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Note

Spiders everywhere

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ABSTRACT

A *spider* is a tree with at most one branch (a vertex of degree at least 3) *centred* at the branch if it exists, and centred at any vertex otherwise. A graph G is *arachnoid* if for any vertex v of G, there exists a spanning spider of G centred at v—in other words: there are spiders everywhere! *Hypotraceable* graphs are non-traceable graphs in which all vertex-deleted subgraphs are traceable. Gargano et al. (2004) defined arachnoid graphs as natural generalisations of traceable graphs and asked for the existence of arachnoid graphs that are (i) non-traceable and non-hypotraceable, or (ii) in which some vertex is the centre of only spiders with more than three legs. An affirmative answer to (ii) implies an affirmative answer to (i). While non-traceable, non-hypotraceable arachnoid graphs were described in Wiener (2017), (ii) remained open. In this paper we give an affirmative answer to this question and discuss spanning spiders whose legs must have some minimum length.

1. Introduction

A graph is *traceable* if it contains a hamiltonian path. A non-traceable graph in which all vertex-deleted subgraphs are traceable is called *hypotraceable*. In this paper, a *spider* shall be either a path, or a tree with one vertex of degree at least 3 and all others with degree at most 2. A spider is *centred* at the vertex of degree at least 3 if there is such a vertex, and centred at any vertex otherwise. A spider S in a graph G is *spanning* if $V(S) = V(G)$.

Motivated by an optical network design problem, Gargano, Hammar, Hell, Stacho, and Vaccaro [5] introduced the following. A graph G is *arachnoid* if for any vertex v of G, there exists a spanning spider of G centred at v—in other words, there are spiders everywhere! Besides proving various results concerning spanning spiders, they showed that it is NP-complete to decide whether a given graph is arachnoid.

Arachnoid graphs are natural generalisations of traceable graphs. Gargano et al. observed that all hypotraceable graphs are arachnoid, but were unable to find other non-traceable arachnoid graphs and therefore raised the question whether such graphs exist. This was answered affirmatively in [8,9]; the smallest example has order 73 and all examples contain a vertex of high degree (more than $\frac{32}{33}n$, where n is the order of the graph). In [10], among others, cubic examples appear. The smallest construction in [10] – which happens to be cubic – has only 28 vertices. It is the smallest known non-traceable arachnoid graph (since the smallest known hypotraceable graph, found by Thomassen [6], has 34 vertices). Ref. [10] also relates arachnoid graphs to Gallai's famous question whether in a connected graph there always is a vertex lying on all

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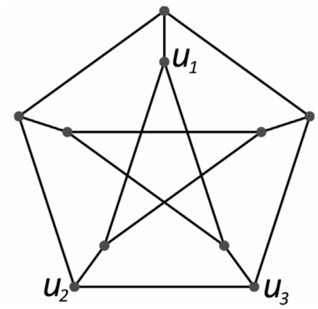

Fig. 1. The Petersen graph.

longest paths [4]; it turns out that the answer is negative, as shown by Walther [7], but determining in which graph classes the answer is positive has led to intriguing results, see e.g. [1].

We will call a path starting at a vertex v a *v-path*, and a v-path ending at a vertex $w \neq v$ a *vw-path*. For a graph G, a subgraph H of G, and a vertex $v \in V(G)$, let $\deg_H(v)$ denote the number of vertices in H adjacent to v—note that v itself need not lie in H. We put $\deg(v) = \deg_G(v)$.

Gargano et al. raised a second interesting question [5, p. 93]: do arachnoid graphs exist in which some vertex is the centre only to spanning spiders with more than three legs?—formally, a *leg* of a spider S is a path in S whose end-vertices are the centre and a leaf of S. Observe that while their first question, discussed above, could be solved by an approach very much related to hypotraceability (although the graphs used in the solution are not hypotraceable themselves, all of their vertices have a neighbour whose deletion gives a traceable graph), in the second problem, this strategy is out of question. In the following we present an affirmative solution to the second Gargano et al. problem. It is worth mentioning that the same construction also solves an open problem of [8,9], namely whether there are arachnoid graphs containing *several* vertices v, such that for all spanning spiders S centred at v, we have $\deg_S(v) \geq d$ for some fixed $d \geq 4$.

