The proof of Theorem 2 is very similar to the proof above. Indeed, the only modification needed in the proof is the profile to go from one scaled ground state from another. In order to do so, we use the simple remark that the solution to (1) with the initial data $\psi_0 = (1+\epsilon)Q_\lambda$ satisfies the requirement for $\epsilon > 0$ sufficiently small. Indeed, we have $\psi_0 \in \mathcal{B}_{\alpha^*}$ using that the $L^2(\mathbb{R}^d)$-norm is invariant under the scaling in the mass critical case. Moreover, using Pohozaev's identity $E(Q_\lambda) = \lambda E(Q) = 0$,

$$\begin{aligned} E(\psi_0) &= \frac{(1+\epsilon)^2}{2}\int |\nabla Q_\lambda|^2 - \frac{(1+\epsilon)^{2+d/4}}{2+d/4}\int |Q_\lambda|^{2+d/4} \\ &= E(Q_\lambda) + \frac{(1+\epsilon)^2-(1+\epsilon)^{2+d/4}}{2+d/4}\int |Q_\lambda|^{2+d/4} \\ &< 0 \end{aligned}$$

Therefore the solution ψ to (1) starting from $\psi_0 = (1+\epsilon)Q_\lambda$ blow-up in finite time and belongs to $C([0,T);H^1(\mathbb{R}^d))$. We use the exact local controllability to reach this blow-up profile from the initial and to the final data in arbitrarly small time.

□

6 Conclusion

In lights of the results presented here, we obtained the controllability of initial and final states close to ground state solitary waves with different scaling. In some sense, this strategy is close to the return method, as a trajectory with "good control properties" [9] was used to connect two different states. However, the small-time global controllability of (11) remains an open question.

Moreover, using the results presented here, we are able to address the question of finite time blow-up in the mass subcritical case on bounded domain. Indeed, consider

$$\begin{cases} i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0, & (x,t) \in \Omega\times\mathbb{R}^+ \\ \psi(x,t) = v(x,t), & (x,t) \in \partial\Omega\times\mathbb{R}^+ \\ \psi(x,0) = \psi_0, & x \in \Omega, \end{cases} \tag{58}$$

with $\psi_0 \in H^1(\Omega)$ and $v \in L^2(\mathbb{R}^+;H^{1/2^-}(\partial\Omega))$. Then

Theorem 13. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p < p^*$ for $0 < p^* - p \ll 1$. Then there exists $\psi_0 \in H^1(\Omega)$ close to Q in $H^1(\Omega)$ and $v \in L^2(\mathbb{R}^+;H^{1/2^-}(\partial\Omega))$ such that the solution blow-up in finite time with*

$$\|\nabla\psi\|_{L^2(\Omega)} \simeq \frac{1}{\sqrt{b(T-t)}}, \quad t \to T^-.$$

The framework given by (58) allows to properly define blow-up solutions in $H^1(\Omega)$ in the mass subcritical regime, that is, without having solutions not belonging to $L^2(|x| > R)$ for $R > 0$ large. We highlight that Theorem 13 is close to the blow-up phenomenon exhibited in for (58) in the mass critical case with $v \equiv 0$ where the ground state was defined as the unique positive steady state.

Acknowledgements

The author would like to thanks Pierre Raphaël for the enlightening discussions and advices.

References

[1] Y. Bahri, Y. Martel, and P. Raphaël. Self-similar blow-up profiles for slightly supercritical nonlinear schrödinger equations. *Preprint*, pages 1–42.

[2] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[3] K. Beauchard and M. Mirrahimi. Practical stabilization of a quantum particle in a one-dimensional infinite square potential well. *SIAM J. Control Optim.*, 48(2):1179–1205, 2009.

[4] N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(7):749–752, 1997.

[5] N. Burq and M. Zworski. Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, 17(2):443–471, 2004.

[6] R. Castelli and H. Teismann. Rigorous numerics for nls: Bound states, spectra, and controllability. *Physica D: Nonlinear Phenomena*, 334:158–173, 11 2016.

[7] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[8] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982.

[9] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[10] B. Dehman, P. Gérard, and G. Lebeau. Stabilization and control for the nonlinear Schrödinger equation on a compact surface. *Math. Z.*, 254(4):729–749, 2006.

[11] G. Fibich, F. Merle, and P. Raphaël. Proof of a spectral property related to the singularity formation for the L^2 critical nonlinear Schrödinger equation. *Phys. D*, 220(1):1–13, 2006.

[12] L. Gagnon. Lagrangian controllability of the 1-dimensional korteweg–de vries equation. *SIAM J. Control Optim.*, 54(6):3152–3173, 2016.

