Choquet type L^1-spaces of a vector capacity

O. Delgado [a], E.A. Sánchez Pérez [b,*]

[a] Departamento de Matemática Aplicada I, E. T. S. de Ingeniería de Edificación, Universidad de Sevilla, Avenida de Reina Mercedes, 4 A, Sevilla 41012, Spain

[b] Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain

Abstract

Given a set function Λ with values in a Banach space X, we construct an integration theory for scalar functions with respect to Λ by using duality on X and Choquet scalar integrals. Our construction extends the classical Bartle–Dunford–Schwartz integration for vector measures. Since just the minimal necessary conditions on Λ are required, several L^1-spaces of integrable functions associated to Λ appear in such a way that the integration map can be defined in them. We study the properties of these spaces and how they are related. We show that the behavior of the L^1-spaces and the integration map can be improved in the case when X is an order continuous Banach lattice, providing new tools for (non-linear) operator theory and information sciences.

Keywords: Non-additive measures; Measures of information; Choquet integral; Fuzzy measure

1. Introduction

Lebesgue type integration of scalar functions with respect to a vector measure was originally developed by Bartle, Dunford and Schwartz [3] in order to extend the classical Riesz representation theorem for vector valued operators. Later, an equivalent integration theory was constructed by Lewis [23] using the duality of the vector measure's codomain. Nowadays this theory is well understood and has found many important applications in functional analysis and operator theory, among others the extension of linear operators to larger domains, see [29] and the references therein for an outlook of this topic. Similar and other applications would be desirable in the case of non-finitely additive vector valued set functions, as for instance the study of non-linear operators or the construction of some measuring tools in information science (e.g. [4,18]). The aim of this paper is to create an integration theory for vector-valued capacities—sometimes also called *fuzzy capacities*—and the corresponding spaces of integrable functions, which fits with the integration with respect to vector measures and allows to address this kind of applications in next works. As a first step in this process, Choquet type integrals will provide the appropriated framework and will play a fundamental role for our goal.

* Corresponding author. Fax. +34 963877669.

E-mail addresses: olvido@us.es (O. Delgado), easancpe@mat.upv.es (E.A. Sánchez Pérez).

Let us explain our motivation regarding possible applications of our work. Our interest in this topic is twofold. On the one hand, the relevance of our paper from the point of view of the mathematical analysis may be found in the aim of finding extensions of recent vector-measure-based developments in operator theory and harmonic analysis. Indeed, Bartle–Dunford–Schwartz integration has found an upturn of interest due to its use for computation of maximal domains of operators, with applications in several applied problems (see for example [10,12,11,29]). In this direction, similar techniques would be applied for non-linear operators using the tools developed here, where linearity—vector measures—is substituted by non-linearity—fuzzy vector capacities—. Concretely, we are thinking about fundamental non-linear operators as the Hardy–Littlewood maximal operator. On the other hand, as a continuation of our research in collaboration with information scientists, we are also interested in the theoretical development of mathematical instruments for the bibliometric analysis of the impact of the scientific research and its multiple applications—altmetrics, research assessment, big data analysis, just to mention some of them (see [6,16])—. Current research on the topic shows that two lines of research make sense. The first one stresses the fact that non-additive set functions and integration are more and more needed for constructing such measuring tools (see [4,18,21,26]). The second one highlights the fact that multiple scalar indexes—that is, vector-valued indexes—are sometimes needed for a suitable mathematical treatment of the information (see [6,17] and the references therein).

Integration with respect to general set functions has a long tradition, and current contributions on this subject are hot research topics in several branches of both pure and applied mathematics. In the Introduction of [22], an excellent historical review on this integration can be found; see also [24] for an overview of non-additive monotonic measures and their properties. There are a lot of mathematical developments related to non-additive integration; the references in both papers just mentioned provide a nice selections of works regarding this subject and its broad class of applications. We must also refer to the so called pseudo analysis, that is based in the study of the properties of the pseudo measures. Different terms are used for related types of set functions, as for example fuzzy measures, capacities, non-monotonic measures, pseudo-additive measures and null-additive set functions. Starting from the early work of Choquet ([9]), Sugeno and others which may be mentioned here (see for example [27,28] and the references therein), solid integration theories have been constructed for related classes of non-additive measures (see the books [13,30,31]; also [33,35]). In particular, the Choquet integral has been deeply studied in the case of scalar functions and scalar positive capacities (e.g. [13]) and even there are studies about the L^1 and other function spaces associated to this integral (e.g. [7]). Regarding general (topological) spaces of scalar functions that are integrable, a lot of work has also been made in recent years, also from different—theoretical and applied—points of view. Although the literature on the subject is really broad, let us mention here the works that are more related to our developments, that are the papers [7,8] for the classical Choquet integral for scalar functions, and [32] for general pseudo measures.

Regarding the vector valued case, also vector capacities with values in a Riesz space has been studied (e.g. [20]). In this case, thanks to the order structure, a Choquet type vector integral can be defined by using vector Riemann integration. Our approach will be different. Starting with a Banach space valued capacity (without any order) we will use Choquet scalar integrals and duality in a similar way as in Lewis integration to define a vector-valued integral. Several L^1-spaces associated to the vector capacity appear along the paper as soon as some specific conditions are required. We study the relation among them and the role played by the Dunford and Bochner integration when the distribution functions with respect to the vector capacity are considered; we must mention here the paper [15] for a similar study in the case of vector measures, that in some sense inspired our work. Finally, we will see how much the behavior and the properties of the L^1-spaces and the integration map improve when the vector capacity take values in an order continuous Banach lattice.

We must remark that *the novelty of our results mainly concerns the vector nature of the proposed Choquet integral, and the fact that we are directly interested in the structure of the associated spaces of integrable functions, besides the properties of the integrals.* This is so because spaces of integrable functions are central in both motivations that led us to start this study. From the pure-analytic point of view, domains of (linear and non-linear) classical operators are normally function spaces, and so the maximal domain must be expected to be such an space too. From the point of view of the applications, all the functions that can be used for representing a measuring tool with some fixed requirements in information science are also elements of such an space.

The contents of this paper are structured as follows. In the preliminaries we collect the basic concepts and facts on scalar capacities and function spaces that are needed. Section 3 is devoted to the space $L^1(\lambda)$ of a scalar positive capacity λ, that is, the space of (λ-a.e. classes of) measurable functions f such that $\int_I \lambda_{|f|}\, dm < \infty$ where m is the Lebesgue measure on the interval $I = [0, \infty)$ and $\lambda_{|f|}$ is the distribution function of $|f|$ with respect to λ. Although this space has already been studied in [7], as it will be our main tool we have preferred to include here a detailed outline in which minimal conditions on λ are required taking care that the identification of functions which are equal λ-a.e. is correct.

In Section 4 we consider a family $\mathcal{F} = (\lambda_\alpha)$ of scalar positive capacities and construct two quasi-Banach function spaces associated to $\mathcal{F}$, namely, the space $L^1(\|\mathcal{F}\|)$ of the capacity $\|\mathcal{F}\| = \sup_\alpha \lambda_\alpha$ and the space $L^1(\mathcal{F})$ of ($\|\mathcal{F}\|$-a.e. classes of) measurable functions f such that $\sup_\alpha \int_I (\lambda_\alpha)_{|f|}\, dm < \infty$. A Banach space valued capacity $\Lambda\colon \Sigma \to X$ come into play in Section 5. As particular cases of the results of Section 4, we obtain the quasi-Banach function spaces $L^1(\|\Lambda\|)$, w-$L^1_v(\Lambda)$, $L^1(|||\Lambda|||)$, w-$L^1_{qv}(\Lambda)$ associated to the families $(|x^*\Lambda|)_{x^*\in B_{X^*}}$ and $(q_{x^*\Lambda})_{x^*\in B_{X^*}}$, where $|x^*\Lambda|$ and $q_{x^*\Lambda}$ are the variation and quasi-variation respectively of the scalar capacity $x^*\Lambda$ given by the composition of Λ with the element x^* in the closed unit ball B_{X^*} of the topological dual X^* of X.

Under the appropriate conditions, in Section 6 we associate to Λ an integration map $I_\Lambda\colon w$-$L^1_{qv}(\Lambda) \to X^{**}$, where $\langle I_\Lambda(f), x^*\rangle = \int_I x^*\Lambda_f\, dm$ for all positive $f \in w$-$L^1_{qv}(\Lambda)$ and $x^* \in X^*$. For non-positive f we use its positive and negative parts. Then two new sets appear in the case when X is non-reflexive: the set $L^1_{qv}(\Lambda)$ of functions $f \in w$-$L^1_{qv}(\Lambda)$ such that $I_\Lambda(f\chi_A) \in j(X)$ for all $A \in \Sigma$, where j is the canonical embedding of X into X^{**}, and $L^1_v(\Lambda) = L^1_{qv}(\Lambda) \cap w$-$L^1_v(\Lambda)$. We find the following containment relations

$$\begin{array}{ccccc}
L^1(\|\Lambda\|) & \subset & L^1_v(\Lambda) & \subset & w\text{-}L^1_v(\Lambda)\\
\cap & & \cap & & \cap\\
L^1(|||\Lambda|||) & \subset & L^1_{qv}(\Lambda) & \subset & w\text{-}L^1_{qv}(\Lambda).
\end{array}$$

In the case when Λ is a vector measure the vertical inclusions are equalities, w-$L^1_v(\Lambda)$ and $L^1_v(\Lambda)$ coincide with the spaces of weakly integrable and integrable functions (in the sense of Lewis) with respect to Λ respectively, and I_Λ is the integration operator with respect to Λ.

In general, since I_Λ is not additive, we cannot know even if $L^1_{qv}(\Lambda)$ and $L^1_v(\Lambda)$ are vector spaces. Section 7 gives conditions on X and Λ under which these two sets are Banach function spaces with the norms of w-$L^1_{qv}(\Lambda)$ and w-$L^1_v(\Lambda)$ respectively. The key is that under these conditions X is a Banach lattice satisfying that $j(X)$ is an ideal of X^{**} and the map I_Λ is increasing and subadditive on positive functions. Moreover, I_Λ turns out to be continuous, the space w-$L^1_{qv}(\Lambda)$ coincides with the space of (Λ-a.e. classes of) measurable functions such that $\Lambda_{|f|}$ is Dunford integrable with respect to m and $L^1(|||\Lambda|||)$ with the space of functions such that $\Lambda_{|f|}$ is Bochner integrable.

We end with Section 8 by showing an example of a vector capacity which satisfies all the conditions required along the paper and giving easier descriptions of its associated L^1-spaces.

2. Preliminaries

Throughout this paper (Ω, Σ) will denote a measurable space. Let $\lambda\colon \Sigma \to [0, \infty]$ be a set function satisfying that $\lambda(\emptyset) = 0$. Such a set function λ will be called a *capacity*. A set $Z \in \Sigma$ is λ*-null* if $\lambda(A) = 0$ for all $A \in \Sigma$ such that $A \subset Z$. Note that every measurable subset of a λ-null set is λ-null. A property holds λ-a.e. if it holds except on a λ-null set. If $(A_n) \subset \Sigma$ is an increasing sequence with $A = \cup A_n$ we will write $A_n \uparrow A$. If the sequence is decreasing with $A = \cap A_n$ we will write $A_n \downarrow A$. The following properties of a capacity will be used in the sequel:

(P1) λ is *increasing* if $\lambda(A) \le \lambda(B)$ for every $A, B \in \Sigma$ such that $A \subset B$.
(P2) λ is *null-additive* if $\lambda(A \cup Z) = \lambda(A)$ for all $A, Z \in \Sigma$ with Z being λ-null.
(P3) λ is *quasi-subadditive* if there exists a constant $K \ge 1$ such that

$$\lambda(A \cup B) \le K(\lambda(A) + \lambda(B))$$

for every disjoint sets $A, B \in \Sigma$. If $K = 1$ it is called *subadditive*.
(P4) λ is *superadditive* if $\lambda(A) + \lambda(B) \le \lambda(A \cup B)$ for every disjoint sets $A, B \in \Sigma$.
(P5) λ is *submodular* if $\lambda(A \cup B) + \lambda(A \cap B) \le \lambda(A) + \lambda(B)$ for all $A, B \in \Sigma$.
(P6) λ is *continuous from below* if $\lambda(A_n) \to \lambda(A)$ whenever $A_n, A \in \Sigma$ with $A_n \uparrow A$.
(P7) λ is *continuous from above at* $\emptyset$ if $\lambda(A_n) \to 0$ whenever $A_n \in \Sigma$ with $A_n \downarrow \emptyset$ and $\lambda(A_1) < \infty$.

Denote by $\mathcal{L}^0(\Omega)$ the space of all measurable functions $f\colon\Omega\to\mathbb{R}$ and by $\mathcal{L}^0(\Omega)^+$ the *positive cone* of $\mathcal{L}^0(\Omega)$, that is the set of functions in $\mathcal{L}^0(\Omega)$ which take values in $[0,\infty)$. Write $\mathcal{N}_\lambda$ for the set of functions $f\in\mathcal{L}^0(\Omega)$ such that $f=0$ λ-a.e. In what follows we require that $\cup Z_n$ *is* λ*-null whenever* (Z_n) *is a sequence of* λ*-null sets*. This fact is obtained for instance if λ is continuous from below and has any of the properties (P2,3,5). Under this requirement $\mathcal{N}_\lambda$ is a vector space and for $f_n-g_n, f-g\in\mathcal{N}_\lambda$ with $f_n\to f$ λ-a.e. it follows that $g_n\to g$ λ-a.e. The *support* of a function $f\in\mathcal{L}^0(\Omega)$ will be denoted by $\mathrm{supp}(f)$. Note that $f\in\mathcal{N}_\lambda$ if and only if $\mathrm{supp}(f)$ is λ-null. Denote by $L^0(\lambda)$ the quotient space $\mathcal{L}^0(\Omega)/\mathcal{N}_\lambda$. That is, $L^0(\lambda)$ is the space of all real measurable functions f defined on Ω, where functions which are equal λ-a.e. are identified. For $f\in\mathcal{L}^0(\Omega)$ we will denote by f^+ and f^- the positive and negative parts of f respectively, that is, $f^+=f\chi_{P_f}$ and $f^-=(-f)\chi_{N_f}$ where $P_f=\{\omega\in\Omega:\ f(\omega)>0\}$ and $N_f=\{\omega\in\Omega:\ f(\omega)<0\}$. We will write $\mathcal{S}$ for the space of simple functions on Σ and $\mathcal{S}_\lambda=\{\varphi\in\mathcal{S}:\ \lambda(\mathrm{supp}(\varphi))<\infty\}$.

By a λ*-quasi-Banach function space* (briefly, λ-quasi-B.f.s.) we mean a quasi-Banach space $E\subset L^0(\lambda)$ with quasi-norm $\|\cdot\|_E$, satisfying that if $f\in L^0(\lambda)$, $g\in E$ and $|f|\le|g|$ λ-a.e. then $f\in E$ with $\|f\|_E\le\|g\|_E$. If E is a Banach space we will refer it as a λ*-Banach function space* (briefly, λ-B.f.s.). In particular, a λ-quasi-B.f.s. is a quasi-Banach lattice for the λ-a.e. pointwise order. Note that all inclusions between λ-quasi-B.f.s. are continuous, see the argument given in [25, p. 2]. A λ-quasi-B.f.s. E is σ*-order continuous* if for every sequence $(f_n)\subset E$ with $f_n\downarrow 0$ λ-a.e. it follows that $\|f_n\|_E\downarrow 0$. In this case $\mathcal{S}\cap E$ is dense in E. It is said that E has the σ*-Fatou property* if for every $(f_n)\subset E$ such that $0\le f_n\uparrow f$ λ-a.e. and $\sup\|f_n\|_E<\infty$ we have that $f\in E$ and $\|f_n\|_E\uparrow\|f\|_E$.

Let ρ be a λ*-quasi-norm function*, that is a map $\rho\colon\mathcal{L}^0(\Omega)\to[0,\infty]$ satisfying the following conditions:

(C1) $\rho(f)\le\rho(g)$ whenever $f,g\in\mathcal{L}^0(\Omega)$ with $|f|\le|g|$ λ-a.e.
(C2) $\rho(f)=0$ if and only if $f=0$ λ-a.e.
(C3) $\rho(af)=|a|\rho(f)$[1] for all $a\in\mathbb{R}$ and $f\in\mathcal{L}^0(\Omega)$.
(C4) There exists $K\ge 1$ such that $\rho(f+g)\le K(\rho(f)+\rho(g))$ for all $f,g\in\mathcal{L}^0(\Omega)$.

Condition (C1) guarantees that $\rho\colon L^0(\lambda)\to[0,\infty]$ is well defined. Then

$$X_\rho=\{f\in L^0(\lambda):\ \rho(f)<\infty\}$$

is a vector space and ρ is a quasi-norm on it. Moreover, if $f\in L^0(\lambda)$, $g\in X_\rho$ and $|f|\le|g|$ λ-a.e. then $f\in X_\rho$ with $\rho(f)\le\rho(g)$.

The λ-quasi-norm function ρ is said to have the σ*-Fatou property* if $\rho(f_n)\uparrow\rho(f)$ whenever $f_n,f\in\mathcal{L}^0(\Omega)$ with $0\le f_n\uparrow f$ λ-a.e. In this case, it is known that X_ρ is complete. For the sake of completeness we include a proof of this fact; it can be obtained by adapting the proof of Theorem 1.6 in [5], taking into account that in this book the definition of function norm includes the σ-Fatou property (see also [7] and Sections 2 and 3 in [8]).