Consider a graph G and a spanning spider S in G. A spanning spider centred at the vertex v is a *v-spider*. If S is a v-spider with leaves L, then we call S a *(v, L)-spider*. We say that a vertex v *has a (k)-spider S* if S is a v-spider and $\deg_S(v) = k$, and that v *has a $(\leq k)$-spider S* if S is a v-spider and $\deg_S(v) \leq k$.

2. Arachnoid graphs in which some spanning spiders must have more than three legs

Theorem 1. *For $n = 120$ and every $n \geq 124$ there exists a graph of order n in which 108 vertices have a (3)-spider, while all other vertices have a (4)-spider but no (≤ 3)-spider.*

Proof. Consider the Petersen graph P shown in Fig. 1.

Denote three vertices of P with u_1, u_2, u_3 as indicated in Fig. 1 and put $U = \{u_1, u_2, u_3\}$.

Claim 1. (1.1) *There exists a hamiltonian u_1u_2-path in P, as well as a hamiltonian u_1u_3-path in P, but no hamiltonian u_2u_3-path in P.*
(1.2) *For every $v \in V(G) \setminus U$ there exists a (v, U)-spider.*
(1.3) *For every $i \in \{1, 2, 3\}$ there is a $(u_i, \{z\} \cup U \setminus \{u_i\})$-spider for some $z \in V(P) \setminus U$.*

Lemma 1 (*Clark and Entringer [3]*)**.** *P is non-hamiltonian, yet there is a hamiltonian path between any pair of non-adjacent vertices of P.*

Proof of Claim 1. Property (1.1) follows directly from Lemma 1. For (1.2) and (1.3), see Figs. 2 and 3, respectively (symmetric cases are omitted). □

We use three copies of P to construct a new graph Δ as shown in Fig. 4. We denote three particular vertices of Δ with w_1, w_2, w_3 as indicated in Fig. 4 and put $W = \{w_1, w_2, w_3\}$.

Claim 2.
(2.1) *For all $i, j \in \{1, 2, 3\}$ with $i \neq j$ there exists no hamiltonian w_iw_j-path in Δ.*
(2.2) *For every $v \in V(\Delta) \setminus W$ there exists a (v, W)-spider.*
(2.3) *For every $i \in \{1, 2, 3\}$ there is a $(w_i, \{z\} \cup W \setminus \{w_i\})$-spider for some $z \in V(\Delta) \setminus W$.*
(2.4) *For any pairwise different $i, j, k \in \{1, 2, 3\}$ there is a hamiltonian w_jw_k-path in $\Delta - w_i$.*

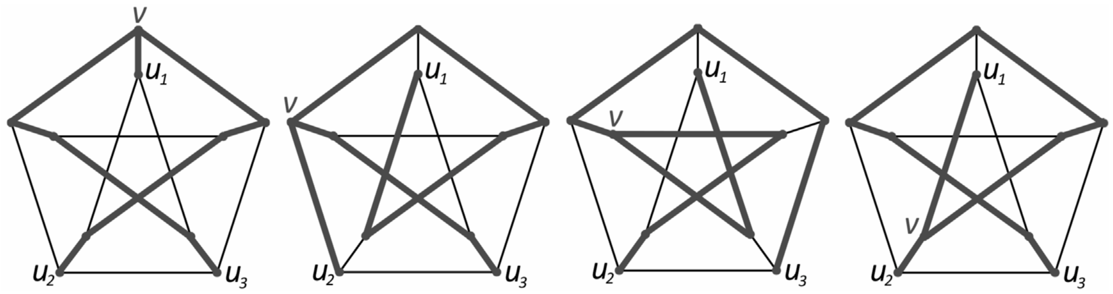

Fig. 2. Proof of property (1.2).

