



Representations of Involutory Subalgebras of Affine Kac–Moody Algebras

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Abstract: We consider the subalgebras of split real, non-twisted affine Kac–Moody Lie algebras that are fixed by the Cartan–Chevalley involution. These infinite-dimensional Lie algebras are not of Kac–Moody type and admit finite-dimensional unfaithful representations. We exhibit a formulation of these algebras in terms of \mathbb{N} -graded Lie algebras that allows the construction of a large class of representations using the techniques of induced representations. We study how these representations relate to previously established spinor representations as they arise in the theory of supergravity and work out a detailed example in the case of the affine extension of \mathfrak{e}_8 .

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1. Introduction

Every (split real) Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ for an indecomposable generalised Cartan matrix A has a Cartan–Chevalley involution ω that defines an involutory (‘maximal compact’) Lie subalgebra $\mathfrak{k} = \mathfrak{k}(\mathfrak{g}) < \mathfrak{g}$ as its subalgebra of fixed points [K]. If \mathfrak{g} is of finite type, then \mathfrak{k} is compact, whence reductive; its structure and representation theory play a key role in studies of symmetric spaces and automorphic representations.

For Kac–Moody algebras \mathfrak{g} , generators and relations for \mathfrak{k} were given in the early work [B], but, curiously, very little is known about the representation theory of \mathfrak{k} in the infinite-dimensional case. One reason for this is that these algebras do not fit into more standard frameworks, as they do not exhibit a graded, but rather a filtered structure; in particular, they are not Kac–Moody algebras [KN1], unlike the algebras from which they descend. This means also that the customary tools of representation theory (lowest and highest weight representations, character formulas, *etc.*) are not applicable. This being said, we point out (see “Appendix A”) that the complexification of \mathfrak{k} is a quotient of a GIM algebra as defined by Slodowy [S]; therefore, the representation theory of the latter, once developed, should help with understanding the (real and complex) representations of \mathfrak{k} .

A remarkable property of \mathfrak{k} is that it inherits an invariant bilinear form from \mathfrak{g} . For split real \mathfrak{g} the bilinear form on \mathfrak{g} is indefinite but its restriction to \mathfrak{k} is (negative) definite and for this reason \mathfrak{k} is sometimes referred to as maximal compact. One may thus consider the Hilbert space completion $\widehat{\mathfrak{k}}$ of \mathfrak{k} with respect to the norm defined by the invariant bilinear form. One of the results of our paper is that, for untwisted affine \mathfrak{g} , the Lie bracket does not close on $\widehat{\mathfrak{k}}$, so that $\widehat{\mathfrak{k}}$ is not a Hilbert Lie algebra with respect to the standard invariant bilinear form; in fact, it is not even a Lie algebra (see “Appendix B”).

The Lie algebra \mathfrak{k} and their associated groups are potentially important in physics applications, namely as infinite-dimensional generalisations of the R symmetries which govern the fermionic sectors of certain supersymmetric unified models of the fundamental interactions. Perhaps the most surprising result to come out of these physics motivated studies is that the infinite-dimensional algebras \mathfrak{k} , unlike the Kac–Moody algebras they are embedded into, admit non-trivial *finite*-dimensional, hence unfaithful, representations, in particular fermionic (double-valued) ones, which cannot be obtained by decomposing representations of the ancestor Kac–Moody algebra \mathfrak{g} under its subalgebra \mathfrak{k} . These representations were originally found by analysing the action of the infinite-dimensional duality groups arising in the dimensional reduction of (super-)gravities to space-time dimensions $D \leq 2$ on the fermions. For the affine case it was realised already long ago [N,NS] that these actions correspond to evaluation representations of a novel type, involving not just the evaluation of loop group elements at one of two distinguished points in the complex spectral parameter plane, but also its higher derivatives. Likewise, for the indefinite, and more specifically hyperbolic, extensions of affine algebras and their involutory subalgebras, several concrete examples were found in terms of actions on the supersymmetry parameters and the gravitino fields (at a given spatial point) [dBHP1,DKN1,dBHP2,DKN2,KN2,DH]. These actions extend the so-called ‘generalised holonomy’ groups in the physics literature [DS,DL,H] to an infinite-dimensional context. Furthermore, and most relevant for the present work, some new finite-dimensional representations *beyond* supergravity were identified in [KN3,KN5], and their structure was further analysed in [KNV], where surprising features were discovered, such as the generic non-compactness of the quotient algebras (and quotient groups) obtained by dividing the original algebra by the annihilating ideal of the given representation. These representations were further studied in [HKL,LK] where in particular the action of the corresponding (covering) group representation of $\tilde{K}(G)$ [GHKW] was clarified; moreover, in complete analogy to the finite-dimensional situation it turns out that in the simple-laced case this two-fold covering group $\tilde{K}(G)$ is simply connected with respect to its natural topology [HK] (induced from the Kac–Peterson topology on the corresponding Kac–Moody group). All these studies of Lie algebra representations and corresponding covering group representations so far, while applicable to large classes of generalised Cartan matrices (simply-laced; often even any symmetrisable generalised

Cartan matrix) have been limited to a small number of explicitly known examples of concrete representations, as all efforts to find a larger class of examples of representations and to understand their underlying structure and representation theory in a more systematic way have failed until now.

In this paper, we study the simplest case, corresponding to untwisted affine Kac–Moody algebras over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We show that there do exist infinitely many such unfaithful representations of ever increasing dimensions. Our construction finally provides a systematic *raison d'être* for such representations. Here, we aim for an *ab initio* construction of \mathfrak{k} representations (in particular of spinorial nature) and not for representations that are obtained by branching representations of the affine \mathfrak{g} . The main method, which is inspired by the supergravity realisation on fermions [NS], is to map the filtered structure to a graded one, by replacing Laurent polynomials $\mathbb{K}[t, t^{-1}]$ in a variable t by power series $\mathbb{K}[[u]]$ in another variable u that is related to t by (3.3). We shall refer to the graded structure as the *parabolic model* of \mathfrak{k} . In the parabolic model based on u , representations can be constructed easily by means of the Poincaré–Birkhoff–Witt (PBW) basis of the enveloping algebra of the underlying Lie algebra graded by powers of u . We will present several explicit examples (related to maximal supergravity) to illustrate the construction, which puts in evidence the rapid growth of the associated representations. Our results can be viewed as a prelude to the construction of similar representations for involutory subalgebras of hyperbolic Kac–Moody algebras, that will be required for a better understanding of the fermionic sector of unified models, and perhaps pave the way for an embedding of the Standard Model fermions into a unified framework [MN1, KN4, MN2].

In a more global view, the results of this paper afford not only a completely new perspective on the representation theory of such algebras, but even more importantly, open new avenues towards exploring the structure of the associated *groups*. This would be especially relevant for infinite-dimensional Kac–Moody algebras of *indefinite* type for which, however, a representation theory extending the present results remains to be developed. While the loop group approach has proven very useful for affine Kac–Moody groups [PS] there is no comparable tool available for studying indefinite Kac–Moody groups, where often one has to resort to ‘local-to-global’ methods. Namely, many mathematical observations concerning (two-spherical) Kac–Moody groups rely upon ‘gluing’ the $\mathrm{SL}(2, \mathbb{R})$ subgroups associated to the simple real roots of the underlying Kac–Moody algebra \mathfrak{g} and on exploiting Tits’ theory of buildings and extensions thereof [T, KP, AM, CFF, M]. However, in this way one does not gain a truly ‘global’ perspective on the associated groups. The same statement applies to an analogous construction of the groups $K(G)$ associated to involutory subalgebras $\mathfrak{k} \subset \mathfrak{g}$ for indefinite \mathfrak{g} , which would proceed by similarly ‘gluing’ compact $\mathrm{SO}(2) \subset \mathrm{SL}(2, \mathbb{R})$ subgroups associated to the simple real roots of \mathfrak{g} [GHKW] (which, contrary to the Kac–Moody group G , in fact also works in the non-two-spherical case). By contrast, we here propose a fundamentally different approach which would be based on the construction of infinite sequences of larger and larger, but still finite-dimensional, quotient groups that, while remaining infinitely degenerate at each step of the sequence, capture ‘more and more’ of the infinite-dimensional group (and in particular the information contained in the root spaces associated to imaginary roots), and in such a way that the group action is fully under control at each step of the construction—in more mathematical terms, we propose to investigate whether the involutory Lie subalgebra \mathfrak{k} and its corresponding group $K(G)$ might be residually finite-dimensional. In the affine case this is true for \mathfrak{g}

and therefore for \mathfrak{k} as well. One of our results is that our construction recovers this for \mathfrak{k} without reference to \mathfrak{g} .

