Representation of Banach lattices as L^1_w spaces of a vector measure defined on a δ-ring

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Abstract

In this paper we prove that every Banach lattice having the Fatou property and having its σ-order continuous part as an order dense subset, can be represented as the space $L^1_w(\nu)$ of weakly integrable functions with respect to some vector measure ν defined on a δ-ring.

1 Introduction

The interplay among the properties of a vector measure ν, its range and its integration operator allows us to understand the behavior of the space $L^1(\nu)$ of integrable functions with respect to ν. This makes desirable to know which spaces can be described as such L^1 spaces. In [2, Theorem 8], Curbera proves that every order continuous Banach lattice E with a weak unit is order isometric to a space $L^1(\nu)$ where ν is a vector measure defined on a σ-algebra. The result remains true if E has not a weak unit but for ν defined on a δ-ring. This was stated in [1, pp. 22-23] but the proof there is just outlined. We present here a proof of this fact in full detail. Note that the differences between the integration theory with respect to vector measures on σ-algebras and the integration theory with respect to vector measures on δ-rings are significant. For instance, bounded functions are always integrable for the first one while they are not necessarily integrable for the second one.

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Associated to ν there is another interesting space whose properties can be studied through the properties of ν. Namely, the space $L^1_w(\nu)$ of weakly integrable functions. In [3, Theorem 2.5], Curbera and Ricker show that every Banach lattice E satisfying the σ-Fatou property and with a weak unit belonging to the σ-order continuous part E_a of E is order isometric to a space $L^1_w(\nu)$ for a vector measure ν defined on a σ-algebra. The aim of this paper is to prove the corresponding result in the case when E has not a weak unit by using a vector measure defined on a δ-ring.

Given an order continuous Banach lattice E, Section 3 is devoted to the construction of a vector measure ν defined on a δ-ring associated to E. In Section 4, we show that $L^1(\nu)$ is order isometric to E via the integration operator. This fact is the starting point for proving our main result in Section 6, namely, every Banach lattice E with the Fatou property such that its σ-order continuous part E_a is order dense in E is order isometric to the L^1_w space of the vector measure associated to E_a which in this case is also order continuous. This L^1_w space is studied first in Section 5. We end with two examples of Banach lattices which can be represented as $L^1_w(\nu)$ with ν defined on a δ-ring, but cannot be represented in the same way for any vector measure defined on a σ-algebra.

2 Preliminaries

2.1 Banach lattices.

Let E be a Banach lattice with norm $\|\cdot\|$ and order $\leq$. A *weak unit* of E is an element $0 \leq e \in E$ such that $x \wedge e = 0$ implies $x = 0$. A closed subspace F of E is an *ideal* of E if $y \in F$ whenever $y \in E$ with $|y| \leq |x|$ for some $x \in F$. An ideal F in E is said to be *order dense* if for every $0 \leq x \in E$ there exists an upwards directed system $0 \leq x_\tau \uparrow x$ such that $(x_\tau)_\tau \subset F$. We will say that E has the *Fatou property* if for every $(x_\tau)_\tau \subset E$ upwards directed system $0 \leq x_\tau \uparrow$ such that $\sup_\tau \|x_\tau\| < \infty$ it follows that there exists $x = \sup_\tau x_\tau$ in E and $\|x\| = \sup_\tau \|x_\tau\|$. We will say that E has the *σ-Fatou property* if for every $(x_n)_{n\geq 1} \subset E$ increasing sequence $0 \leq x_n \uparrow$ such that $\sup_{n\geq 1} \|x_n\| < \infty$ it follows that there exists $x = \sup_{n\geq 1} x_n$ in E and $\|x\| = \sup_{n\geq 1} \|x_n\|$. The Banach lattice E is *order continuous* if for every $(x_\tau)_\tau \subset E$ downwards directed system $x_\tau \downarrow 0$ it follows that $\|x_\tau\| \downarrow 0$. If $\|x_n\| \downarrow 0$ for any $(x_n)_{n\geq 1} \subset E$ decreasing sequence $x_n \downarrow 0$, then E is said to be *σ-order continuous*. We call *order continuous part E_{an}* of E to the largest order continuous ideal in E. It can be described as

$$E_{an} = \{x \in E : \ |x| \geq x_\tau \downarrow 0 \ \text{ implies } \ \|x_\tau\| \downarrow 0\}.$$

Similarly, the *σ-order continuous part E_a* of E is the largest σ-order continuous ideal in E, which can be described as

$$E_a = \{x \in E : \ |x| \geq x_n \downarrow 0 \ \text{ implies } \ \|x_n\| \downarrow 0\}.$$

The Banach lattice E is said to be *σ-complete* if every order bounded sequence has a supremum.

An operator $T\colon E \to F$ between Banach lattices is said to be an *order isometry* if it is a linear isometry which is also an order isomorphism, that is, T is linear, one to one, onto, $\|Tx\|_F = \|x\|_E$ for all $x \in E$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$.

For these and other issues related to Banach lattices, see for instance [6], [7] and [10].

2.2 Integration with respect to vector measures on δ-rings.

This integration theory is due to Lewis [5] and Masani and Niemi [8], [9]. See also [4].

Let $\mathcal{R}$ be a *δ-ring* of subsets of an abstract set Ω (i.e. a ring of sets closed under countable intersections). Associated to $\mathcal{R}$ we have the σ-algebra $\mathcal{R}^{loc}$ of subsets A of Ω such that $A \cap B \in \mathcal{R}$ for every $B \in \mathcal{R}$. The space of measurable real functions on $(\Omega, \mathcal{R}^{loc})$ will be denoted by $\mathcal{M}(\mathcal{R}^{loc})$ and the space of simple functions by $\mathcal{S}(\mathcal{R}^{loc})$. A special role will be played by the simple functions based on $\mathcal{R}$. The space of these functions will be denoted by $\mathcal{S}(\mathcal{R})$.

Let $\lambda\colon \mathcal{R} \to \mathbb{R}$ be a countably additive measure, that is, $\sum_{n\geq 1} \lambda(A_n)$ converges to $\lambda(\cup_{n\geq 1} A_n)$ whenever $(A_n)_{n\geq 1}$ are pairwise disjoint sets in $\mathcal{R}$ with $\cup_{n\geq 1} A_n \in \mathcal{R}$. The *variation* of λ is the countably additive measure $|\lambda|\colon \mathcal{R}^{loc} \to [0, \infty]$ given by

$$|\lambda|(A) = \sup\Big\{ \sum |\lambda(A_i)| : (A_i) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A \Big\}.$$

The space $L^1(\lambda)$ of integrable functions with respect to λ is defined just as $L^1(|\lambda|)$ with the same norm. The space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\lambda)$. For each $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i} \in \mathcal{S}(\mathcal{R})$, the integral of φ with respect to λ is defined as usual, $\int \varphi \, d\lambda = \sum_{i=1}^n \alpha_i \lambda(A_i)$. For every $f \in L^1(\lambda)$, the integral of f with respect to λ is defined as $\int f \, d\lambda = \lim_{n\to\infty} \int \varphi_n \, d\lambda$ for any sequence $(\varphi_n)_{n\geq 1} \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\lambda)$.

Let $\nu\colon \mathcal{R} \to X$ be a *vector measure* with values in a real Banach space X, that is, $\sum_{n\geq 1} \nu(A_n)$ converges to $\nu(\cup_{n\geq 1} A_n)$ in X whenever $(A_n)_{n\geq 1}$ are pairwise disjoint sets in $\mathcal{R}$ with $\cup_{n\geq 1} A_n \in \mathcal{R}$. Denoting by X^* the dual space of X and by B_{X^*} the unit ball of X^*, the *semivariation* of ν is the map $\|\nu\|\colon \mathcal{R}^{loc} \to [0, \infty]$ given by $\|\nu\|(A) = \sup\{|x^*\nu|(A) : x^* \in B_{X^*}\}$ for all $A \in \mathcal{R}^{loc}$, where $|x^*\nu|$ is the variation of the measure $x^*\nu\colon \mathcal{R} \to \mathbb{R}$. A set $B \in \mathcal{R}^{loc}$ is *ν-null* if $\|\nu\|(B) = 0$. A property holds *ν-almost everywhere* (ν-a.e.) if it holds except on a ν-null set.

We will denote by $L^1_w(\nu)$ the space of functions in $\mathcal{M}(\mathcal{R}^{loc})$ which are integrable with respect to $|x^*\nu|$ for all $x^* \in X^*$. Functions which are equal ν-a.e. are identified. The space $L^1_w(\nu)$ is a Banach space with the norm

$$\|f\|_\nu = \sup\Big\{ \int |f| \, d|x^*\nu| : x^* \in B_{X^*} \Big\}.$$

Moreover, it is a Banach lattice having the σ-Fatou property for the ν-a.e. pointwise order and it is an ideal of measurable functions, that is, if $|f| \leq |g|$ ν-a.e. with $f \in \mathcal{M}(\mathcal{R}^{loc})$ and $g \in L^1_w(\nu)$, then $f \in L^1_w(\nu)$. Also, note that convergence in norm of a sequence implies ν-a.e. convergence of some subsequence. A function

$f \in L^1_w(\nu)$ is *integrable with respect to* ν if for each $A \in \mathcal{R}^{loc}$ there exists a vector denoted by $\int_A f d\nu \in X$, such that

$$x^*\Big(\int_A f\, d\nu\Big) = \int_A f\, dx^*\nu \;\text{ for all } x^* \in X^*.$$

We will write $\int f d\nu$ for $\int_\Omega f d\nu$. We will denote by $L^1(\nu)$ the space of integrable functions with respect to ν. It is an order continuous Banach lattice when endowed with the norm and the order structure of $L^1_w(\nu)$. Even more, it is an ideal of measurable functions and so an ideal of $L^1_w(\nu)$. Note that if $\varphi = \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{S}(\mathcal{R})$ then $\varphi \in L^1(\nu)$ with $\int_A \varphi\, d\nu = \sum_{i=1}^n a_i \nu(A_i \cap A)$ for all $A \in \mathcal{R}^{loc}$. The space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu)$. The integration operator $I_\nu \colon L^1(\nu) \to X$ given by $I_\nu(f) = \int f\, d\nu$ is linear and continuous with $\|I_\nu(f)\| \leq \|f\|_\nu$.

