Metric Dimension: from Graphs to Oriented Graphs

Fionn Mc Inerney, Julien Bensmail, Nicolas Nisse

Metric Dimension: from Graphs to Oriented Graphs*

Julien Bensmail[1], Fionn Mc Inerney[1], and Nicolas Nisse[1]

[1]Université Côte d'Azur, CNRS, Inria, I3S, France

Abstract

The metric dimension $\mathrm{MD}(G)$ of an undirected graph G is the cardinality of a smallest set of vertices that allows, through their distances to all vertices, to distinguish any two vertices of G. Many aspects of this notion have been investigated since its introduction in the 70's, including its generalization to digraphs.

In this work, we study, for particular graph families, the maximum metric dimension over all strongly-connected orientations, by exhibiting lower and upper bounds on this value. We first exhibit general bounds for graphs with bounded maximum degree. In particular, we prove that, in the case of subcubic n-node graphs (i.e., with maximum degree 3), all strongly-connected orientations asymptotically have metric dimension at most $\frac{n}{2}$, and that there are such orientations having metric dimension $\frac{2n}{5}$. We then consider strongly-connected orientations of grids. For a torus with n rows and m columns, we show that the maximum value of the metric dimension of a strongly-connected Eulerian orientation is asymptotically $\frac{nm}{2}$ (the equality holding when n, m are even, which is best possible). For a grid with n rows and m columns, we prove that all strongly-connected orientations asymptotically have metric dimension at most $\frac{2nm}{3}$, and that there are such orientations having metric dimension $\frac{nm}{2}$.

Keywords: Resolving sets; Metric dimension; Strongly-connected orientations.

1 Introduction

1.1 Resolving sets and metric dimension in undirected graphs

The *distance* $\mathrm{dist}_G(u, v)$ (or simply $\mathrm{dist}(u, v)$ when no ambiguity is possible) between two vertices u, v of an undirected graph G is the length of a shortest path from u to v. A *resolving set* R of G is a subset of vertices that permits to distinguish all vertices of G according to their distances to the vertices of R. In other words, R is a resolving set if and only if, for every two distinct vertices u, v of G, there exists $w \in R$ such that $\mathrm{dist}_G(w, u) \neq \mathrm{dist}_G(w, v)$. The *metric dimension* $\mathrm{MD}(G)$ of G is the minimum cardinality of a resolving set of G. Since $V(G) \setminus \{v\}$ is a resolving set of G for every $v \in V(G)$, this parameter $\mathrm{MD}(G)$ is defined for every undirected graph G.

The notions of resolving sets and metric dimension have been widely studied since their introduction in the 70's by Harary and Melter [9], and Slater [15], notably because they can be used to model many real-life problems. Many related aspects have been investigated to

*This work has been partially supported by ANR project MultiMod and ANR program "Investments for the Future" under reference ANR-11-LABX-0031-01. It has also been supported by the project GALOP STIC AMSUD 19-STIC-05. An extended abstract of parts of this paper has been presented in [5].

date, including algorithmic and complexity aspects, and bounds on the metric dimension of particular graph families. Our main focus in this paper being the metric dimension of oriented graphs, we refer the interested reader to surveys (e.g. [1, 2]) for more details about investigations in the undirected context.

1.2 Resolving sets and metric dimension in digraphs

A natural way of generalizing graph theoretical problems is to consider their directed counterparts. In the context of the metric dimension of graphs, this was first considered by Chartrand, Rains, and Zhang in [3], before receiving further consideration in several works (see [6, 7, 10, 12, 13]). It is worthwhile recalling that, in digraphs, distances have behaviours that differ from those in undirected graphs. Notably, an important point that should be addressed is that, in the context of general digraphs D, we might have $\mathrm{dist}(u, v) \neq \mathrm{dist}(v, u)$ for any two vertices u, v, where $\mathrm{dist}(u, v)$ here refers to the length of a shortest directed path from u to v. A digraph D is *strongly-connected* (or *strong* for short) if, for every $u, v \in V(D)$, there is a directed path from u to v, and conversely one from v to u. Hence, if D is not strong, then there are vertices $u, v \in V(D)$ such that no directed paths from u to v exist. In such a case, we set $\mathrm{dist}(u, v) = +\infty$.

These peculiar aspects of distances in digraphs must be taken into account when defining directed notions of resolving sets and metric dimension. Throughout this work, the notions of resolving sets and metric dimension in digraphs are with respect to the following definitions. Let R be a subset of vertices of a digraph D. Two vertices u, v of D are said to be *distinguished*, denoted by $u \nsim_R v$, if there exists $w \in R$ such that $\mathrm{dist}(w, u) \neq \mathrm{dist}(w, v)$. Otherwise, u and v are *undistinguished* by R, which is denoted by $u \sim_R v$. In particular, if $\mathrm{dist}(w, u)$ is finite and $\mathrm{dist}(w, v)$ is not for some $w \in R$, then $u \nsim_R v$. A set $R \subseteq V(D)$ is called *resolving* if all pairs of vertices of D are distinguished by R. The *metric dimension* $\mathrm{MD}(D)$ of D is then the smallest size of a resolving set. Note that $\mathrm{MD}(D)$ is defined for every digraph D; in particular, we have $\mathrm{MD}(D) < |V(D)|$ since $R = V(D) \setminus \{v\}$ is a resolving set for any $v \in V(D)$ (as having any vertex in a resolving set makes it distinguished from all other vertices).

Our definitions of directed resolving sets and metric dimension actually differ from those originally introduced by Chartrand, Rains, and Zhang. On the one hand, in their definition of resolving sets, they consider the distances from each of the vertices not in R to the vertices in R in order to distinguish the vertices of D. In our definition, the distances from each of the vertices in R to the vertices not in R are considered. Note that both definitions are equivalent on that point, as, given a digraph D, if we reverse the direction of all arcs, resulting in a digraph $\widetilde{D}$, then any shortest path from u to v in D becomes a shortest path from v to u in $\widetilde{D}$.

On the other hand, their definition of resolving sets requires that the distances from each pair of distinct vertices to the vertices in R which distinguish them have to be defined, while our definition (with distances from vertices in R to the other vertices) allows for undefined distances ($+\infty$) to be used as well. Contrary to our definition, this implies that their definition of metric dimension is not defined for all digraphs. As far as we know, the characterization of digraphs that admit a metric dimension (following their definition) is still an open problem [3].

Although our definitions and those of Chartrand, Rains, and Zhang are different, it is worthwhile mentioning that most of our investigations in this paper also apply to their context, as we mainly focus on strong digraphs, in which case our definitions and theirs are equivalent (up to reversing all arcs).

To date, the investigations on the metric dimension of digraphs have thus been with respect to the definitions originally introduced by Chartrand, Rains, and Zhang. As a first step, they notably gave in [3], a characterization of digraphs with metric dimension 1. Complexity aspects were considered in [13], where it was proved that determining the metric dimension of a strong digraph is NP-complete. Bounds on the metric dimension of various digraph families were later exhibited (Cayley digraphs [6], line digraphs [7], tournaments [10], digraphs with cyclic covering [12], De Bruijn and Kautz digraphs [13], etc.).

1.3 From undirected graphs to oriented graphs

To avoid any confusion, let us recall that an *orientation* D of an undirected graph G is obtained when every edge uv of G is oriented either from u to v (resulting in the arc (u, v)) or conversely (resulting in the arc (v, u)). An *oriented graph* D is a directed graph that is an orientation of a simple graph. Note that when G is simple, D cannot have two vertices u, v such that (u, v) and (v, u) are arcs. Such symmetric arcs are allowed in digraphs, which is the main difference between oriented graphs and digraphs. Throughout this paper, when simply referring to a *graph*, we mean an undirected graph.

In [4], Chartrand, Rains, and Zhang considered the following way of linking resolving sets of undirected graphs and digraphs. They considered, for a given graph G, the worst orientations of G for the metric dimension, i.e., orientations of G with maximum metric dimension. Looking at our definition of resolving sets and metric dimension, this is a legitimate question as it has to be pointed out that, for a graph, the metric dimension might or might not be preserved when orienting its edges. An interesting example (reported e.g., in [3, 10]) is the case of a graph G with a Hamiltonian path: while $\mathrm{MD}(G)$ can be arbitrarily large in general (consider e.g., any complete graph), there is an orientation D of G verifying $\mathrm{MD}(D) = 1$ (just orient all edges of a Hamiltonian path from the first vertex towards the last vertex, and all remaining edges in the opposite direction). Conversely, there exist orientations D of G for which $\mathrm{MD}(D)$ can be much larger than $\mathrm{MD}(G)$. As an example, let us consider any path P with $2n + 1$ vertices $v_0, ..., v_{2n}$. Clearly, $\mathrm{MD}(P) = 1$; however, the orientation D of P obtained by making every vertex v_{2k+1} $(k = 0, ..., n - 1)$ become a source (i.e., orienting its incident edges away) verifies $\mathrm{MD}(D) = n$. As shown in this paper, this phenomenon occurs for strong orientations as well.

In [4], the authors proved that, for every positive integer k, there exist infinitely many graphs for which the metric dimension of any of its strongly-connected orientations is exactly k. They have also proved that there is no constant k such that the metric dimension of any tournament is at most k.

1.4 Our results

Motivated by these observations, we investigate, throughout this work, the parameter WOMD (worst oriented metric dimension) defined as follows. For any connected graph G, let $\mathrm{WOMD}(G)$ denote the maximum value of $\mathrm{MD}(D)$ over all strong orientations D of G. Let us extend this definition to graph families as follows. For any family $\mathcal{G}$ of 2-edge-connected graphs[1], let $\mathrm{WOMD}(\mathcal{G}) = \max_{G \in \mathcal{G}} \frac{\mathrm{WOMD}(G)}{|V(G)|}$. Section 2 first introduces tools and results that will be used in the next sections. In Section 3, bounds on $\mathrm{WOMD}(\mathcal{G}_\Delta)$ are

[1] The edge-connectivity requirement, here and further, is to guarantee the good definition of $\mathrm{WOMD}(G)$ for every $G \in \mathcal{G}_\Delta$, as it is a well-known fact that a graph has strong orientations if and only if it is 2-edge-connected (see [14]).

proved, where $\mathcal{G}_\Delta$ refers to the family of 2-edge-connected graphs with maximum degree Δ. In particular, we prove that we asymptotically have $\frac{2}{5} \leq \text{WOMD}(\mathcal{G}_3) \leq \frac{1}{2}$. In Section 4, we then consider the families of grids and tori. For the family $\mathcal{T}$ of tori, we prove that we asymptotically have $\text{WEOMD}(\mathcal{T}) = \frac{1}{2}$, where the parameter $\text{WEOMD}(\mathcal{T})$ (worst Eulerian oriented metric dimension) is defined similarly to $\text{WOMD}(\mathcal{T})$ except that only strong Eulerian orientations of tori (i.e., all vertices have in-degree and out-degree 2) are considered. For the family $\mathcal{G}$ of grids, we then prove that asymptotically $\frac{1}{2} \leq \text{WOMD}(\mathcal{G}) \leq \frac{2}{3}$. Remaining open questions and problems are gathered in Section 5.