Our main results are the following two theorems:

Theorem A. *Denote by $\mathbb{P} := \mathbb{K}[[u]]$ the ring of formal power series and let \mathbb{P}^\pm denote the \mathbb{K} -submodules of formal even (resp. odd) power series in u . Let $\mathfrak{g} := \mathfrak{g}(A)(\mathbb{K})$ be a split untwisted affine Kac–Moody algebra over \mathbb{K} with ‘maximal compact’ subalgebra \mathfrak{k} fixed by the Cartan–Chevalley involution, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the distinguished classical subalgebra \mathfrak{g} of \mathfrak{g} and set*

$$\mathfrak{N}(\mathbb{P}) := \left(\mathbb{P}^+ \otimes \mathfrak{k} \right) \oplus \left(\mathbb{P}^- \otimes \mathfrak{p} \right), \quad [P \otimes x, Q \otimes y] = (P \cdot Q) \otimes [x, y],$$

where the bracket on the right-hand side denotes the \mathfrak{g} -bracket. Then there exist nonsurjective monomorphisms $\rho_\pm : \mathfrak{k} \rightarrow \mathfrak{N}(\mathbb{P})$, whose precise formula is given in (3.9).

Set $\mathbb{P}_N := \mathbb{P}/\mathcal{I}_N$, where $\mathcal{I}_N := (u^{N+1})$ is the ideal generated by the element u^{N+1} . Projection from \mathbb{P} to \mathbb{P}_N induces homomorphisms of Lie algebras $\pi_N : \mathfrak{N}(\mathbb{P}) \rightarrow \mathfrak{N}(\mathbb{P}_N)$ such that the $\rho_\pm^{(N)} := \rho_\pm \circ \pi_N : \mathfrak{k} \rightarrow \mathfrak{N}(\mathbb{P}_N)$ are epimorphisms.

Theorem B. *Let V be a faithful \mathfrak{k} -module and denote by \mathfrak{V} the induced $\mathfrak{N}(\mathbb{K}[[u]])$ -module. It inherits an \mathbb{N} -grading $\mathfrak{V} = \bigoplus_{i=0}^{\infty} \mathfrak{V}_i$ from $\mathfrak{N}(\mathbb{K}[[u]])$ such that $\mathfrak{V}^{(N)} = \bigoplus_{i=N}^{\infty} \mathfrak{V}_i$ is an invariant submodule. The quotient $\mathfrak{V}/\mathfrak{V}^{(N)}$ is a finite-dimensional $\mathfrak{N}(\mathbb{P}_N)$ -module and therefore provides a representation of \mathfrak{k} .*

On $\overline{\mathfrak{V}} := \{(v_i) \mid v_i \in \mathfrak{V}_i\}$, the formal completion of \mathfrak{V} , there exists a natural, faithful $\mathfrak{N}(\mathbb{K}[[u]])$ -action. Furthermore, $\overline{\mathfrak{V}}$ is the inverse limit of the quotients $\mathfrak{V}/\mathfrak{V}^{(N)}$ which shows that \mathfrak{k} is residually finite-dimensional.

The paper is organised as follows: In Sect. 2 we fix our notation for the involutory subalgebra \mathfrak{k} of an untwisted affine Kac–Moody algebra. We construct the parabolic analogues $\mathfrak{N}(\mathbb{K}[[u]])$ and $\mathfrak{N}(\mathbb{P}_N)$ of \mathfrak{k} over different rings in Sect. 3 and show that there exist homomorphisms from \mathfrak{k} to them. Furthermore we establish that the homomorphism from \mathfrak{k} to $\mathfrak{N}(\mathbb{K}[[u]])$ is injective and thus proving Theorem A. In Sect. 4 we take a slight detour in first constructing induced representations of $\mathfrak{N}(\mathbb{K}[[u]])$ from representations of its classical subalgebra \mathfrak{k} . Although there is no homomorphism from \mathfrak{k} to $\mathfrak{N}(\mathbb{K}[[u]])$ we will show that these induced representations can be *truncated* to provide finite-dimensional representations of $\mathfrak{N}(\mathbb{P}_N)$ and \mathfrak{k} . We will conclude Sect. 4 by proving that the inverse limit of these representations provides a faithful infinite-dimensional representation of \mathfrak{k} , thus proving Theorem B. We specialise our results to the case $\mathfrak{k}(\mathfrak{e}_9)$ in Sect. 5 where we also describe the problem of analysing the finite-dimensional representations’ structure in more detail.

2. Affine Kac–Moody Algebras and Cartan–Chevalley Involution

Let A be an indecomposable generalised Cartan matrix of untwisted affine type and let \mathbb{K} denote \mathbb{R} or \mathbb{C} , see [K, §4] for a complete list of the associated diagrams $\mathcal{D}(A)$. The Kac–Moody algebra $\mathfrak{g} := \mathfrak{g}(A)(\mathbb{K})$ can be constructed explicitly as an extension of the loop algebra $\mathfrak{L}(\hat{\mathfrak{g}})$, where $\hat{\mathfrak{g}}$ denotes the classical Lie-subalgebra $\hat{\mathfrak{g}} < \mathfrak{g}(A)(\mathbb{K})$ that one obtains by deleting the affine node in the generalised Dynkin diagram $\mathcal{D}(A)$. Denote by ω the Cartan–Chevalley involution on $\mathfrak{g}(A)(\mathbb{K})$ [K, §1] and set

$$\mathfrak{k} := \mathfrak{k}(A)(\mathbb{K}) = \text{Fix}_\omega(\mathfrak{g}(A)(\mathbb{K})). \quad (2.1)$$

$\mathfrak{k}(A)(\mathbb{K})$ is called the maximal compact subalgebra of $\mathfrak{g}(A)(\mathbb{K})$ since the restriction of the standard invariant bilinear form on \mathfrak{g} to \mathfrak{k} is (negative) definite. Let us start with a description of \mathfrak{k} that is adapted to the presentation of \mathfrak{g} in terms of the loop algebra $\mathcal{L}(\hat{\mathfrak{g}})$. In Sect. 5, we will complement this by a collection of correspondences to the basis described in [KNP, §4] for the particular case $\mathfrak{g} = \mathfrak{e}_9$, the affine extension of \mathfrak{e}_8 .

Denote by $\hat{\Delta}$ the root system of $\hat{\mathfrak{g}}$ and its Cartan–Chevalley involution¹ by $\hat{\omega}$. The ± 1 eigenspaces of $\hat{\omega}$ provide the Cartan decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{p}}$. In terms of a Cartan basis $\{E_\gamma \mid \gamma \in \hat{\Delta}\} \cup \{h_1, \dots, h_d \in \hat{\mathfrak{h}}\}$, this decomposition is realised as

$$\hat{\mathfrak{k}} = \text{span} \left\{ E_\gamma - E_{-\gamma} \mid \gamma \in \hat{\Delta} \right\}, \quad \hat{\mathfrak{p}} = \text{span} \left(\left\{ E_\gamma + E_{-\gamma} \mid \gamma \in \hat{\Delta} \right\} \cup \hat{\mathfrak{h}} \right), \quad (2.2)$$

where $\hat{\mathfrak{h}}$ denotes the Cartan subalgebra of $\hat{\mathfrak{g}}$. Recall that $\hat{\mathfrak{p}}$ is a $\hat{\mathfrak{k}}$ -module but not a Lie-algebra since $[\hat{\mathfrak{p}}, \hat{\mathfrak{p}}] = \hat{\mathfrak{k}}$. Denote by \mathcal{L} the ring of Laurent polynomials over \mathbb{K} . Then the loop algebra of $\mathcal{L}(\hat{\mathfrak{g}})$ is given by the tensor product $\mathcal{L} \otimes \hat{\mathfrak{g}}$ with the commutator given by the bilinear extension of

$$[P \otimes x, Q \otimes y] = (PQ) \otimes [x, y], \quad (2.3)$$

where $[\cdot, \cdot]$ on the right-hand side denotes the bracket on $\hat{\mathfrak{g}}$. As t^n, t^{-n} (for $n \in \mathbb{N} = \{0, 1, \dots\}$) span \mathcal{L} , the Lie algebra $\mathcal{L}(\hat{\mathfrak{g}})$ is spanned by $\left\{ t^{\pm n} \otimes E_\gamma \mid \gamma \in \hat{\Delta} \right\} \cup \left\{ t^{\pm n} \otimes h \mid h \in \hat{\mathfrak{h}} \right\}$ for $n \in \mathbb{N}$.