A vector measure $\nu \colon \mathcal{R} \to E$ with values in a Banach lattice E is *positive* if $\nu(A) \geq 0$ for all $A \in \mathcal{R}$. In this case, the integration operator $I_\nu \colon L^1(\nu) \to E$ is positive (i.e. $I_\nu(f) \geq 0$ whenever $0 \leq f \in L^1(\nu)$) and it can be checked that $\|f\|_\nu = \|I_\nu(|f|)\|$ for all $f \in L^1(\nu)$.

We know that the space $L^1_w(\nu)$ has the σ-Fatou property for every vector measure $\nu \colon \mathcal{R} \to X$, but what about the Fatou property? The following proposition, which will be needed later on, gives a sufficient condition for $L^1_w(\nu)$ to have the Fatou property.

Proposition 1. *If $\nu \colon \mathcal{R} \to X$ is a σ-finite vector measure, that is, there exists a sequence $(A_n)_{n\geq 1} \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{loc}$ such that $\Omega = (\cup_{n\geq 1} A_n) \cup N$, then $L^1_w(\nu)$ has the Fatou property.*

Proof. Let $\nu \colon \mathcal{R} \to X$ be a σ-finite vector measure. Then, by [4, Remark 3.4], there exists $x_0^* \in B_{X^*}$ such that $|x_0^*\nu|$ is a local control measure for ν, that is, $|x_0^*\nu|$ has the same null sets as ν.

Let $(f_\tau)_\tau \subset L^1_w(\nu)$ be such that $0 \leq f_\tau \uparrow \nu$-a.e. and $\sup_\tau \|f_\tau\|_\nu < \infty$. Then, $0 \leq f_\tau \uparrow |x_0^*\nu|$-a.e. and $\sup_\tau \int f_\tau\, d|x_0^*\nu| \leq \sup_\tau \|f_\tau\|_\nu < \infty$. Since $L^1(x_0^*\nu)$ has the Fatou property, there exists $f = \sup_\tau f_\tau$ in $L^1(x_0^*\nu)$. On the other hand $L^1(x_0^*\nu)$ is order separable, so we can take a sequence $f_{\tau_n} \uparrow f$ in $L^1(x_0^*\nu)$. Then, $f_{\tau_n} \uparrow f$ $|x_0^*\nu|$-a.e. (equivalently ν-a.e.) and so $|x^*\nu|$-a.e. for all $x^* \in X^*$. By using the monotone convergence theorem, we have that

$$\int |f|\, d|x^*\nu| = \lim_n \int |f_{\tau_n}|\, d|x^*\nu| \leq \|x^*\| \cdot \sup_\tau \|f_\tau\|_\nu < \infty,$$

and so $f \in L^1(x^*\nu)$ for all $x^* \in X^*$. Hence, $f \in L^1_w(\nu)$ and $\|f\|_\nu \leq \sup_\tau \|f_\tau\|_\nu$.

Since the $|x_0^*\nu|$-a.e. pointwise order coincides with the ν-a.e. one and $0 \leq f_\tau \uparrow f$ in $L^1(|x_0^*\nu|)$, it follows that $0 \leq f_\tau \uparrow f$ in $L^1_w(\nu)$. Indeed if $g \in L^1_w(\nu)$ is such that $f_\tau \leq g$ ν-a.e. for all τ, then $g \in L^1_w(|x_0^*\nu|)$ is such that $f_\tau \leq g$ $|x_0^*\nu|$-a.e. for all τ, and so $f \leq g$ $|x_0^*\nu|$-a.e. or equivalently ν-a.e. Moreover, since $\|f_\tau\|_\nu \leq \|f\|_\nu$ for all τ, we have that $\|f\|_\nu = \sup_\tau \|f_\tau\|_\nu$. Therefore, $L^1_w(\nu)$ has the Fatou property. ∎

In particular, from Proposition 1, we have that $L^1_w(\nu)$ has the Fatou property for every vector measure ν defined on a σ-algebra.

3 Vector measure associated to an order continuous Banach lattice

Let E be an order continuous Banach lattice. We will prove that there exists a vector measure ν defined on a δ-ring and with values in E, such that the space $L^1(\nu)$ of integrable functions with respect to ν is order isometric to E. More precisely, the integration operator $I_\nu \colon L^1(\nu) \to E$ is an order isometry.

As it has been remarked in the Introduction, in the case when E has a weak unit this result was proved in [2, Theorem 8] with ν defined in a σ-algebra. In the general case, there is an outlined proof in [1, pp. 22-23]. For completeness, we include in this paper a detailed proof.

In this section, we construct a vector measure ν for which we will see in Section 4 that the order isometry works.

The key for constructing our vector measure is the following result of Lindenstrauss and Tzafriri [6, Proposition 1.a.9]: *E can be decomposed into an unconditionally direct sum of a family of mutually disjoints ideals $\{E_\alpha\}_{\alpha\in\Delta}$, each E_α having a weak unit. That is, every $e \in E$ has a unique representation $e = \sum_{\alpha\in\Delta} e_\alpha$ with $e_\alpha \in E_\alpha$, only countably many $e_\alpha \neq 0$ and the series converging unconditionally.*

Each E_α is an order continuous Banach lattice with a weak unit. Then, from [2, Theorem 8], there is a σ-algebra Σ_α of subsets of an abstract set Ω_α and a positive vector measure $\nu_\alpha \colon \Sigma_\alpha \to E_\alpha$ such that the integration operator $I_{\nu_\alpha} \colon L^1(\nu_\alpha) \to E_\alpha$ is an order isometry.

Consider the set $\Omega = \cup_{\alpha\in\Delta}\big(\{\alpha\} \times \Omega_\alpha\big)$, that is

$$\Omega = \big\{(\alpha,\omega) : \alpha \in \Delta \text{ and } \omega \in \Omega_\alpha\big\}.$$

In a similar way, we denote $\cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha = \big\{(\alpha,\omega) : \alpha \in \Delta \text{ and } \omega \in A_\alpha\big\}$, where $A_\alpha \subset \Omega_\alpha$ for all $\alpha \in \Delta$. For every $\Gamma \subset \Delta$ we write $\cup_{\alpha\in\Gamma}\{\alpha\} \times A_\alpha = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha$ whenever $A_\alpha = \emptyset$ for all $\alpha \in \Delta\backslash\Gamma$. Note that if $A_n = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha^n$ for $n \geq 1$,

$$\bigcup_{n\geq 1} A_n = \bigcup_{\alpha\in\Delta}\Big(\{\alpha\} \times \bigcup_{n\geq 1} A_\alpha^n\Big) \text{ and } \bigcap_{n\geq 1} A_n = \bigcup_{\alpha\in\Delta}\Big(\{\alpha\} \times \bigcap_{n\geq 1} A_\alpha^n\Big).$$

Also, if $A = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha$ and $B = \cup_{\alpha\in\Delta}\{\alpha\} \times B_\alpha$,

$$A\backslash B = \bigcup_{\alpha\in\Delta}\big(\{\alpha\} \times A_\alpha\backslash B_\alpha\big).$$

Then the family $\mathcal{R}$ of sets $\cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha$ satisfying that $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \Delta$ and there exists a finite set $I \subset \Delta$ such that A_α is ν_α-null for all $\alpha \in \Delta\backslash I$, is a δ-ring of subsets of Ω. Moreover,

$$\mathcal{R}^{loc} = \big\{ \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha : A_\alpha \in \Sigma_\alpha \text{ for all } \alpha \in \Delta\big\}.$$

Indeed, given $A \in \mathcal{R}^{loc}$, if we take $B_\alpha = \{\omega \in \Omega_\alpha : (\alpha,\omega) \in A\}$ we have that

$$A = \cup_{\alpha\in\Delta}\{\alpha\} \times B_\alpha,$$

where $\{\alpha\} \times B_\alpha = A \cap (\{\alpha\} \times \Omega_\alpha) \in \mathcal{R}$ (as $\{\alpha\} \times \Omega_\alpha \in \mathcal{R}$). So, $B_\alpha \in \Sigma_\alpha$.

Conversely, take $A = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha$ with $A_\alpha \in \Sigma_\alpha$ for every $\alpha \in \Delta$. If $B = \cup_{\alpha\in\Delta}\{\alpha\} \times B_\alpha \in \mathcal{R}$,

$$A \cap B = \bigcup_{\alpha\in\Delta} \big(\{\alpha\} \times A_\alpha \cap B_\alpha\big) \in \mathcal{R}$$

and so $A \in \mathcal{R}^{loc}$.