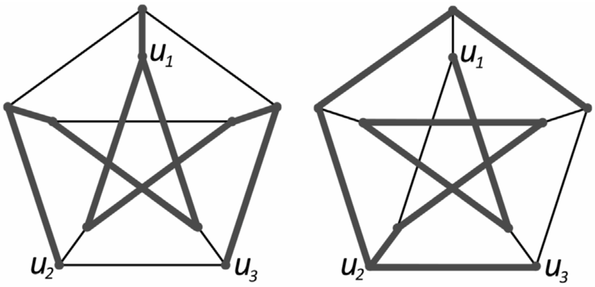

Fig. 3. Proof of property (1.3).

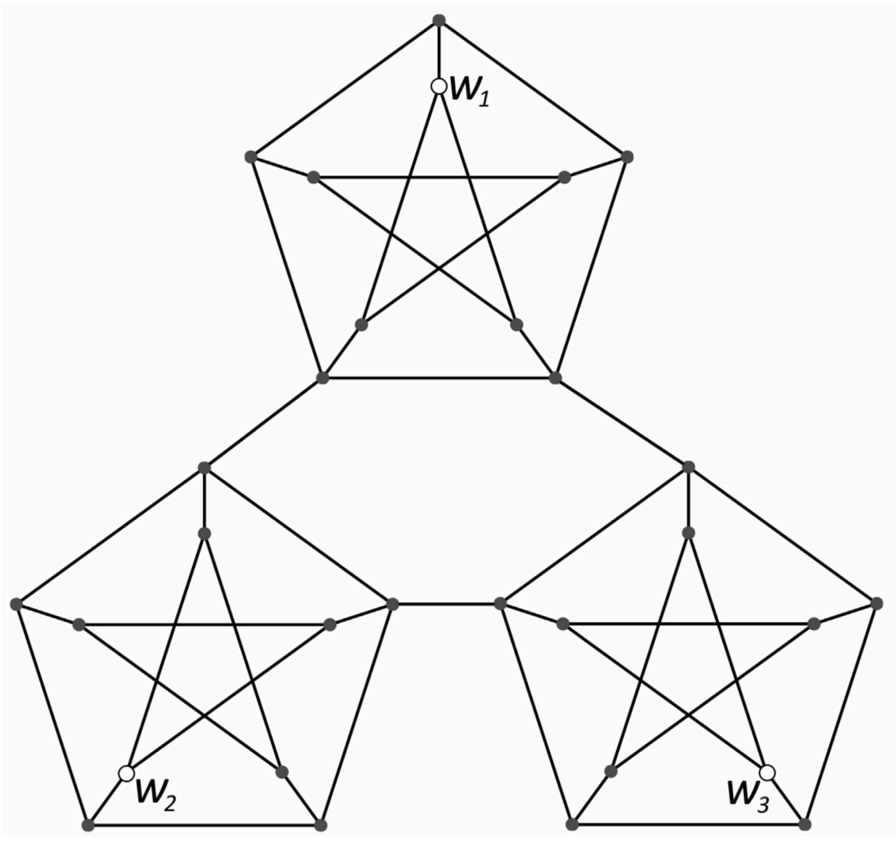

Fig. 4. Three copies of the Petersen graph joined as depicted above to obtain the graph Δ.

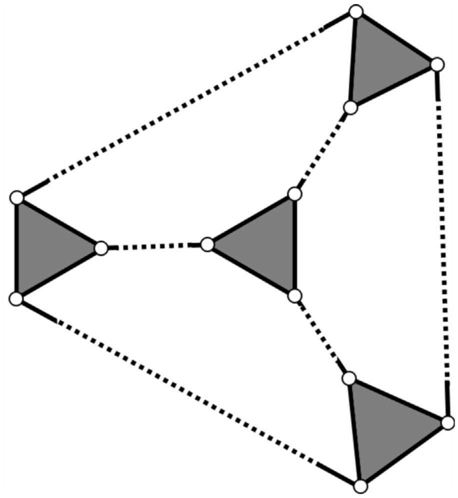

Fig. 5. The graph G, constructed from four copies of Δ.