Now (see [K, §7] or [GO])

$$\mathfrak{g} = \mathcal{L}(\hat{\mathfrak{g}}) \oplus \mathbb{K} \cdot K \oplus \mathbb{K} \cdot d \quad (2.4)$$

with $K, d \in \mathfrak{h}$ so that $\mathfrak{k} \subset \mathcal{L}(\hat{\mathfrak{g}})$ because $\omega(h) = -h \forall h \in \mathfrak{h}$. Thus, in order to describe $\mathfrak{k} = \mathfrak{k}(A)(\mathbb{K})$ we only need to study the loop algebra $\mathcal{L}(\hat{\mathfrak{g}})$ and in particular do not need to consider the central extension of $\mathcal{L}(\hat{\mathfrak{g}})$ by a 2-cocycle, giving rise to K . The Cartan–Chevalley involution restricted to $\mathcal{L}(\hat{\mathfrak{g}})$ is given by the linear extension of

$$\omega : P(t) \otimes x \mapsto P(t^{-1}) \otimes \hat{\omega}(x) \quad \forall P \in \mathcal{L}. \quad (2.5)$$

Denote by \mathcal{L}^\pm the ± 1 eigenspaces of \mathcal{L} under the involution $\eta : t \mapsto t^{-1}$. Observe that evaluation of Laurent polynomials at $t = \pm 1$ is the only evaluation map that commutes with η which is why we refer to $t = \pm 1$ as the fixed points of η . The fixed point set of ω can be constructed from $\mathcal{L}^\pm, \hat{\mathfrak{k}}$ and $\hat{\mathfrak{p}}$:

$$\omega(P \otimes x) = P \otimes x \quad \forall P \in \mathcal{L}^+, x \in \hat{\mathfrak{k}}, \quad \omega(Q \otimes y) = Q \otimes y \quad \forall Q \in \mathcal{L}^-, y \in \hat{\mathfrak{p}}$$

From this we arrive at the following explicit realisation of \mathfrak{k} :

$$\Rightarrow \mathfrak{k}(A)(\mathbb{K}) = \left(\mathcal{L}^+ \otimes \hat{\mathfrak{k}} \right) \oplus \left(\mathcal{L}^- \otimes \hat{\mathfrak{p}} \right). \quad (2.6)$$

¹ Note that the abstract Cartan–Chevalley involution of $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{A})$ as a Kac–Moody algebra of finite type agrees with the restriction of $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ to $\hat{\mathfrak{g}} < \mathfrak{g}$ because the Dynkin diagram of $\hat{\mathfrak{g}}$ is a sub-diagram of $\mathcal{D}(A)$ due to our assumption that A is untwisted.

Remark 1. (i) The Laurent polynomials in \mathfrak{L}^\pm are spanned by $t^n \pm t^{-n}$ for $n \in \mathbb{N}$. The product of two such basis Laurent polynomials is for example

$$(t^m + t^{-m})(t^n - t^{-n}) = \left(t^{m+n} - t^{-(m+n)}\right) - \operatorname{sgn}(m-n) \left(t^{|m-n|} - t^{-|m-n|}\right). \quad (2.7)$$

We shall refer to this as a *filtered* structure on \mathfrak{L} that by (2.6) induces a filtered structure on \mathfrak{k} .

- (ii) The construction in (2.6) can be generalised to the case where \mathfrak{L} denotes a finitely generated commutative \mathbb{K} -algebra with involution and \mathfrak{L}^\pm denote the ± 1 eigenspaces with respect to this involution. This produces an analogue of maximal compact subalgebras to the case where \mathfrak{g} is a generalised current algebra. However, our interest lies in studying different models of \mathfrak{k} . In particular, we will explore the relation of (2.6) to the cases $\mathfrak{L} = \mathbb{K}[[u]]$, $\mathbb{K}[u]$ and $\mathbb{K}[u]/\mathcal{I}_N$, where \mathcal{I}_N is the ideal generated by the monomial u^{N+1} .
- (iii) It is well known that one can form the semi-direct sum of any affine Lie algebra with the Virasoro algebra of centrally extended infinitesimal diffeomorphisms of the circle [K,GO], where both central extensions are identified. This structure descends to \mathfrak{k} and a ‘maximal compact’ subalgebra of the Virasoro algebra [JN]. As this will play no role in our general analysis, we defer its discussion to Sect. 5.
- (iv) The Lie algebra \mathfrak{k} comes with a definite and invariant bilinear form that is inherited from \mathfrak{g} . In ‘Appendix B’, we show that the Hilbert space completion with respect to this norm is not compatible with the Lie algebra structure to form a Hilbert Lie algebra.

3. Reformulation in Terms of Parabolic Algebras

In this section, we construct a Lie algebra monomorphism from \mathfrak{k} to a larger Lie algebra by replacing Laurent polynomials with formal power series. This corresponds to an expansion around the fixed points $t = \pm 1$ of η instead of 0 as described in [NS,KNP].

Denote by $\mathbb{P} := \mathbb{K}[[u]]$ the ring of formal power series together with the involutive ring automorphism $\eta : u^k \mapsto (-1)^k u^k$. We refer to a formal power series in \mathbb{P} either as $\sum_{k=0}^{\infty} a_k u^k$ or just as $(a_k)_{k \geq 0}$ with $a_k \in \mathbb{K}$. Denote the ± 1 eigenspaces of \mathbb{P} with respect to η by \mathbb{P}^\pm , i.e., the formally even/odd power series in u . One has

$$\begin{aligned} \mathbb{P}^+ &:= \{(a_k)_{k \in \mathbb{N}_0} \mid a_{2k} \in \mathbb{K}, a_{2k+1} = 0 \forall k \in \mathbb{N}\}, \\ \mathbb{P}^- &:= \{(a_k)_{k \in \mathbb{N}_0} \mid a_{2k+1} \in \mathbb{K}, a_{2k} = 0 \forall k \in \mathbb{N}\}. \end{aligned}$$

We mimic the loop algebra construction by setting

$$\mathfrak{N}(\mathbb{K}[[u]]) := \left(\mathbb{P}^+ \otimes \mathfrak{k}\right) \oplus \left(\mathbb{P}^- \otimes \mathfrak{p}\right), \quad [P \otimes x, Q \otimes y] = (P \cdot Q) \otimes [x, y], \quad (3.1)$$

where the bracket on the right-hand side again denotes the $\hat{\mathfrak{g}}$ -bracket. Recall that multiplication in the ring of formal power series is given by convolution, i.e., for $P = \sum_{n=0}^{\infty} a_n u^n$ and $Q = \sum_{n=0}^{\infty} b_n u^n$ one has

$$P \cdot Q = \sum_{n=0}^{\infty} c_n u^n, \quad c_n = \sum_{k=0}^n a_k b_{n-k}. \quad (3.2)$$

Remark 2. One could also consider replacing \mathbb{P} by the ring of polynomials $\mathbb{K}[u]$. In contrast to \mathcal{L}^\pm which have a filtered structure, $\mathbb{K}[u]$ has a gradation given by the degree of polynomials. This extends to a gradation on $(\mathbb{K}[u]^+ \otimes \hat{\mathfrak{k}}) \oplus (\mathbb{K}[u]^- \otimes \hat{\mathfrak{p}})$. If we were to construct a monomorphism from \mathfrak{k} using only $\mathbb{K}[u]$ one would expect to be able to pull this gradation back to $\mathfrak{k}(A)(\mathbb{K})$. Since we doubt this to be possible we work over $\mathbb{K}[[u]]$ which arises as the formal completion of $\mathbb{K}[u]$. There we do not have a gradation by degree of polynomials any more in the proper sense because for this any element in $\mathbb{K}[[u]]$ would have to decompose into the sum of finitely many homogeneous elements. This is true for polynomials but not for formal power series. Thus, we sacrifice a graded structure in order to achieve injectivity. However, in Sect. 4 we will consider this case as a preliminary step.