Proposition 1. *Let ρ be a λ-quasi-norm function with the σ-Fatou property. Then X_ρ is a λ-quasi-B.f.s. with the σ-Fatou property and ρ is a quasi-norm on it.*

Proof. Let $r>0$ be such that $2K=2^{\frac{1}{r}}$ where K is the constant of condition (C4). Then

$$\rho\Big(\sum_{j=1}^n f_j\Big)\le 4^{\frac{1}{r}}\Big(\sum_{j=1}^n\rho(f_j)^r\Big)^{\frac{1}{r}} \tag{1}$$

for all finite subset $(f_j)_{j=1}^n\subset X_\rho$, see [19, Lemma 1.1]. Consider a Cauchy sequence $(f_n)\subset X_\rho$ and take (n_k) strictly increasing such that $\rho(f_{n_{k+1}}-f_{n_k})\le\frac{1}{2^k}$. Denote $g_k=f_{n_{k+1}}-f_{n_k}$, $A=\{\omega\in\Omega:\ \sum_{k\ge1}|g_k(\omega)|<\infty\}$ and $g=\sum_{k\ge1}g_k\chi_A$. Since $\sum_{k=1}^m|g_k|\chi_A\uparrow\sum_{k\ge1}|g_k|\chi_A$ pointwise, it follows that

$$\rho(g)\le\rho\Big(\sum_{k\ge1}|g_k|\chi_A\Big)=\lim_{m\to\infty}\rho\Big(\sum_{k=1}^m|g_k|\chi_A\Big)\le 4^{\frac{1}{r}}\Big(\sum_{k\ge1}\rho(g_k)^r\Big)^{\frac{1}{r}}<\infty$$

[1] We use the convention $0\cdot\infty=\infty\cdot 0=0$.

and so $g \in X_\rho$. Similarly, $\rho(\sum_{k\geq m} g_k\chi_A) \leq 4^{\frac{1}{r}}(\sum_{k\geq m}\rho(g_k)^r)^{\frac{1}{r}}$ for each $m \geq 1$. Consider the sets $A_m^N = \{\omega \in \Omega : \sum_{k=1}^m |g_k(\omega)| > N\}$ for $m, N \geq 1$ and note that $\chi_{A_m^N} \leq \frac{1}{N}\sum_{k=1}^m |g_k|$ pointwise and $\chi_{A_m^N} \uparrow \chi_{\cup_{m\geq 1} A_m^N}$ as $m \to \infty$. Then,

$$\rho(\chi_{\cup_{m\geq 1} A_m^N}) = \lim_{m\to\infty} \rho(\chi_{A_m^N}) \leq \frac{1}{N}\lim_{m\to\infty}\rho\Big(\sum_{k=1}^m |g_k|\Big) \leq \frac{4^{\frac{1}{r}}}{N}\Big(\sum_{k\geq 1}\rho(g_k)^r\Big)^{\frac{1}{r}}.$$

Since $\Omega\backslash A = \cap_{N\geq 1}\cup_{m\geq 1} A_m^N$ and so $\rho(\chi_{\Omega\backslash A}) \leq \rho(\chi_{\cup_{m\geq 1}A_m^N})$ for all N, taking $N \to \infty$ we have that $\rho(\chi_{\Omega\backslash A}) = 0$. This implies that $\Omega\backslash A$ is λ-null. Noting that $g + f_{n_1} - f_{n_m} = \sum_{k\geq m} g_k\chi_A$ λ-a.e., given $\varepsilon > 0$ it follows that

$$\rho(g + f_{n_1} - f_n) \leq K\Big(\rho\Big(\sum_{k\geq m} g_k\chi_A\Big) + \rho(f_{n_m} - f_n)\Big) < \varepsilon$$

for large enough n and m. Hence, $f_n \to g + f_{n_1}$ in X_ρ and so X_ρ is complete.

The σ-Fatou property of X_ρ follows clearly from the σ-Fatou property of ρ. $\square$

Let $\xi: \Sigma \to \mathbb{R}$ be a set function satisfying that $\xi(\emptyset) = 0$. Such a set function ξ will be called a *real capacity*. A set $Z \in \Sigma$ is ξ*-null* if $\xi(A) = 0$ for all $A \in \Sigma$ such that $A \subset Z$. The *variation* of ξ is the set function $|\xi|: \Sigma \to [0, \infty]$ defined by

$$|\xi|(A) = \sup\Big\{\sum_{i=1}^n |\xi(A_i)| : (A_i)_{i=1}^n \subset \Sigma \text{ is a partition of } A\Big\}$$

for $A \in \Sigma$. The *quasi-variation* of ξ is the set function $q_\xi: \Sigma \to [0, \infty]$ defined by

$$q_\xi(A) = \sup\{|\xi(B)| : B \in \Sigma \text{ with } B \subset A\}$$

for $A \in \Sigma$. Note that both $|\xi|$ and q_ξ are increasing capacities. We also consider the capacity $|\xi(\cdot)|: \Sigma \to [0, \infty)$ given by $|\xi(\cdot)|(A) = |\xi(A)|$ for $A \in \Sigma$. The following lemma collects several properties involving the capacities $|\xi|$, q_ξ and $|\xi(\cdot)|$ which can be routinely checked. The reader can find more information on the variation of non-additive set functions in [30,36,37].

Lemma 2. *Let ξ be a real capacity on Σ. The following statements hold:*

(a) $|\xi(\cdot)| \leq q_\xi \leq |\xi|$.
(b) ξ-null, $|\xi|$-null, q_ξ-null and $|\xi(\cdot)|$-null sets coincide.
(c) $|\xi|$ is superadditive.
(d) $|\xi(\cdot)| = q_\xi \Leftrightarrow |\xi(\cdot)|$ is increasing, and $q_\xi = |\xi| \Leftrightarrow q_\xi$ is superadditive.
(e) If $\Phi(\xi)$ denotes any one of $|\xi|$, q_ξ, $|\xi(\cdot)|$, then
  (e1) $\Phi(a\,\xi) = |a|\,\Phi(\xi)$ for all $a \in \mathbb{R}$,
  (e2) $\Phi(\xi + \eta) \leq \Phi(\xi) + \Phi(\eta)$ for η being another real capacity on Σ, and
  (e3) if (ξ_n) is a sequence of real capacities on Σ such that $\xi_n(A) \to \xi(A)$ for all $A \in \Sigma$ then $\Phi(\xi) \leq \liminf \Phi(\xi_n)$.
(f) If $|\xi(\cdot)|$ has any of the properties (P2,3,6) then q_ξ has the same property.
(g) If q_ξ has any of the properties (P2,3,6) then $|\xi|$ has the same property.

The quasi-additivity constant is preserved in (f) and (g).

Note that in the case when ξ takes values in $[0, \infty)$, from Lemma 2.(d), in general ξ does not coincide with any of q_ξ or $|\xi|$.

3. L^1-space of a capacity

Let $\lambda: \Sigma \to [0, \infty]$ be an increasing capacity and denote by m the Lebesgue measure on the interval $I = [0, \infty)$. For $f \in \mathcal{L}^0(\Omega)^+$, the *distribution function* of f with respect to λ is the map $\lambda_f: I \to [0, \infty]$ defined by

$$\lambda_f(t) = \lambda(\{\omega \in \Omega : f(\omega) > t\})$$

for $t \in I$. Since λ is increasing we have that λ_f is decreasing and so measurable. Then we can consider the Lebesgue integral

$$I_\lambda(f) = \int_I \lambda_f \, dm \in [0, \infty].$$

Remark 3. If $f \in \mathcal{L}^0(\Omega)^+$ is such that $\lambda(\mathrm{supp}(f)) < \infty$ then λ_f is bounded and so Riemann integrable in every interval $[0, a]$ with $0 < a < \infty$. Then

$$I_\lambda(f) = \int_0^\infty \lambda_f(t) \, dt$$

is the *Choquet integral* of f with respect to λ created in [9].

A positive function $\varphi \in \mathcal{S}$ always can be written in its *standard representation*, that is $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j}$ where $(A_j)_{j=1}^n \subset \Sigma$ is a finite collection of pairwise disjoint sets and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. Setting $\alpha_0 = 0$, we have that

$$\lambda_\varphi = \sum_{k=1}^n \lambda\Big(\bigcup_{j=k}^n A_j\Big) \chi_{[\alpha_{k-1}, \alpha_k)}$$

and so

$$I_\lambda(\varphi) = \sum_{k=1}^n \lambda\Big(\bigcup_{j=k}^n A_j\Big)(\alpha_k - \alpha_{k-1}). \tag{2}$$

In particular, $I_\lambda(\chi_A) = \lambda(A)$ for all $A \in \Sigma$.

Let us show some properties of the *integration map* $I_\lambda : \mathcal{L}^0(\Omega)^+ \to [0, \infty]$ which will be needed later on. Similar result in a slightly different context can be found in recent papers on the Choquet integral. (See [22, Proposition 3.4], and in general Section 3 in this paper; Section 4 gives similar results for other non-additive integrals. See also the references in this paper for more information on these matters. The reader can find more information on continuous monotone set functions in [24] and the papers quoted in it.)

Lemma 4. *The following statements hold:*

(a) $I_\lambda(af) = aI_\lambda(f)$ for all $f \in \mathcal{L}^0(\Omega)^+$ and $0 \le a \in \mathbb{R}$.
(b) $I_\lambda(f) \le I_\lambda(g)$ for every $f, g \in \mathcal{L}^0(\Omega)^+$ such that $f \le g$ pointwise.
(c) $I_\lambda(f) = I_\lambda(g)$ whenever $f, g \in \mathcal{L}^0(\Omega)^+$ with $f = g$ λ-a.e. if and only if λ is null-additive.
(d) $I_\lambda(f_n) \uparrow I_\lambda(f)$ whenever $f_n, f \in \mathcal{L}^0(\Omega)^+$ with $f_n \uparrow f$ pointwise if and only if λ is continuous from below.
(e) $I_\lambda(f_n) \downarrow 0$ whenever $f_n \in \mathcal{L}^0(\Omega)^+$ with $f_n \downarrow 0$ pointwise and $I_\lambda(f_1) < \infty$ if and only if λ is continuous from above at $\emptyset$.

Proof. An appropriate change of variables gives (a). Part (b) is also clear as λ is increasing.

(c) Suppose that $I_\lambda(f) = I_\lambda(g)$ for every $f, g \in \mathcal{L}^0(\Omega)^+$ such that $f = g$ λ-a.e. For $A, Z \in \Sigma$ with Z being λ-null, taking $f = \chi_{A \cup Z}$ and $g = \chi_A$ we have that $f = g$ λ-a.e. and so $\lambda(A \cup Z) = I_\lambda(f) = I_\lambda(g) = \lambda(A)$. Conversely, suppose that λ is null-additive and consider $f, g \in \mathcal{L}^0(\Omega)^+$ with $f = g$ except on a λ-null set Z. For every $t \in I$, denote $A_t = \{\omega \in \Omega : f(\omega) > t\}$ and $B_t = \{\omega \in \Omega : g(\omega) > t\}$. Noting that $A_t \cap Z$, $B_t \cap Z$ are λ-null and $A_t \cap \Omega \backslash Z = B_t \cap \Omega \backslash Z$, we have that

$$\lambda(A_t) = \lambda(A_t \cap \Omega \backslash Z) = \lambda(B_t \cap \Omega \backslash Z) = \lambda(B_t).$$

Then $\lambda_f = \lambda_g$ pointwise and so $I_\lambda(f) = I_\lambda(g)$.

(d) Suppose that $I_\lambda(f_n) \uparrow I_\lambda(f)$ for every $f_n, f \in \mathcal{L}^0(\Omega)^+$ such that $f_n \uparrow f$ pointwise. For $A_n, A \in \Sigma$ with $A_n \uparrow A$, taking $f_n = \chi_{A_n}$ and $f = \chi_A$ we have that $f_n \uparrow f$ pointwise and so $\lambda(A_n) = I_\lambda(f_n) \uparrow I_\lambda(f) = \lambda(A)$. Conversely, suppose that λ is continuous from below and consider $f_n, f \in \mathcal{L}^0(\Omega)^+$ with $f_n \uparrow f$ pointwise. Noting that $\{\omega \in \Omega : f_n(\omega) > t\} \uparrow \{\omega \in \Omega : f(\omega) > t\}$ for all $t \in I$, we have that $\lambda_{f_n} \uparrow \lambda_f$ pointwise. Then, by applying the monotone convergence theorem for the Lebesgue integral with respect to m, it follows that $I_\lambda(f_n) \uparrow I_\lambda(f)$.

(e) Suppose that $I_\lambda(f_n) \downarrow 0$ for every $f_n \in \mathcal{L}^0(\Omega)^+$ such that $f_n \downarrow 0$ pointwise and $I_\lambda(f_1) < \infty$. For $A_n \in \Sigma$ with $A_n \downarrow \emptyset$ and $\lambda(A_1) < \infty$, taking $f_n = \chi_{A_n}$ we have that $f_n \downarrow 0$ pointwise and $I_\lambda(f_1) = \lambda(A_1) < \infty$. So $\lambda(A_n) = I_\lambda(f_n) \downarrow 0$. Conversely, suppose that λ is continuous from above at $\emptyset$ and consider $f_n \in \mathcal{L}^0(\Omega)^+$ with $f_n \downarrow 0$ pointwise and $I_\lambda(f_1) < \infty$. Note that $\lambda_{f_1}(t) < \infty$ for all $t > 0$ as λ_{f_1} is decreasing. Since $\{\omega \in \Omega : f_n(\omega) > t\} \downarrow \emptyset$ for all $t > 0$, we have that $\lambda_{f_n} \downarrow 0$ m-a.e. Then, by applying the dominated convergence theorem for the Lebesgue integral with respect to m, it follows that $I_\lambda(f_n) \downarrow 0$. □

Related results on the additivity properties of the integral under further requirements on the capacity are known; see for example Theorem 4.1 in [7], and Section 4 in this work. See also [30] for a systematic approach to non-additive set functions.

Consider the map $\|\cdot\|_\lambda : \mathcal{L}^0(\Omega) \to [0, \infty]$ defined by $\|f\|_\lambda = I_\lambda(|f|)$ for $f \in \mathcal{L}^0(\Omega)$. The following proposition gives conditions under which $\|\cdot\|_\lambda$ is a λ-quasi-norm function with the σ-Fatou property.

Proposition 5. *Suppose that λ is null-additive, quasi-subadditive and continuous from below. Then, the following statements hold:*

(a) $\|f\|_\lambda \le \|g\|_\lambda$ whenever $f, g \in \mathcal{L}^0(\Omega)$ with $|f| \le |g|$ λ-a.e.
(b) $\|f\|_\lambda = 0$ if and only if $f = 0$ λ-a.e.
(c) $\|af\|_\lambda = |a|\,\|f\|_\lambda$ for all $a \in \mathbb{R}$ and $f \in \mathcal{L}^0(\Omega)$.
(d) $\|f + g\|_\lambda \le 2K(\|f\|_\lambda + \|g\|_\lambda)$ for all $f, g \in \mathcal{L}^0(\Omega)$, with K being the constant of the quasi-subadditivity of λ.
(e) $\|f_n\|_\lambda \uparrow \|f\|_\lambda$ whenever $f_n, f \in \mathcal{L}^0(\Omega)$ with $0 \le f_n \uparrow f$ λ-a.e.

Proof. (a) Let $f, g \in \mathcal{L}^0(\Omega)$ be such that $|f|\chi_A \le |g|\chi_A$ pointwise for some $A \in \Sigma$ with $\Omega \backslash A$ being λ-null. Since $|f|\chi_A = |f|$ and $|g|\chi_A = |g|$ λ-a.e., from Lemma 4.(b) and (c) it follows that

$$\|f\|_\lambda = I_\lambda(|f|) = I_\lambda(|f|\chi_A) \le I_\lambda(|g|\chi_A) = I_\lambda(|g|) = \|g\|_\lambda.$$

(b) If $f = 0$ except on a λ-null set Z, then $\{\omega \in \Omega : |f(\omega)| > t\} \subset Z$ for all $t \in I$ and so $\lambda_{|f|} = 0$ pointwise. Hence $\|f\|_\lambda = 0$. Conversely, suppose that $\|f\|_\lambda = 0$ and denote $A_n = \{\omega \in \Omega : |f(\omega)| > \frac{1}{n}\}$. Since $\frac{1}{n}\chi_{A_n} \le |f|$ pointwise we have that

$$0 = \|f\|_\lambda \ge \left\| \frac{1}{n}\chi_{A_n} \right\|_\lambda = I_\lambda\left(\frac{1}{n}\chi_{A_n}\right) = \frac{1}{n}\lambda(A_n)$$

and so $\lambda(A_n) = 0$. Note that $A_n \uparrow \mathrm{supp}(f)$ from which $\lambda(\mathrm{supp}(f)) = 0$ as λ is continuous from below. Then $f = 0$ λ-a.e.

(c) Clear from Lemma 4.(a).

(d) For every $f, g \in \mathcal{L}^0(\Omega)$, noting that

$$\{\omega \in \Omega : |(f+g)(\omega)| > t\} \subset \{\omega \in \Omega : |2f(\omega)| > t\} \cup \{\omega \in \Omega : |2g(\omega)| > t\}$$

for all $t \in I$ and since λ is increasing and quasi-subadditive with constant K, it follows that $\lambda_{|f+g|} \le K(\lambda_{2|f|} + \lambda_{2|g|})$ pointwise. Then,

$$\|f + g\|_\lambda \le K(\|2f\|_\lambda + \|2g\|_\lambda) = 2K(\|f\|_\lambda + \|g\|_\lambda).$$

(e) Let $f_n, f \in \mathcal{L}^0(\Omega)$ be such that $0 \le f_n\chi_A \uparrow f\chi_A$ pointwise for some $A \in \Sigma$ with $\Omega \backslash A$ being λ-null. Since $f_n\chi_A = |f_n|$ and $f\chi_A = |f|$ λ-a.e., from Lemma 4.(c) and (d) it follows that

$$\|f_n\|_\lambda = I_\lambda(|f_n|) = I_\lambda(f_n\chi_A) \uparrow I_\lambda(f\chi_A) = I_\lambda(|f|) = \|f\|_\lambda. \qquad \square$$

In the remainder of this section we will assume that the capacity λ satisfies the properties (P1,2,3,6). These properties guarantee the good behavior of the L^1*-space of* λ defined as

$$L^1(\lambda) = \{f \in L^0(\lambda) : \|f\|_\lambda < \infty\}.$$

Note that a simple function $\varphi \in L^1(\lambda)$ if and only if $\varphi \in \mathcal{S}_\lambda$, see (2).