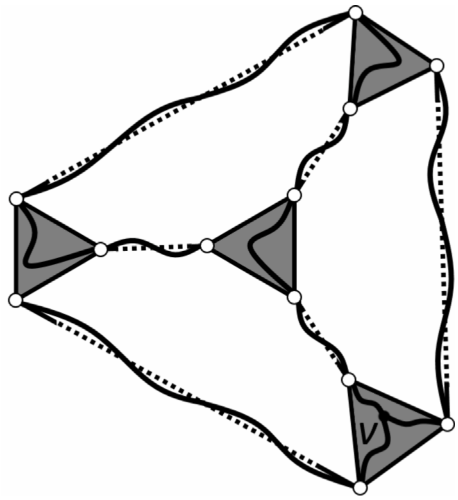

Fig. 6. A (3)-spider (drawn in wavy lines) in G centred at v.

Proof of Claim 2. We first prove the validity of Property (2.1). W.l.o.g. let $i = 1$ and $j = 2$. If there would exist a hamiltonian w_1w_2-path $\mathfrak{p}$ in Δ, necessarily we would have to traverse the copy P_3 of the Petersen graph containing w_3. By adding to $\mathfrak{p} \cap P_3$ the appropriate edge, $\mathfrak{p} \cap P_3$ can be extended to a hamiltonian cycle in P_3, a contradiction, since the Petersen graph is non-hamiltonian. Property (2.2) follows from Properties (1.2) and (1.1). Property (2.3) follows from Property (1.3) and Lemma 1. The final property, (2.4), follows from Lemma 1 and the existence of a hamiltonian cycle C in $P_i - w_i$, where P_i indicates the copy of the Petersen graph containing w_i; notice that C uses the edge between the two tetravalent vertices of P_i. □

We construct the graph G as shown in Fig. 5, where each of the grey triangles represents a copy of Δ, the triple of white vertices in each copy of Δ are the respective copies of w_1, w_2, w_3, and the dashed lines between white vertices represent edges (referred to as *dashed edges* in the sequel).

We now show the statement for order $|V(G)| = 120$. Let v be an arbitrary non-white vertex in G and let Δ' be that one of the four copies of Δ, which contains v. Furthermore, let w_1, w_2, w_3 be the set of white vertices of Δ'. By Property (2.2), Δ' contains a $(v, \{w_1, w_2, w_3\})$-spider. Adding the dashed edges incident with w_1, w_2, w_3 to reach the other copies of Δ and using Property (2.4), we can find a (3)-spider in G centred at v as illustrated in Fig. 6.

Let v be a white vertex of G. Using Property (2.3) instead of Property (2.2) in the aforementioned construction, we obtain a (4)-spider centred at v. In order to show that there is no (≤ 3)-spider centred at the white vertices, we need the following property.

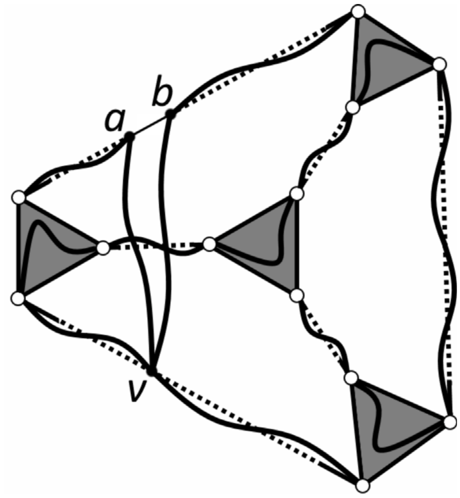

Fig. 7. A (4)-spider (drawn in wavy lines) in G_k centred at v.

Claim 3. *In a spanning spider S of G, each copy of Δ contains the centre of S or a leaf of S (possibly both).*

Proof of Claim 3. Let us assume to the contrary that there exists a copy Δ' of Δ, say the one depicted in Fig. 4, containing neither the centre of S nor a leaf of S. Thus, since S spans G, there must exist a w_iw_j-path that spans Δ', for some i and j. In order for such a path to exist we need to fully traverse one of the copies of the Petersen graph present in Δ'—however, this leads to a contradiction with the fact that the Petersen graph is non-hamiltonian (since the endvertices of the traversal would be neighbours in the Petersen graph). This completes the proof of Claim 3. □

A very similar argument yields that G is non-traceable, so G contains no (≤ 2)-spider.