Terminology and notation Let D be a digraph. For a vertex v of D, we denote by $d_D^-(v)$ (resp. $d_D^+(v)$) the *in-degree* (resp. *out-degree*) of v which is the number of in-coming (resp. out-going) arcs incident to v. For every arc (v,u) (resp. (u,v)) in-coming to (resp. out-going from) u, we call u an *out-neighbour* (resp. *in-neighbour*) of v. The set of all in-neighbours (resp. out-neighbours) of v is denoted by $N_D^-(v)$ (resp. $N_D^+(v)$). The subscripts in the previous notations might be dropped when no ambiguity is possible. We denote by $\Delta^+(D)$ (resp., $\Delta^-(D)$) the maximum out-degree (resp., maximum in-degree) of a vertex in D. Note that, in an oriented graph D, the in-degree (resp. out-degree) of a vertex corresponds to the cardinality of its in-neighbourhood (resp. out-neighbourhood).

2 Tools and preliminary results

We start off by pointing out the following property of resolving sets in digraphs having vertices with the same in-neighbourhood. This result will be one of our main tools for building digraphs with large metric dimension.

Lemma 2.1. *Let D be a digraph and $S \subseteq V(D)$ be a subset of $|S| \geq 2$ vertices such that, for every $u, v \in S$, we have $N^-(u) = N^-(v)$. Then, any resolving set of D contains at least $|S| - 1$ vertices of S.*

Proof. If two vertices $u, v \in S$ do not belong to a resolving set R, then $\text{dist}(w,u) = \text{dist}(w,v)$ for every $w \in R$, contradicting that R is a resolving set. □

We now introduce a technique that will be used in the next sections for exhibiting upper bounds on the metric dimension of strong digraphs with maximum out-degree at least 2. The technique is based on a connection between the resolving sets of a such digraph and the vertex covers of a particular graph associated to it. A *vertex cover* of a graph G is a subset $S \subseteq V(G)$ of vertices such that, for every edge uv of G, at least one of u and v belongs to S. To any digraph D we associate an *auxiliary (undirected) graph* D_{aux} constructed as follows:

- the vertices of D_{aux} are those of D;
- for every two distinct vertices u, v of D such that $N_D^-(u) \cap N_D^-(v) \neq \emptyset$, let us add the edge uv to D_{aux}.

In other words, D_{aux} is the simple undirected graph depicting the pairs of distinct vertices of D sharing an in-neighbour. By construction, note that, in D_{aux}, every two distinct vertices are joined by at most one edge.

It turns out that, for strong digraphs D with maximum out-degree at least 2, a vertex cover of D_{aux} is resolving in D.

Lemma 2.2. *Let D be a strong digraph with $\Delta^+(D) \geq 2$. Then, any vertex cover of D_{aux} is a resolving set of D.*

Proof. Towards a contradiction, assume the claim is false, i.e., there exists a set $S \subseteq V(D)$ which is a vertex cover of D_{aux} but not a resolving set of D. Since $\Delta^+(D) \geq 2$, there are edges in D_{aux} and thus $S \neq \emptyset$. Let v_1, v_2 be two vertices that cannot be distinguished through their distances from S; in other words, for every $w \in S$ (note that $w \neq v_1, v_2$), we have $\text{dist}_D(w, v_1) = \text{dist}_D(w, v_2)$, and that distance is finite since D is strong. Now consider such a vertex $w \in S$ at minimum distance from v_1 and v_2. In D, any shortest path P_1 from w to v_1 has the same length as any shortest path P_2 from w to v_2.

Because $v_1 \neq v_2$ and P_1, P_2 are shortest paths, note that all vertices of P_1 and P_2 cannot be the same; let thus x_1 denote the first vertex of P_1 that does not belong to P_2, and, similarly, let thus x_2 denote the first vertex of P_2 that does not belong to P_1. In other words, the first vertices of P_1 and P_2 coincide up to some vertex x, but the next vertices x_1 (in P_1) and x_2 (in P_2) are different. So, D_{aux} contains the edge x_1x_2, and at least one of x_1, x_2 belongs to S. Furthermore, x_1 and x_2 are closer to v_1, v_2 than w is; this is a contradiction to the original choice of w. □

Lemma 2.2 shows that a resolving set of any strong digraph (with maximum out-degree at least 2) can be obtained by considering every vertex and choosing at least all of its out-neighbours but one. The proof suggests that this is because this is a way to distinguish all shortest paths from a vertex to other ones.

Corollary 2.3. *For every strong digraph D with $\Delta^+(D) \geq 2$, the metric dimension* $\text{MD}(D)$ *of D is at most the size of a minimum vertex cover of D_{aux}.*

Unfortunately, determining the minimum size of a vertex cover of a given graph is an NP-complete problem in general [8]. However, in the context of Corollary 2.3, we are mostly interested in having reasonable upper bounds on the size of a minimum vertex cover of D_{aux}. Such upper bounds can be exhibited when D has particular additional properties, as will be shown in the next sections.

3 Strong oriented graphs with bounded maximum degree

By the *maximum degree* $\Delta(D)$ of a given oriented graph D, we mean the maximum degree of its underlying undirected graph (i.e., the maximum value of $d^-(v) + d^+(v)$ over the vertices v of D). In this section, we investigate the maximum value that $\text{MD}(D)$ can take among all strong orientations D of a graph with given maximum degree. Since a strong oriented graph D with $\Delta(D) = 2$ is a directed cycle, in which case $\text{MD}(D)$ is trivially 1, we focus on cases where $\Delta(D) \geq 3$.

All our lower bounds in this section are obtained through the following constructions. For any $k \in \mathbb{N}$ and $\Delta \geq 2$, we denote by $T_{\Delta,k}$ the rooted Δ-ary complete tree with depth k. More precisely, $T_{\Delta,k}$ is a rooted tree such that every non-leaf vertex has Δ children and all leaves are at distance k from the root. Note that $|V(T_{\Delta,k})| = \frac{\Delta^{k+1}-1}{\Delta-1}$ and $T_{\Delta,k}$ has Δ^k leaves and maximum degree $\Delta + 1$. For any $k \in \mathbb{N}$ and $\Delta, i \geq 2$, let $D_{\Delta,k,i}$ be the oriented graph defined as follows (see Figure 1 for an illustration). Start with T being a copy of $T_{\Delta,k-1}$ with all edges oriented from the root to the leaves. Let $v_1^{k-1}, \dots, v_{\Delta^{k-1}}^{k-1}$ be the leaves of T and let r be its root. For every $1 \leq j \leq \Delta^{k-1}$, add i out-neighbours $u_1^j, \dots, u_i^j$ to v_j^{k-1}. Then, for $1 \leq j \leq \Delta^{k-1}$ and $1 \leq \ell < i$, add the arc (u_ℓ^j, u_i^j). Then, add a copy T' of $T_{\Delta,k-2}$ where all edges are oriented from the leaves to the root. Let $v'_1, \dots, v'_{\Delta^{k-2}}$ be the leaves of T' and let r' be its root. For every $1 \leq j \leq \Delta^{k-2}$ and for every $1 \leq \ell \leq \Delta$, add the arc $(u_i^{\Delta(j-1)+\ell}, v'_j)$. Finally, add the arc (r', r); note that this ensures that $D_{\Delta,k,i}$ is strong.

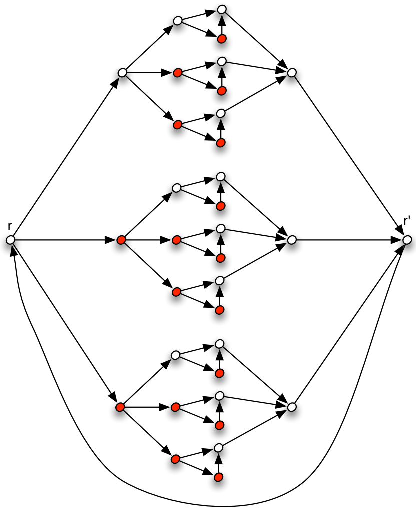


Figure 1: The oriented graph $D_{3,3,2}$. The set of red vertices is an example of an optimal resolving set.

Lemma 3.1. *For every $k \in \mathbb{N}$ and $\Delta, i \geq 2$, the graph $D_{\Delta,k,i}$ is a strong oriented graph with maximum degree* $\max\{\Delta+1, i+1\}$,

$$|V(D_{\Delta,k,i})| = \frac{\Delta^k - 1}{\Delta - 1} + i\Delta^{k-1} + \frac{\Delta^{k-1} - 1}{\Delta - 1}$$

and

$$\mathrm{MD}(D_{\Delta,k,i}) \geq \Delta^{k-1} - 1 + \Delta^{k-1} \max\{1, i-2\}.$$

Proof. We only need to prove the last statement. For every $1 \leq \ell \leq k-1$, let $v_1^\ell, \ldots, v_{\Delta^\ell}^\ell$ denote the vertices of $D_{\Delta,k,i}$ at distance ℓ from $r = v_1^0$, and let $v_1^k, \ldots, v_{i\Delta^{k-1}}^k$ denote the vertices of $D_{\Delta,k,i}$ at distance k from $r = v_1^0$. Note that, for every $0 \leq \ell \leq k-2$ and $1 \leq j \leq \Delta^\ell$, the vertices $v_{\Delta(j-1)+1}^{\ell+1}, \ldots, v_{\Delta(j-1)+\Delta}^{\ell+1}$ have the same in-neighbourhood $\{v_j^\ell\}$. By Lemma 2.1, every resolving set of $D_{\Delta,k,i}$ thus has to include at least $\Delta - 1$ of these vertices. For every $1 \leq j \leq \Delta^{k-1}$, the vertices $v_{i(j-1)+1}^k, \ldots, v_{i(j-1)+i}^k$ have the same in-neighbourhood $\{v_j^{k-1}\}$. Again by Lemma 2.1, every resolving set of $D_{\Delta,k,i}$ must thus include at least $i-2$ of these vertices. Moreover, it can be checked that, when $i = 2$, every resolving set of $D_{\Delta,k,i}$ must include at least one of $v_{2(j-1)+1}^k, v_{2(j-1)+2}^k$. Figure 1 shows an example of a resolving set of $D_{3,3,2}$.

