7th GAMM Juniors' Summer School on Applied Mathematics and Mechanics

Learning Models from Data: Model Reduction, System Identification and Machine Learning

Data-driven modeling from noisy measurements

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PAMM · Proc. Appl. Math. Mech. 2020;20:S1 e202000358. doi:10.1002/pamm.202000358.

Abstract

In this contribution, we review the robustness of two data-driven reduced-order modeling methods (the Loewner framework and the AAA algorithm) in the context of noisy data (frequency response measurements). Both methods are based on interpolation, with the mention that the AAA algorithm also enforces least squares approximation. It is shown that AAA is more robust to noise than Loewner is and that data partitioning plays an important role for the latter method, since it considerably influences robustness/approximation properties.

Approaches

- Finite element/difference schemes usually lead to large-scale high fidelity models and expensive computations in memory/time.
- Data-driven methods are used to learn/reveal reduced-order models to be employed as surrogates with cheap computation time/memory.

Type of measurements

- **Time domain:** Easy to collect, commonly used in flow control problems described by nonlinear PDEs, gas/energy network control and simulation, etc.
- **Frequency domain:** Typically inferred from DNS (direct numerical simulations) or measured with special equipment, e.g., the S (scattering) parameters.

The Loewner Framework

Setup: Consider linear LTI systems described by: $\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \quad y(t) = \mathbf{C}\mathbf{x}(t)$.
Aim: Construct reduced-order models directly from measurements (**Loewner framework**).
Frequency domain: linear [1] & bilinear [2] & **Time domain:** linear [3] & bilinear [4].

1. Given measurements $\{(\xi_k, f(\xi_k)) : k = 1, \ldots, 2n\} \rightarrow$, divide the data into 2 disjoint sets:
2. The associated **Loewner** & **shifted-Loewner** matrices $\mathbb{L}$ & $\mathbb{L}_s$ are introduced:

$$\mathbb{L}_{(i,j)} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}, \; \mathbb{L}_{s(i,j)} = \frac{\mu_i \mathbf{v}_i - \lambda_j \mathbf{w}_j}{\mu_i - \lambda_j}, \; i, j = 1, \ldots, n.$$

- **Exact amount of data - regular Loewner pencil:**

$$H(s) = \mathbb{W}(\mathbb{L}_s - s\mathbb{L})^{-1}\mathbb{V} = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{pmatrix}^T \left(\begin{bmatrix} \frac{\mu_1\mathbf{v}_1 - \mathbf{w}_1\lambda_1}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1 - \mathbf{w}_n\lambda_n}{\mu_1-\lambda_n} \\ \vdots & \ddots & \vdots \\ \frac{\mu_n\mathbf{v}_n - \mathbf{w}_1\lambda_1}{\mu_n-\lambda_1} & \cdots & \frac{\mu_n\mathbf{v}_n - \mathbf{w}_n\lambda_n}{\mu_n-\lambda_n} \end{bmatrix} - s \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_n}{\mu_1-\lambda_n} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_n - \mathbf{w}_1}{\mu_n-\lambda_1} & \cdots & \frac{\mathbf{v}_n - \mathbf{w}_n}{\mu_n-\lambda_n} \end{bmatrix} \right)^{-1} \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = f(s).$$

- **Redundant data - singular Loewner pencil:**
Project using the singular vectors of the Loewner matrix, i.e., $[\mathbf{Y}, \mathbf{S}, \mathbf{X}] = \mathrm{svd}(\mathbb{L})$:

$$\underbrace{\{\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B}\}}_{\text{model}} = \underbrace{\{\mathbb{W}, -\mathbb{L}, -\mathbb{L}_s, \mathbb{V}\}}_{\text{original}} \underset{\text{SVD}}{\Rightarrow} \underbrace{\{\mathbb{W}\mathbf{X}, -\mathbf{Y}^*\mathbb{L}\mathbf{X}, -\mathbf{Y}^*\mathbb{L}_s\mathbf{X}, \mathbf{Y}^*\mathbb{V}\}}_{\text{reduced}} = \underbrace{\{\mathbf{C}_r, \mathbf{E}_r, \mathbf{A}_r, \mathbf{B}_r\}}_{\text{ROM}}.$$

Data partitioning: a crucial step in the Loewner framework since it influences the quality of the reduced order model (ROM); consider two types of partitioning schemes:

$$\textbf{Loewner alternate}: \begin{cases} \text{Sampling points}: \{\mu_1, \lambda_1, \mu_2, \lambda_2, \ldots, \mu_n, \lambda_n\}, \\ \text{Sampling values}: \{\mathbf{v}_1, \mathbf{w}_1, \mathbf{v}_2, \mathbf{w}_2, \ldots, \mathbf{v}_n, \mathbf{w}_n\} \end{cases},$$

$$\textbf{Loewner half-half}: \begin{cases} \text{Sampling points}: \{\mu_1, \ldots, \mu_n, \lambda_1, \ldots, \lambda_n\}, \\ \text{Sampling values}: \{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_n\} \end{cases}$$

The Adaptive Antoulas-Anderson (AAA) Algorithm

The AAA algorithm introduced in [5] is an adaptive and iterative extension of the interpolation-based method of Anderson and Antoulas in [6]. The main steps are:

1. Express rational approximants in a barycentric representation.
2. Select the interpolation points (support points) via a Greedy scheme.
3. Compute the other variables (weights) to enforce least squares approximation.