Let $\nu\colon \mathcal{R} \to E$ be the set function defined by

$$\nu\big(\cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha\big) = \sum_{\alpha\in\Delta} \nu_\alpha(A_\alpha).$$

Let us see that ν is a vector measure. Given $A_n = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha^n \in \mathcal{R}$ for $n \geq 1$ mutually disjoint sets such that $\cup_{n\geq 1}A_n \in \mathcal{R}$, we have that

$$\bigcup_{n\geq 1} A_n = \bigcup_{\alpha\in\Delta} \Big(\{\alpha\} \times \bigcup_{n\geq 1} A_\alpha^n\Big)$$

where $\bigcup_{n\geq 1} A_\alpha^n$ is a disjoint union for every $\alpha \in \Delta$ and there exists a finite set $I \subset \Delta$ such that $\bigcup_{n\geq 1} A_\alpha^n$ is ν_α-null for all $\alpha \in \Delta\backslash I$. Since for each $\alpha \in \Delta$ the sum $\sum_{n\geq 1}\nu_\alpha(A_\alpha^n)$ converges to $\nu_\alpha(\cup_{n\geq 1}A_\alpha^n)$ in E_α and so in E, then we have that

$$\nu\Big(\bigcup_{n\geq 1} A_n\Big) = \sum_{\alpha\in I} \nu_\alpha\Big(\bigcup_{n\geq 1} A_\alpha^n\Big) = \sum_{\alpha\in I}\sum_{n\geq 1} \nu_\alpha(A_\alpha^n) = \sum_{n\geq 1}\sum_{\alpha\in I} \nu_\alpha(A_\alpha^n) = \sum_{n\geq 1} \nu(A_n).$$

Note that a set $A = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha \in \mathcal{R}^{loc}$ is ν-null if and only if A_α is ν_α-null for all $\alpha \in \Delta$. Also note that ν is positive as every ν_α is so.

Remark 2. Let $f \in \mathcal{M}(\mathcal{R}^{loc})$. For each $\alpha \in \Delta$, we denote by f_α the function $f_\alpha\colon \Omega_\alpha \to \mathbb{R}$ given by $f_\alpha(\omega) = f(\alpha, \omega)$ for all $\omega \in \Omega_\alpha$. Since for every Borel set B on $\mathbb{R}$ we have that

$$f^{-1}(B) = \cup_{\alpha\in\Delta}\{\alpha\} \times f_\alpha^{-1}(B) \in \mathcal{R}^{loc},$$

then $f_\alpha^{-1}(B) \in \Sigma_\alpha$ for each $\alpha \in \Delta$. Hence, $f_\alpha \in \mathcal{M}(\Sigma_\alpha)$ for each $\alpha \in \Delta$. In particular, if $\varphi = \sum_{j=1}^n a_j\chi_{A_j}$ with $A_j = \cup_{\alpha\in\Delta}\{\alpha\} \times A_\alpha^j \in \mathcal{R}^{loc}$, then $\varphi_\alpha = \sum_{j=1}^n a_j\chi_{A_\alpha^j} \in \mathcal{S}(\Sigma_\alpha)$.

From now and on, f_α will denote the functions defined in Remark 2 for some function $f \in \mathcal{M}(\mathcal{R}^{loc})$. The following lemma will allow us to give useful descriptions of the spaces $L^1(\nu)$ and $L^1_w(\nu)$ in next sections.

Lemma 3. *Let $f \in \mathcal{M}(\mathcal{R}^{loc})$ and $\alpha \in \Delta$. Then,*

a) $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$ if and only if $f_\alpha \in L^1_w(\nu_\alpha)$.

b) $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(\nu)$ if and only if $f_\alpha \in L^1(\nu_\alpha)$. In this case

$$\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu = \int f_\alpha\, d\nu_\alpha.$$

Proof. Let $x^* \in E^*$ and $x^*_\alpha \in E^*_\alpha$ be the restriction of x^* to E_α. For each function $\varphi = \sum_{j=1}^n a_j \chi_{A_j} \in \mathcal{S}(\mathcal{R}^{loc})$ with $A_j = \cup_{\beta\in\Delta}\{\beta\} \times A^j_\beta$, we have that $\varphi\chi_{\{\alpha\}\times\Omega_\alpha} = \sum_{j=1}^n a_j \chi_{\{\alpha\}\times A^j_\alpha} \in \mathcal{S}(\mathcal{R})$ and $\varphi_\alpha = \sum_{j=1}^n a_j \chi_{A^j_\alpha} \in \mathcal{S}(\Sigma_\alpha)$, then

$$\begin{aligned}\int \varphi\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu &= \sum_{j=1}^n a_j x^*\nu(\{\alpha\} \times A^j_\alpha) = \sum_{j=1}^n a_j x^*\nu_\alpha(A^j_\alpha) \\ &= \sum_{j=1}^n a_j x^*_\alpha\nu_\alpha(A^j_\alpha) = \int \varphi_\alpha\, dx^*_\alpha\nu_\alpha.\end{aligned}$$

It is routine to check that $|x^*\nu|(\{\alpha\} \times A_\alpha) = |x^*_\alpha\nu_\alpha|(A_\alpha)$ for every $A_\alpha \in \Sigma_\alpha$. Then, in a similar way as for $x^*\nu$, we have that $\int \varphi\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| = \int \varphi_\alpha\, d|x^*_\alpha\nu_\alpha|$.

Let $(\varphi_n)_{n\geq1} \subset \mathcal{S}(\mathcal{R}^{loc})$ be a sequence such that $0 \leq \varphi_n \uparrow |f|$ pointwise. Then, $0 \leq \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha} \uparrow |f|\chi_{\{\alpha\}\times\Omega_\alpha}$ and $0 \leq (\varphi_n)_\alpha \uparrow |f_\alpha|$ pointwise. Using the monotone convergence theorem, we have that

$$\begin{aligned}\int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| &= \lim_{n\to\infty} \int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| \qquad (1)\\ &= \lim_{n\to\infty} \int (\varphi_n)_\alpha\, d|x^*_\alpha\nu_\alpha| = \int |f_\alpha|\, d|x^*_\alpha\nu_\alpha|.\end{aligned}$$

Then, $f_\alpha \in L^1_w(\nu_\alpha)$ implies $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$.

Let now $y^* \in E^*_\alpha$ and define $\tilde{y}^* \colon E \to \mathbb{R}$ as $\tilde{y}^*(e) = y^*(e_\alpha)$ for $e = \sum_{\beta\in\Delta} e_\beta$. Then, $\tilde{y}^* \in E^*$ and the restriction of $\tilde{y}^*$ to E_α coincides with y^*. So, by (1),

$$\int |f_\alpha|\, d|y^*\nu_\alpha| = \int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|\tilde{y}^*\nu|.$$

Hence, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$ implies $f_\alpha \in L^1_w(\nu_\alpha)$. Therefore, a) holds.

In the case when $\int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| < \infty$, that is, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(x^*\nu)$, there exists a sequence $(\varphi_n)_{n\geq1} \subset \mathcal{S}(\mathcal{R})$ such that $\varphi_n \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(x^*\nu)$ and so $\varphi_n\chi_{\{\alpha\}\times\Omega_\alpha} \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(x^*\nu)$. Also, by (1), which holds for every function in $\mathcal{M}(\mathcal{R}^{loc})$, we have that $\int |f_\alpha - (\varphi_n)_\alpha|\, d|x^*_\alpha\nu_\alpha| = \int |f - \varphi_n|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu|$, and so $(\varphi_n)_\alpha \to f_\alpha$ in $L^1(x^*_\alpha\nu_\alpha)$. Hence,

$$\begin{aligned}\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu &= \lim_{n\to\infty} \int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu \qquad (2)\\ &= \lim_{n\to\infty} \int (\varphi_n)_\alpha\, dx^*_\alpha\nu_\alpha = \int f_\alpha\, dx^*_\alpha\nu_\alpha.\end{aligned}$$

Suppose that $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(\nu)$. In particular, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$ and so, by a), $f_\alpha \in L^1_w(\nu_\alpha)$. On other hand, taking a sequence $(\varphi_n)_{n\geq1} \subset \mathcal{S}(\mathcal{R})$ such that $\varphi_n \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(\nu)$ and so $\varphi_n\chi_{\{\alpha\}\times\Omega_\alpha} \to f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1(\nu)$, we have that $\int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu$ converges to $\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu$ in E. Since $\int \varphi_n\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu = \int (\varphi_n)_\alpha\, d\nu_\alpha \in E_\alpha$ and E_α is closed in E, we have that $\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu \in E_\alpha$. Given $y^* \in E^*_\alpha$ and $\tilde{y}^* \in E^*$ defined as above, it follows

$$y^*\Big(\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu\Big) = \tilde{y}^*\Big(\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\nu\Big) = \int f\chi_{\{\alpha\}\times\Omega_\alpha}\, d\tilde{y}^*\nu = \int f_\alpha\, dy^*\nu_\alpha,$$

where we have used (2) in the last equality. Hence, $f_\alpha \in L^1(\nu_\alpha)$ and $\int f_\alpha \, d\nu_\alpha = \int f\chi_{\{\alpha\}\times\Omega_\alpha} \, d\nu$.