Hence, any resolving set R of $D_{\Delta,k,i}$ verifies

$$|R| \geq \left(\sum_{\ell=0}^{k-2} \Delta^\ell (\Delta - 1) \right) + \Delta^{k-1} \max\{1, i-2\}$$

which can be manipulated into the claimed lower bound. □

All of our upper bounds on $\mathrm{MD}(D)$ for oriented graphs D with bounded maximum degree (some of which are close to lower bounds that can be established using Lemma 3.1) in this section are obtained from the following theorem.

Theorem 3.2. *Let $\mathcal{G}_{\Delta+1}$ be the family of 2-edge-connected graphs with maximum degree $\Delta+1 \geq 3$. Then,*

$$\mathrm{WOMD}(\mathcal{G}_{\Delta+1}) \leq \frac{\Delta(\Delta-1)}{\Delta(\Delta-1)+2}.$$

Proof. Let G be an oriented graph with maximum degree $\Delta+1$. Set $n = |V(G)|$. We first show that $|E(D_{\text{aux}}(G))| \leq \frac{\Delta(\Delta-1)}{4}n$. By definition, $|E(D_{\text{aux}}(G))| = \sum_{v \in V(G)} \frac{d^+(v)(d^+(v)-1)}{2}$. Since G is strongly-connected and the maximum degree of G is $\Delta+1$, then, for all $v \in V(G)$, we have that $1 \leq d^+(v) \leq \Delta$ and $\sum_{v \in V(G)} d^+(v) = |A(G)| \leq (\Delta+1)n/2$, where $A(G)$ is the set of arcs of G. We will show that the maximum of the objective function $\sum_{v \in V(G)} \frac{d^+(v)(d^+(v)-1)}{2}$ subject to the constraints that, for all $v \in V(G)$, $1 \leq d^+(v) \leq \Delta$, and $\sum_{v \in V(G)} d^+(v) \leq (\Delta+1)n/2$, is obtained when half of the vertices ($(n-1)/2$ if n is odd) have out-degree Δ and the other half ($(n-1)/2$ if n is odd) have out-degree 1 (and one vertex has out-degree at most $\frac{\Delta+1}{2}$ if n is odd). Note that we are just seeking an upper bound on this objective function and thus, it may be that no such oriented graph exists. With regards to proving the aforementioned statement, take any set of out-degrees that satisfy the constraints. If there are two vertices u, v which do not have out-degree 1 or Δ, and w.l.o.g., $d^+(u) \geq d^+(v)$, then increasing the out-degree of u by 1 and decreasing the out-degree of v by 1 still satisfies the constraints but has increased the value of the objective function since

$$\frac{(d^+(u)+1)(d^+(u))}{2} + \frac{(d^+(v)-1)(d^+(v)-2)}{2} > \frac{d^+(u)(d^+(u)-1)}{2} + \frac{d^+(v)(d^+(v)-1)}{2}.$$

Therefore, we will converge to a maximum of the objective function when half of the vertices have out-degree Δ and the other half out-degree 1 if n is even. If n is odd, then $(n-1)/2$ vertices will have out-degree Δ, $(n-1)/2$ vertices out-degree 1, and the last vertex out-degree at most $\frac{\Delta+1}{2}$ due to the constraint that $\sum_{v \in V(G)} d^+(v) \leq (\Delta+1)n/2$.

Thus, $|E(D_{\text{aux}}(G))| \leq \frac{\Delta(\Delta-1)}{4}n$. Hence, $\overline{d}(D_{\text{aux}}(G)) \leq \frac{\Delta(\Delta-1)}{2}$, where $\overline{d}(D_{\text{aux}}(G)) = 2|E(D_{\text{aux}}(G))|/n$ is the average degree of $D_{\text{aux}}(G)$. By a result of Turán [16], we know that, for any graph H, $\alpha(H) \leq n/(\overline{d}(H)+1)$, where $\alpha(H)$ is the size of a maximum independent set in H. Since, for any graph H and any maximal independent set S of H, the set $V(H) \setminus S$ is a vertex cover of H (easy to see by contradiction), there exists a vertex cover of $D_{\text{aux}}(G)$ of size at most

$$n - \frac{n}{\overline{d}(D_{\text{aux}}(G))+1} = \frac{\overline{d}(D_{\text{aux}}(G))}{\overline{d}(D_{\text{aux}}(G))+1}n \leq \frac{\frac{\Delta(\Delta-1)}{2}}{\frac{\Delta(\Delta-1)}{2}+1}n = \frac{\Delta(\Delta-1)}{\Delta(\Delta-1)+2}n.$$

The result then follows by Corollary 2.3. □

From Lemma 3.1 and Theorem 3.2, we get the following result for graphs with maximum degree 3 and 4 respectively.

Corollary 3.3. *Let $\mathcal{G}_3$ ($\mathcal{G}_4$ resp.) be the family of 2-edge-connected graphs with maximum degree 3 (maximum degree 4 resp.). For any $\epsilon > 0$, we have*

$$\frac{2}{5} - \epsilon \leq \mathrm{WOMD}(\mathcal{G}_3) \leq \frac{1}{2}$$

and

$$\frac{1}{2} - \epsilon \leq \mathrm{WOMD}(\mathcal{G}_4) \leq \frac{3}{4}.$$

Proof. The upper bounds follow from Theorem 3.2. The lower bounds follow from Lemma 3.1 by considering the oriented graphs $D_{2,k,2}$ and $D_{3,k,2}$ respectively. Indeed, any resolving set of $D_{2,k,2}$ has at least $2 * 2^{k-1} - 1$ vertices and $n = |V(D_{2,k,2})| = 5 * 2^{k-1} - 2$. Hence, $\lim_{k\to\infty} \mathrm{MD}(D_{2,k,2}) \geq \frac{2n}{5}$. Furthermore, for all $k \in \mathbb{N}$, $\mathrm{MD}(D_{3,k,2}) \geq 2 * 3^{k-1} - 1$ and $n = |V(D_{3,k,2})| = 4 * 3^{k-1} - 1$. Hence, $\lim_{k\to\infty} \mathrm{MD}(D_{3,k,2}) \geq \frac{n}{2}$. □

More generally, i.e., for larger values of the maximum degree, the construction in Lemma 3.1 is asymptotically optimal:

Corollary 3.4. *Let $\mathcal{G}_{\Delta+1}$ be the family of 2-edge-connected graphs with maximum degree $\Delta + 1$. Then,*

$$\lim_{\Delta\to\infty} \mathrm{WOMD}(\mathcal{G}_{\Delta+1}) = 1.$$

Proof. By definition, $\mathrm{WOMD}(\mathcal{G}_{\Delta+1}) \leq 1$ for every Δ. To prove the claim, it is sufficient to show that $\lim_{\Delta\to\infty} \mathrm{WOMD}(D_{\Delta,k,\Delta}) = 1$. By Lemma 3.1, for $\Delta \geq 3$,

$$\mathrm{MD}(D_{\Delta,k,\Delta}) \geq (\Delta - 1)\Delta^{k-1} - 1.$$

Moreover, $|V(D_{\Delta,k,\Delta})|(\Delta - 1) = \Delta^{k+1} + \Delta^{k-1} - 2$. Hence,

$$\frac{\mathrm{MD}(D_{\Delta,k,\Delta})}{|V(D_{\Delta,k,\Delta})|} \geq \frac{(\Delta-1)^2\Delta^{k-1} - (\Delta - 1)}{\Delta^{k+1} + \Delta^{k-1} - 2} = \frac{1 - \frac{1}{\Delta}\left(2 - \frac{1}{\Delta} + \frac{1}{\Delta^{k-1}} - \frac{1}{\Delta^k}\right)}{1 + \frac{1}{\Delta^2}\left(1 - \frac{2}{\Delta^{k-1}}\right)} \underset{\Delta\to\infty}{\to} 1.$$

□

4 Strong orientations of grids and tori

By a *grid* $G_{n,m}$, we refer to the Cartesian product $P_n \square P_m$ of two paths P_n, P_m. A *torus* $T_{n,m}$ is the Cartesian product $C_n \square C_m$ of two cycles C_n, C_m. In the undirected context, it is easy to see that $\mathrm{MD}(G_{n,m}) = 2$ while $\mathrm{MD}(T_{n,m}) = 3$ (see e.g., [11]); however, things get a bit more tricky in the directed context.

Grids and tori have maximum degree 4; thus, bounds on the maximum metric dimension of a strong oriented grid or torus can be derived from our results in Section 3. In this section, we improve these bounds through dedicated proofs and arguments. We first consider strong Eulerian oriented tori (all vertices have in-degree and out-degree 2), for which we exhibit the maximum value of the metric dimension. We then consider strong oriented grids, for which we provide improved bounds.

4.1 Strong Eulerian orientations of tori

Let $0 < n \leq m$ be two integers, and let $T_{n,m}$ be the torus on nm vertices. That is, $V(T_{n,m}) = \{(i,j) \mid 0 \leq i < n, 0 \leq j < m\}$, and $(i,j),(k,\ell) \in E(T_{n,m})$ if and only if $|i-k| \in \{1, n-1\}$ and $j = \ell$, or $|j - \ell| \in \{1, m-1\}$ and $i = k$. By convention, the vertex $(0,0)$ is regarded as the topmost, leftmost vertex of the torus. That is, $\{(0,j) \in V(T_{n,m}) \mid 0 \leq j < m\}$ is the topmost (or first) row, and $\{(i,0) \in V(T_{n,m}) \mid 0 \leq i < n\}$ is the leftmost (or first) column.

