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Trigonometric functions in the p-norm

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Abstract

Trigonometry is the study of circular functions, which are functions defined on the unit circle $x^2+y^2=1$, where distances are measured using the Euclidean norm. We explore trigonometric functions using the p-norm. These are functions defined on the unit p-circle $|x|^p+|y|^p=1$. This approach revealed interesting connections involving transcendental periods, Bell polynomials, Lagrange inversion, Gamma functions, associahedra, and Stirling numbers.

p-Trigonometric Functions

A unit circle in the p-norm (squircle) is the curve defined by the equation $|x|^p+|y|^p=1$, where $p=2$ gives the unit circle. While the standard 2-trigonometric functions are objects of study in many math classes, we wished to examine generalized p-trigonometric functions for any real $p \geq 1$ resulting from the following Coupled Initial Value Problem (CIVP):

$$x'(t) = -y(t)^{p-1} \qquad y'(t) = x(t)^{p-1}$$
$$x(0) = 1 \qquad y(0) = 0$$

We call the unique functions, $x(t)$, $y(t)$, that satisfy this system $\cos_p(t)$, $\sin_p(t)$ respectively.

Graphing these functions and finding their derivatives is computationally intensive. To circumvent this, we use their inverse functions, which are well defined:

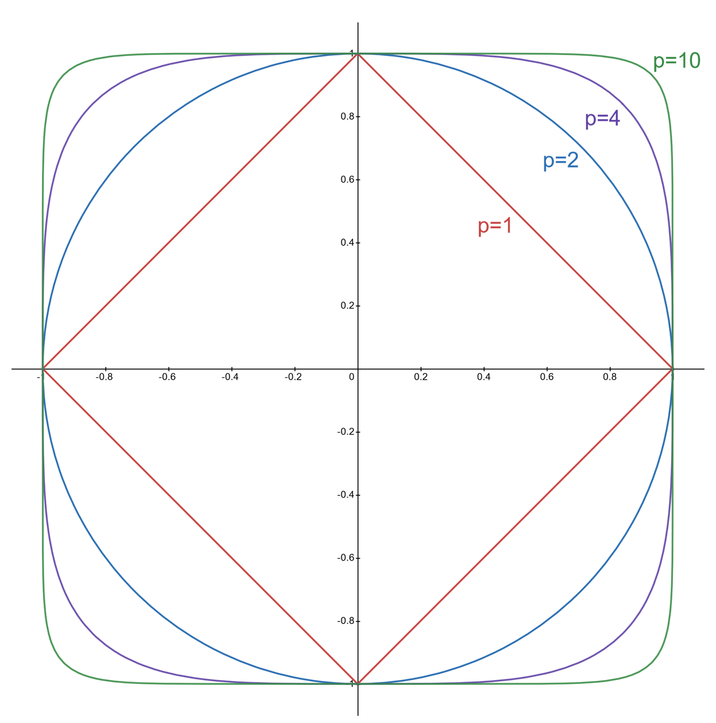

Figure 1. Squircles with multiple values of p

$$\sin_p^{-1} x = \int_0^x \frac{1}{(1-t^p)^{\frac{p-1}{p}}}\,dt \qquad \cos_p^{-1} x = \int_x^1 \frac{1}{(1-t^p)^{\frac{p-1}{p}}}\,dt$$