Now let us assume that G contains a (3)-spider S centred at a white vertex v. By Claim 3, there would have to be a leaf of S in each of the copies of Δ different from Δ'. Since S is a (3)-spider, no leaf of S may lie in Δ'. As v is a white vertex, this would imply that there is a path between two white vertices of Δ' that spans Δ'—this however contradicts Property (2.1).

We have shown the statement for order 120, and now prove it for $n \geq 124$. Consider the graph G depicted in Fig. 5. Let us subdivide at least two dashed edges of G with at least two new vertices for each edge. We denote the set of new vertices by K and add an edge between any two non-adjacent vertices of K to obtain a new graph G_k, where $k = |V(K)|$. Obviously, $G_k[K]$ is a complete graph on k vertices. The arguments for $n \geq 124$ very much resemble those given for $n = 120$, so we will be succinct.

The crucial observation here is that for every spanning spider S of G_k, each of the grey triangles from Fig. 5 contains at least one leaf or the vertex at which S is centred. (This follows from the non-hamiltonicity of the Petersen graph, just like Claim 3.) There are essentially three types of vertices acting as the centre of a spider.

(i) The spider is centred at one of the white vertices of a grey triangle. Then that triangle must also contain a leaf of the spider, again due to Property (2.1).

(ii) The spider is centred at one of the non-white vertices of a grey triangle. Then, following the same arguments as above, we see that there is a spanning spider centred at that vertex with three leaves.

(iii) The centre v of the spanning spider is a vertex of K, see Fig. 7. Clearly, this spider cannot be a (≤ 3)-spider, since it must have a leaf in all 4 grey triangles. How to find a (4)-spider centred at v is depicted in Fig. 7 (it is important to note that $ab \in E(G_k)$—this also motivates the requirement that on each dashed edge we need 0 or at least two new vertices).

This completes the proof. □

3. Open problems

For a spider S, we denote with $L(S)$ the set of leaves of S and put $\ell(S) = |L(S)|$. Let G be an arachnoid graph. For $v \in V(G)$, put

$$\mathrm{ml}(v) = \min_{S \text{ is a } v\text{-spider}} \ell(S).$$

Consider the function

$$\sigma : V(G) \to \mathbb{N}, \quad v \mapsto \begin{cases} 0 & \text{if } G \text{ is traceable} \\ \displaystyle\min_{\substack{S \text{ is a } v\text{-spider} \\ \text{s.t. } \ell(S) = \mathrm{ml}(v)}} \min_{w \in L(S)} \mathrm{dist}_S(v, w) & \text{else,} \end{cases}$$

where $\mathrm{dist}_S(v, w)$ denotes the length of a shortest vw-path in S; the *length* of a path P is defined as $|E(P)|$. We now extend the observation that for hypotraceable graphs we have $\sigma(v) = 1$ for all $v \in V(G)$ and simultaneously generalise [10, Prop. 3] as well as [5, Prop. 6]. But first, we need a definition from [10]. Consider a graph G whose longest path has length $|V(G)| - 2$ and let $W \subset V(G)$ be the set of all vertices w such that $G - w$ is non-traceable. We then call every vertex from W *exceptional*, and say that G is $|W|$-*hypotraceable*.

Proposition 1. *Let G be a k-hypotraceable graph with exceptional vertices W, and let $\deg_W(v) < \deg(v)$ hold for every $v \in V(G)$. Then G is arachnoid. In particular, if $|W| < \delta(G)$, then G is arachnoid and $\sigma(v) = 1$ for all $v \in V(G)$.*

Proof. Let W be the set of exceptional vertices of G, and $v \in V(G)$ arbitrary. Since $|N(v) \cap W| < \deg(v)$, v has a neighbour $u \notin W$. As u is non-exceptional, there exists a hamiltonian path $\mathfrak{p}$ in $G - u$. Now $\mathfrak{p} \cup uv$ is a spanning spider of G centred at v.