As a main result in this section, we determine the maximum metric dimension of a strong Eulerian oriented torus. More precisely, we study the following slight modifications of the parameter WOMD. For a connected graph G, we denote by $\mathrm{WEOMD}(G)$ the maximum value of $\mathrm{MD}(D)$ over all strong Eulerian orientations D of G. For a family $\mathcal{G}$ of 2-edge-connected graphs, we set $\mathrm{WEOMD}(\mathcal{G}) = \max_{G\in\mathcal{G}} \frac{\mathrm{WEOMD}(G)}{|V(G)|}$.

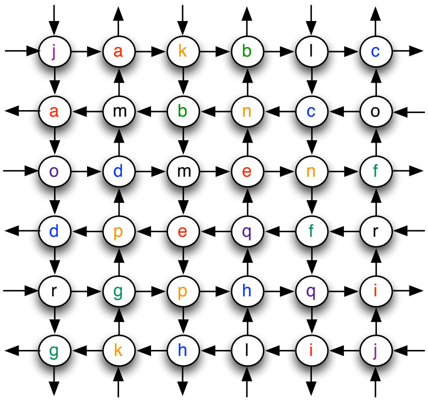


Figure 2: An orientation $\vec{T}^*$ of the $6 * 6$ torus $T_{6,6}$ verifying $\mathrm{MD}(\vec{T}^*) = |V(T_{6,6})|/2$. Every two vertices marked with a same letter have the same in-neighbourhood; thus, every resolving set must contain at least one of them.

Theorem 4.1. *For the family $\mathcal{T}$ of tori, we have* $\mathrm{WEOMD}(\mathcal{T}) = \frac{1}{2}$.

We first show that there exist strong Eulerian oriented tori D with $\mathrm{MD}(D) \geq \frac{nm}{2}$.

Lemma 4.2. *For every $n_0, m_0 \in \mathbb{N}$, there is $n \geq n_0, m \geq m_0$, and a strong Eulerian orientation $\vec{T}^*$ of the torus $T_{n,m}$ such that* $\mathrm{MD}(\vec{T}^*) \geq \frac{nm}{2}$.

Proof. Let n (resp., m) be the smallest even integer greater or equal to n_0 (resp., m_0). We orient $T_{n,m}$ in the following way, resulting in $\vec{T}^*$ (see Figure 2 for an illustration). The edges of the even rows of $T_{n,m}$ are oriented from left to right, i.e., $((2i,j)(2i,j+1 \mod m))$ is an arc for every $0 \leq j < m$ and $0 \leq i < n/2$. The edges of the odd rows are oriented from right to left, i.e., $((2i+1,j)(2i+1,j-1 \mod m))$ is an arc for every $0 \leq j < m$ and $0 \leq i < n/2$. The edges of the even columns are oriented from top to bottom, i.e., $((i,2j)(i+1 \mod n, 2j))$ is an arc for every $0 \leq j < m/2$ and $0 \leq i < n$. The edges of the odd columns are oriented from bottom to top, i.e., $((i,2j+1)(i-1 \mod n, 2j+1))$ is an arc for every $0 \leq j < m/2$ and $0 \leq i < n$.

For every $0 \leq i < n/2$ and $0 \leq j < m/2$, vertices $(2i, 2j+1)$ and $(2i+1, 2j)$ have the same in-neighbourhood. Moreover, $(2i, 2j)$ and $(2i-1 \mod n, 2j-1 \mod m)$ have the same in-neighbourhood. By Lemma 2.1, any resolving set of $\vec{T}^*$ must contain at least one vertex of each of these $\frac{nm}{2}$ pairs of vertices. Hence, $\mathrm{MD}(\vec{T}^*) \geq \frac{nm}{2}$. □

We now prove the upper bound.

Lemma 4.3. *For every strong Eulerian oriented torus $\vec{T}_{n,m}$ with n rows and m columns,*

$$\mathrm{MD}(\vec{T}_{n,m}) \leq \frac{n'm'}{2} + n'' + m'',$$

where, for $x \in \{n, m\}$, (x', x'') equals $(x, 0)$ if x is even and $(x-1, x)$ otherwise. In particular, if both n and m are even, then $\mathrm{MD}(\vec{T}_{n,m}) \leq \frac{nm}{2}$.

Proof. Let us first consider the case when n and m are even. The proof is constructive and provides a resolving set of size at most $\frac{nm}{2}$. The algorithm starts with the set $R = \{(i,j) \in V(\vec{T}_{n,m}) \mid i+j \text{ even}\}$ (note that it is a minimum vertex cover and a stable set of

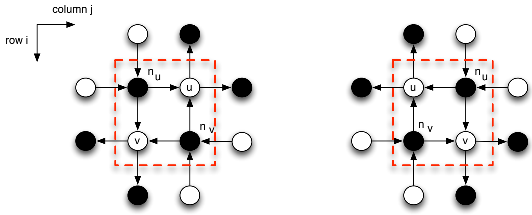


Figure 3: The two cases of “bad squares". Black vertices are the ones in the initial set R.

size $\frac{nm}{2}$) and iteratively performs local modifications (swaps one vertex in R with one of its neighbours not in R) without changing the size of R until R becomes a resolving set R^*.

Let us assume that $R = \{(i,j) \in V(\vec{T}_{n,m}) \mid i+j \text{ even}\}$ is not a resolving set (otherwise, we are done). This means that at least two vertices are not distinguishable by their distances to the vertices in R. Let u and v be two such vertices. Recall that we denote this relationship by $u \sim_R v$.

Necessarily, if $u \sim_R v$ then $u, v \notin R$ (since any vertex $w \in R$ is the only one at distance 0 from itself, it can be distinguished from every other vertex). Moreover, since R is a vertex cover and $d^-(u) = d^-(v) = 2$, each of u and v must have two in-neighbours in R. Since u and v are not distinguishable, they must have the same in-neighbours, denoted by $n_u, n_v \in R$. That is, since each vertex has exactly two in-neighbours and two out-neighbours (by Eulerianity), $N^+(n_u) = N^+(n_v) = \{u, v\}$ and $N^-(u) = N^-(v) = \{n_u, n_v\}$. In what follows, by convention, let us assume that u and n_u are in the same row, say $r \in \{0, ..., n-1\}$, and v and n_v are in row $r+1 \mod n$ (note that the row numbers increase from the top of the torus to the bottom). There are two cases (depending on whether u is on the “left” or on the “right” of the “square” (u, v, n_u, n_v)), as depicted in Figure 3.

If $u \sim_R v$, then this implies that u (and similarly v) can be distinguished from any vertex different from v (resp. u). Moreover, if there are four vertices u, v, x, y such that $u \sim_R v$ and $x \sim_R y$, then $\{u, v, n_u, n_v\} \cap \{x, y, n_x, n_y\} = \emptyset$. Each such (u, v, n_u, n_v), where $u \sim_R v$, is called a *bad square*. Formally, this discussion implies:

Claim 4.4. *For every $u, v \in V(\vec{T}_{n,m})$, if $u \sim_R v$, then u and v belong to the same bad square $\{u, v, n_u, n_v\}$. Moreover, all bad squares are vertex-disjoint.*

Let $\{Q_i = (u^i, v^i, n_u^i, n_v^i) \mid 1 \le i \le p\}$ be the set of all (vertex-disjoint) bad squares such that $u^i \sim_R v^i$ for every $1 \le i \le p$, where p is the number of pairs of undistinguishable vertices. Let $Q = \bigcup_{1 \le i \le p} Q_i$.

The algorithm that computes R^* from R is very simple. Start with $R^* = R$. For every $1 \le i \le p$, remove n_v^i from R^* and add u^i to R^*. For every $1 \le i \le p$, let R_i be the set obtained after swapping n_v^j and u^j for every $1 \le j \le i$ (and $R = R_0$ and $R^* = R_p$). Note that, since all bad squares are disjoint, $|R^*| = |R_i| = |R|$, for every $1 \le i \le p$.

Remark. *Any vertex in R that either does not belong to a bad square or that belongs to the upper row of a bad square is also in R^*.*

The remainder of this proof aims at proving that the obtained set R^* is a resolving set, containing clearly half of the vertices of $\vec{T}_{n,m}$.

Claim 4.5. *For any $x, y \in V(\vec{T}_{n,m}) \setminus Q$ that are distinguishable by R, x and y are distinguishable by R^*.*