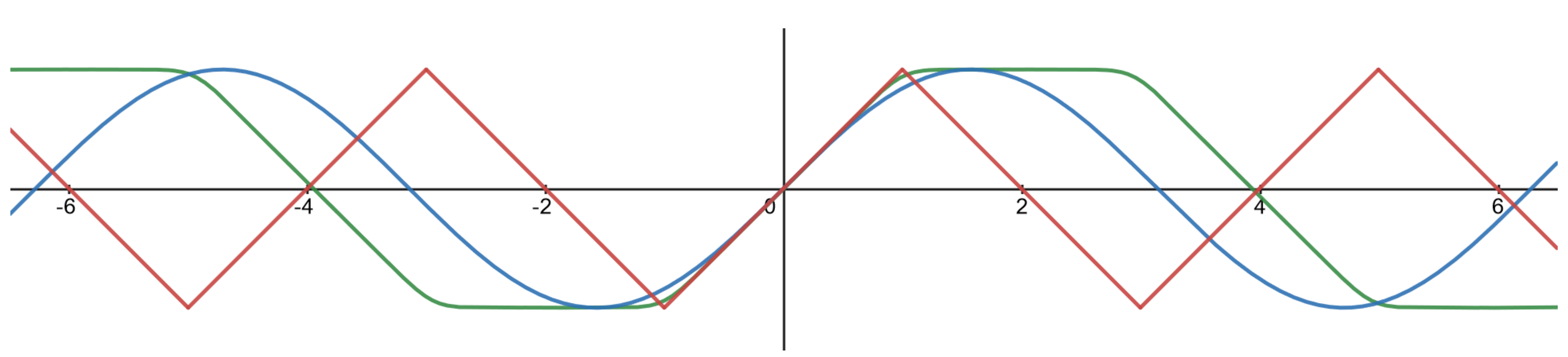

Figure 2. Graph of $\sin_p x$ where red is $p=1$, blue is $p=2$, and green is $p=10$

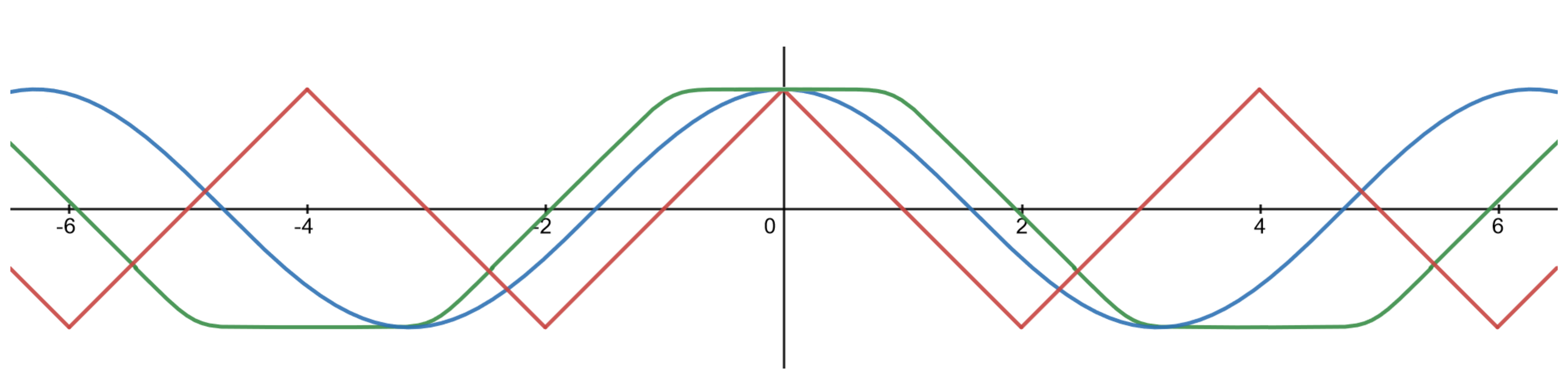

Figure 3. Graph of $\cos_p x$ where red is $p=1$, blue is $p=2$, and green is $p=10$

Lagrange Inversion Theorem

For an equation $z=f(w)$, where f is analytic at c and $f'(c)\neq 0$, the Lagrange Inversion Theorem can be used to find the equation's inverse, $w=g(z)$, in a neighborhood of 0.

Specifically, when f and g are formal power series expressed as

$$f(w)=\sum_{k=0}^{\infty} f_k \frac{w^k}{k!} \quad \text{and} \quad g(z)=\sum_{k=0}^{\infty} g_k \frac{z^k}{k!}$$

with $f_0=0$ and $f_1\neq 0$, applying the Lagrange Inversion Theorem gives us the following [2]:

$$g(z) = c+\sum_{n=1}^{\infty} g_n \frac{(z-f(c))^n}{n!}, \quad \text{with}$$

$$g_n = \frac{1}{f_1^n}\sum_{k=1}^{n-1}(-1)^k n^{(k)} B_{n-1,k}(\hat{f}_1,\hat{f}_2,\ldots,\hat{f}_{n-k}), \quad n\geq 2, \quad \text{where}$$

$$\hat{f}_k = \frac{f_{k+1}}{(k+1)f_1}, \quad g_1=\frac{1}{f_1}, \quad n^{(k)} = n(n+1)\cdots(n+k-1), \text{ and}$$

$$B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) = \sum \frac{n!}{j_1!j_2!\ldots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\ldots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where this sum is taken over all sequences $j_1,j_2,j_3,\ldots,j_{n-k+1}$ of non-negative integers that satisfy $j_1+j_2+\ldots+j_{n-k+1}=k$ and $j_1+2j_2+3j_3+\ldots+(n-k+1)j_{n-k+1}=n$. These are the Bell polynomials.