Suppose now that $f_\alpha \in L^1(\nu_\alpha)$. In particular, $f_\alpha \in L^1_w(\nu_\alpha)$ and so, by a), $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$. Since $\int f_\alpha \, d\nu_\alpha \in E_\alpha \subset E$, for every $x^* \in E^*$ we have that

$$x^*\Big(\int f_\alpha \, d\nu_\alpha\Big) = x^*_\alpha\Big(\int f_\alpha \, d\nu_\alpha\Big) = \int f_\alpha \, dx^*_\alpha\nu_\alpha = \int f\chi_{\{\alpha\}\times\Omega_\alpha} \, dx^*\nu,$$

where $x^*_\alpha \in E^*_\alpha$ is the restriction of x^* to E_α. Then, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(\nu)$. Therefore, b) holds. ∎

4 Description of an order continuous Banach lattice as an $L^1(\nu)$

Let E be an order continuous Banach lattice and ν the associated vector measure constructed in Section 3. Let us give a description of the space $L^1(\nu)$ which will be helpful to prove that E is order isometric to $L^1(\nu)$.

Proposition 4. *The space $L^1(\nu)$ can be described as the space of all functions $f \in \mathcal{M}(\mathcal{R}^{loc})$ such that $f_\alpha \in L^1(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta} \int |f_\alpha| \, d\nu_\alpha$ is unconditionally convergent in E, where f_α is defined as in Remark 2. Moreover, if $f \in L^1(\nu)$ we have that*

$$\int f \, d\nu = \sum_{\alpha\in\Delta} \int f_\alpha \, d\nu_\alpha.$$

Proof. Let $f \in L^1(\nu)$. Then, for every $\alpha \in \Delta$, we have that $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1(\nu)$ and so, by Lemma 3.b), $f_\alpha \in L^1(\nu_\alpha)$. Let $(\varphi_n)_{n\geq1} \subset \mathcal{S}(\mathcal{R})$ be a sequence such that $\varphi_n \to f$ in $L^1(\nu)$ and ν-a.e. Since each φ_n is supported in $\mathcal{R}$, we can write $\operatorname{Supp} \varphi_n = \cup_{\alpha\in\Delta}\{\alpha\} \times A^n_\alpha$ where A^n_α is ν_α-null for all $\alpha \in \Delta\backslash I_n$ with $I_n \subset \Delta$ finite. Then,

$$\operatorname{Supp} f \subset \bigcup_{n\geq1} \operatorname{Supp} \varphi_n = \bigcup_{n\geq1}\bigcup_{\alpha\in\Delta} \{\alpha\} \times A^n_\alpha = \bigcup_{\alpha\in\Delta} \{\alpha\} \times \Big(\bigcup_{n\geq1} A^n_\alpha\Big).$$

Note that $\cup_{n\geq1}A^n_\alpha$ is ν_α-null for every $\alpha \notin I = \cup_n I_n$. So, $\cup_{\alpha\in\Delta\backslash I}\{\alpha\} \times \big(\cup_{n\geq1} A^n_\alpha\big)$ is ν-null and thus

$$f = f\chi_{\cup_{\alpha\in I}\{\alpha\}\times(\cup_{n\geq1}A^n_\alpha)} \quad \nu\text{-a.e.}$$

For every $\alpha \in \Delta\backslash I$, from Lemma 3.b) and since $f\chi_{\{\alpha\}\times\Omega_\alpha} = 0$ ν-a.e., we have that

$$\int |f_\alpha| \, d\nu_\alpha = \int |f|\chi_{\{\alpha\}\times\Omega_\alpha} \, d\nu = 0.$$

Write $I = \{\alpha_j\}_{j\geq1}$ and $g_n = \sum_{j=1}^n |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}$. Note that $0 \leq g_n \uparrow |f| \in L^1(\nu)$. Then, since $L^1(\nu)$ is order continuous, $g_n \to |f|$ in $L^1(\nu)$ and so

$$\sum_{j=1}^n \int |f_{\alpha_j}| \, d\nu_{\alpha_j} = \sum_{j=1}^n \int |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}} \, d\nu = \int g_n \, d\nu \to \int |f| \, d\nu \;\text{ in } E.$$

Therefore, $\sum_{\alpha\in\Delta} \int |f_\alpha| \, d\nu_\alpha$ is unconditionally convergent in E.

Conversely, let $f \in \mathcal{M}(\mathcal{R}^{loc})$ be a function such that $f_\alpha \in L^1(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta} \int |f_\alpha|\, d\nu_\alpha$ is unconditionally convergent in E. From this and since ν_α is positive, we have that there exists a countable set $N \subset \Delta$ such that

$$\|f_\alpha\|_{\nu_\alpha} = \Big\| \int |f_\alpha|\, d\nu_\alpha \Big\|_E = 0 \ \text{ for all } \alpha \in \Delta \backslash N.$$

That is, $f_\alpha = 0$ ν_α-a.e. for all $\alpha \in \Delta\backslash N$. So, for each $\alpha \in \Delta\backslash N$, there exists a ν_α-null set Z_α such that

$$f_\alpha(\omega) = 0 \ \text{ for all } \omega \in \Omega_\alpha \backslash Z_\alpha.$$

Note that the set $\cup_{\alpha\in\Delta\backslash N}\{\alpha\} \times Z_\alpha \in \mathcal{R}^{loc}$ is ν-null, then

$$f = \sum_{\alpha\in N} f\chi_{\{\alpha\}\times\Omega_\alpha} \ \ \nu\text{-a.e.}$$

Write $N = \{\alpha_j\}_{j\geq 1}$ and take $f_n = \sum_{j=1}^n f\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}$ which belongs to $L^1(\nu)$ from Lemma 3.b). Then, for $m < n$,

$$\begin{aligned} \|f_n - f_m\|_\nu &= \Big\| \int |f_n - f_m|\, d\nu \Big\|_E \\ &= \Big\| \sum_{j=m+1}^{n} \int |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}\, d\nu \Big\|_E \\ &= \Big\| \sum_{j=m+1}^{n} \int |f_{\alpha_j}|\, d\nu_{\alpha_j} \Big\|_E \to 0 \end{aligned}$$

as $m, n \to \infty$. Since $f_n \to f$ ν-a.e., it follows that $f \in L^1(\nu)$. Moreover, $f_n \to f$ in $L^1(\nu)$, so

$$\int f\, d\nu = \lim_n \int f_n\, d\nu = \sum_{\alpha\in\Delta} \int f_\alpha\, d\nu_\alpha. \qquad \blacksquare$$

We go on now to show that $L^1(\nu)$ and E are order isometric.

Theorem 5. *The space $L^1(\nu)$ is order isometric to E. Even more, the integration operator $I_\nu\colon L^1(\nu) \to E$ is an order isometry.*

Proof. The integration operator $I_\nu\colon L^1(\nu) \to E$ is a positive (as ν is positive) continuous linear operator satisfying that $\|I_\nu(f)\|_E \leq \|f\|_\nu = \|I_\nu(|f|)\|_E$ for every $f \in L^1(\nu)$. Let us see that I_ν is an isometry. Fix $f \in L^1(\nu)$. From Proposition 4, it follows

$$\begin{aligned} \|f\|_\nu &= \Big\| \int |f|\, d\nu \Big\|_E = \sup_{x^*\in B_{E^*}} \Big| x^*\Big(\int |f|\, d\nu \Big) \Big| \\ &= \sup_{x^*\in B_{E^*}} \Big| x^*\Big(\sum_{\alpha\in\Delta} \int |f_\alpha|\, d\nu_\alpha \Big) \Big| \\ &= \sup_{x^*\in B_{E^*}} \Big| \sum_{\alpha\in\Delta} x^*\Big(\int |f_\alpha|\, d\nu_\alpha \Big) \Big|. \end{aligned} \tag{3}$$

Let $x^* \in E^*$. Note that $x^* \circ I_{\nu_\alpha} \in L^1(\nu_\alpha)^*$ for all $\alpha \in \Delta$ (recall $I_{\nu_\alpha} \colon L^1(\nu_\alpha) \to E_\alpha$ is an order isometry). Taking $\xi_\alpha = \chi_{\{f_\alpha \geq 0\}} - \chi_{\{f_\alpha < 0\}}$, we define $\tilde{x}^* \colon E \to \mathbb{R}$ by

$$\tilde{x}^*(e) = \sum_{\alpha \in \Delta} x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I_{\nu_\alpha}^{-1}(e_\alpha)\big)$$

for all $e \in E$ with $e = \sum_{\alpha \in \Delta} e_\alpha$ such that $e_\alpha \in E_\alpha$ and the sum is unconditionally convergent. Let us see that $\tilde{x}^*$ is well defined and belongs to E^*. Take an element $e = \sum_{\alpha \in \Delta} e_\alpha \in E$ as above. Then, $|e| = \sum_{\alpha \in \Delta} |e_\alpha|$ where the sum is also unconditionally convergent. Let $N \subset \Delta$ be a countable set such that $e_\alpha = 0$ for all $\alpha \in \Delta \backslash N$. Then, $\xi_\alpha I_{\nu_\alpha}^{-1}(e_\alpha) = 0$ and so $x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I_{\nu_\alpha}^{-1}(e_\alpha)\big) = 0$ for all $\alpha \in \Delta \backslash N$. Writing $N = \{\alpha_j\}_{j \geq 1}$ we have that

$$\begin{aligned} \Big| \sum_{j=n}^{m} x^* \circ I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})\big) \Big| &= \Big| x^*\Big(\sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})\big) \Big) \Big| \\ &\leq \|x^*\| \cdot \Big\| \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})\big) \Big\|_E. \end{aligned}$$