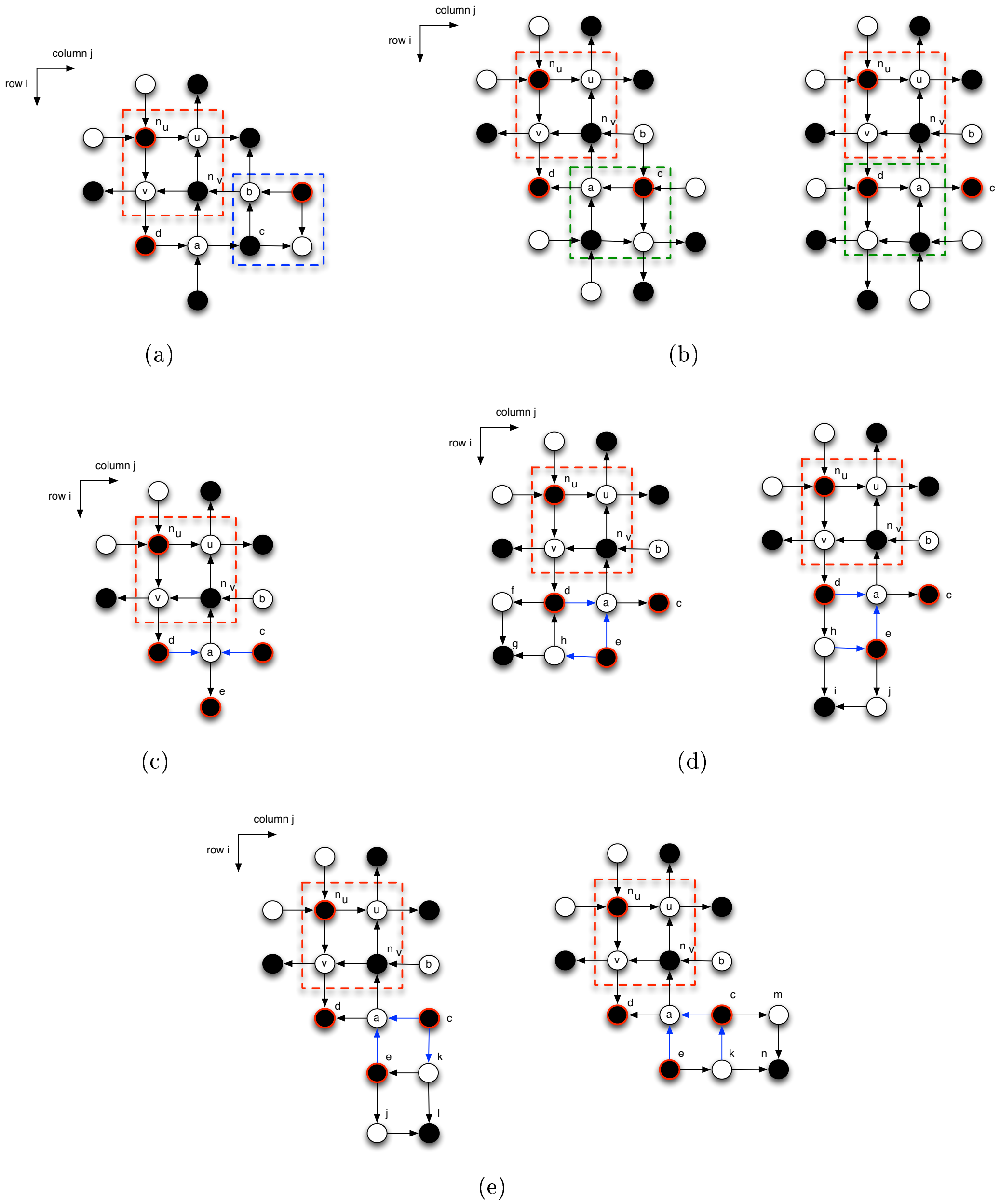


Figure 4: Different cases considered in the proof of Lemma 4.3. Black vertices are the ones of R. A black vertex circled in red is in $R \cap R^*$. Dotted squares are bad squares. Blue arcs are the ones whose orientation has been fixed depending on the cases, while the orientations of black arcs are forced.

Proof of the claim. Let $x, y \in V(\vec{T}_{n,m}) \setminus Q$ be distinguished by $R = R_0$, and let us prove by induction on $1 \leq i \leq p$ that x and y are distinguishable by R_i. Let $i \geq 1$; by the induction hypothesis, x and y are distinguishable in R_{i-1}, so there is a vertex $q \in R_{i-1}$ such that $\text{dist}(q, x) \neq \text{dist}(q, y)$. If $q \in R_i$, then x and y are distinguished. Otherwise, $q = n_v^i$. Note that, for every vertex $w \notin Q_i$, $\text{dist}(n_v^i, w) = \text{dist}(n_u^i, w)$. Hence, $\text{dist}(n_u^i, x) \neq \text{dist}(n_u^i, y)$ and x and y can be distinguished by R_i. ◇

Claim 4.6. *For every $1 \leq i \leq p$, every vertex in Q_i can be distinguished from any other* *vertex by R^*.*

Proof of the claim. Indeed, $n_u^i, u^i \in R^*$, and v^i is the only vertex not in R^* at distance 1 from $n_u^i \in R^*$. It remains to prove that n_v^i can be distinguished from any other vertex. Let us consider the case when n_v^i is the bottom-right vertex of Q_i (the case when n_v^i is the bottom-left vertex of Q_i is symmetric). Let a and b be the two in-neighbours of n_v^i. Note that $a, b \notin R$. Let c be the vertex ($\neq n_v^i$) adjacent to a and b. Let d be the out-neighbour of v which is adjacent to a (via either (a, d) or (d, a)). Since the bad squares are disjoint, d cannot be in the lower row of a bad square and, so, by the above remark, $d \in R \cap R^*$; see Figure 4. There are several cases to be considered.

- **Case 1:** $c \notin R^*$.
  This is the case where b and c are in a same bad square (depicted in blue in Figure 4a). Therefore, $b \in R^*$. Moreover, this bad square and the fact that the in-degree and out-degree of every vertex is 2 force the orientation of the arcs to be in such a way that n_v^i is the only vertex at distance 1 from b and at distance 2 from d. Hence, n_v^i is distinguishable from any other vertex.
- **Case 2:** $c \in R^*$.
  - **Case 2.1:** $a \in R^*$.
    So a must be in a bad square. There are two cases depicted by the green dotted squares in Figures 4b. In both cases, since c and d are in R^*, n_v^i is the only vertex not in R^* that is at distance 1 from a. Hence, n_v^i is distinguishable from any other vertex.
  - **Case 2.2:** $a \notin R^*$
    Therefore, the vertex $e \notin \{c, d, n_v^i\}$ is adjacent to a and belongs to R^*. Let h be the vertex different from a that is adjacent to d and e.
    We now consider the possible values of $N^-(a)$.
    * **Case 2.2.1:** $N^-(a) = \{c, d\}$ (see Figure 4c).
      Since $e \in R^*$, n_v^i is the only vertex not in R^* that is at distance 2 from c and d.
    * **Case 2.2.2:** $N^-(a) = \{d, e\}$ and (e, h) is an arc (see Figure 4d, left).
      Since $\{a, e, h, d\}$ is not a bad square (since $a \notin R^*$), there is an arc from h to d. Note that n_v^i is at distance 2 from d and e. The only other vertex not in R^* that may be at distance 2 from d and e is the vertex g (on the left of h). In that case, the vertex $f \neq h$ that is adjacent to d and g must be such that there is an arc from f to g. Since either f or g belongs to R^*, g and n_v^i can be distinguished.
    * **Case 2.2.3:** $N^-(a) = \{d, e\}$ and (h, e) is an arc (see Figure 4d, right).
      Since $\{a, e, h, d\}$ is not a bad square (since $a \notin R^*$), there is an arc from d to h. Note that n_v^i is at distance 2 from d and e. The only other vertex not in R^* that may be at distance 2 from d and e is the vertex i (below h). In that case, there is an arc from h to i. Since either i or h belongs to R^*, i and n_v^i can be distinguished.
    * **Case 2.2.4:** $N^-(a) = \{c, e\}$ (see Figures 4e).
      Let $k \neq a$ be the vertex adjacent to c and e. The two cases, depending on whether there is the arc (c, k) or (k, c), are similar to the previous Cases 2.2.2 and 2.2.3.

This concludes the proof of the claim. $\diamond$

Hence, in the case n, m even, R^* is a resolving set of size $\frac{nm}{2}$. In the cases when n (resp., m) is odd, we first add all the vertices of the first row (resp., of the first column) to the resolving set. The remaining vertices induce a grid with even sides on which we proceed as above. □

4.2 Strong oriented grids

In this section, we consider the maximum metric dimension of a strong oriented grid. For every such grid, we deal with its vertices using the same terminology introduced in Section 4.1 for tori (i.e., the vertices of the topmost row have first coordinate 0, and the vertices of the leftmost column have second coordinate 0). Our main result to be proved in this section is the following.

Theorem 4.7. *Let $\mathcal{G}$ be the family of grids. For any $\epsilon > 0$, we have*

$$\frac{1}{2} - \epsilon \leq \text{WOMD}(\mathcal{G}) \leq \frac{2}{3} + \epsilon.$$

We start off by exhibiting strong orientations of grids for which the metric dimension is about half of the vertices.

Lemma 4.8. *For every $n_0, m_0 \in \mathbb{N}$, there is $n \geq n_0$, $m \geq m_0$ and a strong orientation $\vec{G}^*$ of the grid $G_{n,m}$ such that $\text{MD}(\vec{G}^*) \geq \frac{nm}{2} - \frac{n+m}{2}$.*

Proof. Let n (resp., m) be the smallest even integer greater or equal to n_0 (resp., m_0). We orient $G_{n,m}$ as follows, resulting in $\vec{G}^*$. All edges of the even rows are oriented from right to left, while all edges of the odd rows are oriented from left to right. All edges of the even columns are oriented from top to bottom, while all edges of the odd columns are oriented from bottom to top. Note that $\vec{G}^*$ is indeed strong under the assumption that n and m are even (in particular, no corner vertex is a source or sink).

For every even $0 \leq i < n$ and odd $1 \leq j < m-1$, the vertices (i,j) and $(i+1, j+1)$ have the same in-neighbourhood. Similarly, for every odd $1 \leq i < n-1$ and odd $1 \leq j < m$, the vertices (i,j) and $(i+1, j-1)$ have the same in-neighbourhood. For each of these pairs of vertices, Lemma 2.1 implies that at least one of the two vertices must belong to any resolving set of $\vec{G}^*$. The only vertices that do not appear in these pairs are those of the form $(0, 2k)$, $(2k+1, 0)$, $(2k, m-1)$, and $(n-1, 2k+1)$ for $k \in \mathbb{N}$ and the vertices $(n-1, 0)$ and $(n-1, m-1)$. There are $n+m$ such vertices. The bound then follows. □

We now prove that every strong oriented grid has a resolving set including $\frac{2}{3}$ of the vertices.

Lemma 4.9. *For every strong oriented grid $\vec{G}_{n,m}$ with n rows and m columns, if $m \equiv 0 \mod 3$ or $n \equiv 0 \mod 3$, then $\text{MD}(\vec{G}_{n,m}) \leq \frac{2nm}{3}$, and $\text{MD}(\vec{G}_{n,m}) \leq \lfloor \frac{2nm}{3} \rfloor + 2m$ otherwise.*

Proof. Let us first consider the case when $m \mod 3 = 0$ (the case $n \mod 3 = 0$ is similar up to rotation). The algorithm starts with the set $R = \{V(G_{n,m}) \setminus (i, 3j-1) | 0 \leq i \leq n-1, 1 \leq j \leq m/3\}$ (i.e., R contains the first 2 out of every 3 columns from left to right in the grid) and iteratively performs local modifications (swaps one vertex in R with one of its neighbours not in R) without changing the size of R until R becomes a resolving set R^*. Note that $|R| = \frac{2nm}{3}$.