Theorem 6. *The space $L^1(\lambda)$ is a λ-quasi-B.f.s. with the σ-Fatou property and $\|\cdot\|_\lambda$ is a quasi-norm on it. Moreover, $L^1(\lambda)$ is σ-order continuous if and only if λ is continuous from above at $\emptyset$. In this last case, $\mathcal{S}_\lambda$ is dense in $L^1(\lambda)$.*

Proof. The first part follows from Propositions 1 and 5.

Suppose that $L^1(\lambda)$ is σ-order continuous and let $f_n \in \mathcal{L}^0(\Omega)^+$ with $f_n \downarrow 0$ pointwise and $I_\lambda(f_1) < \infty$. Since $f_n \in L^1(\lambda)$ as $\|f_n\|_\lambda \le \|f_1\|_\lambda = I_\lambda(f_1)$, we have that $I_\lambda(f_n) = \|f_n\|_\lambda \downarrow 0$. Then λ is continuous from above at $\emptyset$ by Lemma 4.(e).

Conversely, suppose that λ is continuous from above at $\emptyset$ and let $f_n \in L^1(\lambda)$ be such that $f_n \downarrow 0$ λ-a.e. Taking $A \in \Sigma$ such that $\Omega \backslash A$ is λ-null and $f_n \chi_A \downarrow 0$ pointwise and noting that $I_\lambda(f_1\chi_A) = \|f_1\chi_A\|_\lambda = \|f_1\|_\lambda < \infty$, from Lemma 4.(e) we have that $\|f_n\|_\lambda = \|f_n\chi_A\|_\lambda = I_\lambda(f_n\chi_A) \downarrow 0$. So $L^1(\lambda)$ is σ-order continuous.

Noting that $\mathcal{S}_\lambda = \mathcal{S} \cap L^1(\lambda)$ we conclude the proof. □

Remark 7. Let $\mu\colon \Sigma \to [0,\infty]$ be a measure. Obviously μ is a capacity satisfying all the properties (P1–7). For every positive $\varphi \in \mathcal{S}$, from (2) it follows that $I_\mu(\varphi)$ coincides with the Lebesgue integral $\int_\Omega \varphi\, d\mu$. For $f \in \mathcal{L}^0(\Omega)^+$, taking a sequence $(\varphi_n) \subset \mathcal{S}$ such that $0 \le \varphi_n \uparrow f$ pointwise, from Lemma 4.(d) and applying the monotone convergence theorem for the Lebesgue integral with respect to μ, it follows that

$$I_\mu(f) = \lim I_\mu(\varphi_n) = \lim \int_\Omega \varphi_n\, d\mu = \int_\Omega f\, d\mu.$$

Then our space $L^1(\mu)$ is just the classical space of Lebesgue integrable functions with respect to μ.

Now we want to extend the integration map I_λ to non-positive functions of $L^1(\lambda)$. In order to obtain the Lebesgue integral in the case when λ is a measure, for a function $f \in L^1(\lambda)$ we define $I_\lambda(f) = I_\lambda(f^+) - I_\lambda(f^-)$. Note that the definition is meaningful as $I_\lambda(f^+), I_\lambda(f^-) < \infty$ and if $g \in L^1(\lambda)$ is such that $f = g$ λ-a.e., since $f^+ = g^+$ and $f^- = g^-$ λ-a.e., from Lemma 4.(c) it follows that $I_\lambda(f) = I_\lambda(g)$. It is important to emphasize that the integration map $I_\lambda\colon \mathcal{L}^0(\Omega)^+ \to [0,\infty]$ is not additive in general. In fact this is only the case when λ is a measure, since finite additivity and continuity from below imply countable additivity. So the definition of I_λ for a non-positive function depend on its positive and negative parts.

From Lemma 4.(a) it follows that $I_\lambda\colon L^1(\lambda) \to \mathbb{R}$ is homogeneous, that is, $I_\lambda(af) = aI_\lambda(f)$ for all $f \in L^1(\lambda)$ and $a \in \mathbb{R}$. Indeed, we only have to note that $(af)^+ = af^+$, $(af)^- = af^-$ if $a \ge 0$ and $(af)^+ = -af^-$, $(af)^- = -af^+$ if $a < 0$.

It is well known that I_λ is subadditive on the set of positive simple functions if and only if λ is submodular, see for instance [13, Ch. 6] and the references therein or [1] for a nice proof. In this case, since λ is continuous from below, it follows that I_λ is subadditive on all $\mathcal{L}^0(\Omega)^+$ and so $\|\cdot\|_\lambda$ is a norm, that is $L^1(\lambda)$ is a λ-B.f.s.

Proposition 8. *If λ is submodular then $I_\lambda\colon L^1(\lambda) \to \mathbb{R}$ is continuous.*

Proof. Suppose that λ is submodular and so I_λ is subadditive on $\mathcal{L}^0(\Omega)^+$. Given $f, g \in \mathcal{L}^0(\Omega)^+$, applying Lemma 4.(b) we have that

$$I_\lambda(f) \le I_\lambda(|f-g| + g) \le I_\lambda(|f-g|) + I_\lambda(g),$$

from which it follows that $|I_\lambda(f) - I_\lambda(g)| \le I_\lambda(|f-g|)$. For $f, g \in L^1(\lambda)$, since $|f^+ - g^+| \le |f-g|$ and $|f^- - g^-| \le |f-g|$, we obtain that

$$\begin{aligned}|I_\lambda(f) - I_\lambda(g)| &\le |I_\lambda(f^+) - I_\lambda(g^+)| + |I_\lambda(f^-) - I_\lambda(g^-)| \\ &\le I_\lambda(|f^+ - g^+|) + I_\lambda(|f^- - g^-|) \\ &\le 2\, I_\lambda(|f-g|) = 2\,\|f-g\|_\lambda. \qquad \square\end{aligned}$$

We end this section by constructing a class of capacities which satisfy all the necessary properties to obtain a good L^1-space.

Example 9. Let $\mu\colon\Sigma\to[0,\infty]$ be a measure and $\Phi\colon I\to I$ an increasing function vanishing only at zero, with $\lim_{x\to 0^+}\Phi(x)=0$ and being derivable in $(0,\infty)$ with a decreasing derivative Φ'. For instance $\Phi(x)=x^p$ with $0<p<1$, $\Phi(x)=1-e^{-x}$ or $\Phi(x)=\ln(1+x)$. Note that

$$\Phi(a+b-c)+\Phi(c)\leq\Phi(a)+\Phi(b) \tag{3}$$

for all $0\leq c\leq a,b<\infty$. Consider the capacity $\lambda\colon\Sigma\to[0,\infty]$ given by

$$\lambda(A)=\Phi(\mu(A))$$

for all $A\in\Sigma$. As usual, $\Phi(\infty)=\lim_{x\to\infty}\Phi(x)$. Note that the λ-null and μ-null sets coincide and so $L^0(\lambda)=L^0(\mu)$. It is direct to check that λ is increasing, null-additive and continuous from below. Let us see that λ is submodular and so subadditive. For every $A,B\in\Sigma$ with $\mu(A),\mu(B)<\infty$, since μ is a measure we have that

$$\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B).$$

Applying (3) for $a=\mu(A)$, $b=\mu(B)$ and $c=\mu(A\cap B)$, we obtain that

$$\begin{aligned}\lambda(A\cup B)+\lambda(A\cap B)&=\Phi(\mu(A)+\mu(B)-\mu(A\cap B))+\Phi(\mu(A\cap B))\\&\leq\Phi(\mu(A))+\Phi(\mu(B))=\lambda(A)+\lambda(B).\end{aligned}$$

If any of A or B has infinite μ measure then the submodular inequality is clear. Therefore

$$L^1(\lambda)=\Big\{f\in L^0(\mu):\ \|f\|_\lambda=\int_I\Phi(\mu_{|f|})\,dm<\infty\Big\}$$

is a μ-B.f.s. with the σ-Fatou property and $\|\cdot\|_\lambda$ is a norm on it. Moreover, the integration map $I_\lambda\colon L^1(\lambda)\to\mathbb{R}$ is continuous. The space $L^1(\lambda)$ turns out to be an *intermediate space* between $L^\infty(\mu)$ and $L^1(\mu)$, that is,

$$L^\infty(\mu)\cap L^1(\mu)\subset L^1(\lambda)\subset L^\infty(\mu)+L^1(\mu).$$

Let us show this fact. For every positive $\varphi\in\mathcal{S}$ with standard representation $\varphi=\sum_{j=1}^n\alpha_j\chi_{A_j}$ such that $\alpha_n\leq 1$, since Φ is concave (as Φ' decreases) and $\sum_{k=1}^n(\alpha_k-\alpha_{k-1})=\alpha_n\leq 1$, it follows that

$$\begin{aligned}\|\varphi\|_\lambda&=\sum_{k=1}^n\Phi\Big(\mu\Big(\bigcup_{j=k}^n A_j\Big)\Big)(\alpha_k-\alpha_{k-1})\\&\leq\Phi\Big(\sum_{k=1}^n\mu\Big(\bigcup_{j=k}^n A_j\Big)(\alpha_k-\alpha_{k-1})\Big)=\Phi\Big(\int_\Omega\varphi\,d\mu\Big).\end{aligned}$$

Note that the concave inequality holds even if some $\cup_{j=k}^n A_j$ has infinite μ measure. Then, if $f\in L^0(\mu)$ is such that $|f|\leq 1$ μ-a.e., taking a sequence $(\varphi_n)\subset\mathcal{S}$ such that $0\leq\varphi_n\uparrow|f|$ μ-a.e., we have that

$$\|f\|_\lambda=\lim\|\varphi_n\|_\lambda\leq\lim\Phi\Big(\int_\Omega\varphi_n\,d\mu\Big)=\Phi\Big(\int_\Omega|f|\,d\mu\Big).$$

Hence every non-null function $f\in L^\infty(\mu)\cap L^1(\mu)$ satisfies that

$$\|f\|_\lambda\leq\|f\|_\infty\Phi\Big(\frac{\|f\|_1}{\|f\|_\infty}\Big)$$

and so $L^\infty(\mu)\cap L^1(\mu)\subset L^1(\lambda)$. On the other hand, since Φ' is decreasing we have that

$$\Phi'(x_0)\,x\leq\Phi(x) \tag{4}$$

for all $0\leq x<x_0<\infty$. Let $f\in L^1(\lambda)$ and denote $A_n=\{\omega\in\Omega:\ |f(\omega)|>n\}$. Note that there exists n such that $\mu(A_n)<\infty$ as in other case $\Phi(\mu_{|f|})=\Phi(\infty)$ which contradicts $\|f\|_\lambda<\infty$. Then $\lim\mu(A_n)=0$ as $A_n\downarrow\{\omega\in\Omega:\ |f(\omega)|=\infty\}$. Take $x_0>0$ such that $\Phi'(x_0)>0$ and n_0 such that $\mu(A_{n_0})<x_0$. Then, since $\mu_{|f|\chi_{A_{n_0}}}\leq\mu(A_{n_0})$ pointwise, by (4) it follows that

$$\|f\chi_{A_{n_0}}\|_\lambda = \int_I \Phi(\mu_{|f|\chi_{A_{n_0}}})\,dm \geq \Phi'(x_0)\int_I \mu_{|f|\chi_{A_{n_0}}}\,dm = \Phi'(x_0)\int_\Omega |f|\chi_{A_{n_0}}\,d\mu.$$

Hence $f = f\chi_{\Omega\setminus A_{n_0}} + f\chi_{A_{n_0}}$ with $f\chi_{\Omega\setminus A_{n_0}} \in L^\infty(\mu)$ and $f\chi_{A_{n_0}} \in L^1(\mu)$ and so we have that $L^1(\lambda) \subset L^\infty(\mu) + L^1(\mu)$.

Finally note that in the case when $\Phi(\infty) = \infty$ or μ is finite it follows that λ is continuous from above at $\emptyset$ and so $L^1(\lambda)$ is σ-order continuous having $\mathcal{S}_\mu$ ($\mathcal{S}$ if μ is finite) as a dense subspace.

4. L^1-spaces associated to a family of capacities

Let $\mathcal{F} = (\lambda_\alpha)_{\alpha\in\Delta}$ be a family of capacities on Σ satisfying the properties (P1,2,6) and being *uniformly quasi-subadditive*, that is, there is a constant $K \geq 1$ such that

$$\lambda_\alpha(A \cup B) \leq K\big(\lambda_\alpha(A) + \lambda_\alpha(B)\big)$$

for all $A, B \in \Sigma$ and $\alpha \in \Delta$. For each $\alpha \in \Delta$, by Theorem 6, we have that

$$L^1(\lambda_\alpha) = \Big\{ f \in L^0(\lambda_\alpha) : \|f\|_{\lambda_\alpha} = \int_I (\lambda_\alpha)_{|f|}\,dm < \infty \Big\}$$

is a λ_α-quasi-B.f.s. with the σ-Fatou property and $\|\cdot\|_{\lambda_\alpha}$ is a quasi-norm on it.

Consider the set function $\|\mathcal{F}\|\colon \Sigma \to [0,\infty]$ given by

$$\|\mathcal{F}\|(A) = \sup_{\alpha\in\Delta} \lambda_\alpha(A)$$

for $A \in \Sigma$. Note that $\|\mathcal{F}\|$ is a capacity and a set is $\|\mathcal{F}\|$-null if and only if it is λ_α-null for all $\alpha \in \Delta$.

Proposition 10. *The capacity $\|\mathcal{F}\|$ satisfies the properties (P1,2,3,6).*

Proof. It is clear that $\|\mathcal{F}\|$ is increasing (as each λ_α is so) and quasi-subadditive with the constant K of the uniform quasi-subadditivity of $\mathcal{F}$.

Let $A, Z \in \Sigma$ with Z being $\|\mathcal{F}\|$-null. Since each λ_α is null-additive and Z is λ_α-null, we have that $\lambda_\alpha(A \cup Z) = \lambda_\alpha(A)$. Then $\|\mathcal{F}\|(A \cup Z) = \|\mathcal{F}\|(A)$ and hence $\|\mathcal{F}\|$ is null-additive.

Let $A_n, A \in \Sigma$ be such that $A_n \uparrow A$. Since $\|\mathcal{F}\|$ is increasing we have that $\|\mathcal{F}\|(A_n) \uparrow$ and $\|\mathcal{F}\|(A_n) \leq \|\mathcal{F}\|(A)$ for all n. Then $\lim \|\mathcal{F}\|(A_n) \leq \|\mathcal{F}\|(A)$. On the other hand, since each λ_α is continuous from below, we have that

$$\lambda_\alpha(A) = \lim \lambda_\alpha(A_n) \leq \lim \|\mathcal{F}\|(A_n).$$

Then $\|\mathcal{F}\|(A) \leq \lim \|\mathcal{F}\|(A_n)$. That is, $\|\mathcal{F}\|(A_n) \uparrow \|\mathcal{F}\|(A)$ and so $\|\mathcal{F}\|$ is continuous from below. □

From Theorem 6 we have that

$$L^1(\|\mathcal{F}\|) = \Big\{ f \in L^0(\|\mathcal{F}\|) : \|f\|_{\|\mathcal{F}\|} = \int_I \|\mathcal{F}\|_{|f|}\,dm < \infty \Big\}$$

is a $\|\mathcal{F}\|$-quasi-B.f.s. with the σ-Fatou property and $\|\cdot\|_{\|\mathcal{F}\|}$ is a quasi-norm on it. For each $\alpha \in \Delta$ and $f \in \mathcal{L}^0(\Omega)$, since $(\lambda_\alpha)_{|f|} \leq \|\mathcal{F}\|_{|f|}$ pointwise, we have that $\|f\|_{\lambda_\alpha} \leq \|f\|_{\|\mathcal{F}\|}$. On the other hand, since every $\|\mathcal{F}\|$-null set is λ_α-null, the map $[i]$ which takes a $\|\mathcal{F}\|$-a.e. class in $L^0(\|\mathcal{F}\|)$ represented by f into the λ_α-a.e. class in $L^0(\lambda_\alpha)$ represented by the same f is well-defined. Then, $[i]$ takes $L^1(\|\mathcal{F}\|)$ into $L^1(\lambda_\alpha)$ and we will write $L^1(\|\mathcal{F}\|) \subset_{[i]} L^1(\lambda_\alpha)$.

Let us create an intermediate space between $L^1(\|\mathcal{F}\|)$ and each $L^1(\lambda_\alpha)$. Consider the map $\|\cdot\|_{\mathcal{F}}\colon \mathcal{L}^0(\Omega) \to [0,\infty]$ defined by

$$\|f\|_{\mathcal{F}} = \sup_{\alpha\in\Delta} \|f\|_{\lambda_\alpha}$$

for $f \in \mathcal{L}^0(\Omega)$. The following proposition shows that $\|\cdot\|_{\mathcal{F}}$ is a $\|\mathcal{F}\|$-quasi-norm function with the σ-Fatou property.