As the power series $\mathbb{K}[[u]]$ are associated with the expansion around the fixed point $t = \pm 1$ rather than $t = 0$, we construct a homomorphism by relating the expansion via a (Möbius-type) transformation

$$u = \frac{1 \mp t}{1 \pm t} \Leftrightarrow t = \pm \frac{1 - u}{1 + u} \quad (3.3)$$

through the Taylor series ($n \geq 0$)

$$t^n + t^{-n} = (\pm 1)^n \sum_{k \geq 0} a_{2k}^{(n)} u^{2k}, \quad t^n - t^{-n} = (\pm 1)^n \sum_{k \geq 0} a_{2k+1}^{(n)} u^{2k+1}. \quad (3.4)$$

The filtered multiplication of the Laurent polynomials in \mathcal{L}^\pm is then captured by the following lemma.

Lemma 3. For each $n \in \mathbb{N}$ the sequences $(a_{2k}^{(n)})_{k \in \mathbb{N}}$ and $(a_{2k+1}^{(n)})_{k \in \mathbb{N}}$ given by given by

$$\begin{aligned} a_{2k}^{(n)} &= 2 \sum_{\ell=0}^n \binom{2n}{2\ell} \binom{k - \ell + n - 1}{k - \ell}, \\ a_{2k+1}^{(n)} &= -2 \sum_{\ell=0}^{n-1} \binom{2n}{2\ell+1} \binom{k - \ell + n - 1}{k - \ell} \end{aligned} \quad (3.5)$$

satisfy

$$\sum_{\ell=0}^k a_{2\ell}^{(m)} a_{2(k-\ell)}^{(n)} = a_{2k}^{(m+n)} + a_{2k}^{(|m-n|)} \quad (3.6a)$$

$$\sum_{\ell=0}^k a_{2\ell}^{(m)} a_{2(k-\ell)+1}^{(n)} = a_{2k+1}^{(m+n)} + \operatorname{sgn}(n - m) a_{2k+1}^{(|m-n|)} \quad (3.6b)$$

$$\sum_{\ell=0}^{k-1} a_{2\ell+1}^{(m)} a_{2(k-\ell)-1}^{(n)} = a_{2k}^{(m+n)} - a_{2k}^{(|m-n|)}. \quad (3.6c)$$

The coefficients $(-1)^n a_{2k}^{(n)}$ and $(-1)^n a_{2k+1}^{(n)}$ also satisfy Eqs. (3.6a), (3.6b) and (3.6c). Furthermore, for fixed $n \in \mathbb{N}^* = \{1, 2, \dots\}$, the values of $a_{2k}^{(n)}$ and $a_{2k+1}^{(n)}$ are given by polynomials in k of degree $n - 1$ for all $k \in \mathbb{N}^*$.

Remark 4. The transformations (3.3) have the property that they interchange the points 0 and ∞ (that are exchanged by the involution $t \leftrightarrow t^{-1}$) with the fixed points ± 1 of η . The maps $t \mapsto u(t)$ in (3.3) are fixed uniquely by the requirement that $(0, \infty, +1, -1) \mapsto (+1, -1, 0, \infty)$ and $(0, \infty, +1, -1) \mapsto (+1, -1, \infty, 0)$, respectively.

Proof. We first compute the Taylor series for the Möbius transformation (3.3) for $n \in \mathbb{N}$

$$t^n + t^{-n} = (\pm 1)^n \frac{2}{(1 - u^2)^n} \sum_{\ell=0}^n \binom{2n}{2\ell} u^{2\ell} = 2(\pm 1)^n \sum_{k \geq 0} \sum_{\ell=0}^n \binom{2n}{2\ell} \binom{n+k-1}{k} u^{2(k+\ell)},$$

from which the first formula in (3.5) follows. The second identity in (3.5) is deduced similarly from the expansion of $t^n - t^{-n}$. The order of the polynomial follows from the binomial coefficient $\binom{n+k-\ell-1}{k-\ell}$.

The convolution properties (3.6) follow from multiplying out

$$(t^m + t^{-m})(t^n + t^{-n}) = t^{m+n} + t^{-(m+n)} + t^{|m-n|} + t^{-|m-n|}$$

for (3.6a) for the first one upon using (3.2). The identities (3.6b) and (3.6c) follow in the same way by changing the appropriate signs. \square

We collect some formulae concerning the coefficients $a_{2k}^{(n)}$ and $a_{2k+1}^{(n)}$. One has $a_0^{(n)} = 2$ for all $n \in \mathbb{N}$. For $k > 0$ the first few coefficients $a_{2k}^{(n)}$ are given by the following polynomials in k (note that these expressions are not valid for $k = 0$):

$$a_{2k}^{(0)} = 0, \quad a_{2k}^{(1)} = 4, \quad a_{2k}^{(2)} = 16k, \quad a_{2k}^{(3)} = 32k^2 + 4, \quad a_{2k}^{(4)} = \frac{128}{3}k^3 + \frac{64}{3}k, \quad (3.7)$$

while the first few coefficients $a_{2k+1}^{(n)}$ are given by polynomials in k as well:

$$\begin{aligned} a_{2k+1}^{(1)} &= -4, \quad a_{2k+1}^{(2)} = -16k - 8, \quad a_{2k+1}^{(3)} = -32k^2 - 32k - 12, \\ a_{2k+1}^{(4)} &= -\frac{128}{3}k^3 - 64k^2 - \frac{160}{3}k - 16. \end{aligned} \quad (3.8)$$

With Lemma 3 above one can now construct a Lie algebra homomorphism from \mathfrak{k} to $\mathfrak{N}(\mathbb{K}[[u]])$, defined in (3.1), that is constructed using (3.4).

Proposition 5. *The linear map $\rho_{\pm} : \mathfrak{k}(A)(\mathbb{K}) \rightarrow \mathfrak{N}(\mathbb{K}[[u]])$ defined by*

$$(t^n + t^{-n}) \otimes x \mapsto (\pm 1)^n \sum_{k=0}^{\infty} a_{2k}^{(n)} u^{2k} \otimes x \quad \forall x \in \mathring{\mathfrak{k}} \quad (3.9a)$$

$$(t^n - t^{-n}) \otimes y \mapsto (\pm 1)^n \sum_{k=0}^{\infty} a_{2k+1}^{(n)} u^{2k+1} \otimes y \quad \forall y \in \mathring{\mathfrak{p}} \quad (3.9b)$$

with $a_{2k}^{(n)}, a_{2k+1}^{(n)}$ as in (3.5) extends to a homomorphism of Lie algebras.

Proof. This can be checked case by case for pairs $(a, b) \in (\mathfrak{L}^+ \otimes \mathfrak{k}) \times (\mathfrak{L}^+ \otimes \mathfrak{k})$, $(\mathfrak{L}^+ \otimes \mathfrak{k}) \times (\mathfrak{L}^- \otimes \mathfrak{p})$, $(\mathfrak{L}^- \otimes \mathfrak{p}) \times (\mathfrak{L}^- \otimes \mathfrak{p})$ and extending the result by (bi-)linearity. We demonstrate this for the first pair, the other pairs work analogously.