Assume R is not a resolving set (otherwise, we are done). This means that at least two vertices are not distinguishable by their distances from the vertices in R. Let u and v be two such vertices. Clearly, $u, v \notin R$ as otherwise they are distinguishable since one of them is the only vertex at distance 0 from itself.

Claim 4.10. *For every $u, v \in V(G_{n,m})$, if $u \sim_R v$, then u and v belong to the same column in $\vec{G}_{n,m}$.*

Proof of the claim. For purpose of contradiction, let us assume that $u \sim_R v$ with $u \in C_1$ and $v \in C_2$ where C_1 and C_2 are two distinct columns of $V(\vec{G}_{n,m})$ which contain no vertices in R. W.l.o.g., C_1 is to the left of C_2. Let C_1^r be the column just to the right of C_1 and let C_2^ℓ be the column just to the left of C_2. Note that all the vertices of C_1^r and C_2^ℓ are in R and that C_1^r and C_2^ℓ are distinct. Since only strong orientations are considered and C_1^r separates u from every vertex in the columns to the right of C_1^r, there exists a vertex a in C_1^r such that, for every vertex x in a column to the right of C_1^r (in particular, for every vertex in C_2^ℓ), $\text{dist}(a, u) < \text{dist}(x, u)$. Similarly, there exists a vertex b in C_2^ℓ such that, for every vertex x in a column to the left of C_2^ℓ (in particular, for every vertex in C_1^r), $\text{dist}(b, v) < \text{dist}(x, v)$. Therefore, $\text{dist}(b, v) < \text{dist}(a, v)$ and $\text{dist}(a, u) < \text{dist}(b, u)$ and, it is not possible to have both $\text{dist}(b, v) = \text{dist}(b, u)$ and $\text{dist}(a, u) = \text{dist}(a, v)$ simultaneously. Since $a, b \in R$, u and v are distinguished, a contradiction. ◇

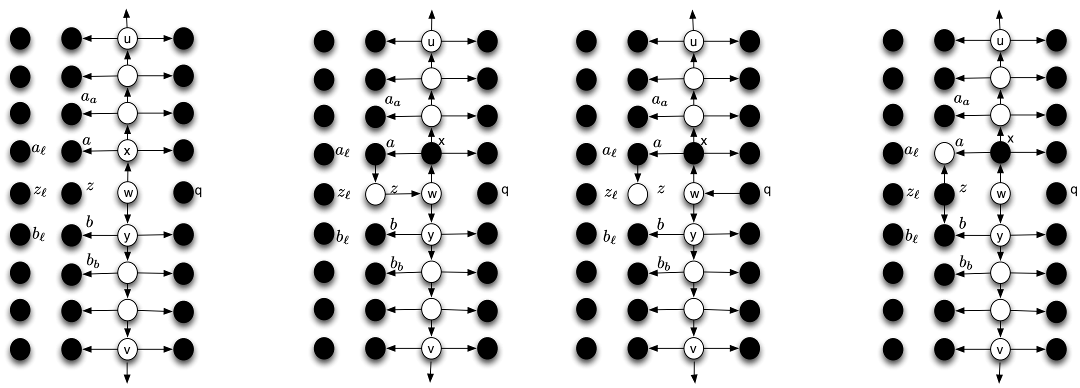
Claim 4.11. *For every $u, v \in V(\vec{G}_{n,m})$ such that $u \sim_R v$, in a column C (containing no vertices in R), there is a unique vertex $w \in C$ at the same distance from u and v such that, for any $z \in R$, every shortest path from z to u (to v resp.) passes through w.*

Proof of the claim. W.l.o.g., let us assume that u is in a row above v. Since u and v are not distinguishable, $\text{dist}(x, u) = \text{dist}(x, v)$ for any vertex $x \in R$. Let $z \in R$ be a vertex of R that minimizes its distance to u (and so to v). Let P_u (resp., P_v) be a shortest path from z to u (resp., v). All vertices of P_u (resp., of P_v) are not in R (by the minimality of the distance between z and u) and so are in C. The only possibility then, is that both P_u and P_v start with a common arc (z, w) (with $w \in C$ uniquely defined) and then P_u goes up to u, while P_v goes down to v.

Now, let x be any vertex in R and let Q be any shortest path from x to u. For purpose of contradiction, let us assume that Q does not pass through w. Let y be the last vertex of Q in R (possibly $y = x$). Therefore, the path Q' from y to u has all its vertices (but y) in C. In particular, if y is above u (or in the same row), Q' enters C and goes down to u, and if u is above y, Q' enters C and goes up to u. In all cases, if Q (and so Q') does not pass through w, then y must be closer to u than to v, contradicting that u and v are not distinguished. The same proof holds for any path from x to v. ◇

The vertex $w \notin R$ defined in the previous claim is called the *last common vertex* (LCV) of the two undistinguished vertices u and v. Let Q be the set of all vertices $w \in V(\vec{G}_{n,m}) \setminus R$ such that w is an LCV for two vertices $u, v \in V(\vec{G}_{n,m})$ such that $u \sim_R v$. Note that one of these vertices w may be an LCV for multiple pairs of vertices that are not distinguishable; but in these cases, the local modifications the algorithm makes are sufficient to distinguish all the vertices in all the pairs with the same LCV.

The algorithm computes R^* from R as follows. Start with $R^* = R$. For every $w \in Q$, the algorithm proceeds as follows. Let u and v be two undistinguished vertices such that $w \in Q$ is their LCV (u and v exist by definition of $w \in Q$). W.l.o.g., let us assume that u is above v. Let z^w be the neighbour to the left of w, x^w be the neighbour above w, and y^w be the neighbour below w (it may be that $x^w = u$, in which case $y^w = v$) in the grid underlying $\vec{G}_{n,m}$. Also, let a^w and b^w be the neighbours above and below z^w resp. in the underlying grid. Note that any column with no vertices in R has two columns on its left, so it is the case for the column of w and so, a^w, z^w, and b^w exist. Then, the algorithm proceeds to do the following swap between a vertex in R (either z^w or a^w) and a vertex not in R (the vertex x^w):



(a) (b) (c)

Figure 5: Configurations with two undistinguished vertices u and v. In (a) (resp., in $(b),(c)$), black vertices are those in R (resp., in R^*), and white ones are in $V(\vec{G}_{n,m}) \setminus R$ (resp., in $V(\vec{G}_{n,m}) \setminus R^*$). The vertex w is the LCV of u and v. Figures (b) depict the cases when $z \notin R^*$. Figure (c) is the case when $a \notin R^*$.

- If (a^w, z^w) or (b^w, z^w) is an arc, then remove z^w from R^* and add x^w to R^*.
- Else, remove a^w from R^* and add x^w to R^*.

The remainder of this proof aims at proving that the obtained set R^* is a resolving set. For this purpose, we need further notations. Let $w \in Q$ be the LCV of two undistinguished vertices u and v, and let x^w, y^w, z^w, a^w, b^w be defined relative to w as above. In addition, let q^w be the neighbour to the right of w (if it exists, i.e., if w is not in the rightmost column) in the underlying grid. Also, let a^w_ℓ, b^w_ℓ, and z^w_ℓ be the neighbours to the left of a^w, b^w, and z^w resp. (note that any column with no vertices in R has two columns on its left, so it is the case for the column of w and so, a^w_ℓ, b^w_ℓ, and z^w_ℓ exist) and let a^w_a and b^w_b be the neighbours above and below a^w and b^w resp. (if they exist, that is, they do not surpass the dimensions of the grid) in the underlying grid. Finally, let $H_w = \{w, z^w, a^w, b^w, a^w_\ell, b^w_\ell, z^w_\ell, a^w_a, b^w_b, q^w\} \cup \{u, v \mid u \sim_R v, w \text{ } LCV \text{ of } u \text{ and } v\}$. All superscripts ^{w} will be omitted if there is no ambiguity.

Claim 4.12. *For any $w, w' \in Q$, we have $(H_w \setminus \{a^w_a, b^w_b, q^w\}) \cap (H_{w'} \setminus \{q^{w'}\}) = \emptyset$ and $(H_w \setminus \{q^w\}) \cap (H_{w'} \setminus \{a^{w'}_a, b^{w'}_b, q^{w'}\}) = \emptyset$. In particular, the modifications done by the algorithm (relative to each $w \in Q$) are independent of each other.*

Proof of the claim. Let $u \sim_R v$ with w as their LCV and such that $\text{dist}(w,u) = \text{dist}(w,v)$ is maximum. Let C be the column of w, u, and v. As mentioned in the proof of Claim 4.11, there must be a directed (shortest and included in C) path from w to u and a directed (shortest and included in C) path from w to v. Moreover, Claim 4.11 implies that all the vertices of these paths (but w) have out-degree 3 (since all shortest paths from R to u and v go through w). In particular, u and v have out-degree 3 (unless they are in the first or last row). It is then easy to see that, if $(H_w \setminus \{a^w_a, b^w_b, q^w\}) \cap (H_{w'} \setminus \{q^{w'}\}) \neq \emptyset$ or $(H_w \setminus \{q^w\}) \cap (H_{w'} \setminus \{a^{w'}_a, b^{w'}_b, q^{w'}\}) \neq \emptyset$, then this would contradict the orientations of these arcs (see Figure 5). ◇

Claim 4.13. *For any $w \in Q$, any $s \in H_w$, and any $t \in V(\vec{G}_{n,m})$, we have $s \nsim_{R^*} t$.*

Proof of the claim. For any $w \in Q$, let $H_w = \{w, z, a, b, a_\ell, b_\ell, z_\ell, a_a, b_b, q\} \cup \{u, v \mid u \sim_R v, w \text{ } LCV \text{ of } u \text{ and } v\}$ (the superscript w's are omitted here as there is no ambiguity). Note that $x, b, a_\ell, b_\ell, z_\ell, a_a, b_b, q \in R^*$ (a_a, b_b, and q are only in R^* if they exist of course) due to the algorithm and Claim 4.12, and so $x, b, a_\ell, b_\ell, z_\ell, a_a, b_b, q \in R^*$ are distinguishable from all other vertices.