Proposition 11. *The following statements hold:*

(a) $\|f\|_{\mathcal{F}} \le \|g\|_{\mathcal{F}}$ *whenever* $f, g \in \mathcal{L}^0(\Omega)$ *with* $|f| \le |g|$ $\|\mathcal{F}\|$*-a.e.*
(b) $\|f\|_{\mathcal{F}} = 0$ *if and only if* $f = 0$ $\|\mathcal{F}\|$*-a.e.*
(c) $\|af\|_{\mathcal{F}} = |a|\,\|f\|_{\mathcal{F}}$ *for all* $a \in \mathbb{R}$ *and* $f \in \mathcal{L}^0(\Omega)$.
(d) $\|f + g\|_{\mathcal{F}} \le 2K\big(\|f\|_{\mathcal{F}} + \|g\|_{\mathcal{F}}\big)$ *for all* $f, g \in \mathcal{L}^0(\Omega)$, *with* K *being the constant of the uniform quasi-subadditivity of* $\mathcal{F}$.
(e) $\|f_n\|_{\mathcal{F}} \uparrow \|f\|_{\mathcal{F}}$ *whenever* $f_n, f \in \mathcal{L}^0(\Omega)$ *with* $0 \le f_n \uparrow f$ $\|\mathcal{F}\|$*-a.e.*

Proof. By Proposition 5, each map $\|\cdot\|_{\lambda_\alpha} : \mathcal{L}^0(\Omega) \to [0,\infty]$ is a λ_α-quasi-norm function with the σ-Fatou property.

(a) If $f, g \in \mathcal{L}^0(\Omega)$ are such that $|f| \le |g|$ $\|\mathcal{F}\|$-a.e., then for each $\alpha \in \Delta$ we have that $|f| \le |g|$ λ_α-a.e. and so $\|f\|_{\lambda_\alpha} \le \|g\|_{\lambda_\alpha}$. Hence, $\|f\|_{\mathcal{F}} \le \|g\|_{\mathcal{F}}$.

(b) Recall that for any capacity λ we have that $f = 0$ λ-a.e. if and only if $\mathrm{supp}(f)$ is λ-null. Then, $\|f\|_{\mathcal{F}} = 0$ or equivalently $\|f\|_{\lambda_\alpha} = 0$ for all $\alpha \in \Delta$, if and only if $\mathrm{supp}(f)$ is λ_α-null for all $\alpha \in \Delta$, that is, $\mathrm{supp}(f)$ is $\|\mathcal{F}\|$-null.

(c) Clear as $\|af\|_{\lambda_\alpha} = |a|\,\|f\|_{\lambda_\alpha}$ for all $a \in \mathbb{R}$, $f \in \mathcal{L}^0(\Omega)$ and $\alpha \in \Delta$.

(d) Given $f, g \in \mathcal{L}^0(\Omega)$, since the constant K of the uniform quasi-subadditivity of $\mathcal{F}$ is the constant of the quasi-subadditivity of each λ_α, we have that $\|f+g\|_{\lambda_\alpha} \le 2K(\|f\|_{\lambda_\alpha} + \|g\|_{\lambda_\alpha})$. Hence, $\|f+g\|_{\mathcal{F}} \le 2K(\|f\|_{\mathcal{F}} + \|g\|_{\mathcal{F}})$.

(e) Let $f_n, f \in \mathcal{L}^0(\Omega)$ be such that $0 \le f_n \uparrow f$ $\|\mathcal{F}\|$-a.e. Since $\|f_n\|_{\mathcal{F}} \uparrow$ and $\|f_n\|_{\mathcal{F}} \le \|f\|_{\mathcal{F}}$, we have that $\lim \|f_n\|_{\mathcal{F}} \le \|f\|_{\mathcal{F}}$. On the other hand, for each $\alpha \in \Delta$ we have that $0 \le f_n \uparrow f$ λ_α-a.e. and so

$$\|f\|_{\lambda_\alpha} = \lim \|f_n\|_{\lambda_\alpha} \le \lim \|f_n\|_{\mathcal{F}}.$$

Then $\|f\|_{\mathcal{F}} \le \lim \|f_n\|_{\mathcal{F}}$. That is, $\|f_n\|_{\mathcal{F}} \uparrow \|f\|_{\mathcal{F}}$. $\square$

Propositions 1 and 11 yield the following result.

Theorem 12. *The space*

$$L^1(\mathcal{F}) = \Big\{ f \in L^0(\|\mathcal{F}\|) :\ \|f\|_{\mathcal{F}} = \sup_{\alpha\in\Delta} \|f\|_{\lambda_\alpha} < \infty \Big\}$$

is a $\|\mathcal{F}\|$*-quasi-B.f.s. with the* σ*-Fatou property and* $\|\cdot\|_{\mathcal{F}}$ *is a quasi-norm on it.*

For every $\alpha \in \Delta$ and $f \in \mathcal{L}^0(\Omega)$ we have that $\|f\|_{\lambda_\alpha} \le \|f\|_{\mathcal{F}} \le \|f\|_{\|\mathcal{F}\|}$. So,

$$\mathcal{S}_{\|\mathcal{F}\|} \subset L^1(\|\mathcal{F}\|) \subset L^1(\mathcal{F}) \subset_{[i]} L^1(\lambda_\alpha).$$

Note that $\|\chi_A\|_{\mathcal{F}} = \|\chi_A\|_{\|\mathcal{F}\|} = \|\mathcal{F}\|(A)$ for all $A \in \Sigma$, as $\|\chi_A\|_\lambda = \lambda(A)$ for any capacity λ.

Let us see that particular conditions on the family $\mathcal{F}$ make that $L^1(\mathcal{F})$ is the "intersection" of all the spaces $L^1(\lambda_\alpha)$ in the sense:

$$L^1(\mathcal{F}) = \{ f \in L^0(\|\mathcal{F}\|) :\ f \in L^1(\lambda_\alpha)\ \text{ for all } \alpha \in \Delta \}.$$

Let E be a real Banach space with norm $\|\cdot\|_E$ and denote by B_E the closed unit ball of E. Assume that each $\alpha \in E$ is associated to an increasing capacity λ_α on Σ in a way that:

(E1) $\lambda_{a\alpha} = |a|\lambda_\alpha$ for all $a \in \mathbb{R}$ and $\alpha \in E$.
(E2) $\lambda_{\alpha+\beta} \le \lambda_\alpha + \lambda_\beta$ for all $\alpha, \beta \in E$.
(E3) If $\alpha_n \to \alpha$ in E then $\lambda_\alpha \le \liminf \lambda_{\alpha_n}$.

Suppose that $N \subset B_E$ satisfies the following properties:

(N1) For every $\alpha \in B_E$ there exists $\beta \in N$ such that $\lambda_\alpha \le \lambda_\beta$.
(N2) For each $\alpha \in N$ the capacity λ_α is null-additive and continuous from below.
(N3) There exists $K \ge 1$ such that $\lambda_\alpha(A \cup B) \le K(\lambda_\alpha(A) + \lambda_\alpha(B))$ for all $A, B \in \Sigma$ and $\alpha \in N$.

Consider the uniformly quasi-subadditive family $\mathcal{F} = (\lambda_\alpha)_{\alpha\in N}$ of capacities which satisfy the properties (P1,2,6). Note that the map $\|\cdot\|_{\lambda_\alpha}\colon \mathcal{L}^0(\Omega) \to [0,\infty]$ can be considered for all $\alpha \in E$ but we only can assure that $L^1(\lambda_\alpha)$ is a λ_α-quasi-B.f.s. with quasi-norm $\|\cdot\|_{\lambda_\alpha}$ if $\alpha \in N$.

Theorem 13. *For $f \in \mathcal{L}^0(\Omega)$ the following statements are equivalent:*

(a) $\|f\|_{\mathcal{F}} < \infty$.
(b) $\|f\|_{\lambda_\alpha} < \infty$ *for all* $\alpha \in N$.
(c) $\|f\|_{\lambda_\alpha} < \infty$ *for all* $\alpha \in E$.

Proof. (a) $\Rightarrow$ (b) Clear as $\|f\|_{\mathcal{F}} = \sup_{\alpha\in N}\|f\|_{\lambda_\alpha}$.

(b) $\Rightarrow$ (c) For each non-null $\alpha \in E$, by (E1) and (N1) we have that

$$\lambda_\alpha = \|\alpha\|_E \lambda_{\frac{\alpha}{\|\alpha\|_E}} \le \|\alpha\|_E \lambda_\beta$$

for some $\beta \in N$ and so $\|f\|_{\lambda_\alpha} \le \|\alpha\|_E \cdot \|f\|_{\lambda_\beta} < \infty$. Note that for $\alpha = 0$, by (E1) we have that $\lambda_\alpha = 0$ and so $\|\cdot\|_{\lambda_\alpha} = 0$.

(c) $\Rightarrow$ (a) For every $k \in \mathbb{N}$ denote

$$F_k = \{\alpha \in E : \|f\|_{\lambda_\alpha} \le k\}.$$

Let us see that F_k is closed in E. Consider a sequence $(\alpha_n) \subset F_k$ satisfying that $\alpha_n \to \alpha$ in E. By (E3) we have that $(\lambda_\alpha)_{|f|} \le \liminf(\lambda_{\alpha_n})_{|f|}$ pointwise. Then, applying the Fatou lemma for the Lebesgue integral with respect to m, it follows that

$$\|f\|_{\lambda_\alpha} = \int_I (\lambda_\alpha)_{|f|}\, dm \le \int_I \liminf(\lambda_{\alpha_n})_{|f|}\, dm \le \liminf \int_I (\lambda_{\alpha_n})_{|f|}\, dm \le k$$

and so $\alpha \in F_k$. On the other hand, since $E = \cup F_k$, from the Baire theorem there exists F_{k_0} with a non-void interior, that is, $B(\alpha_0, r_0) \subset F_{k_0}$ for some closed ball of E centered at α_0 with radius $r_0 > 0$. Let $\alpha \in N \subset B_E$ and take $\beta = \alpha_0 + r_0\alpha \in B(\alpha_0, r_0)$. From (E1) and (E2) it follows that $(\lambda_\alpha)_{|f|} \le \frac{1}{r_0}((\lambda_\beta)_{|f|} + (\lambda_{\alpha_0})_{|f|})$ pointwise. Since $\beta, \alpha_0 \in F_{k_0}$, we have that

$$\|f\|_{\lambda_\alpha} = \int_I (\lambda_\alpha)_{|f|}\, dm \le \frac{1}{r_0}\Big(\int_I (\lambda_\beta)_{|f|}\, dm + \int_I (\lambda_{\alpha_0})_{|f|}\, dm\Big) \le \frac{2k_0}{r_0}$$

and so $\|f\|_{\mathcal{F}} \le \frac{2k_0}{r_0} < \infty$. $\square$

From Theorem 13 we have that

$$\begin{aligned} L^1(\mathcal{F}) &= \{f \in L^0(\|\mathcal{F}\|) : \|f\|_{\lambda_\alpha} < \infty \ \text{ for all } \alpha \in N\} \\ &= \{f \in L^0(\|\mathcal{F}\|) : \|f\|_{\lambda_\alpha} < \infty \ \text{ for all } \alpha \in E\}. \end{aligned}$$

Moreover, from (N1) it follows that $\|f\|_{\mathcal{F}} = \sup_{\alpha\in B_E}\|f\|_{\lambda_\alpha}$ for all $f \in \mathcal{L}^0(\Omega)$. This means that $L^1(\mathcal{F})$ is independent of N, that is, any other subset of B_E with the same properties as N gives the same space. Also $L^1(\|\mathcal{F}\|)$ and $S_{\|\mathcal{F}\|}$ are independent of N, as $\|\mathcal{F}\|(A) = \sup_{\alpha\in B_E}\lambda_\alpha(A)$ for all $A \in \Sigma$.

Remark 14. (I) The only simple functions in $L^1(\mathcal{F})$ are those of $S_{\|\mathcal{F}\|}$. Indeed, if $\varphi \in \mathcal{S}$ is such that $\|\varphi\|_{\lambda_\alpha} < \infty$ for all $\alpha \in N$, then $\|\chi_{\mathrm{supp}(\varphi)}\|_{\lambda_\alpha} = \lambda_\alpha(\mathrm{supp}(\varphi)) < \infty$ for all $\alpha \in N$ and so $\|\mathcal{F}\|(\mathrm{supp}(\varphi)) = \|\chi_{\mathrm{supp}(\varphi)}\|_{\mathcal{F}} < \infty$.

(II) Let $a \in \mathbb{R}$ and $\alpha \in N$. By (E1) the increasing capacity $\lambda_{a\alpha}$ is null-additive, continuous from below and satisfies (N3). Then we can consider the $\lambda_{a\alpha}$-quasi-B.f.s. $L^1(\lambda_{a\alpha})$ which satisfies $L^1(\mathcal{F}) \subset_{[i]} L^1(\lambda_{a\alpha})$ with $\|f\|_{\lambda_{a\alpha}} \le |a|\,\|f\|_{\mathcal{F}}$.

5. w-L^1-spaces associated to a vector capacity

Let X be a real Banach space with norm $\|\cdot\|_X$ and $\Lambda\colon\Sigma\to X$ a set function satisfying that $\Lambda(\emptyset)=0$. Such a set function Λ will be called a *vector capacity*. A set $Z\in\Sigma$ is Λ-*null* if $\Lambda(A)=0$ for all $A\in\Sigma$ such that $A\subset Z$. For each x^* belonging to the topological dual X^* of X we consider the real capacity $x^*\Lambda\colon\Sigma\to\mathbb{R}$ defined by $x^*\Lambda(A)=\langle x^*,\Lambda(A)\rangle$ for $A\in\Sigma$. The *semivariation* of Λ is the increasing capacity $\|\Lambda\|\colon\Sigma\to[0,\infty]$ defined by

$$\|\Lambda\|(A)=\sup_{x^*\in B_{X^*}}|x^*\Lambda|(A)$$

for $A\in\Sigma$, where B_{X^*} denotes the closed unit ball of X^* and $|x^*\Lambda|$ is the variation of $x^*\Lambda$. The *quasi-variation* of Λ is the increasing capacity $|||\Lambda|||\colon\Sigma\to[0,\infty]$ defined by

$$|||\Lambda|||(A)=\sup\{\|\Lambda(B)\|_X : B\in\Sigma \text{ with } B\subset A\}$$

for $A\in\Sigma$. Note that $|||\Lambda|||(A)\le\|\Lambda\|(A)$ and $|||\Lambda|||(A)=\sup_{x^*\in B_{X^*}}q_{x^*\Lambda}(A)$ for all $A\in\Sigma$, where $q_{x^*\Lambda}$ is the quasi-variation of $x^*\Lambda$. It is routine to check that the Λ-null, $\|\Lambda\|$-null and $|||\Lambda|||$-null sets are the same. So, in the case when the identification works we denote $L^0(\|\Lambda\|)=L^0(|||\Lambda|||)$ by $L^0(\Lambda)$. Moreover, since the notions of $\|\Lambda\|$-quasi-B.f.s. and $|||\Lambda|||$-quasi-B.f.s. coincide, we will refer to them as Λ-quasi-B.f.s. In the case of a real capacity $\xi\colon\Sigma\to\mathbb{R}$ it follows that $\|\xi\|=|\xi|$ and $|||\xi|||=q_\xi$.

5.1. The space w-$L^1_v(\Lambda)$

Let $N\subset B_{X^*}$ satisfy the following properties:

(vN1) For every $x^*\in B_{X^*}$ there exists $y^*\in N$ such that $|x^*\Lambda|\le|y^*\Lambda|$.
(vN2) $|x^*\Lambda|$ is null-additive and continuous from below for all $x^*\in N$.
(vN3) There exists a constant $K\ge1$ such that

$$|x^*\Lambda|(A\cup B)\le K(|x^*\Lambda|(A)+|x^*\Lambda|(B))$$

for all $A,B\in\Sigma$ and $x^*\in N$.

Each $x^*\in X^*$ is associated to the increasing capacity $|x^*\Lambda|$. From Lemma 2.(e) it follows that:

(vE1) $|(ax^*)\Lambda|=|a|\,|x^*\Lambda|$ for all $a\in\mathbb{R}$ and $x^*\in X^*$.
(vE2) $|(x^*+y^*)\Lambda|\le|x^*\Lambda|+|y^*\Lambda|$ for all $x^*,y^*\in X^*$.
(vE3) If $x^*_n\to x^*$ in X^* then $|x^*\Lambda|\le\liminf|x^*_n\Lambda|$.

Consider the uniformly quasi-subadditive family $\mathcal{F}=(|x^*\Lambda|)_{x^*\in N}$ of capacities satisfying the properties (P1,2,6). Noting that $\|\mathcal{F}\|=\|\Lambda\|$ and denoting by w-$L^1_v(\Lambda)$ the space $L^1(\mathcal{F})$, from all what we have seen in Section 4 we obtain the next conclusions.

Theorem 15. *The following statements hold:*

(a) $L^1(|x^*\Lambda|)=\{f\in L^0(|x^*\Lambda|) : \|f\|_{|x^*\Lambda|}=\int_I|x^*\Lambda|_{|f|}\,dm<\infty\}$ *is a* $|x^*\Lambda|$*-quasi-B.f.s. with the* σ*-Fatou property and* $\|\cdot\|_{|x^*\Lambda|}$ *is a quasi-norm on it for every* $x^*\in\widehat{N}=\{ay^* : a\in\mathbb{R},\ y^*\in N\}$.
(b) $\|\Lambda\|$ *is a capacity satisfying the properties (P1,2,3,6).*
(c) $L^1(\|\Lambda\|)=\{f\in L^0(\Lambda) : \|f\|_{\|\Lambda\|}=\int_I\|\Lambda\|_{|f|}\,dm<\infty\}$ *is a* Λ*-quasi-B.f.s. with the* σ*-Fatou property and* $\|\cdot\|_{\|\Lambda\|}$ *is a quasi-norm on it.*
(d) w-$L^1_v(\Lambda)=\{f\in L^0(\Lambda) : \|f\|_{|x^*\Lambda|}<\infty$ *for all* $x^*\in X^*\}$ *is a* Λ*-quasi-B.f.s. with the* σ*-Fatou property and a quasi-norm on it is given by*

$$\|f\|_v=\sup_{x^*\in B_{X^*}}\|f\|_{|x^*\Lambda|}.$$

(e) $\|x^*\|^{-1}_{X^*}\|f\|_{|x^*\Lambda|}\le\|f\|_v\le\|f\|_{\|\Lambda\|}$ *for all* $f\in\mathcal{L}^0(\Omega)$*, and* $x^*\in X^*$ *non-null.*

(f) $\mathcal{S}_{\|\Lambda\|} \subset L^1(\|\Lambda\|) \subset w\text{-}L^1_v(\Lambda) \subset_{[i]} L^1(|x^*\Lambda|)$ *for all* $x^* \in \widehat{N}$.