[13] J. Ginibre and G. Velo. On a class of nonlinear schrödinger equations. i. the cauchy problem, general case. *Journal of Functional Analysis*, 32(1):1 – 32, 1979.

[14] N. Godet. Blow up on a curve for a nonlinear Schrödinger equation on Riemannian surfaces. *Dyn. Partial Differ. Equ.*, 10(2):99–155, 2013.

[15] A. Hasegawa and M. Matsumoto. *Optical Solitons in Fibers*. Springer Series in Photonics. Springer Berlin Heidelberg, 2002.

[16] S. Jaffard. Contrôle interne exact des vibrations d'une plaque rectangulaire. *Portugal. Math.*, 47(4):423–429, 1990.

[17] L. Jin. Control for Schrödinger equation on hyperbolic surfaces. *Math. Res. Lett.*, 25(6):1865–1877, 2018.

[18] T. Kato. On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.*, 46(1):113–129, 1987.

[19] H. Lange and H. Teismann. Controllability of the nonlinear Schrödinger equation in the vicinity of the ground state. *Math. Methods Appl. Sci.*, 30(13):1483–1505, 2007.

[20] C. Laurent. Global controllability and stabilization for the nonlinear Schrödinger equation on an interval. *ESAIM Control Optim. Calc. Var.*, 16(2):356–379, 2010.

[21] C. Laurent. Global controllability and stabilization for the nonlinear Schrödinger equation on some compact manifolds of dimension 3. *SIAM J. Math. Anal.*, 42(2):785–832, 2010.

[22] C. Laurent. Internal control of the Schrödinger equation. *Math. Control Relat. Fields*, 4(2):161–186, 2014.

[23] G. Lebeau. Contrôle de l'équation de Schrödinger. *J. Math. Pures Appl. (9)*, 71(3):267–291, 1992.

[24] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988.

[25] W. Littman and S. Taylor. Smoothing evolution equations and boundary control theory. volume 59, pages 117–131. 1992. Festschrift on the occasion of the 70th birthday of Shmuel Agmon.

[26] F. Merle. Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.*, 69(2):427–454, 1993.

[27] F. Merle and P. Raphaël. Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation. *Geom. Funct. Anal.*, 13(3):591–642, 2003.

[28] F. Merle and P. Raphaël. On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation. *Invent. Math.*, 156(3):565–672, 2004.

[29] F. Merle and P. Raphaël. The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Ann. of Math. (2)*, 161(1):157–222, 2005.

[30] F. Merle and P. Raphaël. Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation. *Comm. Math. Phys.*, 253(3):675–704, 2005.

[31] F. Merle and P. Raphaël. On a sharp lower bound on the blow-up rate for the L^2 critical nonlinear Schrödinger equation. *J. Amer. Math. Soc.*, 19(1):37–90, 2006.

[32] F. Merle, P. Raphaël, and J. Szeftel. Stable self-similar blow-up dynamics for slightly L^2 super-critical NLS equations. *Geom. Funct. Anal.*, 20(4):1028–1071, 2010.

[33] M. Mirrahimi. Lyapunov control of a quantum particle in a decaying potential. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1743–1765, 2009.

[34] C. Muñoz. On approximate controllability of generalized KdV solitons. *SIAM J. Control Optim.*, 52(1):52–81, 2014.

[35] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[36] P. Raphaël. Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation. *Math. Ann.*, 331(3):577–609, 2005.

[37] L. Rosier and B.-Y. Zhang. Exact boundary controllability of the nonlinear Schrödinger equation. *J. Differential Equations*, 246(10):4129–4153, 2009.

[38] D. L. Russell. Boundary value control of the higher-dimensional wave equation. *SIAM J. Control*, 9:29–42, 1971.

[39] D. L. Russell. Boundary value control theory of the higher-dimensional wave equation. II. *SIAM J. Control*, 9:401–419, 1971.

[40] A. Soffer. Soliton dynamics and scattering. In *International Congress of Mathematicians. Vol. III*, pages 459–471. Eur. Math. Soc., Zürich, 2006.

[41] C. Sulem and P.-L. Sulem. Focusing nonlinear Schrödinger equation and wave-packet collapse. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996)*, volume 30, pages 833–844, 1997.

[42] C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

[43] T. Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[44] M. I. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.*, 87(4):567–576, 1982/83.

[45] V. Zakharov. Collapse of self-focusing of langmuir waves. Number 35, pages 908–914, 1972.

[46] E. Zuazua. Remarks on the controllability of the Schrödinger equation. In *Quantum control: mathematical and numerical challenges*, volume 33 of *CRM Proc. Lecture Notes*, pages 193–211. Amer. Math. Soc., Providence, RI, 2003.