The Taylor series expansion of $\sin_p(x)$ is obtained when the above theorem is applied to

$$\sin_p^{-1}(x) = \int_0^x \frac{1}{(1-t^p)^{\frac{p-1}{p}}}\,dt = \sum_{k=0}^{\infty}\left(\frac{p-1}{p}\right)^{(k)} \frac{x^{kp+1}}{k!(kp+1)}.$$

For example, when $p=4$, applying this theorem gives the first few terms of $\sin_4(x)$:

$$\sin_4(x) = x-\frac{18}{5!}x^5+\frac{14364}{9!}x^9-\ldots$$

Note that when $\sin_2^{-1}(x) := \sin^{-1}(x)$ is evaluated at $x=1$, we get the value $\frac{\pi}{2}$. Thus, we define $\pi_p := 2\sin_p^{-1}(1)$, which is also the area of the unit p-circle.

Associahedra

We can reformulate the Lagrange Inversion Theorem in a much more geometric setting using the language of associahedra. The associahedron K_n is a convex $(n-2)$-polytope where distinct vertices correspond to distinct parenthetical groupings of n symbols, and edges are drawn between vertices if they are obtainable through applying the associative law once. If we have the same setup as above and $f_1=1$, we can find each g_n by

$$g_n = \sum_{F \text{ face of } K_n} (-1)^{n-\dim F} f_F$$

where $f_F = f_{i_1}\cdots f_{i_m}$ for any face $F=K_{i_1}\times\cdots\times K_{i_m}$ of K_n.

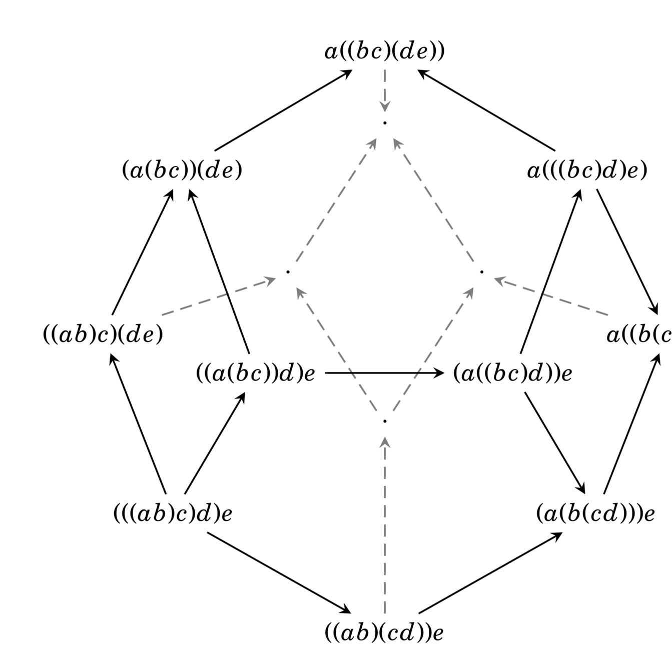

Figure 4. Image Credits: Niles Johnson [4]

Computing π_p

We can calculate π_p using $\pi_p = 2\int_0^1 \frac{1}{(1-t^p)^{\frac{p-1}{p}}}\,dt$.

Letting $u=t^p$, we get the beta integral function, which we can then put in terms of gamma using the identity $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, giving us

$$\pi_p = \frac{2\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}.$$

Using this equation, we can find values of π_3 and π_4 to be

$$\pi_3 = \frac{2\Gamma^2(\frac{1}{3})}{3\Gamma(\frac{2}{3})} \approx 3.533 \quad \text{and} \quad \pi_4 = \frac{2\Gamma^2(\frac{1}{4})}{4\Gamma(\frac{2}{4})} \approx 3.708$$

When $p\approx 3.162$, the area of the unit squircle is approximately halfway between the areas of a unit circle and a square with side length 2. This gives us $\pi_{3.162} \approx (\pi_2+4)/2$; see figure 5.