For a real capacity $\xi\colon \Sigma \to \mathbb{R}$ the existence of a set N satisfying (vN1,2,3) is equivalent to $|\xi|$ having the properties (P2,3,6). Moreover, in this case it follows that $L^1(\|\xi\|) = w\text{-}L^1_v(\xi) = L^1(|\xi|)$ with equals norms.

5.2. *The space* $w\text{-}L^1_{qv}(\Lambda)$

Let $N \subset B_{X^*}$ satisfy the following properties:

(qvN1) For every $x^* \in B_{X^*}$ there exists $y^* \in N$ such that $q_{x^*\Lambda} \le q_{y^*\Lambda}$.
(qvN2) $q_{x^*\Lambda}$ is null-additive and continuous from below for all $x^* \in N$.
(qvN3) There exists a constant $K \ge 1$ such that

$$q_{x^*\Lambda}(A \cup B) \le K(q_{x^*\Lambda}(A) + q_{x^*\Lambda}(B))$$

for all $A, B \in \Sigma$ and $x^* \in N$.

Each $x^* \in X^*$ is associated to the increasing capacity $q_{x^*\Lambda}$. From Lemma 2.(e) it follows that:

(qvE1) $q_{(ax^*)\Lambda} = |a|\, q_{x^*\Lambda}$ for all $a \in \mathbb{R}$ and $x^* \in X^*$.
(qvE2) $q_{(x^*+y^*)\Lambda} \le q_{x^*\Lambda} + q_{y^*\Lambda}$ for all $x^*, y^* \in X^*$.
(qvE3) If $x_n^* \to x^*$ in X^* then $q_{x^*\Lambda} \le \liminf q_{x_n^*\Lambda}$.

Consider the uniformly quasi-subadditive family $\mathcal{F} = (q_{x^*\Lambda})_{x^* \in N}$ of capacities satisfying the properties (P1,2,6). Noting that the $q_{x^*\Lambda}$-null and the $|x^*\Lambda|$-null sets coincide and $\|\mathcal{F}\| = |||\Lambda|||$, denoting by $w\text{-}L^1_{qv}(\Lambda)$ the space $L^1(\mathcal{F})$, from all what we have seen in Section 4 we obtain the next conclusions.

Theorem 16. *The following statements hold:*

(a) $L^1(q_{x^*\Lambda}) = \{f \in L^0(|x^*\Lambda|) : \|f\|_{q_{x^*\Lambda}} = \int_I (q_{x^*\Lambda})_{|f|}\, dm < \infty\}$ *is a* $|x^*\Lambda|$*-quasi-B.f.s. with the* σ*-Fatou property and* $\|\cdot\|_{q_{x^*\Lambda}}$ *is a quasi-norm on it for every* $x^* \in \widehat{N} = \{ay^* : a \in \mathbb{R},\ y^* \in N\}$.
(b) $|||\Lambda|||$ *is a capacity satisfying the properties (P1,2,3,6).*
(c) $L^1(|||\Lambda|||) = \{f \in L^0(\Lambda) : \|f\|_{|||\Lambda|||} = \int_I |||\Lambda|||_{|f|}\, dm < \infty\}$ *is a* Λ*-quasi-B.f.s. with the* σ*-Fatou property and* $\|\cdot\|_{|||\Lambda|||}$ *is a quasi-norm on it.*
(d) $w\text{-}L^1_{qv}(\Lambda) = \{f \in L^0(\Lambda) : \|f\|_{q_{x^*\Lambda}} < \infty$ *for all* $x^* \in X^*\}$ *is a* Λ*-quasi-B.f.s. with the* σ*-Fatou property and a quasi-norm on it is given by*

$$\|f\|_{qv} = \sup_{x^* \in B_{X^*}} \|f\|_{q_{x^*\Lambda}}.$$

(e) $\|x^*\|_{X^*}^{-1}\|f\|_{q_{x^*\Lambda}} \le \|f\|_{qv} \le \|f\|_{|||\Lambda|||}$ *for all* $f \in \mathcal{L}^0(\Omega)$ *and* $x^* \in X^*$ *non-null.*
(f) $\mathcal{S}_{|||\Lambda|||} \subset L^1(|||\Lambda|||) \subset w\text{-}L^1_{qv}(\Lambda) \subset_{[i]} L^1(q_{x^*\Lambda})$ *for all* $x^* \in \widehat{N}$.

For a real capacity $\xi\colon \Sigma \to \mathbb{R}$ the existence of a set N satisfying (qvN1,2,3) is equivalent to q_ξ having the properties (P2,3,6). Moreover, in this case it follows that $L^1(|||\xi|||) = w\text{-}L^1_{qv}(\xi) = L^1(q_\xi)$ with equals norms. Note that if ξ is positive and satisfies (P1,2,3,6), then $q_\xi = \xi$ and so $L^1(q_\xi) = L^1(\xi)$.

5.3. *Relation between* $w\text{-}L^1_v(\Lambda)$ *and* $w\text{-}L^1_{qv}(\Lambda)$

Let $N \subset B_{X^*}$ satisfy the properties (qvN1,2,3). By Lemma 2.(g) we have that N has the properties (vN2,3). Let us see that (vN1) also holds. Let $x^* \in B_{X^*}$ and $y^* \in N$ be such that $q_{x^*\Lambda} \le q_{y^*\Lambda}$. For every partition $(A_i)_{i=1}^n \subset \Sigma$ of a set $A \in \Sigma$, recalling that the variation of a real capacity is always superadditive, we have that

$$\sum_{i=1}^n |x^*\Lambda(A_i)| \le \sum_{i=1}^n q_{x^*\Lambda}(A_i) \le \sum_{i=1}^n q_{y^*\Lambda}(A_i) \le \sum_{i=1}^n |y^*\Lambda|(A_i) \le |y^*\Lambda|(A)$$

and so $|x^*\Lambda|(A) \le |y^*\Lambda|(A)$. Then, we can consider all the spaces given by Theorems 15 and 16. For $f \in \mathcal{L}^0(\Omega)$ and $x^* \in X^*$, since $q_{x^*\Lambda} \le |x^*\Lambda|$, we have that $\|f\|_{q_{x^*\Lambda}} \le \|f\|_{|x^*\Lambda|}$ and so $\|f\|_{qv} \le \|f\|_v$. On the other hand, since $|||\Lambda||| \le \|\Lambda\|$, we have that $\|f\|_{|||\Lambda|||} \le \|f\|_{\|\Lambda\|}$. Therefore, the following containments hold:

$$\begin{array}{ccccccc} \mathcal{S}_{\|\Lambda\|} & \subset & L^1(\|\Lambda\|) & \subset & w\text{-}L^1_v(\Lambda) & \subset_{[i]} & L^1(|x^*\Lambda|) \\ \cap & & \cap & & \cap & & \cap \\ \mathcal{S}_{|||\Lambda|||} & \subset & L^1(|||\Lambda|||) & \subset & w\text{-}L^1_{qv}(\Lambda) & \subset_{[i]} & L^1(q_{x^*\Lambda}) \end{array} \tag{5}$$

with equality in the vertical inclusions $\cap$ if there exists $C \ge 1$ such that $|x^*\Lambda| \le C\, q_{x^*\Lambda}$ for every $x^* \in N$.

For a real capacity $\xi\colon \Sigma \to \mathbb{R}$ with quasi-variation q_ξ satisfying the properties (P2,3,6) it follows that $L^1(|\xi|) \subset L^1(q_\xi)$, with equality only in the case when there exists $C \ge 1$ such that $|\xi| \le C q_\xi$ (for instance if ξ is a measure).

Remark 17. In the case when Λ is a vector measure, it is direct to check that the set $N = B_{X^*}$ satisfies the properties (qvN1,2,3). Since $|x^*\Lambda| \le 2\, q_{x^*\Lambda}$ for all $x^* \in X^*$, all the vertical inclusions in (5) are equalities. From Remark 7 it follows that $w\text{-}L^1_v(\Lambda) = w\text{-}L^1_{qv}(\Lambda)$ is just the space of *weakly integrable* functions with respect to Λ, see [34]. Note that in this case $\|\Lambda\|$ is finite and so $\mathcal{S}_{\|\Lambda\|} = \mathcal{S}$.

6. Integral map for a vector capacity

Let $\Lambda\colon \Sigma \to X$ be a vector capacity and let $N \subset B_{X^*}$ satisfy the properties (qvN1,2,3). Assume that Λ has the following properties:

(Λ1) Λ is *weakly continuous from below*, that is, $\Lambda(A_n) \to \Lambda(A)$ weakly in X whenever $A_n, A \in \Sigma$ with $A_n \uparrow A$.
(Λ2) Λ is *null-additive*, that is, $\Lambda(A \cup Z) = \Lambda(A)$ for all $A, Z \in \Sigma$ with Z Λ-null.

For $f \in \mathcal{L}^0(\Omega)^+$ the *distribution function* of f with respect to Λ is defined as the map $\Lambda_f\colon I \to X$ given by

$$\Lambda_f(t) = \Lambda(\{\omega \in \Omega : f(\omega) > t\})$$

for $t \in I$. Every $x^* \in X^*$ yields the map $x^*\Lambda_f\colon I \to \mathbb{R}$ defined by $x^*\Lambda_f(t) = \langle x^*, \Lambda_f(t)\rangle$ for $t \in I$. Our goal is to construct an integration map for Λ on $w\text{-}L^1_{qv}(\Lambda)$ through the Lebesgue integrals of $x^*\Lambda_f$. Note that if N satisfies (vN1,2,3) instead of (qvN1,2,3), the same construction works on $w\text{-}L^1_v(\Lambda)$. For this aim the properties (Λ1,2) are crucial as they guarantee the next results.

Lemma 18. *The following statements hold:*

(a) The map $x^*\Lambda_f$ is measurable for all $f \in \mathcal{L}^0(\Omega)^+$ and $x^* \in X^*$.
(b) If $f, g \in \mathcal{L}^0(\Omega)^+$ are such that $f = g$ Λ-a.e. then $\Lambda_f = \Lambda_g$ pointwise.

Proof. (a) Let $\xi\colon \Sigma \to \mathbb{R}$ be a real capacity continuous from below, i.e. $\xi(A_n) \to \xi(A)$ whenever $A_n, A \in \Sigma$ with $A_n \uparrow A$. Every positive $\varphi \in \mathcal{S}$ with standard representation $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j}$ satisfies that

$$\xi_\varphi = \sum_{k=1}^n \xi\Big(\bigcup_{j=k}^n A_j\Big) \chi_{[\alpha_{k-1},\alpha_k)}$$

is measurable. For a general $f \in \mathcal{L}^0(\Omega)^+$, taking a sequence $(\varphi_n) \subset \mathcal{S}$ such that $0 \le \varphi_n \uparrow f$ pointwise, since

$$\{\omega \in \Omega : \varphi_n(\omega) > t\} \uparrow \{\omega \in \Omega : f(\omega) > t\}$$

for all $t \in I$, we have that $\xi_{\varphi_n} \to \xi_f$ pointwise with ξ_{φ_n} being measurable. So, ξ_f is measurable. By (Λ1) it follows that the real capacity $x^*\Lambda$ is continuous from below for each $x^* \in X^*$ and since $x^*\Lambda_f$ coincides with the distribution function $(x^*\Lambda)_f$ of f with respect to $x^*\Lambda$, we have the conclusion.

(b) Let $f, g \in \mathcal{L}^0(\Omega)^+$ be such that $f = g$ except on a Λ-null set Z. For every $t \in I$, denote $A_t = \{\omega \in \Omega : f(\omega) > t\}$ and $B_t = \{\omega \in \Omega : g(\omega) > t\}$. Noting that $A_t \cap Z$, $B_t \cap Z$ are Λ-null and $A_t \cap \Omega\backslash Z = B_t \cap \Omega\backslash Z$, by ($\Lambda$2) we have that

$$\Lambda(A_t) = \Lambda(A_t \cap \Omega \backslash Z) = \Lambda(B_t \cap \Omega \backslash Z) = \Lambda(B_t).$$

Then $\Lambda_f = \Lambda_g$ pointwise. $\square$

Denote

$$\mathcal{D}(\Lambda) = \left\{ f \in L^0(\Lambda) : \int_I |x^* \Lambda_{|f|}| \, dm < \infty \text{ for all } x^* \in X^* \right\},$$

that is, $\mathcal{D}(\Lambda)$ is the set of functions $f \in L^0(\Lambda)$ such that $\Lambda_{|f|}$ is *Dunford integrable* with respect to m, see [14, Ch. II, § 3]. For every $f \in \mathcal{D}(\Lambda)$ it follows that

$$\|f\|_{\mathcal{D}(\Lambda)} = \sup_{x^* \in B_{X^*}} \int_I |x^* \Lambda_{|f|}| \, dm < \infty,$$

see the proof of (c) $\Rightarrow$ (a) in Theorem 13.

Remark 19. Note that in general $\mathcal{D}(\Lambda)$ is not even a vector space. If we can find $M \subset B_{X^*}$ satisfying that for every $x^* \in B_{X^*}$ there exists $y^* \in M$ such that $|x^*\Lambda(\cdot)| \leq |y^*\Lambda(\cdot)|$ and $(|x^*\Lambda(\cdot)|)_{x^* \in M}$ is a uniformly quasi-subadditive family of capacities with the properties (P1,2,6), from Theorems 12 and 13 it follows that $\mathcal{D}(\Lambda)$ is a Λ-quasi-B.f.s. with quasi-norm $\|\cdot\|_{\mathcal{D}(\Lambda)}$. But in this case, $q_{x^*\Lambda} = |x^*\Lambda(\cdot)|$ for all $x^* \in M$ as $|x^*\Lambda(\cdot)|$ is increasing and M satisfies the properties (qvN1,2,3), so $w\text{-}L^1_{qv}(\Lambda) = \mathcal{D}(\Lambda)$ with equal norms. This is the reason why we do not consider this space in Section 5.

For a positive function $f \in \mathcal{D}(\Lambda)$ we define the *integral of f with respect to* Λ as the element $I_\Lambda(f) \in X^{**}$ given by

$$\langle I_\Lambda(f), x^* \rangle = \int_I x^* \Lambda_f \, dm$$

for all $x^* \in X^*$. That is, $I_\Lambda(f)$ is the *Dunford integral* of Λ_f with respect to m. Note that

$$|\langle I_\Lambda(f), x^* \rangle| \leq \|x^*\|_{X^*} \|f\|_{\mathcal{D}(\Lambda)}.$$

For a general function $f \in \mathcal{D}(\Lambda)$ we cannot use its positive and negative parts for defining $I_\Lambda(f)$ as they could not be in $\mathcal{D}(\Lambda)$, so we will consider functions in $w\text{-}L^1_{qv}(\Lambda)$. Note that $w\text{-}L^1_{qv}(\Lambda) \subset \mathcal{D}(\Lambda)$ with $\|f\|_{\mathcal{D}(\Lambda)} \leq \|f\|_{qv}$ for $f \in w\text{-}L^1_{qv}(\Lambda)$, as $|x^*\Lambda(\cdot)| \leq q_{x^*\Lambda}$ for all $x^* \in X^*$. For a general function $f \in w\text{-}L^1_{qv}(\Lambda)$ we define $I_\Lambda(f) = I_\Lambda(f^+) - I_\Lambda(f^-)$. By Lemma 18.(b) we have that $I_\Lambda(f) = I_\Lambda(g)$ whenever $f, g \in w\text{-}L^1_{qv}(\Lambda)$ with $f = g$ Λ-a.e. So the *integration map* $I_\Lambda \colon w\text{-}L^1_{qv}(\Lambda) \to X^{**}$ is well defined. However, the definition of I_Λ for a non positive function depends on its positive and negative parts as I_Λ is not additive in general. In fact it can be proved that the additivity is obtained only in the case when Λ is a vector measure on the δ-ring of sets $A \in \Sigma$ with $|||\Lambda|||(A) < \infty$. The following properties of I_Λ will be used later.

Lemma 20. *The following statements hold:*

(a) I_Λ is homogeneous, that is, $I_\Lambda(af) = aI_\Lambda(f)$ for all $f \in w\text{-}L^1_{qv}(\Lambda)$ and $a \in \mathbb{R}$.
(b) If $f_n, f \in w\text{-}L^1_{qv}(\Lambda)$ are such that $0 \leq f_n \uparrow f$ Λ-a.e. then $I_\Lambda(f_n) \to I_\Lambda(f)$ in the weak topology of X^{**}.*

Proof. (a) For a positive function $f \in w\text{-}L^1_{qv}(\Lambda)$ and $a \geq 0$, by making an appropriate change of variables it follows that $\int_I x^*\Lambda_{af} \, dm = a \int_I x^*\Lambda_f \, dm$ for all $x^* \in X^*$ and so $I_\Lambda(af) = aI_\Lambda(f)$. For general f and a only note that $(af)^+ = af^+$, $(af)^- = af^-$ if $a \geq 0$ and $(af)^+ = -af^-$, $(af)^- = -af^+$ if $a < 0$.