Note that, since I_{ν_α} is an order isometry, $|I_{\nu_\alpha}(h)| = I_{\nu_\alpha}(|h|)$ for all $h \in L^1(\nu_\alpha)$ and $I_{\nu_\alpha}(\tilde{h}) \leq I_{\nu_\alpha}(h)$ whenever $\tilde{h} \leq h \in L^1(\nu_\alpha)$ (the same holds for $I_{\nu_\alpha}^{-1}$) . Then,

$$\begin{aligned} \Big| \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})\big) \Big| &\leq \sum_{j=n}^{m} \big| I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})\big) \big| \\ &= \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(|\xi_{\alpha_j} I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})|\big) \\ &\leq \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(|I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})|\big) \\ &= \sum_{j=n}^{m} I_{\nu_{\alpha_j}}\big(I_{\nu_{\alpha_j}}^{-1}(|e_{\alpha_j}|)\big) = \sum_{j=n}^{m} |e_{\alpha_j}|. \end{aligned}$$

Therefore,

$$\Big| \sum_{j=n}^{m} x^* \circ I_{\nu_{\alpha_j}}\big(\xi_{\alpha_j} I_{\nu_{\alpha_j}}^{-1}(e_{\alpha_j})\big) \Big| \leq \|x^*\| \cdot \Big\| \sum_{j=n}^{m} |e_{\alpha_j}| \Big\|_E \to 0$$

as $n, m \to \infty$. So, $\tilde{x}^*$ is well defined, obviously linear and continuous as $|\tilde{x}^*(e)| \leq \|x^*\| \cdot \|e\|_E$ for all $e \in E$, that is, $\tilde{x}^* \in E^*$ and $\|\tilde{x}^*\| \leq \|x^*\|$. Moreover,

$$x^*\Big(\int |f_\alpha| \, d\nu_\alpha \Big) = x^* \circ I_{\nu_\alpha}(|f_\alpha|) = x^* \circ I_{\nu_\alpha}(\xi_\alpha f_\alpha) = x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I_{\nu_\alpha}^{-1}\big(I_{\nu_\alpha}(f_\alpha)\big)\big)$$

for all $\alpha \in \Delta$. From Proposition 4, we have that $I_\nu(f) = \sum_{\alpha \in \Delta} I_{\nu_\alpha}(f_\alpha)$ and so,

$$\tilde{x}^*\big(I_\nu(f)\big) = \sum_{\alpha \in \Delta} x^* \circ I_{\nu_\alpha}\big(\xi_\alpha I_{\nu_\alpha}^{-1}\big(I_{\nu_\alpha}(f_\alpha)\big)\big) = \sum_{\alpha \in \Delta} x^*\Big(\int |f_\alpha| \, d\nu_\alpha \Big).$$

Hence, we have proved that for every $x^* \in B_{E^*}$ there exists $\tilde{x}^* \in B_{E^*}$ such that $\sum_{\alpha\in\Delta} x^*\Big(\int |f_\alpha|\, d\nu_\alpha\Big) = \tilde{x}^*\big(I_\nu(f)\big)$. Then, from (3), $\|f\|_\nu \le \|I_\nu(f)\|_E$. Therefore, I_ν is a linear isometry.

Let us see now that I_ν is onto. Let $e = \sum_{\alpha\in\Delta} e_\alpha \in E$. Since each $e_\alpha \in E_\alpha$, there exists $h_\alpha \in L^1(\nu_\alpha)$ such that $e_\alpha = I_{\nu_\alpha}(h_\alpha)$. Define $f\colon \Omega \to \mathbb{R}$ by $f(\alpha,\omega) = h_\alpha(\omega)$ for all $(\alpha,\omega)\in\Omega$. Then, $f \in \mathcal{M}(\mathcal{R}^{loc})$ (as $f^{-1}(B) = \cup_{\alpha\in\Delta}\{\alpha\}\times h_\alpha^{-1}(B)$ for every Borel set B on $\mathbb{R}$), $f_\alpha = h_\alpha \in L^1(\nu_\alpha)$ for all $\alpha\in\Delta$ and

$$\sum_{\alpha\in\Delta} I_{\nu_\alpha}(f_\alpha) = \sum_{\alpha\in\Delta} I_{\nu_\alpha}(h_\alpha) = \sum_{\alpha\in\Delta} e_\alpha$$

is unconditionally convergent in E. So, by Proposition 4, we have that $f \in L^1(\nu)$ and $I_\nu(f) = \sum_{\alpha\in\Delta} I_{\nu_\alpha}(f_\alpha) = e$. Note that if $e \ge 0$, that is, $e_\alpha \ge 0$ for all $\alpha\in\Delta$, then $h_\alpha \ge 0$ for all $\alpha\in\Delta$ and so $f \ge 0$. Hence, I_ν^{-1} is positive.

So, I_ν is positive, linear, one to one and onto with I_ν^{-1} positive. Then, by [6, p. 2], I_ν is an order isomorphism. ∎

Let us show an example of the representation as an $L^1(\nu)$ of an order continuous Banach lattice without weak unit. This example has been already studied in [1, p. 23] and [4, Example 2.2].

Example 6. Consider an uncountable set Γ and the δ-ring $\mathcal{R} = \{A \subset \Gamma :\ A \text{ is finite}\}$. The space $\ell^1(\Gamma)$ is order continuous, so, by Theorem 5, $\ell^1(\Gamma)$ is order isometric to $L^1(\nu)$ for some vector measure ν defined on a δ-ring, via the integration operator. The vector measure $\nu\colon \mathcal{R} \to \ell^1(\Gamma)$ can be defined as $\nu(A) = \sum_{\gamma\in A} e_\gamma$, where e_γ is the characteristic function of the point γ. In this case, the integration operator is the identity map. Note that $\ell^1(\Gamma)$ cannot be represented as $L^1(\nu)$ with ν defined on a σ-algebra, as it has no weak unit.

5 $L^1_w(\nu)$ for ν associated to an order continuous Banach lattice

Until now, we have considered an order continuous Banach lattice E. If we forget about the order continuity property, descriptions of E by means of a vector measure could exist. For instance, if E is a Banach lattice satisfying the σ-Fatou property with a weak unit belonging to the σ-order continuous part E_a of E, then there exists a vector measure ν defined on a σ-algebra such that E is order isometric to $L^1_w(\nu)$, see [3, Theorem 2.5]. In this reference, it is noted that in this case E_a is also order continuous. Indeed, E_a is an ideal of E which is σ-complete as it is σ-Fatou ([10, Theorem 113.1]). Then, E_a is also σ-complete and, as it is σ-order continuous, it follows that it is order continuous ([6, Proposition 1.a.8]). The proof of the representation of E as an $L^1_w(\nu)$ consists in taking a vector measure ν such that $L^1(\nu)$ is order isometric to E_a via the integration operator I_ν, and extending I_ν to $L^1_w(\nu)$. The result is that this extension is an order isometry from $L^1_w(\nu)$ onto E. Our question now is if a similar result is possible if we forget about the weak unit and consider vector measures defined on a δ-ring, as it happens in the case when E is order continuous. For solving this question, we will need a description of $L^1_w(\nu)$ along the lines of Proposition 4.

Let E be again an order continuous Banach lattice and ν the associated vector measure constructed in Section 3.

Proposition 7. *The space $L^1_w(\nu)$ can be described as the space of all functions $f \in \mathcal{M}(\mathcal{R}^{loc})$ such that $f_\alpha \in L^1_w(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta} \int |f_\alpha|\, d|x^*\nu_\alpha|$ converges for all $x^* \in E^*$, where f_α is defined as in Remark 2. Moreover, if $f \in L^1_w(\nu)$ and $x^* \in E^*$, then*

$$\int f\, dx^*\nu = \sum_{\alpha\in\Delta} \int f_\alpha\, dx^*\nu_\alpha \quad \text{and} \quad \int f\, d|x^*\nu| = \sum_{\alpha\in\Delta} \int f_\alpha\, d|x^*\nu_\alpha|.$$

Proof. Let $f \in L^1_w(\nu)$. Then, $f\chi_{\{\alpha\}\times\Omega_\alpha} \in L^1_w(\nu)$ and so, by Lemma 3.a), $f_\alpha \in L^1_w(\nu_\alpha)$ for every $\alpha \in \Delta$. Take $x^* \in E^*$. For every $I \subset \Delta$ finite, by (1), we have that

$$\begin{aligned} \sum_{\alpha\in I} \int |f_\alpha|\, d|x^*\nu_\alpha| &= \sum_{\alpha\in I} \int |f|\chi_{\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| \\ &= \int |f|\chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}\, d|x^*\nu| \le \|x^*\| \cdot \|f\|_\nu. \end{aligned}$$

So, $\sum_{\alpha\in\Delta} \int |f_\alpha|\, d|x^*\nu_\alpha|$ is convergent.