First, set $x^{(n)} := (t^n + t^{-n}) \otimes x$ for all $x \in \mathfrak{k}$. Then by definition

$$\left[x_1^{(m)}, x_2^{(n)} \right] = \left(t^{m+n} + t^{-(m+n)} \right) \otimes [x_1, x_2] + \left(t^{|m-n|} + t^{-|m-n|} \right) \otimes [x_1, x_2],$$

so that from (3.9a)

$$\rho_{\pm} \left(\left[x_1^{(m)}, x_2^{(n)} \right] \right) = (\pm 1)^{m+n} \sum_{k=0}^{\infty} \left(a_{2k}^{(m+n)} + a_{2k}^{(|m-n|)} \right) u^{2k} \otimes [x_1, x_2].$$

On the other hand

$$\begin{aligned} \left[\rho \left(x_1^{(m)} \right), \rho \left(x_2^{(n)} \right) \right] &= (\pm 1)^{m+n} \sum_{k=0}^{\infty} \sum_{\ell=0}^k a_{2(k-\ell)}^{(m)} a_{2\ell}^{(n)} u^{2k} \otimes [x_1, x_2] \\ &= (\pm 1)^{m+n} \sum_{k=0}^{\infty} \left(a_{2k}^{(m+n)} + a_{2k}^{(|m-n|)} \right) u^{2k} \otimes [x_1, x_2], \end{aligned}$$

by (3.6a), proving the claim. For the other pairs one proceeds similarly, using in particular $a_0^{(n)} = 2$ for all n , and this shows that the map ρ_{\pm} extends to a homomorphism of Lie algebras. \square

It is now possible to introduce a cutoff in (3.9) in order to make all expressions finite sums. Denote by

$$\mathcal{I}_N := \left(u^{N+1} \right) \quad (3.10)$$

the ideal in $\mathbb{K}[[u]]$ that is generated by the element u^{N+1} and set $\mathbb{P}_N := \mathbb{K}[[u]]/\mathcal{I}_N$. Furthermore, define the even and odd parts as

$$\mathbb{P}_N^+ := \text{span}_{\mathbb{K}} \left\{ u^{2k} + \mathcal{I}_N \mid k = 0, \dots, \left\lfloor \frac{N}{2} \right\rfloor \right\}, \quad \mathbb{P}_N^- := \text{span}_{\mathbb{K}} \left\{ u^{2k+1} + \mathcal{I}_N \mid k = \left\lfloor \frac{N-1}{2} \right\rfloor \right\}$$

and consider the Lie algebra

$$\mathfrak{N}(\mathbb{P}_N) := \left(\mathbb{P}_N^+ \otimes \mathfrak{k} \right) \oplus \left(\mathbb{P}_N^- \otimes \mathfrak{p} \right), \quad [P \otimes x, Q \otimes y] = (P \cdot Q) \otimes [x, y] \quad (3.11)$$

where the bracket on the right-hand side still denotes the \mathfrak{g} -bracket.

Remark 6. Note that the ring of formal power series \mathbb{P} is isomorphic to the inverse limit of the inverse system $(\mathbb{P}_N)_{N \in \mathbb{N}}$ with the obvious bonding maps [R]. As a consequence $\mathfrak{N}(\mathbb{K}[[u]])$ is the inverse limit of the $\mathfrak{N}(\mathbb{P}_N)$:

$$\mathfrak{N}(\mathbb{K}[[u]]) = \varprojlim_{N \rightarrow \infty} \mathfrak{N}(\mathbb{P}_N). \quad (3.12)$$

In the following we will construct homomorphisms $\rho_{\pm}^{(N)} : \mathfrak{k}(A)(\mathbb{K}) \rightarrow \mathfrak{N}(\mathbb{P}_N)$ which are then shown to be surjective. As $\rho_{\pm}^{(N)}$ is constructed via ρ_{\pm} and the natural projection

π_N from $\mathfrak{N}(\mathbb{K}[[u]])$ to $\mathfrak{N}(\mathbb{P}_N)$ the $\rho_{\pm}^{(N)}$ are compatible with the natural projections from $\mathfrak{N}(\mathbb{P}_{N+M})$ to $\mathfrak{N}(\mathbb{P}_N)$. Assuming we had started with the $\rho_{\pm}^{(N)}$ then by the universal property of inverse limits we would have obtained ρ_{\pm} as the unique homomorphism $\rho_{\pm} : \mathfrak{k}(A)(\mathbb{K}) \rightarrow \mathfrak{N}(\mathbb{K}[[u]])$ such that $\rho_{\pm}^{(N)} = \rho_{\pm} \circ \pi_N$.

Corollary 7. *The homomorphisms of Lie algebras $\rho_{\pm} : \mathfrak{k}(A)(\mathbb{K}) \rightarrow \mathfrak{N}(\mathbb{K}[[u]])$ induce homomorphisms $\rho_{\pm}^{(N)} : \mathfrak{k}(A)(\mathbb{K}) \rightarrow \mathfrak{N}(\mathbb{P}_N)$ which are given explicitly by*

$$(t^n + t^{-n}) \otimes x \mapsto (\pm 1)^n \sum_{k=0}^{\lfloor N/2 \rfloor} a_{2k}^{(n)} (u^{2k} + \mathcal{I}_N) \otimes x \quad \forall x \in \mathring{\mathfrak{k}} \quad (3.13)$$

$$(t^n - t^{-n}) \otimes y \mapsto (\pm 1)^n \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} a_{2k+1}^{(n)} (u^{2k+1} + \mathcal{I}_N) \otimes y \quad \forall y \in \mathring{\mathfrak{p}}. \quad (3.14)$$

Proof. $\mathfrak{N}(\mathbb{P}_N)$ is a quotient of $\mathfrak{N}(\mathbb{K}[[u]])$ because $\mathbb{P}_N = \mathbb{K}[[u]]/\mathcal{I}_N$ is a quotient of \mathbb{P} . \square

Next, we want to show that the homomorphisms ρ_{\pm} are injective but not surjective. Towards this we will need the following fact about matrices whose entries are obtained from evaluation of polynomials:

Lemma 8. *Let $0 \neq p_1, \dots, p_n \in \mathbb{K}[t]$ be linearly independent polynomials. Then there exist $N_1, \dots, N_n \in \mathbb{N}$ such that the matrix $\mathcal{M}(N_1, \dots, N_n) := (p_i(N_j))_{i,j=1}^n$ is regular.*

Proof. This can be achieved by induction on n and an expansion of the determinant which yields a linear combination of linearly independent polynomials. As a nonzero polynomial $p(t)$ is equal to 0 for only finitely many $t \in \mathbb{K}$ this can be used to show regularity of $\mathcal{M}(N_1, \dots, N_n)$ for suitable N_1, \dots, N_n . \square

Proposition 9. *The homomorphism of Lie algebras $\rho_{\pm} : \mathfrak{k}(A)(\mathbb{K}) \rightarrow \mathfrak{N}(\mathbb{K}[[u]])$ from Proposition 5 is injective. Furthermore its image does not contain elements in $\mathfrak{N}(\mathbb{K}[[u]]) = (\mathbb{P}^+ \otimes \mathfrak{k}) \oplus (\mathbb{P}^- \otimes \mathfrak{p})$ whose formal power series contain only finitely many nonzero coefficients. In particular, the elements $u^{2k+2} \otimes x$ for $x \in \mathring{\mathfrak{k}}$ and $u^{2k+1} \otimes y$ for $y \in \mathring{\mathfrak{p}}$ and $k \geq 0$ are not contained in the image of $\mathfrak{k}(A)(\mathbb{K})$ in $\mathfrak{N}(\mathbb{K}[[u]])$.*

Remark 10. Observe that there is a certain asymmetry. It is possible to map elements from $\mathfrak{k}(A)(\mathbb{K})$ to $\mathfrak{N}(\mathbb{K}[[u]])$ by allowing formal power series but in the reverse direction it is not possible to define such a map for elements such as $u^{2k+2} \otimes x$ because the Laurent polynomials do not have a completion that behaves well under the involution η . One can only complete in one direction, i.e., for an element $\sum_{n \in \mathbb{Z}} c_n u^n \otimes x$ in some completion of $\mathfrak{k}(A)(\mathbb{K})$, $c_n \neq 0$ is only possible for either $n > N$ or $n < N$ but not both at the same time. But this does not agree with the demand $c_{-n} = \pm c_n$ unless $c_n \neq 0$ for only finitely many n .