(b) Let $f_n, f \in w\text{-}L^1_{qv}(\Lambda)$ be such that $0 \leq f_n\chi_A \uparrow f\chi_A$ pointwise with $A \in \Sigma$ being such that $\Omega \backslash A$ is Λ-null. Since

$$\{\omega \in \Omega : (f_n\chi_A)(\omega) > t\} \uparrow \{\omega \in \Omega : (f\chi_A)(\omega) > t\}$$

for all $t \in I$, from ($\Lambda 1$) it follows that $x^*\Lambda_{f_n\chi_A} \to x^*\Lambda_{f\chi_A}$ pointwise for every $x^* \in X^*$. On the other hand, since $|x^*\Lambda_{f_n\chi_A}| \le (q_{x^*\Lambda})_{|f|}$ pointwise, by applying the dominated convergence theorem for the Lebesgue integral with respect to m we have that

$$\langle I_\Lambda(f_n\chi_A), x^*\rangle = \int_I x^*\Lambda_{f_n\chi_A}\, dm \to \int_I x^*\Lambda_{f\chi_A}\, dm = \langle I_\Lambda(f\chi_A), x^*\rangle.$$

Since $I_\Lambda(f_n) = I_\Lambda(f_n\chi_A)$ and $I_\Lambda(f) = I_\Lambda(f\chi_A)$ the conclusion follows. $\square$

If $\xi\colon \Sigma \to \mathbb{R}$ is a real capacity continuous from below, null-additive and q_ξ satisfies (P3), in which case q_ξ also satisfies (P2, 6), the integration map $I_\xi\colon L^1(q_\xi) \to \mathbb{R}$ is given by $I_\xi(f) = \int_I \xi_{f^+}\, dm - \int_I \xi_{f^-}\, dm$ for all $f \in L^1(q_\xi)$.

Remark 21. In the case when $x^* \in X^*$ is such that $x^*\Lambda$ is null-additive and $q_{x^*\Lambda}$ satisfies (P3), since $x^*\Lambda$ is continuous from below by ($\Lambda 1$), it follows that $\langle I_\Lambda(f), x^*\rangle = I_{x^*\Lambda}(f)$ for all $f \in w\text{-}L^1_{qv}(\Lambda) \subset L^1(q_{x^*\Lambda})$.

We have defined I_Λ by using Dunford integration, a natural question now is what is the role of the *Bochner integration* in this play. For Bochner integration theory we refer to [2, Ch. 11, § 8]. Denote by j the canonical embedding of X into X^{**} and

$$\mathcal{B}(\Lambda) = \{f \in L^0(\Lambda) :\ \Lambda_{|f|} \text{ is Bochner integrable}\}.$$

Note that

$$\mathcal{B}(\Lambda) \subset \Big\{ f \in L^0(\Lambda) : \int_I \|\Lambda_{|f|}\|_X\, dm < \infty \Big\} \subset \mathcal{D}(\Lambda). \tag{6}$$

Every positive $\varphi \in \mathcal{S}$ with standard representation $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j}$ satisfies that

$$\Lambda_\varphi = \sum_{k=1}^n \Lambda(\cup_{j=k}^n A_j)\chi_{[\alpha_{k-1},\alpha_k)} \tag{7}$$

is an X-step function and so Bochner integrable with respect to m with Bochner integral $\int_I \Lambda_\varphi\, dm = \sum_{k=1}^n \Lambda(\cup_{j=k}^n A_j)(\alpha_k - \alpha_{k-1}) \in X$. Hence, $\varphi \in \mathcal{B}(\Lambda)$ with $I_\Lambda(\varphi) = j(\int_I \Lambda_\varphi\, dm)$. From this it follows that $\mathcal{S} \subset \mathcal{B}(\Lambda)$. In the case when the convergence in the property ($\Lambda 1$) is in X, i.e. Λ is *continuous from below*, we find another functions in $\mathcal{B}(\Lambda)$.

Proposition 22. *If Λ is continuous from below then $L^1(|||\Lambda|||) \subset \mathcal{B}(\Lambda)$ and*

$$I_\Lambda(f) = j\Big(\int_I \Lambda_{f^+}\, dm\Big) - j\Big(\int_I \Lambda_{f^-}\, dm\Big)$$

for every $f \in L^1(|||\Lambda|||)$.

Proof. Let $f \in \mathcal{L}^0(\Omega)^+$ and take a sequence $(\varphi_n) \subset \mathcal{S}$ such that $0 \le \varphi_n \uparrow f$ pointwise. Since Λ is continuous from below we have that

$$\|\Lambda_f(t) - \Lambda_{\varphi_n}(t)\|_X \to 0$$

for all $t \in I$ and so Λ_f is strongly m-measurable. If moreover $f \in L^1(|||\Lambda|||)$, since

$$\|\Lambda_f(t) - \Lambda_{\varphi_n}(t)\|_X \le \|\Lambda_f(t)\|_X + \|\Lambda_{\varphi_n}(t)\|_X \le 2\,|||\Lambda|||_f(t)$$

for all $t \in I$, by applying the dominated convergence theorem for the Lebesgue integral with respect to m we have that $\int_I \|\Lambda_f - \Lambda_{\varphi_n}\|\, dm \to 0$. This means that Λ_f is Bochner integrable with respect to m with Bochner integral $\int_I \Lambda_f\, dm$ such that $\int_I \Lambda_{\varphi_n}\, dm \to \int_I \Lambda_f\, dm$ in X. So, $f \in \mathcal{B}(\Lambda)$ and $I_\Lambda(\varphi_n) \to j(\int_I \Lambda_f\, dm)$ in X^{**}. On the other hand, from Lemma 20.(b) it follows that $I_\Lambda(\varphi_n) \to I_\Lambda(f)$ weakly* in X^{**}. So, $I_\Lambda(f) = j(\int_I \Lambda_f\, dm)$. For a general $f \in L^1(|||\Lambda|||)$, noting that $|f|, f^+, f^- \in L^1(|||\Lambda|||)$ we have the conclusion. $\square$

Remark 23. In the case when Λ is continuous from below, if there exists a constant $C \geq 1$ such that $|||\Lambda||| \leq C\|\Lambda(\cdot)\|_X$, from (6) and Proposition 22 it follows that

$$L^1(|||\Lambda|||) = \mathcal{B}(\Lambda) = \Big\{ f \in L^0(\Lambda) : \int_I \|\Lambda_{|f|}\|_X \, dm < \infty \Big\}.$$

Now, two natural sets arise when we think about *Pettis integration*, see for instance [2, Ch. 11, § 10]. Namely,

$$L^1_{qv}(\Lambda) = \{ f \in w\text{-}L^1_{qv}(\Lambda) : I_\Lambda(f\chi_A) \in j(X) \text{ for all } A \in \Sigma \}$$

and $L^1_v(\Lambda) = L^1_{qv}(\Lambda) \cap w\text{-}L^1_v(\Lambda)$. Since I_Λ is not additive in general we cannot know even if $L^1_{qv}(\Lambda)$ and $L^1_v(\Lambda)$ are vector spaces. Our goal in the next section is to give conditions under which these spaces are closed subspaces of $w\text{-}L^1_{qv}(\Lambda)$ and $w\text{-}L^1_v(\Lambda)$ respectively. Of course, if X is reflexive there is no problem as in this case $L^1_{qv}(\Lambda) = w\text{-}L^1_{qv}(\Lambda)$ and $L^1_v(\Lambda) = w\text{-}L^1_v(\Lambda)$.

Note that $\mathcal{S}_{|||\Lambda|||} \subset L^1_{qv}(\Lambda)$ and $\mathcal{S}_{\|\Lambda\|} \subset L^1_v(\Lambda)$. Even more, in the case when Λ is continuous from below we have that $L^1(|||\Lambda|||) \subset L^1_{qv}(\Lambda)$ and $L^1(\|\Lambda\|) \subset L^1_v(\Lambda)$. All these containments follows from (5), Proposition 22 and the preceding comments.

Remark 24. If Λ is a vector measure, in which case it is continuous from below and null-additive, as we have already pointed out in Remark 17, it follows that $w\text{-}L^1_v(\Lambda) = w\text{-}L^1_{qv}(\Lambda)$ is the space of weakly integrable functions with respect to Λ. Moreover, every $f \in w\text{-}L^1_{qv}(\Lambda)$ satisfies that $I_\Lambda(f) = w\text{-}\int_\Omega f \, d\Lambda$ is the *weak integral* of f with respect to Λ, see [34]. Indeed, for a positive $\varphi \in \mathcal{S}$, by (7) and since $x^*\Lambda$ is a real measure for each $x^* \in X^*$, it follows that $\int_I x^*\Lambda_\varphi \, dm = \int_\Omega \varphi \, dx^*\Lambda$. For a positive function $f \in w\text{-}L^1_{qv}(\Lambda)$, taking a sequence $(\varphi_n) \subset \mathcal{S}$ such that $0 \leq \varphi_n \uparrow f$ Λ-a.e., by Lemma 20.(b) and applying the dominated convergence theorem for the Lebesgue integral with respect to $x^*\Lambda$ we have that

$$\langle I_\Lambda(f), x^* \rangle = \lim \langle I_\Lambda(\varphi_n), x^* \rangle = \lim \int_\Omega \varphi_n \, dx^*\Lambda = \int_\Omega f \, dx^*\Lambda = \Big\langle w\text{-}\int_\Omega f \, d\Lambda, x^* \Big\rangle.$$

For a general $f \in w\text{-}L^1_v(\Lambda)$, since the weak integration map with respect to Λ is a linear operator the conclusion follows. Therefore, $L^1_v(\Lambda) = L^1_{qv}(\Lambda)$ is the space of *integrable* functions with respect to Λ and for every $f \in L^1_{qv}(\Lambda)$ we have that $I_\Lambda(f) = \int_\Omega f \, d\Lambda$ is the *integral* of f with respect to Λ, see [34].

7. L^1-spaces associated to a vector capacity

Let X be an *order continuous* Banach lattice, that is, every order bounded increasing sequence in X converges in the norm of X, see [25, Proposition 1.a.8]. In this case, from [25, Theorem 1.b.16] we have that $j(X)$ is an *ideal* of X^{**}, that is, if $x^{**} \in X^{**}$ and $x \in X$ with $|x^{**}| \leq |j(x)|$ then $x^{**} \in j(X)$. Recall that X^* is also a Banach lattice with the order $x^* \geq 0$ if and only if $\langle x^*, x \rangle \geq 0$ for all $0 \leq x \in X$.

Consider a vector capacity $\Lambda \colon \Sigma \to X$ with the following properties:

($o\Lambda 1$) Λ is *increasing*, that is, $\Lambda(A) \leq \Lambda(B)$ for all $A, B \in \Sigma$ such that $A \subset B$.
($o\Lambda 2$) Λ is *submodular*, that is, $\Lambda(A \cup B) + \Lambda(A \cap B) \leq \Lambda(A) + \Lambda(B)$ for all $A, B \in \Sigma$.
($o\Lambda 3$) Λ is *continuous from below*, that is, $\Lambda(A_n) \to \Lambda(A)$ in X whenever $A_n, A \in \Sigma$ with $A_n \uparrow A$.

Note that by ($o\Lambda 1$) we have that Λ is *positive*, that is, $\Lambda(A) \geq 0$ for all $A \in \Sigma$. Then, from ($o\Lambda 2$) it follows that Λ is *subadditive*, that is, $\Lambda(A \cup B) \leq \Lambda(A) + \Lambda(B)$ for all $A, B \in \Sigma$.

Lemma 25. *The vector capacity Λ has the properties (Λ1,2) and the set $N = \{x^* \in B_{X^*} : x^* \geq 0\}$ satisfies (qvN1,2,3).*

Proof. The property ($\Lambda 1$) is obvious from ($o\Lambda 3$). For every $A, Z \in \Sigma$ with Z Λ-null, by ($o\Lambda 1$) and since Λ is subadditive, it follows that

$$\Lambda(A) \leq \Lambda(A \cup Z) \leq \Lambda(A) + \Lambda(Z) = \Lambda(A)$$

and so ($\Lambda 2$) holds. For every $x^* \in B_{X^*}$ it follows that $|x^*| \in N$ and $q_{x^*\Lambda} \leq q_{|x^*|\Lambda}$, as $|\langle x^*, x\rangle| \leq \langle |x^*|, |x|\rangle$ for all $x \in X$. So, (qvN1) holds. For each $0 \leq x^* \in X^*$, from ($o\Lambda 1$) we have that the real capacity $x^*\Lambda$ is increasing and takes values in $[0, \infty)$. Then, $q_{x^*\Lambda} = x^*\Lambda$, see Lemma 2.(d). Since by ($o\Lambda 2$) we have that $x^*\Lambda$ is submodular and so subadditive, (qvN3) holds for $K = 1$. Moreover, by ($o\Lambda 3$) and since $x^*\Lambda$ is null-additive (as it is increasing and subadditive), it follows that (qvN2) holds. □

From the previous lemma, all what we have seen in Section 6 holds for Λ. In particular,

$$\begin{array}{ccccccccc} \mathcal{S}_{\|\Lambda\|} & \subset & L^1(\|\Lambda\|) & \subset & L^1_v(\Lambda) & \subset & w\text{-}L^1_v(\Lambda) & \subset_{[i]} & L^1(|x^*\Lambda|) \\ \cap & & \cap & & \cap & & \cap & & \cap \\ \mathcal{S}_{|||\Lambda|||} & \subset & L^1(|||\Lambda|||) & \subset & L^1_{qv}(\Lambda) & \subset & w\text{-}L^1_{qv}(\Lambda) & \subset_{[i]} & L^1(x^*\Lambda) \end{array} \tag{8}$$

with $0 \leq x^* \in X^*$. Moreover, we can add the following results.

Proposition 26. *The following statements hold:*

(a) $w\text{-}L^1_{qv}(\Lambda) = \mathcal{D}(\Lambda) = \{f \in L^0(\Lambda) : \int_I x^*\Lambda_{|f|}\, dm < \infty$ *for all* $0 \leq x^* \in X^*\}$ *and* $\|f\|_{qv} = \|I_\Lambda(|f|)\|_{X^{**}}$ *for all* $f \in w\text{-}L^1_{qv}(\Lambda)$.

(b) $L^1(|||\Lambda|||) = \mathcal{B}(\Lambda) = \{f \in L^0(\Lambda) : \int_I \|\Lambda_{|f|}\|_X\, dm < \infty\}$ *and* $\|f\|_{|||\Lambda|||} = \int_I \|\Lambda_{|f|}\|_X\, dm$ *for all* $f \in L^1(|||\Lambda|||)$.

(c) $\mathcal{S}_{|||\Lambda|||} = \mathcal{S}$.

Proof. (a) For the first part only note that $q_{x^*\Lambda} \leq q_{|x^*|\Lambda} = |x^*|\Lambda$ for all $x^* \in X^*$. In the second one we use that $\|y^*\|_{Y^*} = \sup_{0 \leq y \in B_Y} \langle y^*, y\rangle$ for any Banach lattice Y and $0 \leq y^* \in Y^*$. Namely, for every $f \in w\text{-}L^1_{qv}(\Lambda)$ it follows that

$$\begin{aligned} \|f\|_{qv} &= \sup_{x^* \in N} \int_I (q_{x^*\Lambda})_{|f|}\, dm = \sup_{x^* \in N} \int_I x^*\Lambda_{|f|}\, dm \\ &= \sup_{x^* \in N} \langle I_\Lambda(|f|), x^*\rangle = \|I_\Lambda(|f|)\|_{X^{**}}. \end{aligned}$$

(b) Note that $\|y\|_Y = \sup_{0 \leq y^* \in B_{Y^*}} \langle y^*, y\rangle$ for any Banach lattice Y and $0 \leq y \in Y$. Then, for every $A \in \Sigma$ we have that

$$|||\Lambda|||(A) = \sup_{x^* \in N} q_{x^*\Lambda}(A) = \sup_{x^* \in N} x^*\Lambda(A) = \|\Lambda(A)\|_X.$$

From this fact that the conclusion follows, see Remark 23.

(c) Clear as $|||\Lambda||| = \|\Lambda(\cdot)\|_X$. □

For each $0 \leq x^* \in X^*$, the capacity $x^*\Lambda$ satisfies the properties (P1,2,3,5,6) and $\langle I_\Lambda(f), x^*\rangle = I_{x^*\Lambda}(f)$ for all $f \in w\text{-}L^1_{qv}(\Lambda) \subset L^1(x^*\Lambda)$, see Remark 21. This gives the following properties for I_Λ.

Proposition 27. *The following statements hold:*

(a) $0 \leq I_\Lambda(f) \leq I_\Lambda(g)$ *for all* $f, g \in w\text{-}L^1_{qv}(\Lambda)$ *such that* $0 \leq f \leq g$ Λ*-a.e.*

(b) $I_\Lambda(f + g) \leq I_\Lambda(f) + I_\Lambda(g)$ *for all positive functions* $f, g \in w\text{-}L^1_{qv}(\Lambda)$.

(c) $I_\Lambda : w\text{-}L^1_{qv}(\Lambda) \to X^{**}$ *is continuous.*

Proof. (a) Let $f, g \in w\text{-}L^1_{qv}(\Lambda)$ be such that $0 \leq f \leq g$ Λ-a.e. For each $0 \leq x^* \in X^*$, since every Λ-null set is $x^*\Lambda$-null, by Lemma 4.(b) and (c) we have that

$$0 \leq \langle I_\Lambda(f), x^*\rangle = I_{x^*\Lambda}(f) \leq I_{x^*\Lambda}(g) = \langle I_\Lambda(g), x^*\rangle.$$

Since the canonical embedding of X into X^{**} is order preserving (see [25, Proposition 1.a.2]) and so $x \in X$ is such that $x \geq 0$ if and only if $\langle x^*, x\rangle \geq 0$ for all $0 \leq x^* \in X^*$, it follows that $0 \leq I_\Lambda(f) \leq I_\Lambda(g)$.