Conversely, let $f \in \mathcal{M}(\mathcal{R}^{loc})$ be such that $f_\alpha \in L^1_w(\nu_\alpha)$ for all $\alpha \in \Delta$ and $\sum_{\alpha\in\Delta} \int |f_\alpha|\, d|x^*\nu_\alpha|$ converges for all $x^* \in E^*$. Fix $x^* \in E^*$. There exists a countable set $N \subset \Delta$ such that

$$\int |f_\alpha|\, d|x^*\nu_\alpha| = 0 \;\text{ for all } \alpha \in \Delta\backslash N.$$

Then, for every $\alpha \in \Delta\backslash N$, there exists a $|x^*\nu_\alpha|$-null set Z_α such that

$$f_\alpha(\omega) = 0 \;\text{ for all } \omega \in \Omega_\alpha\backslash Z_\alpha.$$

Noting that $\cup_{\alpha\in\Delta\backslash N}\{\alpha\} \times Z_\alpha$ is $|x^*\nu|$-null, it follows

$$f = \sum_{\alpha\in N} f\chi_{\{\alpha\}\times\Omega_\alpha} \;\; |x^*\nu|\text{-a.e.}$$

Write $N = \{\alpha_j\}_{j\ge1}$ and take $f_n = \sum_{j=1}^n f\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}$ which, by Lemma 3.a), is in $L^1_w(\nu)$. Then, for $m < n$, by (1),

$$\int |f_n - f_m|\, d|x^*\nu| = \sum_{j=m+1}^{n} \int |f|\chi_{\{\alpha_j\}\times\Omega_{\alpha_j}}\, d|x^*\nu| = \sum_{j=m+1}^{n} \int |f_{\alpha_j}|\, d|x^*\nu_{\alpha_j}| \to 0$$

as $m, n \to \infty$. Note that $f_n \to f$ $|x^*\nu|$-a.e. So, $f \in L^1(|x^*\nu|)$ and $f_n \to f$ in $L^1(|x^*\nu|)$. Therefore, $f \in L^1_w(\nu)$ and, by (1) and (2),

$$\int f\, dx^*\nu = \sum_{\alpha\in\Delta} \int f_\alpha\, dx^*\nu_\alpha \quad \text{and} \quad \int f\, d|x^*\nu| = \sum_{\alpha\in\Delta} \int f_\alpha\, d|x^*\nu_\alpha| \quad \text{for all } x^* \in E^*.$$

■

For the proof of our main result we will need the following fact which holds for the vector measure ν associated to the order continuous Banach lattice E.

Proposition 8. *The space $L^1_w(\nu)$ has the Fatou property.*

Proof. For every $I \subset \Delta$ finite, consider $\Omega_I = \cup_{\alpha\in I}\{\alpha\} \times \Omega_\alpha$ and the σ-algebra $\Sigma_I = \big\{ \cup_{\alpha\in I} \{\alpha\} \times A_\alpha : A_\alpha \in \Sigma_\alpha \text{ for all } \alpha \in I\big\}$ of parts of Ω_I. Note that $\Omega_I \subset \Omega$ and $\Sigma_I \subset \mathcal{R}$. Denote by $\nu_I \colon \Sigma_I \to E$ the restriction of ν to Σ_I. Since ν_I is a vector measure defined on a σ-algebra, $L^1_w(\nu_I)$ has the Fatou property, see Proposition 1.

For each $f \in \mathcal{M}(\mathcal{R}^{loc})$, denote by f^I the function resulting from the restriction of f to Ω_I. Of course, $f^I \in \mathcal{M}(\Sigma_I)$. For every $x^* \in E^*$, it follows

$$\int |f^I|\, d|x^*\nu_I| = \int |f|\chi_{\Omega_I}\, d|x^*\nu|. \tag{4}$$

Indeed, for every $A \in \Sigma_I$ we have that $|x^*\nu_I|(A) = |x^*\nu|(A)$ and so it is routine to check that (4) holds for $f \in \mathcal{S}(\mathcal{R}^{loc})$. For a general f the result follows by applying the monotone convergence theorem. Then, for every $f \in L^1_w(\nu)$ we have that $f\chi_{\Omega_I} \in L^1_w(\nu)$ and so $f^I \in L^1_w(\nu_I)$ with $\|f^I\|_{\nu_I} = \|f\chi_{\Omega_I}\|_\nu$. Note that if Z is a ν-null set then $Z \cap \Omega_I$ is ν_I-null.

Let $(f_\tau)_\tau \subset L^1_w(\nu)$ be an upwards directed system $0 \le f_\tau \uparrow \nu$-a.e. such that $\sup_\tau \|f_\tau\|_\nu < \infty$. Then, $(f^I_\tau)_\tau \subset L^1_w(\nu_I)$ is an upwards directed system $0 \le f^I_\tau \uparrow$ ν_I-a.e. and $\sup_\tau \|f^I_\tau\|_{\nu_I} = \sup_\tau \|f_\tau\chi_{\Omega_I}\|_\nu \le \sup_\tau \|f_\tau\|_\nu < \infty$. Since $L^1_w(\nu_I)$ has the Fatou property, there exists $f^I = \sup_\tau f^I_\tau$ in $L^1_w(\nu_I)$ and $\|f^I\|_{\nu_I} = \sup_\tau \|f^I_\tau\|_{\nu_I}$.

Now, from each $I = \{\alpha\}$ with $\alpha \in \Delta$, we construct the function $f \colon \Omega \to \mathbb{R}$ given by $f(\alpha, \omega) = f^{\{\alpha\}}(\alpha, \omega)$ for all $(\alpha, \omega) \in \Omega$. Since $f^{-1}(B) = \cup_{\alpha\in\Delta}(f^{\{\alpha\}})^{-1}(B)$ for all Borel set B on $\mathbb{R}$, we have that $f \in \mathcal{M}(\mathcal{R}^{loc})$. Noting that $\cup_{\alpha\in\Delta}\{\alpha\} \times Z_\alpha$ is ν-null whenever $\{\alpha\} \times Z_\alpha$ is $\nu_{\{\alpha\}}$-null for all $\alpha \in \Delta$, we have that $f = \sup_\tau f_\tau$. Let us see that $f \in L^1_w(\nu)$ by using the characterization of Proposition 7. For every $\alpha \in \Delta$ and $y^* \in E^*_\alpha$, taking $\tilde{y}^* \in E^*$ defined as $\tilde{y}^*(e) = y^*(e_\alpha)$ for $e = \sum_{\alpha\in\Delta} e_\alpha$, by (1) and (4), we have that

$$\int |f_\alpha|\, d|y^*\nu_\alpha| = \int |f|\chi_{\Omega_{\{\alpha\}}}\, d|\tilde{y}^*\nu| = \int |f^{\{\alpha\}}|\, d|\tilde{y}^*\nu_{\{\alpha\}}| < \infty.$$

So, $f_\alpha \in L^1_w(\nu_\alpha)$. Moreover, given $x^* \in E^*$, for every $I \subset \Delta$ finite,

$$\begin{aligned}\sum_{\alpha\in I}\int |f_\alpha|\, d|x^*\nu_\alpha| &= \sum_{\alpha\in I}\int |f|\chi_{\Omega_{\{\alpha\}}}\, d|x^*\nu| = \int |f|\chi_{\Omega_I}\, d|x^*\nu| \\ &= \int |f^I|\, d|x^*\nu_I| \le \|f^I\|_{\nu_I} \le \sup_\tau \|f_\tau\|_\nu < \infty.\end{aligned}$$

Then $\sum_{\alpha\in I} \int |f_\alpha|\, d|x^*\nu_\alpha|$ converges and so $f \in L^1_w(\nu)$. Moreover,

$$\int |f|\, d|x^*\nu| = \sum_{\alpha\in\Delta}\int |f_\alpha|\, d|x^*\nu_\alpha| \le \sup_\tau \|f_\tau\|_\nu.$$

Hence, $\|f\|_\nu \le \sup_\tau \|f_\tau\|_\nu$. The equality follows, as $\|f_\tau\| \le \|f\|_\nu$ for all τ. ■

Note that for the proof of Proposition 8 the fact that Ω is an uncountable disjoint union of sets in $\mathcal{R}$ and also the way as the δ-ring $\mathcal{R}$ is defined are crucial. So, $L^1_w(\nu)$ has the Fatou property for the particular vector measure ν constructed in Section 3. But, has $L^1_w(\nu)$ the Fatou property for every vector measure ν defined on a δ-ring? In the case when ν is σ-finite, the answer is yes (Proposition 1), however for the general case this is an open question.

6 Description of a Banach lattice as an $L^1_w(\nu)$

Let E be now a general Banach lattice. We always can consider the order continuous part E_{an} of E. Then, we can take the vector measure ν associated to E_{an} as in Section 3, and so, by Theorem 5, $I_\nu\colon L^1(\nu) \to E_{an}$ is an order isometry. The question is if it is possible to extend I_ν to the space $L^1_w(\nu)$ in a way that the extension is an order isometry between $L^1_w(\nu)$ and E. Note that if this extension is possible, by Proposition 8, E must have the Fatou property. So, we will require E to have this property. In this case, E has the σ-Fatou property and then $E_{an} = E_a$, as we said at the beginning of Section 5.

In order to prove the desired result, we will need the next Lemma. Recall that the order continuous part E_a of E can be decomposed into an unconditionally direct sum of a family of mutually disjoints ideals $\{E_a^\alpha\}_{\alpha\in\Delta}$, each E_a^α having a weak unit u_α (see Section 3).