Then, let P_{xu} be the directed (shortest) path from x to u (with no vertices in R) which exists by the proof of Claim 4.11. Let S_{xu} be the set of out-neighbours in R^* of all the vertices in P_{xu}. Every vertex r in P_{xu} is distinguishable from every other vertex by its distance to x and some other vertex from S_{xu}. Indeed, if $\text{dist}(x, r) = 1$, then r can be distinguished from a since either $a \in R^*$ or a is the single vertex both at distance 1 from x and z. Otherwise, for any vertex $t \neq r$ at distance $\text{dist}(x, r)$ from x, any path from x to t crosses a vertex in $S_{xu} \subseteq R^*$ and so $r \nsim_{R^*} t$.

Now, it remains to show that every vertex in $H_w \setminus (\{x, b, a_\ell, b_\ell, z_\ell, a_a, b_b, q\} \cup V(P_{xu}))$ can be distinguished from all other vertices. There are two cases to be considered depending on whether z or a is not in R^*.

Case $z \notin R^*$. Then, by definition of the algorithm, (a, z) or (b, z) is an arc.

- If (a, z) is an arc, then z is distinguishable from all other vertices as it is the only vertex at distance 1 from $a \in R^*$ that is not in R^* since (x, a) is an arc (see proof of Claim 4.12), and $a_a, a_\ell \in R^*$ (if a_a exists) by Claim 4.12.
- Else, if (b, z) is an arc, then z is distinguishable from all other vertices as it is the only vertex at distance 1 from $b \in R^*$ that is not in R^* since (y, b) is an arc (see proof of Claim 4.12), and $b_b, b_\ell \in R^*$ (if b_b exists) by Claim 4.12.

Therefore, if $z \notin R^*$, it is distinguishable.

Now, we will show that all vertices on the directed (shortest) path from w to v are also distinguishable from every other vertex. Let P_{wv} be the set of vertices of the directed (shortest) path from w to v (w, v included) and let S_{wv} be the set of out-neighbours in R^* of the vertices in P_{wv}. Note that $x \in S_{wv}$. Either q exists and (w, q) or (q, w) is an arc or q does not exist and thus, (z, w) is an arc since $\vec{G}_{n,m}$ is strong. Note that $a_a \in R^*$ ($b_b \in R^*$ resp.) if a_a exists (b_b exists resp.) by Claim 4.12.

- Let us first assume that (w, q) is an arc or q does not exist. Therefore, (z, w) is an arc since $\vec{G}_{n,m}$ is strong.
  - Let us first assume that (a, z) is an arc. Let $T = \{a_\ell, a_a, z_\ell, b\} \cup S_{wv} \subseteq R^*$ (or let $T = \{a_\ell, z_\ell, b\} \cup S_{wv}$ if a_a does not exist). Note that for any $t \in T$, $t \in R^*$. For any vertex $r \in P_{wv}$ and $t \in T$, we have $\text{dist}(a, r) \leq \text{dist}(t, r)$ (since by Claim 4.11, all shortest paths from t to r pass through w). Moreover, for any vertex $h \in V(\vec{G}_{n,m}) \setminus (P_{wv} \cup \{z\})$, there exists $t \in T$ such that $\text{dist}(a, h) > \text{dist}(t, h)$ since any shortest path from a to h passes through a vertex $t \in T$. Thus, all vertices $r \in P_{wv}$ are distinguishable from every vertex in $V(\vec{G}_{n,m}) \setminus P_{wv}$ (since it has already been shown that z is distinguishable from all other vertices). Finally, r is distinguished from every other vertex of P_{wv} by their distances from a. Hence, every vertex $r \in P_{wv}$ can be distinguished by R^* from all other vertices.
  - Let us assume that (b, z) is an arc. Let $T = \{b_\ell, b_b, z_\ell, a\} \cup S_{wv} \subseteq R^*$ (or let $T = \{b_\ell, z_\ell, a\} \cup S_{wv}$ if b_b does not exist). Note that for any $t \in T$, $t \in R^*$. For any vertex $r \in P_{wv}$ and $t \in T$, we have $\text{dist}(b, r) \leq \text{dist}(t, r)$ (since by Claim 4.11, all shortest paths from t to r pass through w). Moreover, for any vertex $h \in V(\vec{G}_{n,m}) \setminus (P_{wv} \cup \{z\})$, there exists $t \in T$ such that

$\text{dist}(b,h) > \text{dist}(t,h)$ since any shortest path from b to h passes through a vertex $t \in T$. Thus, all vertices $r \in P_{wv}$ are distinguishable from every vertex in $V(\vec{G}_{n,m}) \setminus P_{wv}$. Finally, r is distinguished from every other vertex of P_{wv} by their distances from b. Hence, every vertex $r \in P_{wv}$ can be distinguished by R^* from all other vertices.

- Second, let us assume that (q,w) is an arc. Let $N = N^+(q) \setminus \{w\}$. Let q_r be the neighbour to the right of q in $\vec{G}_{n,m}$. Note that for all $p \in (N \setminus \{q_r\})$, $p \in R^*$ due to the algorithm and thus, only q_r may not be in R^*, which is the case if either $q_r = a^{w'}$ or $q_r = z^{w'}$ for another LCV $w' \in Q$.
  - Let us assume that $N \subseteq R^*$. Let $T = N \cup S_{wv} \cup \{a, z_\ell, b\}$. Note that $T \subseteq R^*$. As above, all vertices on the directed (shortest) path P_{wv} from w to v (w, v included) are distinguishable from any other vertex by their distances from q and from the vertices of T.
  - Let us assume that $q_r = a^{w'}$ for another LCV w' and such that $q_r \notin R^*$. Let $T = (N \setminus \{q_r\}) \cup \{a_a^{w'}, z^{w'}, a, z_\ell, b\} \cup S_{wv}$. Note that $(x^{w'}, a^{w'})$ is an arc (by the proof of Claim 4.12 applied to w') and that $a_a^{w'}, z^{w'}, a, z_\ell, b \in R^*$. Then, as above, all vertices on the directed (shortest) path P_{wv} from w to v (w, v included) are distinguishable from any other vertex by their distances from q and from the vertices of T.
  - Let us assume that $q_r = z^{w'}$ for another LCV w' and such that $q_r \notin R^*$. There are two subcases: either $(w', z^{w'})$ is an arc or $(z^{w'}, w')$ is an arc.
    * Let us assume that $(w', z^{w'})$ is an arc. Let $T = (N \setminus \{q_r\}) \cup \{a^{w'}, b^{w'}, a, z_\ell, b\} \cup S_{wv}$. Note that $a^{w'}, b^{w'}, a, z_\ell, b \in R^*$. Then, as above, all vertices on the directed (shortest) path P_{wv} from w to v (w, v included) are distinguishable from any other vertex by their distances from q and from the vertices of T.
    * Let us assume that $(z^{w'}, w')$ is an arc. By the algorithm, since $z^{w'} \notin R^*$, either $(a^{w'}, z^{w'})$ or $(b^{w'}, z^{w'})$ is an arc. W.l.o.g., let $(a^{w'}, z^{w'})$ be an arc. Let $T = (N \setminus \{q_r\}) \cup \{a^{w'}, b^{w'}, a, z_\ell, b\} \cup S_{wv} \cup S_{w'v'}$ where $S_{w'v'}$ is defined analogously to S_{wv} for w' and $v^{w'}$. Then, as above, all vertices on the directed (shortest) path P_{wv} from w to v (w, v included) are distinguishable from any other vertex not in $P_{w'v'}$ (defined respectively to w' and v') by their distances from q and from the vertices of T. Note here that $z^{w'}$ is distinguished from all other vertices as it is the only vertex at distance 1 from both q and $a^{w'}$ that is not in R^*.

      Finally, all the vertices of the directed (shortest) path P_{wv} from w to v (w, v included) are distinguishable from all the vertices of the directed (shortest) path $P_{w'v'}$ by their distances from q and $a^{w'}$. Indeed, for any vertex $r \in P_{wv}$, $\text{dist}(q,r) < \text{dist}(a^{w'}, r)$ and for any vertex $r' \in P_{w'v'}$, $\text{dist}(q, r') \geq \text{dist}(a^{w'}, r')$.

Therefore, for any $w \in Q$, any $s \in H_w$, and any $t \in V(\vec{G}_{n,m})$ such that $z \notin R^*$, we have $s \nsim_{R^*} t$.

Case $a \notin R^*$. In this case, (z,a) and (z,b) are arcs. Then, a is distinguishable from all other vertices as it is the only vertex not in R^* that is at distance 1 from both $z, x \in R^*$.

The proof that all vertices on the directed (shortest) path from w to v are also distinguishable from every other vertex is analogous to the one above when $z \notin R^*$ with z taking on the role that a had in the other case for distinguishing these vertices from the rest, and so is omitted.

Therefore, for any $w \in Q$ such that $a \notin R^*$, any $s \in H_w$, and any $t \in V(\vec{G}_{n,m})$, we have $s \nsim_{R^*} t$.

$\diamond$

Claim 4.14. *For all vertices $s, t \in V(G_{n,m})$ such that $s \nsim_R t$, we have $s \nsim_{R^*} t$.*

Proof of the claim. Let $s \in V(\vec{G}_{n,m}) \setminus \bigcup_{w \in Q} H_w$, let us show that s can be distinguished from every vertex $t \in V(\vec{G}_{n,m}) \setminus \bigcup_{w \in Q} H_w$ (note that, if s and/or $t \in \bigcup_{w \in Q} H_w$, the result follows from Claim 4.13). Note that $s \nsim_R t$ and so, there is $k \in R$ such that $\text{dist}(k, s) \neq \text{dist}(k, t)$. If $k \in R^*$, it is still the case and we are done. Otherwise, there are two cases to be considered.