(b) Let $f, g \in w\text{-}L^1_{qv}(\Lambda)$ be positive functions. For each $0 \le x^* \in X^*$, since $x^*\Lambda$ is submodular and so $I_{x^*\Lambda}$ is subadditive on $\mathcal{L}^0(\Omega)^+$ (see the comments preceding Proposition 8), we have that

$$\begin{aligned}\langle I_\Lambda(f+g), x^*\rangle &= I_{x^*\Lambda}(f+g) \le I_{x^*\Lambda}(f) + I_{x^*\Lambda}(g)\\ &= \langle I_\Lambda(f), x^*\rangle + \langle I_\Lambda(g), x^*\rangle = \langle I_\Lambda(f) + I_\Lambda(g), x^*\rangle.\end{aligned}$$

Then, $I_\Lambda(f+g) \le I_\Lambda(f) + I_\Lambda(g)$.

(c) For every $f, g \in w\text{-}L^1_{qv}(\Lambda)$ and $0 \le x^* \in X^*$ it follows that

$$|I_{x^*\Lambda}(f) - I_{x^*\Lambda}(g)| \le 2\, I_{x^*\Lambda}(|f-g|),$$

see the proof of Proposition 8. Then, using that $\|y^*\|_{Y^*} \le 2\sup_{0\le y\in B_Y} |\langle y^*, y\rangle|$ for any Banach lattice Y and $y^* \in Y^*$, we have that

$$\begin{aligned}\|I_\Lambda(f) - I_\Lambda(g)\|_{X^{**}} &\le 2 \sup_{x^*\in N} |\langle I_\Lambda(f) - I_\Lambda(g), x^*\rangle|\\ &= 2 \sup_{x^*\in N} |I_{x^*\Lambda}(f) - I_{x^*\Lambda}(g)|\\ &\le 4 \sup_{x^*\in N} I_{x^*\Lambda}(|f-g|) = 4\,\|f-g\|_{qv}. \qquad \square\end{aligned}$$

Note that the spaces $w\text{-}L^1_{qv}(\Lambda)$ and $w\text{-}L^1_v(\Lambda)$ are actually Λ-B.f.s.'. Indeed, for $0 \le x^* \in X^*$, since $x^*\Lambda$ is increasing and submodular it can be proved that $|x^*\Lambda|$ is submodular, and so $I_{x^*\Lambda}$ and $I_{|x^*\Lambda|}$ are subadditive on $\mathcal{L}^0(\Omega)^+$. Recalling that

$$\|f\|_{qv} = \sup_{x^*\in N} I_{x^*\Lambda}(|f|) \quad \text{and} \quad \|f\|_v = \sup_{x^*\in N} I_{|x^*\Lambda|}(|f|)$$

for every $f \in L^0(\Lambda)$, it follows that $\|\cdot\|_{qv}$ and $\|\cdot\|_v$ are norms.

The properties of I_Λ shown in Proposition 27 allow us to get the main result of this section.

Theorem 28. *The sets $L^1_{qv}(\Lambda)$ and $L^1_v(\Lambda)$ are Λ-B.f.s.' with norms $\|\cdot\|_{qv}$ and $\|\cdot\|_v$ respectively.*

Proof. From Lemma 20.(a) it is clear that $af \in L^1_{qv}(\Lambda)$ for all $f \in L^1_{qv}(\Lambda)$ and $a \in \mathbb{R}$. If $f, g \in L^1_{qv}(\Lambda)$ are positive functions, since $j(X)$ is an ideal of X^{**}, from Proposition 27.(b) it follows that $f+g \in L^1_{qv}(\Lambda)$. For general $f, g \in L^1_{qv}(\Lambda)$ and $A \in \Sigma$, noting that $(f+g)^+ \le f^+ + g^+$ and $(f+g)^- \le f^- + g^-$, by Proposition 27.(a) and (b) we have that

$$\begin{aligned}|I_\Lambda((f+g)\chi_A)| &= |I_\Lambda((f\chi_A + g\chi_A)^+) - I_\Lambda((f\chi_A + g\chi_A)^-)|\\ &\le I_\Lambda((f\chi_A + g\chi_A)^+) + I_\Lambda((f\chi_A + g\chi_A)^-)\\ &\le I_\Lambda((f\chi_A)^+ + (g\chi_A)^+) + I_\Lambda((f\chi_A)^- + (g\chi_A)^-)\\ &\le I_\Lambda((f\chi_A)^+) + I_\Lambda((g\chi_A)^+) + I_\Lambda((f\chi_A)^-) + I_\Lambda((g\chi_A)^-).\end{aligned}$$

Since $h^+ = h\chi_{P_h}$ and $h^- = (-h)\chi_{N_h}$ for any $h \in \mathcal{L}^0(\Omega)$ (see the Preliminaries), it follows that the last element in the above inequality belongs to $j(X)$ and so $I_\Lambda((f+g)\chi_A) \in j(X)$. Hence, $f+g \in L^1_{qv}(\Lambda)$. Therefore, $L^1_{qv}(\Lambda)$ is a linear subspace of $w\text{-}L^1_{qv}(\Lambda)$, from which it is clear that $L^1_v(\Lambda)$ is a linear subspace of $w\text{-}L^1_v(\Lambda)$.

Since $I_\Lambda\colon w\text{-}L^1_{qv}(\Lambda) \to X^{**}$ is continuous and $j(X)$ is closed in X^{**}, it follows that $L^1_{qv}(\Lambda)$ is closed in $w\text{-}L^1_{qv}(\Lambda)$. From this, since $w\text{-}L^1_v(\Lambda) \subset w\text{-}L^1_{qv}(\Lambda)$ and all the containments between Λ-quasi-B.f.s.' are continuous, it follows that $L^1_v(\Lambda)$ is closed in $w\text{-}L^1_v(\Lambda)$. Then, $L^1_{qv}(\Lambda)$ and $L^1_v(\Lambda)$ are Banach spaces with norms $\|\cdot\|_{qv}$ and $\|\cdot\|_v$ respectively.

On the other hand, if $f \in L^0(\Lambda)$ and $g \in L^1_{qv}(\Lambda)$ are such that $|f| \le |g|$ Λ-a.e. then $f \in w\text{-}L^1_{qv}(\Lambda)$ with $\|f\|_{qv} \le \|g\|_{qv}$. Moreover, since $(f\chi_A)^+, (f\chi_A)^- \le |g|$ Λ-a.e. for every $A \in \Sigma$, by Proposition 27.(a) and (b) we have that

$$I_\Lambda((f\chi_A)^+), I_\Lambda((f\chi_A)^-) \le I_\Lambda(|g|) \le I_\Lambda(g^+) + I_\Lambda(g^-).$$

Then $I_\Lambda((f\chi_A)^+), I_\Lambda((f\chi_A)^-) \in j(X)$ and so $f \in L_{qv}(\Lambda)$. Hence, $L^1_{qv}(\Lambda)$ and $L^1_v(\Lambda)$ are Λ-B.f.s.'. $\square$

Finally we conclude this section by characterizing when the space $L^1_{qv}(\Lambda)$ is σ-order continuous.

Theorem 29. *The Λ-B.f.s. $L^1_{qv}(\Lambda)$ is σ-order continuous if and only if Λ is* continuous from above at $\emptyset$, *i.e.* $\Lambda(A_n) \to 0$ *in X whenever $A_n \in \Sigma$ with $A_n \downarrow \emptyset$. Moreover, in this case $\mathcal{S}$ is dense in $L^1_{qv}(\Lambda)$.*

Proof. First note that $\mathcal{S} \subset L^1_{qv}(\Lambda)$, see (8) and Proposition 26.(c). Moreover, from Proposition 22 it follows that $I_\Lambda(\chi_A) = j(\Lambda(A))$ for all $A \in \Sigma$.

Suppose that $L^1_{qv}(\Lambda)$ is σ-order continuous. Given $A_n \in \Sigma$ with $A_n \downarrow \emptyset$, since $\chi_{A_n} \downarrow 0$ pointwise and $(\chi_{A_n}) \subset L^1_{qv}(\Lambda)$, by using Proposition 26.(a) we have that

$$\|\Lambda(A_n)\|_X = \|I_\Lambda(\chi_{A_n})\|_{X^{**}} = \|\chi_{A_n}\|_{qv} \to 0.$$

Conversely suppose that Λ is continuous from above at $\emptyset$ and let $(f_n) \subset L^1_{qv}(\Lambda)$ be such that $f_n \downarrow 0$ Λ-a.e. For each $0 \le x^* \in X^*$, since $x^*\Lambda$ is continuous from above at $\emptyset$, by Theorem 6 we have that $L^1(x^*\Lambda)$ is σ-order continuous. Then, since $f_n \downarrow 0$ $x^*\Lambda$-a.e. it follows that

$$\langle I_\Lambda(f_n), x^*\rangle = I_{x^*\Lambda}(f_n) = \|f_n\|_{x^*\Lambda} \downarrow 0.$$

Hence, $I_\Lambda(f_n) \downarrow 0$ in the order of X^{**}. Since $j(X)$ is order continuous as X is so, we have that $\|f_n\|_{qv} = \|I_\Lambda(f_n)\|_{X^{**}} \downarrow 0$. $\square$

8. Example

Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), c)$ with c being the counting measure. Note that $L^0(c)$ is just the space ℓ^0 of all real sequences and the c-a.e. pointwise order coincides with the coordinate order. Let X be a σ-order continuous saturated c-B.f.s. Recall that X being *saturated* means that there exists a sequence $x = (x_n) \in X$ such that $x_n > 0$ for all n. In this case, X is a *Köthe function space* in the sense of [25, Definition 1.b.17]. In what follows we collect some properties of X which will be used later in our example.

Denoting the *scalar product* of two sequences $x = (x_n), y = (y_n) \in \ell^0$ by $(x, y) = \sum x_n y_n$ provided the sum exists, the *Köthe dual* of X is given by

$$X' = \Big\{ y = (y_n) \in \ell^0 : (|x|, |y|) < \infty \text{ for all } x = (x_n) \in X \Big\}.$$

The space X' endowed with the norm $\|y\|_{X'} = \sup_{x \in B_X} |(x, y)|$ for $y \in X'$, is a saturated c-B.f.s. The linear isometry $\eta : X' \to X^*$ given by $\langle \eta(y), x\rangle = (x, y)$ for all $y \in X'$ and $x \in X$ is surjective as X is σ-order continuous, see [25, pg. 29]. If $x^* \in X^*$ and $y \in X'$ are such that $x^* = \eta(y)$, then $x^* \ge 0$ if and only if $y \ge 0$. Note that $X \subset X''$. From [25, Proposition 1.b.18] it follows that $\|x\|_X = \|x\|_{X''}$ for all $x \in X$. The equality $X = X''$ holds with equal norms if and only if X has the Fatou property, see [25, pg. 30]. Also we will use the linear isometry $\pi : X'' \to X^{**}$ defined by $\langle \pi(z), x^*\rangle = \big(z, \eta^{-1}(x^*)\big)$ for all $z \in X''$ and $x^* \in X^*$. Note that if $z \in X$ then $\pi(z) = j(z)$.

Let us construct now a vector capacity with values in X. Consider a *σ-finite* measure $\mu : \Sigma \to [0, \infty]$ and let $\Phi : I \to I$ be a function as in Example 9. Then, the capacity $\lambda : \Sigma \to [0, \infty]$ given by $\lambda(A) = \Phi(\mu(A))$ for $A \in \Sigma$, satisfies the properties (P1,2,3,5,6) and the λ-null and μ-null sets coincide. Take a sequence of pairwise disjoint sets $(\Omega_n) \subset \Sigma$ such that $\Omega = \cup \Omega_n$ and $\mu(\Omega_n) < \infty$ for all n and assume that $e = (\lambda(\Omega_n))_n \in X$. We define the vector capacity $\Lambda : \Sigma \to X$ by

$$\Lambda(A) = (\lambda(A \cap \Omega_n))_n$$

for $A \in \Sigma$. Note that Λ is well defined as $\Lambda(A) \le e$.

Remark 30. The condition $e \in X$ holds in many cases. For instance if $\mu(\Omega_n) \to 0$ then $e \in c_0$. This happens whenever μ is finite. An example of a non-finite μ could be the measure m on I with density $h(x) = \frac{1}{1+x}$, that is $\mu(A) = \int_A h\, dm$. In this case, by taking $\Omega_n = [n-1, n)$ we have that $\mu(\Omega_n) = \ln(\frac{1+n}{n}) < \frac{1}{n}$. So $e \in c_0$ and for instance if $\Phi(x) = x^p$ with $0 < p < 1$ then $e \in \ell^q$ for every $\frac{1}{p} < q < \infty$.

Let us see that Λ satisfies the requirements of Section 7.

Lemma 31. *The vector capacity Λ has the properties (oΛ1,2,3) and is continuous from above at $\emptyset$. Moreover the Λ-null and μ-null sets coincide.*

Proof. Note that Λ is increasing and submodular as λ is so. Given $A_k, A \in \Sigma$ such that $A_k \uparrow A$, since λ is increasing and continuous from below, for each fixed n we have that $\lambda(A_k \cap \Omega_n) \uparrow \lambda(A \cap \Omega_n)$. Then $\Lambda(A_k) \uparrow \Lambda(A)$ pointwise and so in X, as X is σ-order continuous. Hence, Λ is continuous from below.

Given $A_k \in \Sigma$ with $A_k \downarrow \emptyset$, since μ is a measure, for each fixed n it follows that $\mu(A_k \cap \Omega_n) \downarrow 0$ and so $\lambda(A_k \cap \Omega_n) \downarrow 0$. That is, $\Lambda(A_k) \downarrow 0$ pointwise and so in X. Hence, Λ is continuous from above at $\emptyset$.

It is direct to check that a set Z is Λ-null if and only if $Z \cap \Omega_n$ is λ-null (equivalently, μ-null) for all n. This happens if and only if $Z = \cup Z \cap \Omega_n$ is μ-null. $\square$

Therefore, all what we have seen in Sections 5, 6 and 7 hold for Λ. Moreover we can give nicer descriptions for the L^1-spaces associated to Λ in terms of λ.

Proposition 32. *The following statements hold:*

(a) $L^1(|||\Lambda|||) = \{f \in L^0(\mu) : \int_I \|(\lambda_{|f|\chi_{\Omega_n}})_n\|_X \, dm < \infty\}$ *and for* $f \in L^1(|||\Lambda|||)$ *we have that* $\|f\|_{|||\Lambda|||} = \int_I \|(\lambda_{|f|\chi_{\Omega_n}})_n\|_X \, dm$.
(b) $w\text{-}L^1_{qv}(\Lambda) = \{f \in L^0(\mu) : (I_\lambda(|f|\chi_{\Omega_n}))_n \in X''\}$ *and for* $f \in w\text{-}L^1_{qv}(\Lambda)$ *we have that* $\|f\|_{qv} = \|(I_\lambda(|f|\chi_{\Omega_n}))_n\|_{X''}$. *Moreover,* $(I_\lambda(f\chi_{\Omega_n}))_n \in X''$ *and* $I_\Lambda(f) = \pi((I_\lambda(f\chi_{\Omega_n}))_n)$.
(c) $L^1_{qv}(\Lambda) = \{f \in L^0(\mu) : (I_\lambda(|f|\chi_{\Omega_n}))_n \in X\}$ *and for* $f \in L^1_{qv}(\Lambda)$ *we have that* $\|f\|_{qv} = \|(I_\lambda(|f|\chi_{\Omega_n}))_n\|_X$. *Moreover,* $(I_\lambda(f\chi_{\Omega_n}))_n \in X$ *and* $I_\Lambda(f) = j((I_\lambda(f\chi_{\Omega_n}))_n)$.

Proof. First note that $L^0(\Lambda) = L^0(\mu)$ as the Λ-null and μ-null sets coincide.

(a) For every $f \in \mathcal{L}^0(\Omega)^+$, since

$$\{\omega \in \Omega : f(\omega) > t\} \cap \Omega_n = \{\omega \in \Omega : (f\chi_{\Omega_n})(\omega) > t\}$$

for all $t \in I$, it follows that $\Lambda_f = (\lambda_{f\chi_{\Omega_n}})_n$. Then, the conclusion follows from Proposition 26.(b).