Lemma 9. *Suppose that E_a is order dense in E. Then, for every $0 \le e \in E$ it follows*

$$e_{(n,I)} = \sum_{\alpha\in I} e \wedge (nu_\alpha) \uparrow e \tag{5}$$

where the indices (n, I) are such that $n \in \mathbb{N}$ and $I \subset \Delta$ is finite. Moreover, in the case when $0 \le e \in E_a$, there exists a countable set $\{\alpha_j\} \subset \Delta$ such that $e \wedge (nu_\alpha) = 0$ for all n and $\alpha \in \Delta \backslash \{\alpha_j\}$, and

$$e = \lim_{n,m} \sum_{j=1}^{m} e \wedge (nu_{\alpha_j}) \text{ in norm.} \tag{6}$$

Proof. Let $0 \le e \in E$ and $e_{(n,I)}$ as in (5). Then $0 \le e_{(n,I)} \uparrow$ and $e_{(n,I)} \le e$ for all (n, I). Note that $\{nu_\alpha : \alpha \in \Delta\}$ is a set of pairwise disjoint elements, so

$$e_{(n,I)} = \sum_{\alpha\in I} e \wedge (nu_\alpha) = e \wedge \Big(\sum_{\alpha\in I} nu_\alpha \Big) \tag{7}$$

(see [7, Theorem 12.5]). Let $z \in E$ be such that $e_{(n,I)} \le z$ for all (n, I). Let us see that $e \le z$. Suppose first that $e \in E_a$ and write $e = \sum_{j\ge1} e_{\alpha_j}$ where $e_{\alpha_j} \in E_a^{\alpha_j}$ and the series converges unconditionally. Note that, since $e \ge 0$ and $\{e_{\alpha_j}\}$ is a set of pairwise disjoint elements, $e_{\alpha_j} \ge 0$ for every j. Then $\sum_{j=1}^m e_{\alpha_j} \uparrow e$ in the lattice order (see [10, Theorem 100.4.(i)]). For a fix j we have that $e_{\alpha_j} \wedge (nu_{\alpha_j}) \uparrow e_{\alpha_j}$ (see [6, pp. 7-8]). Then, for each m it follows that $\sum_{j=1}^m e_{\alpha_j} \wedge (nu_{\alpha_j}) \uparrow \sum_{j=1}^m e_{\alpha_j}$ (see [7, Theorem 15.2]). Since $e_{\alpha_j} \le e$ for all j, taking $I_m = \{\alpha_1, ..., \alpha_m\}$ we have that $\sum_{j=1}^m e_{\alpha_j} \wedge (nu_{\alpha_j}) \le e_{(n,I_m)} \le z$ for all n and so $\sum_{j=1}^m e_{\alpha_j} \le z$. Hence $e \le z$. Note that actually we have proved that $\sum_{j=1}^m e \wedge (nu_{\alpha_j}) \uparrow e$ where the indices are (n, m). Then, by the order continuity of E_{an}, it follows that $e = \lim_{n,m} \sum_{j=1}^m e \wedge (nu_{\alpha_j})$ in norm. Hence, (5) and (6) hold if $e \in E_a$.

In the general case, since E_a is order dense in E, there exists $(e_\tau) \subset E_a$ such that $0 \le e_\tau \uparrow e$. We now know that $\sum_{\alpha\in I} e_\tau \wedge (nu_\alpha) \uparrow e_\tau$ for every τ. Then, since $\sum_{\alpha\in I} e_\tau \wedge (nu_\alpha) \le e_{(n,I)} \le z$, we have that $e_\tau \le z$ for every τ, and so $e \le z$. ∎

Now we can prove our main result by using Lemma 9.

Theorem 10. *If E has the Fatou property and E_a is order dense in E, then E is order isometric to $L^1_w(\nu)$.*

Proof. Let us extend I_ν to $L^1_w(\nu)$. First, consider $0 \leq f \in L^1_w(\nu)$ and choose $(\varphi_n)_{n\geq 1} \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \varphi_n \uparrow f$. For each $n \geq 1$ and $I \subset \Delta$ finite, we define $\xi_{(n,I)} = \varphi_n \chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha} \in \mathcal{S}(\mathcal{R})$. Then, $(\xi_{(n,I)})_{(n,I)} \subset L^1(\nu)$ is an upwards directed system $0 \leq \xi_{(n,I)} \uparrow f$ in $L^1_w(\nu)$ and so, since I_ν is positive, $(I_\nu(\xi_{(n,I)}))_{(n,I)} \subset E_a \subset E$ is an upwards directed system $0 \leq I_\nu(\xi_{(n,I)}) \uparrow$ and $\sup_{(n,I)} \|I_\nu(\xi_{(n,I)})\|_E = \sup_{(n,I)} \|\xi_{(n,I)}\|_\nu \leq \|f\|_\nu < \infty$. Then, by the Fatou property of E, there exists $e = \sup_{(n,I)} I_\nu(\xi_{(n,I)})$ in E and $\|e\|_E = \sup_{(n,I)} \|I_\nu(\xi_{(n,I)}\|_E$. We define $T(f) = e$.

Using an argument similar to the one in [3, Theorem 2.5], we will see that T is well defined. Take another sequence $(\psi_n)_{n\geq 1} \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \psi_n \uparrow f$. Denote $\eta_{(n,I)} = \psi_n \chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}$ and $z = \sup_{(n,I)} I_\nu(\eta_{(n,I)})$. Let $0 \leq x^* \in E^*$ be fixed. Then, $x^*(e) \geq x^*\big(I_\nu(\xi_{(n,I)})\big) = \int \xi_{(n,I)}\, dx^*\nu$ for all $n \geq 1$ and $I \subset \Delta$ finite. It can be proved that also $0 \leq \xi_{(n,I)} \uparrow f$ in $L^1(x^*\nu)$, since $L^1(x^*\nu)$ has the Fatou property, we have that $\sup_{(n,I)} \int \xi_{(n,I)}\, dx^*\nu = \int f\, dx^*\nu$. Consequently, $x^*(e) \geq \int f\, dx^*\nu \geq x^*\big(I_\nu(\xi_{(n,I)})\big)$ for all $n \geq 1$ and $I \subset \Delta$ finite. In a similar way, $x^*(z) \geq \int f\, dx^*\nu \geq x^*\big(I_\nu(\eta_{(n,I)})\big)$ for all $n \geq 1$ and $I \subset \Delta$ finite. Then, it follows that $x^*(e) \geq x^*\big(I_\nu(\eta_{(n,I)})\big)$ and $x^*(z) \geq x^*\big(I_\nu(\xi_{(n,I)})\big)$ for all $n \geq 1$ and $I \subset \Delta$ finite. Since this holds for all $0 \leq x^* \in E^*$, we have that $e \geq I_\nu(\eta_{(n,I)})$ and $z \geq I_\nu(\xi_{(n,I)})$ for all $n \geq 1$ and $I \subset \Delta$ finite. Then, $e \geq z$ and $z \geq e$, and thus $e = z$. So, T is well defined. Moreover,

$$\|T(f)\|_E = \|e\|_E = \sup_{(n,I)} \|I_\nu(\xi_{(n,I)})\|_E = \sup_{(n,I)} \|\xi_{(n,I)}\|_\nu = \|f\|_\nu,$$

where in the last equality we have used that $L^1_w(\nu)$ has the Fatou property (see Proposition 8). Let us see now that $T(f \wedge g) = Tf \wedge Tg$ for every $0 \leq f, g \in L^1_w(\nu)$. Consider sequences $(\varphi_n)_{n\geq 1}, (\psi_n)_{n\geq 1} \subset \mathcal{S}(\mathcal{R}^{loc})$ satisfying that $0 \leq \varphi_n \uparrow f$ and $0 \leq \psi_n \uparrow g$, and denote $\xi_{(n,I)} = \varphi_n \chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}$ and $\eta_{(n,I)} = \psi_n \chi_{\cup_{\alpha\in I}\{\alpha\}\times\Omega_\alpha}$. Then, $Tf = \sup_{(n,I)} I_\nu(\xi_{(n,I)})$ and $Tg = \sup_{(n,I)} I_\nu(\eta_{(n,I)})$. Note that $(\varphi_n \wedge \psi_n)_{n\geq 1}$ which is contained in $\mathcal{S}(\mathcal{R}^{loc})$, satisfies that $0 \leq \varphi_n \wedge \psi_n \uparrow f \wedge g$ (see [7, Theorem 15.3]) and also $(\varphi_n \wedge \psi_n)\chi_{\{\alpha\}\times\Omega_\alpha} = (\xi_{(n,I)} \wedge \eta_{(n,I)})_{(n,I)}$. Then, since I_ν is an order isometry, we have that

$$T(f \wedge g) = \sup_{(n,I)} I_\nu(\xi_{(n,I)} \wedge \eta_{(n,I)}) = \sup_{(n,I)} I_\nu(\xi_{(n,I)}) \wedge I_\nu(\eta_{(n,I)}) = Tf \wedge Tg.$$

For a general $f \in L^1_w(\nu)$, we define $Tf = Tf^+ - Tf^-$ where f^+ and f^- are the positive and negative parts of f respectively. So, $T\colon L^1_w(\nu) \to E$ is a positive linear operator extending I_ν. For the linearity, see for instance [7, Theorem 15.8]. Moreover T is an isometry. Indeed, for $f \in L^1_w(\nu)$, since $f^+ \wedge f^- = 0$, we have that $Tf^+ \wedge Tf^- = T(f^+ \wedge f^-) = 0$. Then, it follows that $|Tf| = |Tf^+ - Tf^-| = Tf^+ + Tf^- = T|f|$, and so, $\|T(f)\|_E = \|T(|f|)\|_E = \|f\|_\nu$.