G is non-traceable, so $\sigma(v) \neq 0$ for all $v \in V(G)$. However, since for each vertex $v \in V(G)$ there is a v-spider with a leg of length 1 we have that $\sigma(v) = 1$ for all $v \in V(G)$. $\square$

As mentioned in the introduction, traceable and hypotraceable graphs were the two families of arachnoid graphs found by Gargano et al. in [5]—they asked whether more exist. In [8,9], the first author presented an infinite family of non-traceable non-hypotraceable arachnoid graphs $\mathcal{G}$, settling the aforementioned question of Gargano et al. affirmatively. By construction, all vertices of each graph $G \in \mathcal{G}$ have a neighbour whose deletion gives a traceable graph, from which $\sigma(v) = 1$ follows for each vertex v in G—in fact, if all spiders have three legs, then this characterises the examples in which each $\sigma(v)$ is equal to 1. He also showed that for any prescribed graph H there exists a non-traceable non-hypotraceable arachnoid graph that contains H as an induced subgraph.

We now compute the σ-values of vertices of graphs lying in the family constructed in our main theorem. Let G be such a graph. We differentiate between three types of vertices according to their role in Fig. 7: $V_1(G)$ shall be the set of white vertices, $V_2(G)$ the set of vertices located on the dashed edges and not in $V_1(G)$, and $V_3(G)$ the set of vertices in the grey triangles and not in $V_1(G)$.

Let $v \in V_1(G)$. The structure of a v-spider (which necessarily has exactly four legs) is clear from the proof of the theorem—see Fig. 6 but consider v to be a white vertex, and recall that a fourth leg appears. By the structural properties of the Petersen graph and the graph Δ, it is now not difficult to see that $\sigma(v) = 1$.

Let $v \in V_2(G)$. We are in the situation depicted in Fig. 7. If v is adjacent to a white vertex, we have $\sigma(v) = 1$. In all other cases $\sigma(v) = 2$, since we can choose the vertex a (see Fig. 7) to be adjacent to a white vertex.

Let $v \in V_3(G)$. We are in the situation shown in Fig. 6. We have

$$\sigma(v) = 1 + \min_{i \in \{1,2,3\}} (\mathrm{dist}_\Delta(v, w_i) + \rho_i),$$

where ρ_i is 0 if no vertices were added to the dashed edge incident with (the white vertex) w_i, and 2 otherwise.

We have established that every graph constructed in Theorem 1 contains vertices whose σ-value is 1. Thus, the following natural question remains open.

Problem 1. Is there an arachnoid graph G with $\sigma(v) \geq 2$ for all $v \in V(G)$?

As we have mentioned earlier, the construction presented in Theorem 1 also solves a problem raised in [8,9], namely whether there exist arachnoid graphs containing *several* vertices that do not have a $(d-1)$-spider for some fixed $d \geq 4$. The construction works only for $d = 4$, obviously, thus for the values $d \geq 5$ this problem is still open—even if we require just one vertex without a $(\leq d-1)$-spider. Now that both questions of Gargano et al. [5] concerning the existence of certain arachnoid graphs have been answered (and actually a bit more) we may go a little further and ask:

Problem 2. Do arachnoid graphs exist in which no vertex has a (≤ 3)-spider (or even a $(\leq d-1)$-spider for some $d \geq 5$)?

We are also interested in a different stronger version of Theorem 1:

Problem 3. Is there a planar analogue of Theorem 1?

We end this paper with a relaxation of Grötschel's question whether bipartite hypotraceable graphs exist [2, p. 54]. It is easy to see that if an arachnoid graph G is bipartite with partite sets A and B, then $\|A|-|B\| \leq 1$, and if G is hypotraceable, then $|A| = |B|$.

Problem 4. Do non-traceable bipartite arachnoid graphs exist?

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