Proof. For a generic element in $\mathfrak{k}(A)(\mathbb{K})$ one can split the analysis into two pieces because elements from $\mathfrak{L}^+ \otimes \mathring{\mathfrak{k}}$ are mapped to elements which only involve even powers u^{2k} while the ones from $\mathfrak{L}^- \otimes \mathring{\mathfrak{p}}$ are mapped to series involving only odd powers u^{2k+1} . Therefore consider the image of

$$\chi := \sum_{i=1}^K (t^{n_i} + t^{-n_i}) \otimes x_i$$

under ρ :

$$\rho(\chi) = \sum_{i=1}^K \sum_{k \geq 0} a_{2k}^{(n_i)} u^{2k} \otimes x_i = \sum_{k \geq 0} \sum_{i=1}^K a_{2k}^{(n_i)} u^{2k} \otimes x_i = 0,$$

so that we need $\sum_{i=1}^K a_{2k}^{(n_i)} x_i = 0$ for all $k \geq 0$. For a basis e_1, \dots, e_d of \mathfrak{k} and $x_i = \sum_{j=1}^d c_i^j e_j$ this yields

$$\sum_{i=1}^K \sum_{j=1}^d a_{2k}^{(n_i)} c_i^j e_j = 0 \quad \forall k \geq 0 \Leftrightarrow \sum_{i=1}^K a_{2k}^{(n_i)} c_i^j = 0 \quad \forall k \geq 0, \quad \forall j = 1, \dots, d.$$

This way one sees that $\rho(\chi) = 0$ admits nontrivial solutions if and only if the infinite linear system of equations

$$\sum_{i=1}^K a_{2k}^{(n_i)} z_i = 0 \quad \forall k \geq 0 \tag{3.15}$$

does. The coefficient $a_{2N}^{(n)}$ is given by the evaluation at N of a nontrivial polynomial $p_n \in \mathbb{K}[x]$ of degree $n-1$ according to Eq. (3.5). Note that the polynomials p_{n_1}, \dots, p_{n_K} are linearly independent if the n_1, \dots, n_K are pairwise distinct because then they are of different degree. Consider a subsystem of linear equations of (3.15) given by

$$\sum_{i=1}^K a_{2k}^{(n_i)} z_i = 0 \quad \forall k \in \{N_1, \dots, N_K\} \Leftrightarrow \sum_{i=1}^K p_{n_i}(k) z_i = 0 \quad \forall k \in \{N_1, \dots, N_K\}.$$

By Lemma 8 it is possible to choose N_1, \dots, N_K such that the matrix $(p_{n_i}(N_j))_{i,j=1}^K$ is regular. Thus, this subsystem of linear equations admits only the trivial solution and therefore so does (3.15). Another way to put this result is that for a basis $\{e_1, \dots, e_d\}$ of \mathfrak{k} the elements of the set

$$\{\rho((t^n + t^{-n}) \otimes e_i) \mid n \geq 0, i = 1, \dots, d\} \in \mathfrak{N}(\mathbb{K}) \tag{3.16}$$

are linearly independent. Since $a_{2N+1}^{(n)}$ is also given by a polynomial the same argument works for

$$\{\rho((t^n - t^{-n}) \otimes f_i) \mid n \geq 1, i = 1, \dots, D\} \in \mathfrak{N}(\mathbb{K}), \tag{3.17}$$

where $\{f_1, \dots, f_D\}$ is a basis of \mathfrak{p} . This shows that ρ_{\pm} is injective.

We next consider the claim that elements $\sum_{k \geq 0} b_{2k} u^{2k} \otimes x_k$ with only finitely many $b_{2k} \neq 0$ are not contained in the image of $\mathfrak{k}(A)(\mathbb{K})$ in $\mathfrak{N}(\mathbb{K}[[u]])$. Any element in the image of $\mathfrak{k}(A)(\mathbb{K})$ in $\mathfrak{N}(\mathbb{K}[[u]])$ can be written as a linear combination of elements in (3.16) and (3.17). It suffices to focus on one set as they are split into even and odd coefficients. For elements of type (3.16) this implies that there exist $b_1, \dots, b_N \in \mathbb{K} \setminus \{0\}$ such that

$$\sum_{j=1}^N b_j a_{2k}^{(n_j)} = 0 \quad \forall k > 0 \Leftrightarrow \sum_{j=1}^N b_j p^{(n_j)}(k) = 0 \quad \forall k > 0.$$

The polynomials $p^{(n_j)}$ are linearly independent and so their sum is a polynomial of fixed degree greater than 0. Then the above equation is a contradiction to the fact that a nonzero polynomial can be equal to 0 only at finitely many points. \square

Propositions 5 and 9 show the first part of Theorem A concerning $\mathfrak{N}(\mathbb{P})$ and we consider the statements concerning $\mathfrak{N}(\mathbb{P}_N)$ next.

Proposition 11. *The homomorphisms $\rho_{\pm}^{(N)} : \mathfrak{k}(A)(\mathbb{K}) \rightarrow \mathfrak{N}(\mathbb{P}_N)$ are surjective.*

Proof. Set

$$\mathring{\mathfrak{k}}_{(2k)} := \text{span}_{\mathbb{K}} \left\{ \left(u^{2k} + \mathcal{I}_N \right) \otimes x \mid x \in \mathring{\mathfrak{k}} \right\} \subset \mathfrak{N}(\mathbb{P}_N), \quad (3.18a)$$

$$\mathring{\mathfrak{p}}_{(2k+1)} := \text{span}_{\mathbb{K}} \left\{ \left(u^{2k+1} + \mathcal{I}_N \right) \otimes y \mid y \in \mathring{\mathfrak{p}} \right\} \subset \mathfrak{N}(\mathbb{P}_N) \quad (3.18b)$$

and note that $\mathfrak{N}(\mathbb{P}_N)$ decomposes into vector spaces as

$$\mathfrak{N}(\mathbb{P}_N) \cong \bigoplus_{k=0}^{\lfloor N/2 \rfloor} \mathring{\mathfrak{k}}_{(2k)} \oplus \bigoplus_{k=0}^{\lfloor (N-1)/2 \rfloor} \mathring{\mathfrak{p}}_{(2k+1)}. \quad (3.19)$$

The image of $\rho_{\pm}^{(N)}$ in $\mathfrak{N}(\mathbb{P}_N)$ is spanned by elements of the form

$$\begin{aligned} & \sum_{k=0}^{\lfloor N/2 \rfloor} a_{2k}^{(n)} \left(u^{2k} + \mathcal{I}_N \right) \otimes x \quad \forall n \in \mathbb{N}, x \in \mathring{\mathfrak{k}}, \\ & \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} a_{2k+1}^{(n)} \left(u^{2k+1} + \mathcal{I}_N \right) \otimes y \quad \forall n \in \mathbb{N}^*, y \in \mathring{\mathfrak{p}}. \end{aligned}$$

Since $a_{2k}^{(0)} = 2\delta_{k,0}$ one already has $\mathring{\mathfrak{k}}_{(0)} = \{1 \otimes x \mid x \in \mathring{\mathfrak{k}}\} \subset \text{im } \rho_{\pm}^{(N)}$. With this it is possible to remove the $\mathring{\mathfrak{k}}_{(0)}$ -part from other elements:

$$\rho_{\pm}^{(N)} \left(x_{(m)} - \frac{1}{2} (\pm 1)^m a_0^{(m)} x_{(0)} \right) = \sum_{k=1}^{\lfloor N/2 \rfloor} a_{2k}^{(n)} \left(u^{2k} + \mathcal{I}_N \right) \otimes x \quad \forall n \in \mathbb{N}, x \in \mathring{\mathfrak{k}}.$$