Let us prove that T is onto. Let $0 \le e \in E$. Since E_a is order dense in E, from Lemma 9 we have that $e_{(n,I)} = \sum_{\alpha\in I} e \wedge (nu_\alpha) \uparrow e$. Fix n and $\beta \in \Delta$. Since $e \wedge (nu_\beta) \in E_a^\beta$ as $0 \le e \wedge (nu_\beta) \le nu_\beta$, there exists $0 \le g_{n,\beta} \in L^1(\nu_\beta)$ such that $e \wedge (nu_\beta) = I_{\nu_\beta}(g_{n,\beta})$. Define $f_{n,\beta}\colon \Omega \to \mathbb{R}$ by $f_{n,\beta}(\alpha,\omega) = g_{n,\beta}(\omega)$ if $\alpha = \beta$ and $f_{n,\beta}(\alpha,\omega) = 0$ in other case. Then, from Proposition 4, we have that $f_{n,\beta} \in L^1(\nu)$ and $I_\nu(f_{n,\beta}) = I_{\nu_\beta}(g_{n,\beta}) = e \wedge (nu_\beta)$. Taking $\xi_{(n,I)} = \sum_{\alpha\in I} f_{n,\alpha} \in L^1(\nu)$, we have that $0 \le \xi_{(n,I)} \uparrow$ as $\xi_{(n,I)} = I_\nu^{-1}(e_{(n,I)})$ and $\sup_{(n,I)} \|\xi_{(n,I)}\|_\nu = \sup_{(n,I)} \|e_{(n,I)}\|_E \le \|e\|_E$. By the Fatou property of $L^1_w(\nu)$, there exists $f = \sup_{(n,I)} \xi_{(n,I)}$ in $L^1_w(\nu)$.

If we prove that $x^*(e) \ge \int f\, dx^*\nu$ for all $0 \le x^* \in X^*$, by the same argument used to see that T is well defined, we will have that $Tf = e$. Fix $\alpha \in \Delta$, since $0 \le \xi_{(n,I)} \uparrow f$ in $L^1_w(\nu)$, it follows that $0 \le \xi_{(n,I)}\chi_{\{\alpha\}\times\Omega_\alpha} \uparrow f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1_w(\nu)$. Since $\xi_{(n,I)}\chi_{\{\alpha\}\times\Omega_\alpha} = \sum_{\beta\in I} f_{n,\beta}\chi_{\{\alpha\}\times\Omega_\alpha} = f_{n,\alpha}\chi_{\{\alpha\}\times\Omega_\alpha}$, actually we deal with a sequence. Writing $h_n^\alpha = f_{n,\alpha}\chi_{\{\alpha\}\times\Omega_\alpha}$, we have that $0 \le h_n^\alpha \uparrow f\chi_{\{\alpha\}\times\Omega_\alpha}$ in $L^1_w(\nu)$ and so ν-a.e. Fix now $0 \le x^* \in X^*$. Since $h_n^\alpha \uparrow f\chi_{\{\alpha\}\times\Omega_\alpha}$ $x^*\nu$-a.e., applying the dominated convergence theorem (see [8, Theorem 2.22]), we have that $\int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu = \lim \int h_n^\alpha\, dx^*\nu$. Noting that $\int h_n^\alpha\, dx^*\nu = x^* I_\nu(f_{n,\alpha}\chi_{\{\alpha\}\times\Omega_\alpha}) \le x^* I_\nu(f_{n,\alpha}) = x^*(e \wedge (nu_\alpha))$, we obtain that

$$\begin{aligned}\sum_{\alpha\in I} \int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu &= \lim \sum_{\alpha\in I} \int h_n^\alpha\, dx^*\nu \le \lim \sum_{\alpha\in I} x^*\big(e \wedge (nu_\alpha)\big) \\ &= \lim x^*(e_{(n,I)}) \le x^*(e)\end{aligned}$$

for all finite $I \subset \Delta$. Therefore, by the description of $L^1_w(\nu)$ given in Proposition 7 and (2),

$$\int f\, dx^*\nu = \sum_{\alpha\in\Delta} \int f\chi_{\{\alpha\}\times\Omega_\alpha}\, dx^*\nu \le x^*(e).$$

For a general $e \in E$, consider e^+ and e^- the positive and negative parts of e. Let $g, h \in L^1_w(\nu)$ be such that $Tg = e^+$ and $Th = e^-$. Then, taking $f = g - h \in L^1_w(\nu)$ we have that $Tf = e$. Note that T^{-1} is positive. So, T is positive, linear, one to one and onto with inverse being positive, then T is an order isomorphism (see [6, p. 2]). ∎

Note that in the first lines of the proof of Theorem 10, we have seen that $L^1(\nu)$ is order dense in $L^1_w(\nu)$. So, the conditions required in this theorem are necessary and sufficient for the extension of $I_\nu\colon L^1(\nu) \to E_a$ to $L^1_w(\nu)$ to be possible in the desired way.

Finally, note that Theorem 10 generalizes [3, Theorem 2.5] where every Banach lattice E with the σ-Fatou property having a weak unit belonging to E_a is represented by means of spaces L^1_w for a vector measure defined on a σ-algebra. Indeed, in this case, E has actually the Fatou property and E_a is order dense in E.

We end by showing two examples of the representation of Banach lattices as $L^1_w(\nu)$ spaces.

Example 11. Consider an uncountable set Γ and the δ-ring $\mathcal{R} = \{A \subset \Gamma : A \text{ is finite}\}$. The space $\ell^\infty(\Gamma)$ has the Fatou property and its σ-order continuous part $c_0(\Gamma)$ is order dense. Then, from Theorem 10, $\ell^\infty(\Gamma)$ is order isometric to $L^1_w(\nu)$ for some vector measure ν defined on a δ-ring. The vector measure $\nu\colon \mathcal{R} \to c_0(\Gamma)$ can be defined as in Example 6 and in this case, the order isometry is the identity map, see [4, Example 2.2]. Note that $\ell^\infty(\Gamma)$ cannot be represented as $L^1_w(\nu)$ with ν defined on a σ-algebra, as its σ-order continuous part has no weak unit.

Example 12. Also, we can find Banach lattices without weak unit satisfying the requirements of Theorem 10. Let Γ and Δ be disjoint uncountable sets and consider the Banach lattice $\ell^1(\Gamma) \times \ell^\infty(\Delta)$ endowed with the norm $\|(x,y)\| = \|x\|_{\ell^1(\Gamma)} + \|y\|_{\ell^\infty(\Delta)}$ and the order $(x,y) \le (\tilde{x},\tilde{y})$ if and only if $x \le \tilde{x}$ and $y \le \tilde{y}$ for $x, \tilde{x} \in \ell^1(\Gamma)$ and $y, \tilde{y} \in \ell^\infty(\Delta)$. This space has the Fatou property and its σ-order continuous part $\ell^1(\Gamma) \times c_0(\Delta)$ is order dense. In this case, taking the δ-ring $\mathcal{R} = \{A \subset \Gamma \cup \Delta : A \text{ is finite}\}$, the vector measure $\nu\colon \mathcal{R} \to \ell^1(\Gamma) \times c_0(\Delta)$ can be defined as $\nu(A) = \big(\nu_1(A \cap \Gamma), \nu_2(A \cap \Delta)\big)$ for all $A \in \mathcal{R}$, where ν_1 and ν_2 are the vector measures defined in Example 6 and Example 11 respectively. Indeed, $\big(\ell^1(\Gamma) \times c_0(\Delta)\big)^*$ is identified with $\big(\ell^1(\Gamma)\big)^* \times \big(c_0(\Delta)\big)^*$ in the way $x^* = (x_1^*, x_2^*)$ such that $x^*(a,b) = x_1^*(a) + x_2^*(b)$ for all $(a,b) \in \ell^1(\Gamma) \times c_0(\Delta)$ and with $\|x^*\| = \max\{\|x_1^*\|, \|x_2^*\|\}$. So, $x^*\nu(A) = x_1^*\nu_1(A \cap \Gamma) + x_2^*\nu_2(A \cap \Delta)$ for all $A \in \mathcal{R}$ and thus

$$|x^*\nu|(B) = |x_1^*\nu_1|(B \cap \Gamma) + |x_2^*\nu_2|(B \cap \Delta) \quad \text{for all } B \in \mathcal{R}^{loc}.$$

Then, for every $f \in \mathcal{M}(\mathcal{R}^{loc})$ we have that

$$\int |f|\, d|x^*\nu| = \int |f|\chi_\Gamma\, d|x_1^*\nu_1| + \int |f|\chi_\Delta\, d|x_2^*\nu_2|.$$

Noting that $L^1_w(\nu_1) \times L^1_w(\nu_2) = \ell^1(\Gamma) \times \ell^\infty(\Delta)$ isometrically, it follows that the operator $T\colon L^1_w(\nu) \to \ell^1(\Gamma) \times \ell^\infty(\Delta)$, defined by $Tf = (f\chi_\Gamma, f\chi_\Delta)$ for all $f \in L^1_w(\nu)$, is an order isometry. Note that T restricted to $L^1(\nu)$ is the integration operator I_ν which is an order isometry between $L^1(\nu)$ and $\ell^1(\Gamma) \times c_0(\Delta)$.

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