The AAA algorithm

Inputs: Discrete set $\Gamma \subset \mathbb{C}$ with N points, function f, error tolerance $\epsilon > 0$
Output: Rational approximant $r_n(s)$ of order (n, n) displayed in a barycentric form.

1. Set $j = 0$, $\Gamma^{(0)} := \Gamma$, and $r_{-1} := N^{-1}\sum_{i=1}^{N} f(\gamma_i)$.
2. Select a point $z_j \in \Gamma^{(j)}$ where $|f(s) - r_{j-1}(s)|$ is maximal, with

$$r_{j-1}(s) := \left(\sum_{k=0}^{j-1} \frac{\omega_k^{(j-1)}}{s - z_k} \right)^{-1} \left(\sum_{k=0}^{j-1} \frac{\omega_k^{(j-1)} f_k}{s - z_k} \right) \quad \text{for } j \geq 1.$$

3. **If** $|f(z_j) - r_{j-1}(z_j)| \leq \varepsilon$, return r_{j-1}; **else**, set $f_j := f(z_j)$ and $\Gamma^{(j+1)} := \Gamma^{(j)} \setminus \{z_j\}$.
4. Find the weights $\boldsymbol{\omega}^{(j)} = [\omega_0^{(j)}, \ldots, \omega_j^{(j)}]$ by solving in a least squares sense over $z \in \Gamma^{(j+1)}$

$$\sum_{k=0}^{j} \frac{\omega_k^{(j)}}{s - z_k} f(s) \approx \sum_{k=0}^{j} \frac{\omega_k^{(j)} f_k}{s - z_k} \Leftrightarrow \left(\sum_{k=0}^{j} \frac{f(s) - f_k}{s - z_k} \right) \omega_k^{(j)} \approx 0 \Leftrightarrow \mathbb{L}^{(j)} \boldsymbol{\omega}^{(j)} = 0.$$

The solution is given by the $(j+1)^{\text{th}}$ right singular vector of Loewner matrix $\mathbb{L}^{(j)} \in \mathbb{C}^{(N-j-1)\times(j+1)}$.

5. Set $j := j + 1$ and go to step 2.

In what follows, we will use a modified version of AAA that enforces **real-valued** and **strictly-proper** rational appoximants.

Numerical experiments

- Consider a structural model of component 1r (Russian service module) of the International Space Station (ISS) from the SLICOT MOR benchmarks in [7].
- The original model has 3 inputs and 3 outputs - here consider the first input and output.
- The transfer function is sampled at 500 points on the imaginary axis, logarithmically-spaced points in the interval $(10^{-2}, 10^3)i$; the dimension of the original model is $\ell = 270$.
- Add normally distributed noise to the 1000 samples from the transfer function (for different values of the standard deviation $\sigma \geqslant 0$); originally, note that $\|\Sigma\|_{\mathcal{H}_\infty} = 1.1556 \cdot 10^{-1}$.

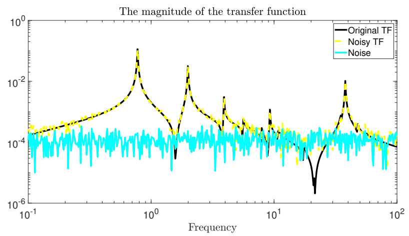

Fig. 1: The transfer function evaluated at the sample points for the case with noise ($\sigma = 10^{-4}$) and the case without noise ($\sigma = 0$).

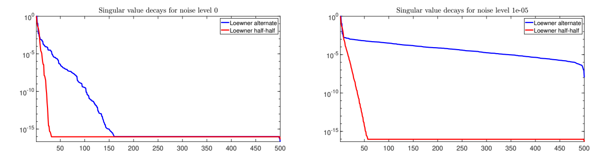

Fig. 2: Singular value decay for the two types of Loewner partitioning and for $\sigma = 0$ (left) & $\sigma = 10^{-5}$ (right).

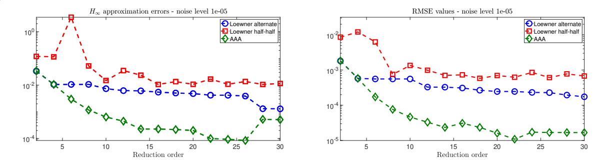

Fig. 3: The $\mathcal{H}_\infty$ errors (left) and RMSE errors (right) for ROMs of orders in $[2, 30]$ for Loewner and AAA.

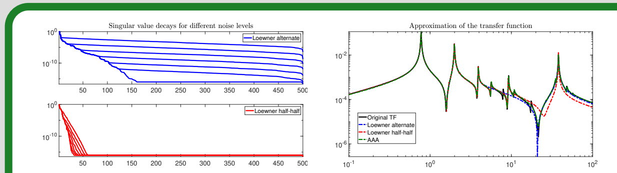

Fig. 4: Singular value decay for the two types of partitioning and for various noise levels ranging from $\sigma = 0$ to $\sigma = 10^{-4}$ (left) + approx. of the transfer function for the Loewner and AAA models when $n = 24$ (right).

Conclusion and outlook

This study has shown new results on how data partitioning affects the quality of data-driven approximants (and also the decay of the Loewner singular values). It was also shown that the AAA algorithm is more robust than the Loewner framework in the context of noisy data (this can be explained since Loewner is purely interpolatory). Future research topics include explicitly quantifying the sensitivity of the data-driven models.

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Acknowledgements Open access funding enabled and organized by Projekt DEAL.