By the properties of the Cartan decomposition (cf. [HN, prop. 13.1.10]) one has that if $\mathring{\mathfrak{g}}$ is simple and non-compact then $\mathring{\mathfrak{k}} = [\mathring{\mathfrak{p}}, \mathring{\mathfrak{p}}]$ and that $\mathring{\mathfrak{p}}$ is a simple $\mathring{\mathfrak{k}}$ -module. Therefore any element $x \in \mathring{\mathfrak{k}}$ can be written as an iterated commutator $[x^{(1)}, [x^{(2)}, \dots, x^{(k)}]]$ for $x^{(1)}, \dots, x^{(k)} \in \mathring{\mathfrak{k}}$ or $\mathring{\mathfrak{p}}$. Choose levels n_1, \dots, n_k such that $n_1 + \dots + n_k = \lfloor N/2 \rfloor$ and set

$$\tilde{x}_{(n_i)}^{(i)} := x_{(n_i)}^{(i)} - \frac{1}{2} (\pm 1)^{n_i} a_0^{(n_i)} x_{(0)}^{(i)}.$$

Then

$$\rho_{\pm}^{(N)} \left(\left[\tilde{x}_{(n_1)}^{(1)}, \left[\dots, \tilde{x}_{(n_k)}^{(k)} \right] \right] \right) = \left(u^{2\lfloor N/2 \rfloor} + \mathcal{I}_N \right) \otimes x$$

which shows that

$$\left\{ \left(u^{2\lfloor N/2 \rfloor} + \mathcal{I}_N \right) \otimes x \mid x \in \mathring{\mathfrak{k}} \right\}$$

is contained in the image of $\rho_{\pm}^{(N)}$. The same procedure works for $(u^{2k+1} + \mathcal{I}_N) \otimes \mathring{\mathfrak{p}}$ because $\mathring{\mathfrak{p}}$ is a simple $\mathring{\mathfrak{k}}$ -module and therefore there exist iterated commutators here as well. Repeat this process for the lower levels as now the level $(u^{2\lfloor N/2 \rfloor} + \mathcal{I}_N)$ can be removed. This shows by induction that each homogeneous space in (3.19) lies in the image of $\rho_{\pm}^{(N)}$ which concludes the proof. \square

With this, we have proven all parts of Theorem A. We conclude this section with some results on the structure of $\mathfrak{N}(\mathbb{P}_N)$ and the kernel of $\rho_{\pm}^{(N)}$.

Proposition 12. *Let $\mathring{\mathfrak{k}}_{(2k)}$ and $\mathring{\mathfrak{p}}_{(2k+1)}$ be as in (3.18a) and (3.18b) and denote by $\mathfrak{z}(\mathring{\mathfrak{k}}_{(0)})$ the center² of $\mathring{\mathfrak{k}}_{(0)}$, then*

$$\mathfrak{J}_{(N)} := \mathfrak{z}(\mathring{\mathfrak{k}}_{(0)}) \oplus \bigoplus_{k=1}^{\lfloor N/2 \rfloor} \mathring{\mathfrak{k}}_{(2k)} \oplus \bigoplus_{k=0}^{\lfloor (N-1)/2 \rfloor} \mathring{\mathfrak{p}}_{(2k+1)} \subset \mathfrak{N}(\mathbb{P}_N)$$

is the radical of $\mathfrak{N}(\mathbb{P}_N)$, i.e., the unique maximal solvable ideal in $\mathfrak{N}(\mathbb{P}_N)$. Hence, the Levi decomposition of $\mathfrak{N}(\mathbb{P}_N)$ is given by

$$\mathfrak{N}(\mathbb{P}_N) \cong \left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)} \right] \ltimes \mathfrak{J}_{(N)}.$$

Proof. The \mathbb{N} -graded structure of $\mathfrak{N}(\mathbb{P}_N)$ is given by its decomposition into vector spaces (3.19). By the gradation one deduces

$$\left[\mathring{\mathfrak{p}}_{(2k-1)}, \mathring{\mathfrak{p}}_{(2\ell-1)} \right] \subseteq \mathring{\mathfrak{k}}_{(2k+2\ell-2)}, \quad \left[\mathring{\mathfrak{k}}_{(2k)}, \mathring{\mathfrak{p}}_{(2\ell-1)} \right] \subseteq \mathring{\mathfrak{p}}_{(2k+2\ell-1)}, \quad \left[\mathring{\mathfrak{k}}_{(2k)}, \mathring{\mathfrak{k}}_{(2\ell)} \right] \subseteq \mathring{\mathfrak{k}}_{(2k+2\ell)},$$

where it is understood that $\mathring{\mathfrak{k}}_{(2k)} = \{0\} = \mathring{\mathfrak{p}}_{(2\ell+1)}$ for $k > \lfloor N/2 \rfloor$ and $\ell > \lfloor (N-1)/2 \rfloor$. From this it follows that $\mathfrak{J}_{(N)}$ is an ideal as in particular

$$\left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(2\ell)} \right] \subseteq \mathring{\mathfrak{k}}_{(2\ell)} \quad \text{and} \quad \left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{p}}_{(2\ell-1)} \right] \subseteq \mathring{\mathfrak{p}}_{(2\ell-1)}.$$

Since $\left[\mathfrak{z}(\mathring{\mathfrak{k}}_{(0)}), \mathfrak{z}(\mathring{\mathfrak{k}}_{(0)}) \right] = \{0\}$ the lowest degree in $[\mathfrak{J}_{(N)}, \mathfrak{J}_{(N)}]$ is 1 and therefore $\mathfrak{J}_{(N)}$ is solvable because $\mathring{\mathfrak{k}}_{(2k)} = \{0\} = \mathring{\mathfrak{p}}_{(2\ell+1)}$ for $k > \lfloor N/2 \rfloor$ and $\ell > \lfloor (N-1)/2 \rfloor$. Consider the ideal generated by $\mathfrak{J}_{(N)} + x$ for $0 \neq x \in \left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)} \right]$. As $\left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)} \right]$ is semi-simple so is the ideal \mathfrak{j}_0 in $\left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)} \right]$ generated by x . Since ideals of semisimple Lie algebras are semisimple, \mathfrak{j}_0 is also perfect. Thus, the upper derived series $\mathfrak{j}_0^{(n+1)} := \left[\mathfrak{j}_0^{(n)}, \mathfrak{j}_0^{(n)} \right]$ becomes constant at $\mathfrak{j}_0 := [\mathfrak{j}_0, \mathfrak{j}_0]$ and therefore the upper derived series $\mathfrak{J}_{(N)}^{(n+1)} := \left[\mathfrak{J}_{(N)}^{(n)}, \mathfrak{J}_{(N)}^{(n)} \right]$ will always contain \mathfrak{j}_0 . Thus, $\mathfrak{J}_{(N)}$ is a maximal solvable ideal and therefore by definition the radical of $\mathfrak{N}(\mathbb{P}_N)$. \square

Remark 13. We call $\mathfrak{N}(\mathbb{K}[[u]])$ a parabolic Lie algebra since it is the inverse limit of the parabolic Lie algebra $\mathfrak{N}(\mathbb{P}_N)$. By Proposition 5 and Corollary 7, we can construct representations of the involutory subalgebra \mathfrak{k} by considering representations of $\mathfrak{N}(\mathbb{K}[[u]])$ or $\mathfrak{N}(\mathbb{P}_N)$, respectively.

² Note that for a generalised Dynkin diagram A of untwisted affine type, the only cases when $\mathfrak{z}(\mathring{\mathfrak{k}}_{(0)})$ is nontrivial are $A = C_l^{(1)}$ and $A = A_1^{(1)}$. For $\mathring{\mathfrak{k}}_{(0)} = \mathfrak{k}(C_l)$ one has $\mathfrak{k}(C_l) \cong \mathfrak{u}_l$ which contains a nontrivial center, whereas for $\mathfrak{k}(A_1) \cong \mathbb{R}$ the center is already all of $\mathring{\mathfrak{k}}_{(0)}$.