- Let us first assume that $k = a^w$ for some $w \in Q$ such that $a^w = a \notin R^*$. In that case, s can still be distinguished from t by one of a_a or a_ℓ. Indeed, all shortest paths from a to any other vertex in $V(\vec{G}_{n,m})$ pass through a_a and/or a_ℓ that are in R^* (recall that, if $a \notin R^*$, it implies that there is the arc (z, a)). Therefore, if $\text{dist}(a, s) \neq \text{dist}(a, t)$ then $\text{dist}(a_a, s) \neq \text{dist}(a_a, t)$ and/or $\text{dist}(a_\ell, s) \neq \text{dist}(a_\ell, t)$.
- Second, let us assume that $k = z^w$ for some $w \in Q$ such that $z^w = z \notin R^*$.
  If all z's out-neighbours are in R^*, then as above, s and t can still be distinguished by one of z's neighbours. So, let us assume that (z, w) is an arc.
  There are four remaining cases to be considered.
  - First, let us assume that there is a vertex $h \in R^*$ that is both on a shortest path from z to s and on a shortest path from z to t. This case is trivial as h distinguishes s and t since z distinguished s and t.
  - Second, let us assume that there are two vertices $h, p \in R^*$ where h is on a shortest path from z to s and p is on a shortest path from z to t where h (p resp.) is not on a shortest path from z to t (z to s resp.) as otherwise, we are in the first case. For purpose of contradiction, assume that neither h nor p can distinguish s and t. Then, $\text{dist}(h, s) = \text{dist}(h, t)$ and $\text{dist}(p, s) = \text{dist}(p, t)$. W.l.o.g., let us assume $\text{dist}(z, s) < \text{dist}(z, t)$. Then $\text{dist}(z, s) = \text{dist}(z, h) + \text{dist}(h, s) = \text{dist}(z, h) + \text{dist}(h, t) \geq \text{dist}(z, t)$, a contradiction. Therefore, h or p can distinguish s and t.
  - Then, let us consider the case when there exist a shortest path from z to s and a shortest path from z to t, both containing no vertices in R^*. In this case, both s and t must be in the same column C as w. Moreover, x cannot be on the path between z and s (resp., t) since then, it would be the first case. Therefore, both s and t are below w and one of s and t must be below the other, w.l.o.g., say t is below s, and there must exist a directed (shortest) path from w to s and from w to t that is entirely contained in C. In this case, as in Claim 4.13, either a or b (depending on which of the arcs (a, z) or (b, z) exists) can distinguish s and t.
  - Finally, let us assume that there is a vertex $h \in R^*$ on every shortest path from z to s and no shortest path from z to t containing a vertex in R^* (or *vice versa*). Then, t must be in the same column as w (and below w since the shortest path from z to t does not cross $x \in R^*$) and the directed shortest path from w to t is entirely contained in C. Let us assume that there is an arc (a, z) (the case when there is an arc (b, z) is similar and at least one of these cases must occur since $z \notin R^*$). Let us emphasize that no shortest path from a to t goes through a vertex in R^* (by the previous cases and since $\text{dist}(a, t) = \text{dist}(z, t) + 1$), therefore, the only shortest path from a to t goes through z and w and goes down along C until t. If there is a shortest path from a to s that passes through z, then a

distinguishes s and t since z did. Otherwise, any shortest path from a to s must go through a_a or a_ℓ. If $\text{dist}(a, s) = \text{dist}(a, t)$ (otherwise, a distinguishes s and t), then $\min\{\text{dist}(a_a, s), (a_\ell, s)\} = \text{dist}(a, s) - 1$. Since clearly $\min\{\text{dist}(a_a, t), (a_\ell, t)\} > \text{dist}(a, t)$, then at least one of a_a and a_ℓ can distinguish s and t.

◇

This concludes the proof that R^* is a resolving set.

Finally, in the case when m is not divisible by 3, we first add all the vertices of the last $x \in \{1, 2\}$ columns if $m \mod 3 = x$ to our resolving set, and then the remaining vertices induce a grid with a number of columns that is divisible by 3 on which we proceed as above. □

5 Conclusion

In this work, we have investigated, for a few families of graphs, the worst strong orientations in terms of metric dimension. In particular settings, such as when considering strong Eulerian orientations of tori, we managed to identify the worst possible orientations (Theorem 4.1). For other families (graphs with bounded maximum degree and grids), we have exhibited both lower and upper bounds on WOMD that are more or less distant apart. As further work on this topic, it would be interesting to lower the gap between our lower and upper bounds, or consider strong orientations of other graph families.

In particular, two appealing directions could be to improve Corollary 3.3 and Theorem 4.7. For graphs with maximum degree 3, we do wonder whether there are strong orientations for which the metric dimension is more than $\frac{2}{5}$ of the vertices. It is also legitimate to ask whether our upper bound ($\frac{1}{2}$ of the vertices), which was obtained from the simple technique described in Corollary 2.3, can be lowered further.

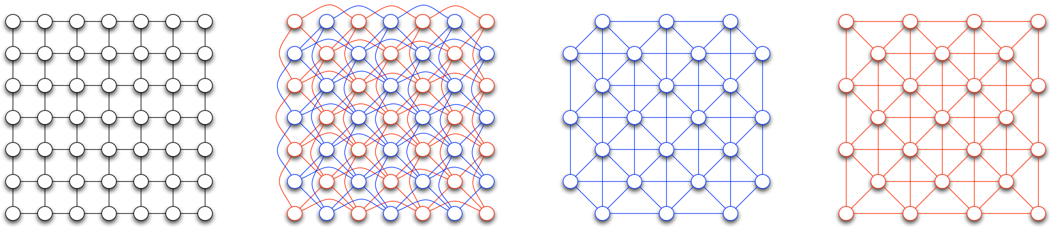
In Lemma 4.9, we proved that any strong orientation of a grid asymptotically has metric dimension at most $\frac{2}{3}$ of the vertices. Towards improving this upper bound, one could consider applying Corollary 2.3, for instance as follows. For a given oriented grid D, let A^* be the graph obtained as follows (where we deal with the vertices of D using the same terminology as in Section 4):

- $V(A^*) = V(D)$.
- We add, in A^*, an edge between two vertices (i, j) and (i', j') if they are joined by a path of length exactly 2 in the grid underlying D. That is, the edge is added whenever (i', j') is of the form $(i-1, j-1)$, $(i-2, j)$, $(i-1, j+1)$, $(i, j+2)$, $(i+1, j+1)$, $(i+2, j)$, $(i+1, j-1)$, or $(i, j-2)$.

Note that A^* has two connected components C_1, C_2 being basically obtained by glueing K_4's along edges. See Figure 6 for an illustration.

It can be noticed that for any oriented grid D, its auxiliary graph D_{aux} is a subgraph of A^*. From Corollary 2.3, any upper bound on the size of a minimum vertex cover of A^* is thus also an upper bound on $\text{MD}(D)$ (assuming D is strong, in which case it necessarily verifies $\Delta^+(D) \geq 2$). Unfortunately, we have observed that any minimum vertex cover of A^* covers $\frac{3}{4}$ of the vertices, which is not better than our upper bound in Lemma 4.9.

There is still hope, however, to improve our upper bound using the vertex cover method. Indeed, under the assumption that D is a strong oriented graph, actually D_{aux} can be far from having all the edges that A^* has. For instance, it can easily be proved that, in D_{aux}, it is not possible that a vertex (i, j) is adjacent to all four vertices $(i-2, j)$, $(i, j+2)$,



(a) $G_{9,9}$ (b) A^* (c) C_1 (d) C_2

Figure 6: The grid $G_{9,9}$ and the associated graph A^*.

$(i+2, j)$, $(i, j-2)$ (if they exist). Using a computer, we were actually able to check on small grids that, for all strong orientations D, the minimum vertex cover of D_{aux} has size at most $\frac{1}{2}$ of the vertices. This leads us to raising the following two questions related to our upper bound in Lemma 4.9:

Question 5.1. *For any strong orientation D of a grid $G_{n,m}$, do the minimum vertex covers of D_{aux} have size at most $\frac{nm}{2}$?*

Question 5.2. *For any strong orientation D of a grid $G_{n,m}$, do we have* $\text{MD}(D) \leq \frac{nm}{2}$*?*

Note that if the upper bound in Question 5.2 held, then it would be quite close to the lower bound we have established in Lemma 4.8.

6 Acknowledgements

We would like to thank an anonymous reviewer for his/her suggestions which allowed us to slightly improve the bound in Theorem 3.2.

References

[1] R.F. Bailey, P.J. Cameron. Base size, metric dimension and other invariants of groups and graphs. *Bulletin of the London Mathematical Society*, 43:209-242, 2011.

[2] G. Chartrand, L. Eroh, M. Johnson, O. Oellermann. Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, 105(1-3):99-113, 2000.

[3] G. Chartrand, M. Rains, P. Zhang. The directed distance dimension of oriented graphs. *Mathematica Bohemica*, 125:155-168, 2000.

[4] G. Chartrand, M. Rains, P. Zhang. On the dimension of oriented graphs. *Utilitas Mathematica*, 60:139-151, 2001.

[5] J. Bensmail, F. Mc Inerney, N. Nisse. Metric Dimension: from Graphs to Oriented Graphs. In *Proceedings of the 10th Latin & American Algorithms, Graphs and Optimization Symposium, LAGOS 2019*, volume 346 of *Electronic Notes in Theoretical Computer Science*, pages 111-123, 2019.

[6] M. Fehr, S. Gosselin, O.R. Oellermann. The metric dimension of Cayley digraphs. *Discrete Mathematics*, 306:31-41, 2006.

[7] M. Feng, M. Xu, K. Wang. On the metric dimension of line graphs. *Discrete Applied Mathematics*, 161:802-805, 2013.

[8] M.R. Garey, D.S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness.* W.H. Freeman Co., 1990.

[9] F. Harary, R.A. Melter. On the metric dimension of a graph. *Ars Combinatoria*, 2:191–195, 1976.

[10] A. Lozano. Symmetry Breaking in Tournaments. *The Electronic Journal of Combinatorics*, 20(1):Paper #P69, 2013.

[11] P. Manuel, B. Rajan, I. Rajasingh, M.C. Monica. Landmarks in torus networks. *Journal of Discrete Mathematical Sciences and Cryptography*, 9(2):263-271, 2013.

[12] S. Pancahayani, R. Simanjuntak. Directed Metric Dimension of Oriented Graphs with Cyclic Covering. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 94:15-25, 2015.

[13] B. Rajan, I. Rajasingh, J.A. Cynthia, P. Manuel. Metric dimension of directed graphs. *International Journal of Computer Mathematics*, 91(7):1397-1406, 2014.

[14] H.E. Robbins. A theorem on graphs, with an application to a problem in traffic control. *American Mathematical Monthly*, 46:281–283, 1939.

[15] P.J. Slater. Leaves of trees. pages 549–559. *Congressus Numerantium*, No. XIV, 1975.

[16] P. Turán. On an extremal problem in graph theory (in Hungarian). *Mat. Fiz. Lapok*, 48:436-452, 1941.