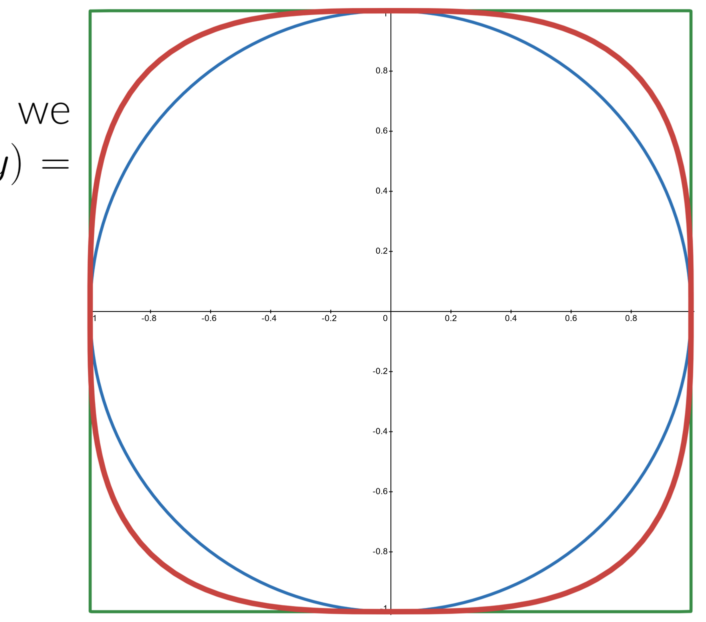

Figure 5. a 3.162-circle (red), a unit circle (blue), and a square (green)

Applications

Squigonometry has several applications, specifically in design. Rather than using rounded rectangles, Apple uses p-circles for their icons, as the curvature continuity leads to a more sleek look, unifying the design of their hardware and icons [6]. Another design application can be found in squircular dinner plates, designed to allow a greater surface area for food while taking up the same amount of cabinet space as their circular counterparts [5].

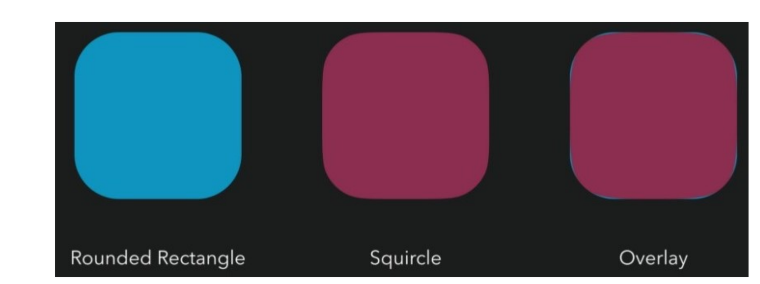

Figure 6. Image Credits: Arthur Van Siclen [6]

Future Research

We have been working towards a closed form solution for the successive derivatives of functions of the form $\cos_p^m(t)\sin_p^n(t)$, which led to a connection to Stirling numbers. Extending from our generalization of the standard trigonometric functions, we plan to investigate other transcendental functions in the p-norm. These include the p-hyperbolic functions and p-exponential functions, which can be related via the equation $\exp_p(t) := \sinh_p(t)+\cosh_p(t)$ that is motivated by the formula $e^t = \cosh(t)+\sinh(t)$.

References

[1] Pamela E. Harris, Erik Insko, Aaron Wootton, editor.
A Project-Based Guide to Undergraduate Research in Mathematics.
Foundations for Undergraduate Research in Mathematics. Springer International Publishing, 2020.

[2] Wikipedia Contributors.
Lagrange inversion theorem, 3 2021.

[3] David E. Edmunds, Petr Gurka, and Jan Lang.
Properties of generalized trigonometric functions.
J. Approx. Theory, 164(1):47–56, January 2012.

[4] Niles Johnson.
Associahedron K5, 3 2021.

[5] Peter Lynch.
Squircles, 7 2016.

[6] Arthur Van Siclen.
Rounded corners in the apple ecosystem, 6 2020.