(b) Let $0 \le x^* \in X^*$ and $0 \le y = (y_n) \in X'$ be such that $x^* = \eta(y)$. For $f \in \mathcal{L}^0(\Omega)^+$ we have that

$$x^*\Lambda_f(t) = \langle x^*, \Lambda_f(t)\rangle = \sum y_n \lambda_{f\chi_{\Omega_n}}(t)$$

for all $t \in I$. By applying the monotone convergence theorem we obtain that

$$\int_I x^*\Lambda_f \, dm = \sum y_n \int_I \lambda_{f\chi_{\Omega_n}} \, dm = \sum y_n I_\lambda(f\chi_{\Omega_n}).$$

From this it follows that $f \in L^0(\mu)$ is such that $\int_I x^*\Lambda_{|f|} \, dm < \infty$ for all $0 \le x^* \in X^*$ if and only if $\big(I_\lambda(|f|\chi_{\Omega_n})\big)_n \in X''$. Then, by Proposition 26.(a), the description of $w\text{-}L^1_{qv}(\Lambda)$ holds. Moreover, for a positive function $f \in w\text{-}L^1_{qv}(\Lambda)$ we have that

$$\langle I_\Lambda(f), x^*\rangle = \int_I x^*\Lambda_f \, dm = \langle \pi((I_\lambda(f\chi_{\Omega_n}))_n), x^*\rangle$$

for all $0 \le x^* \in X^*$. The same equality holds for a general $x^* \in X^*$ by taking positive and negative parts of x^*. Hence, $I_\Lambda(f) = \pi((I_\lambda(f\chi_{\Omega_n}))_n)$. Then, for a general $f \in w\text{-}L^1_{qv}(\Lambda)$ it follows that

$$\|f\|_{qv} = \|I_\Lambda(|f|)\|_{X^{**}} = \big\|\big(I_\lambda(|f|\chi_{\Omega_n})\big)_n\big\|_{X''}.$$

For every n it is clear that $f\chi_{\Omega_n} \in L^1(\lambda)$ as $(I_\lambda(|f|\chi_{\Omega_n}))_n \in X''$. Noting that $(f\chi_A)^+ = f^+\chi_A$ and $(f\chi_A)^- = f^-\chi_A$ for all $A \in \Sigma$, we have that

$$(I_\lambda(f\chi_{\Omega_n}))_n = (I_\lambda(f^+\chi_{\Omega_n}))_n - (I_\lambda(f^-\chi_{\Omega_n}))_n \in X''$$

as $f^+, f^- \in w\text{-}L^1_{qv}(\Lambda)$. Moreover,

$$\begin{aligned} I_\Lambda(f) &= I_\Lambda(f^+) - I_\Lambda(f^-) \\ &= \pi((I_\lambda(f^+\chi_{\Omega_n}))_n) - \pi((I_\lambda(f^-\chi_{\Omega_n}))_n) \\ &= \pi((I_\lambda(f\chi_{\Omega_n}))_n). \end{aligned}$$

(c) For every $f \in L^1_{qv}(\Lambda) \subset w\text{-}L^1_{qv}(\Lambda)$ we have that $(I_\lambda(f\chi_{\Omega_n}))_n \in X''$ and

$$\pi((I_\lambda(f\chi_{\Omega_n}))_n) = I_\Lambda(f) \in j(X).$$

Since π coincides with j on X and π is injective it follows that $(I_\lambda(f\chi_{\Omega_n}))_n \in X$ and $I_\Lambda(f) = j((I_\lambda(f\chi_{\Omega_n}))_n)$. Noting that $|f| \in L^1_{qv}(\Lambda)$, we have that $(I_\lambda(|f|\chi_{\Omega_n}))_n \in X$ and $\|f\|_{qv} = \|(I_\lambda(|f|\chi_{\Omega_n}))_n\|_{X''} = \|(I_\lambda(|f|\chi_{\Omega_n}))_n\|_X$.

On the other hand, if $f \in L^0(\mu)$ is such that $(I_\lambda(|f|\chi_{\Omega_n}))_n \in X \subset X''$ we have that $f \in w\text{-}L^1_{qv}(\Lambda)$. Then, for every $A \in \Sigma$ it follows that $f\chi_A \in w\text{-}L^1_{qv}(\Lambda)$ and

$$\begin{aligned} |I_\Lambda(f\chi_A)| &= |I_\Lambda(f^+\chi_A) - I_\Lambda(f^-\chi_A)| \le I_\Lambda(f^+\chi_A) + I_\Lambda(f^-\chi_A) \\ &\le 2I_\Lambda(|f|) = 2\pi((I_\lambda(|f|\chi_{\Omega_n}))_n) = 2j((I_\lambda(|f|\chi_{\Omega_n}))_n), \end{aligned}$$

and so $I_\Lambda(f\chi_A) \in j(X)$. Hence, $f \in L^1_{qv}(\Lambda)$. □

Note that $L^1_{qv}(\Lambda)$ is σ-order continuous as Λ is continuous from above at $\emptyset$, see Theorem 29.

In order to give a description in terms of λ for the spaces $L^1(\|\Lambda\|)$, $w\text{-}L^1_v(\Lambda)$ and $L^1_v(\Lambda)$, we consider the variation $|\lambda|$ of λ defined as in the real capacity case. Since $|\lambda|$ is superadditive and λ is subadditive it follows that $|\lambda|$ is finitely additive. Moreover, since $|\lambda|$ is continuous from below as λ is so, we have that $|\lambda|$ is a measure. We will need the following result.

Lemma 33. *Let $x^* \in X^*$ and $y = (y_n) \in X'$ be such that $x^* = \eta(y)$. For every $A \in \Sigma$ it follows that*

$$|x^*\Lambda|(A) = \sum |y_n|\,|\lambda|(A \cap \Omega_n).$$

Proof. For every partition $(A_i)_{i=1}^m \subset \Sigma$ of $A \in \Sigma$ we have that

$$\begin{aligned} \sum_{i=1}^m |x^*\Lambda(A_i)| &= \sum_{i=1}^m \Big|\sum_{n\ge1} y_n\lambda(A_i\cap\Omega_n)\Big| \le \sum_{i=1}^m\sum_{n\ge1}|y_n|\lambda(A_i\cap\Omega_n) \\ &= \sum_{n\ge1}|y_n|\sum_{i=1}^m\lambda(A_i\cap\Omega_n) \le \sum_{n\ge1}|y_n||\lambda|(A\cap\Omega_n). \end{aligned}$$

Hence, $|x^*\Lambda|(A) \le \sum |y_n|\,|\lambda|(A\cap\Omega_n)$. On the other hand, for a fixed k take a partition $(B_i^k)_{i=1}^{m_k} \subset \Sigma$ of $A \cap \Omega_k$. Since $B_i^k \cap \Omega_k = B_i^k$ and $B_i^k \cap \Omega_n = \emptyset$ whenever $n \ne k$, we have that

$$|y_k|\sum_{i=1}^{m_k}\lambda(B_i^k) = \sum_{i=1}^{m_k}\Big|\sum_{n\ge1}y_n\lambda(B_i^k\cap\Omega_n)\Big| = \sum_{i=1}^{m_k}|x^*\Lambda(B_i^k)| \le |x^*\Lambda|(A\cap\Omega_k).$$

Then, $|y_k|\,|\lambda|(A\cap\Omega_k) \le |x^*\Lambda|(A\cap\Omega_k)$. Since $|x^*\Lambda|$ is superadditive and increasing it follows that

$$\sum_{k=1}^n |y_k|\,|\lambda|(A\cap\Omega_k) \le \sum_{k=1}^n |x^*\Lambda|(A\cap\Omega_k) \le |x^*\Lambda|(A)$$

for all n and so $\sum |y_n|\,|\lambda|(A\cap\Omega_n) \le |x^*\Lambda|(A)$. □

Now from the previous lemma we can obtain the following conclusions.

Proposition 34. *The following statements hold:*

(a) $L^1(\|\Lambda\|)$ *is the space of functions* $f \in L^0(\mu)$ *such that* $(|\lambda|_{|f|\chi_{\Omega_n}}(t))_n \in X''$ *for all* $t > 0$ *and* $\int_I \|(|\lambda|_{|f|\chi_{\Omega_n}})_n\|_{X''}\, dm < \infty$*, and for* $f \in L^1(\|\Lambda\|)$ *we have that* $\|f\|_{\|\Lambda\|} = \int_I \|(|\lambda|_{|f|\chi_{\Omega_n}})_n\|_{X''}\, dm$*.*

(b) $w\text{-}L^1_v(\Lambda) = \{f \in L^0(\mu) : (I_{|\lambda|}(|f|\chi_{\Omega_n}))_n \in X''\}$ *and for* $f \in w\text{-}L^1_v(\Lambda)$ *we have that* $\|f\|_v = \|(I_{|\lambda|}(|f|\chi_{\Omega_n}))_n\|_{X''}$*.*

(c) $L^1_v(\Lambda) = \{f \in L^0(\mu) : (I_\lambda(|f|\chi_{\Omega_n}))_n \in X$ *and* $(I_{|\lambda|}(|f|\chi_{\Omega_n}))_n \in X''\}$*.*

Note that $I_{|\lambda|}(|f|\chi_{\Omega_n}) = \int_{\Omega_n} |f|\, d|\lambda|$ *for all* n *as* $|\lambda|$ *is a measure.*

Proof. (a) By Lemma 33 every $A \in \Sigma$ satisfies that

$$\|\Lambda\|(A) = \sup_{x^* \in B_{X^*}} |x^*\Lambda|(A) = \sup_{y=(y_n)\in B_{X'}} \sum |y_n|\, |\lambda|(A \cap \Omega_n).$$

Then $\|\Lambda\|(A) < \infty$ if and only if $(|\lambda|(A \cap \Omega_n))_n \in X''$, and in this case

$$\|\Lambda\|(A) = \|(|\lambda|(A \cap \Omega_n))_n\|_{X''}.$$

If $f \in L^1(\|\Lambda\|)$, since $\int_I \|\Lambda\|_{|f|}\, dm < \infty$ and $\|\Lambda\|_{|f|}$ is decreasing, we have that $\|\Lambda\|_{|f|}(t) < \infty$ for all $t > 0$. So $(|\lambda|_{|f|\chi_{\Omega_n}}(t))_n \in X''$ with $\|(|\lambda|_{|f|\chi_{\Omega_n}}(t))_n\|_{X''} = \|\Lambda\|_{|f|}(t)$ for all $t > 0$. The converse inclusion follows as if $f \in L^0(\mu)$ is such that $(|\lambda|_{f\chi_{\Omega_n}}(t))_n \in X''$ for all $t > 0$ then $\|\Lambda\|_{|f|}(t) = \|(|\lambda|_{f\chi_{\Omega_n}}(t))_n\|_{X''}$ for all $t > 0$.

(b) Let $f \in L^0(\mu)$. For every $x^* \in X^*$ and $y = (y_n) \in X'$ with $x^* = \eta(y)$, from Lemma 33 and applying the monotone convergence theorem it follows that

$$\int_I |x^*\Lambda|_{|f|}\, dm = \sum |y_n| \int_I |\lambda|_{|f|\chi_{\Omega_n}}\, dm = \sum |y_n| I_{|\lambda|}(|f|\chi_{\Omega_n}).$$

Then $\int_I |x^*\Lambda|_{|f|}\, dm < \infty$ for all $x^* \in X^*$ if and only if $(I_{|\lambda|}(|f|\chi_{\Omega_n}))_n \in X''$. From this the description of $w\text{-}L^1_v(\Lambda)$ holds. Moreover, for $f \in w\text{-}L^1_v(\Lambda)$ we have that

$$\|f\|_v = \sup_{y=(y_n)\in B_{X'}} \sum |y_n| I_{|\lambda|}(|f|\chi_{\Omega_n}) = \|(I_{|\lambda|}(|f|\chi_{\Omega_n}))_n\|_{X''}.$$

(c) Clear from part (b) of this proposition and Proposition 32.(c). □

Finally note that in the case when X has the Fatou property (i.e. $X = X''$), we have that $L^1_{qv}(\Lambda) = w\text{-}L^1_{qv}(\Lambda)$ and

$$L^1_v(\Lambda) = w\text{-}L^1_v(\Lambda) = \{f \in L^0(\mu) : (I_{|\lambda|}(|f|\chi_{\Omega_n}))_n \in X\}.$$

The first equality above follows as $\lambda \le |\lambda|$ and so $I_\lambda(|f|\chi_{\Omega_n}) \le I_{|\lambda|}(|f|\chi_{\Omega_n})$ for all n. Moreover, $\|f\|_v = \|(I_{|\lambda|}(|f|\chi_{\Omega_n}))_n\|_X$ for all $f \in L^1_v(\Lambda)$. Also in this case we have that $L^1_v(\Lambda)$ is σ-order continuous. Indeed, given $(f_k) \subset L^1_v(\Lambda)$ such that $f_k \downarrow 0$ μ-a.e. (equivalently $|\lambda|$-a.e.), since $L^1(|\lambda|)$ is σ-order continuous as $|\lambda|$ is a measure, for each fixed n it follows that $I_{|\lambda|}(f_k\chi_{\Omega_n}) \downarrow 0$. Then $(I_{|\lambda|}(|f_k|\chi_{\Omega_n}))_n \downarrow 0$ pointwise and so in X.

Acknowledgements

The first and second authors gratefully acknowledge the support of the Ministerio de Economía y Competitividad under projects MTM2015-65888-C4-1-P and MTM2016-77054-C2-1-P, respectively. The first author also acknowledges the support of the Junta de Andalucía (project FQM-7276), Spain.

References

[1] A. Alfonsi, A simple proof for the convexity of the Choquet integral, Stat. Probab. Lett. 104 (2015) 22–25.
[2] C. Aliprantis, K. Border, Infinite Dimensional Analysis: A Hitchhiker's Guide, Springer-Verlag, New York, 1999.
[3] R.G. Bartle, N. Dunford, J. Schwartz, Weak compactness and vector measures, Can. J. Math. 7 (1955) 289–305.
[4] G. Beliakov, S. James, Citation-based journal ranks: the use of fuzzy measures, Fuzzy Sets Syst. 167 (2011) 101–119.
[5] C. Bennett, R.C. Sharpley, Interpolation of Operators, vol. 129, Academic Press, Boston, 1988.
[6] J.M. Calabuig, A. Ferrer-Sapena, E.A. Sánchez-Pérez, Vector-valued impact measures and generation of specific indexes for research assessment, Scientometrics 108 (3) (2016) 1425–1443.

[7] J. Cerdà, Lorentz capacity spaces, Contemp. Math.- Am. Math. Soc. 445 (2007) 49–55.
[8] J. Cerdà, J. Martín, P. Silvestre, Capacitary function spaces, Collect. Math. 62 (1) (2011) 95–118.
[9] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953) 131–295.
[10] G.P. Curbera, W.J. Ricker, Vector measures, integration and applications, in: Positivity, Birkhäuser, Basel, 2007, pp. 127–160.
[11] O. Galdames Bravo, On the norm with respect to vector measures of the solution of an infinite system of ordinary differential equations, Mediterr. J. Math. 12 (3) (2015) 939–956.
[12] O. Delgado, J. Soria, Optimal domain for the Hardy operator, J. Funct. Anal. 244 (1) (2007) 119–133.
[13] D. Denneberg, Non-additive Measure and Integral, Kluwer Academic Publishers, Dordrecht, 1994.
[14] J. Diestel, J.J. Uhl Jr., Vector Measures, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
[15] A. Fernández, F. Mayoral, F. Naranjo, Bartle–Dunford–Schwartz integral versus Bochner, Pettis and Dunford integrals, J. Convex Anal. 20 (2013) 339–353.
[16] A. Ferrer-Sapena, E.A. Sánchez-Pérez, L.M. González, F. Peset, R. Aleixandre-Benavent, Mathematical properties of weighted impact factors based on measures of prestige of the citing journals, Scientometrics 105 (3) (2015) 2089–2108.
[17] M. Gagolewski, R. Mesiar, Aggregating different paper quality measures with a generalized h-index, J. Informetr. 6 (4) (2012) 566–579.
[18] M. Gagolewski, R. Mesiar, Monotone measures and universal integrals in a uniform framework for the scientific impact assessment problem, Inf. Sci. 263 (2014) 166–174.
[19] N.J. Kalton, N.T. Peck, J.W. Roberts, An F-Space Sampler, London Math. Soc. Lecture Notes, vol. 89, Cambridge Univ. Press, Cambridge, 1985.
[20] J. Kawabe, The Choquet integral in Riesz space, Fuzzy Sets Syst. 159 (2008) 629–645.
[21] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (1) (2010) 178–187.
[22] E.P. Klement, J. Li, R. Mesiar, E. Pap, Integrals based on monotone set functions, Fuzzy Sets Syst. 281 (2015) 88–102.
[23] D.R. Lewis, Integration with respect to vector measures, Pac. J. Math. 33 (1970) 157–165.
[24] J. Li, R. Mesiar, E. Pap, E.P. Klement, Convergence theorems for monotone measures, Fuzzy Sets Syst. 281 (2015) 103–127.
[25] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces, vol. II, Springer-Verlag, Berlin, 1979.
[26] R. Mesiar, M. Gagolewski, H-index and other Sugeno integrals: some defects and their compensation, IEEE Trans. Fuzzy Syst. 24 (6) (2016) 1668–1672.
[27] T. Murofushi, M. Sugeno, A theory of fuzzy measures: representations, the Choquet integral, and null sets, J. Math. Anal. Appl. 159 (1991) 532–549.
[28] T. Murofushi, M. Sugeno, M. Machida, Non-monotonic fuzzy measures and the Choquet integral, Fuzzy Sets Syst. 64 (1994) 73–86.
[29] S. Okada, W.J. Ricker, E.A. Sánchez Pérez, Optimal Domain and Integral Extension of Operators Acting in Function Spaces, Oper. Theory, Adv. Appl., vol. 180, Birkhäuser, Basel, 2008.
[30] E. Pap, Null-Additive Set Functions, vol. 337, Kluwer Academic Publishers, Dordrecht, 1995.
[31] E. Pap, Pseudo-Additive Measures and Their Applications, in: Handbook of measure Theory (vol. II), Elsevier, Amsterdam, 2002, Ch. 35.
[32] E. Pap, M. Štrboja, I. Rudas, Pseudo-L^p space and convergence, Fuzzy Sets Syst. 238 (2014) 113–128.
[33] J.R. Sims, W. Zhenyuan, Fuzzy measures and fuzzy integrals: an overview, Int. J. Gen. Syst. 17 (1990) 157–189.
[34] G.F. Stefansson, L_1 of a vector measure, Matematiche 48 (1993) 219–234.
[35] G. Tatjana, I. Štajner-Papuga, M. Štrboja, An approach to pseudo-integration of set-valued functions, Inf. Sci. 181 (2011) 2278–2292.
[36] Q. Zhang, Some properties of the variations of non-additive set functions I, Fuzzy Sets Syst. 118 (2001) 529–538.
[37] Q. Zhang, Z. Gao, Some properties of the variations of non-additive set functions II, Fuzzy Sets Syst. 121 (2001) 257–266.