Remark 14. By a consequence of Lie's theorem one has that every simple representation of $\mathfrak{N}(\mathbb{P}_N)$ over a complex vector space is given by the tensor product of a simple representation of $\mathfrak{N}(\mathbb{P}_N)/\text{rad}(\mathfrak{N}(\mathbb{P}_N)) \cong [\mathring{\mathfrak{k}}, \mathring{\mathfrak{k}}]$ with a one-dimensional representation of $\mathfrak{N}(\mathbb{P}_N)$. Therefore, the simple representations of $\mathfrak{k}(A)(\mathbb{K})$ that factor through $\mathfrak{N}(\mathbb{P}_N)$ are essentially the simple representations of $\mathring{\mathfrak{k}}(\mathbb{K})$. These correspond to truncation at $N = 0$.

The kernels $\ker \rho_{\pm}^{(N)}$ are described by linear systems of equations as the following proposition shows:

Proposition 15. *Let $\ker \rho_{\pm}^{(N)}$ be the kernel of the homomorphism described in Corollary 7. Then with*

$$x_{(m)} := (t^m + t^{-m}) \otimes x \quad \forall x \in \mathring{\mathfrak{k}}, \quad y_{(m)} := (t^m - t^{-m}) \otimes y \quad \forall y \in \mathring{\mathfrak{p}}$$

one has

$$\begin{aligned} \ker \rho_{\pm}^{(N)} = & \text{span} \left\{ \sum_{i=1}^M (\pm 1)^{m_i} b_i x_{(m_i)} \mid \sum_{i=1}^M (\pm 1)^{m_i} b_i a_{2k}^{(m_i)} = 0 \quad \forall k=0, \dots, \lfloor N/2 \rfloor, \quad \forall x \in \mathring{\mathfrak{k}} \right\} \\ & \oplus \text{span} \left\{ \sum_{i=1}^M (\pm 1)^{m_i} b_i y_{(m_i)} \mid \sum_{i=1}^M (\pm 1)^{m_i} b_i a_{2k+1}^{(m_i)} = 0 \quad \forall k=0, \dots, \lfloor (N-1)/2 \rfloor, \quad \forall y \in \mathring{\mathfrak{p}} \right\} \\ \ker \rho_{\pm}^{(N)} \supset & \ker \rho_{\pm}^{(N+1)}. \end{aligned}$$

Proof. In general one has for $x \in \mathring{\mathfrak{k}}, y \in \mathring{\mathfrak{p}}$ that

$$\rho_{\pm}^{(N)}(x_{(m)}) = (\pm 1)^m \sum_{k=0}^{\lfloor N/2 \rfloor} a_{2k}^{(m)} u^{2k} \otimes x, \quad \rho_{\pm}^{(N)}(y_{(m)}) = (\pm 1)^m \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} a_{2k+1}^{(m)} u^{2k+1} \otimes y$$

and one computes for $\sum_{i=1}^M (\pm 1)^{m_i} b_i x_{(m_i)}$ that

$$\begin{aligned} \rho_{\pm}^{(N)} \left(\sum_{i=1}^M (\pm 1)^{m_i} b_i x_{(m_i)} \right) &= \sum_{i=1}^M (\pm 1)^{m_i} b_i \sum_{k=0}^{\lfloor N/2 \rfloor} a_{2k}^{(m_i)} u^{2k} \otimes x \\ &= \sum_{k=0}^{\lfloor N/2 \rfloor} \left(\sum_{i=1}^M (\pm 1)^{m_i} b_i a_{2k}^{(m_i)} \right) u^{2k} \otimes x = 0 \end{aligned}$$

if and only if

$$\sum_{i=1}^M (\pm 1)^{m_i} b_i a_{2k}^{(m_i)} = 0 \quad \forall k = 0, \dots, \lfloor N/2 \rfloor.$$

Similarly one deduces that

$$\rho_{\pm}^{(N)} \left(\sum_{i=1}^M (\pm 1)^{m_i} b_i y_{(m_i)} \right) = 0 \Leftrightarrow \sum_{i=1}^M (\pm 1)^{m_i} b_i a_{2k+1}^{(m_i)} = 0 \quad \forall k = 0, \dots, \lfloor (N-1)/2 \rfloor.$$

These equations remain unaltered by changing $N \mapsto N + N_0, N_0 \in \mathbb{N}$, there only appear additional equations to be satisfied. This shows that

$$\ker \rho_{\pm}^{(N)} \supset \ker \rho_{\pm}^{(N+1)}.$$

□

Remark 16. Specialised to ϵ_9 the ideals $\ker \rho_{\pm}^{(0)}$ in $\mathfrak{k}(\epsilon_9)(\mathbb{K})$ coincide with the Dirac ideals of [KNP]. Furthermore, these ideals can be shown to be principal ideals.

4. Representations from Quotients of Induced Representations

In this section, we consider representations of \mathfrak{k} that are constructed using induced representations of $\mathfrak{N}(\mathbb{K}[u])$, the model of \mathfrak{k} over the polynomial ring which will be defined below. Since representations of $\mathfrak{N}(\mathbb{K}[u])$ in general do not provide representations of \mathfrak{k} , we will then describe how to recover representations of \mathfrak{k} .

Construct the Lie algebra $\mathfrak{N}(\mathbb{K}[u])$ in the same way as $\mathfrak{N}(\mathbb{K}[[u]])$ in (3.1) but replace $\mathbb{K}[[u]]$ by the ring of polynomials $\mathbb{K}[u]$. As mentioned in Remark 2, $\mathfrak{N}(\mathbb{K}[u])$ is \mathbb{N} -graded and its graded decomposition is given by

$$\mathfrak{N}(\mathbb{K}[u]) = \bigoplus_{k=0}^{\infty} \mathfrak{N}^{(k)}, \quad (4.1)$$

where

$$\mathfrak{N}^{(k)} = \begin{cases} \text{span}_{\mathbb{K}} \left\{ u^{2n} \otimes \mathfrak{k} \right\}, & k = 2n \text{ even,} \\ \text{span}_{\mathbb{K}} \left\{ u^{2n+1} \otimes \mathfrak{p} \right\}, & k = 2n + 1 \text{ odd.} \end{cases}$$

We believe that there does not exist any non-trivial Lie algebra homomorphism from \mathfrak{k} into $\mathfrak{N}(\mathbb{K}[u])$. Such homomorphisms, similar to those of Proposition 11, only exist when we quotient by \mathcal{I}_N defined in (3.10) which can also be thought of as an ideal of $\mathbb{K}[u]$. The structure of $\mathfrak{N}(\mathbb{K}[u])$ is analogous to that of $\mathfrak{N}(\mathbb{P}_N)$ described in Proposition 12. In particular, the (now no longer solvable) ideal \mathfrak{N}_+ is given by

$$\mathfrak{N}_+ = \mathfrak{z}(\mathfrak{k}) \oplus \bigoplus_{k>0} \mathfrak{N}^{(k)},$$

where $\mathfrak{z}(\mathfrak{k})$ denotes the center of \mathfrak{k} . The ideal \mathfrak{N}_+ inherits the \mathbb{N} -grading from $\mathfrak{N}(\mathbb{K}[u])$. We shall in the following assume that $\mathfrak{z}(\mathfrak{k}) = 0$ for simplicity. All statements can be generalised straight-forwardly.

The universal enveloping algebras $\mathcal{U}(\mathfrak{N}_+)$ and $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$ also inherit the \mathbb{N} -gradation from the Lie algebra and the Poincaré–Birkhoff–Witt theorem provides a basis of them. One has

$$\mathcal{U}(\mathfrak{N}_+) = \bigoplus_{\ell=0}^{\infty} \mathcal{U}_{\ell}, \quad (4.2)$$

where \mathcal{U}_{ℓ} denotes the (ordered) words in the tensor algebra of degree ℓ with respect to the \mathbb{N} -grading, where $x \leq y$ if $\deg(x) \leq \deg(y)$. For the first few levels, this means

$$\begin{aligned} \mathcal{U}_0 &= 1, & \mathcal{U}_1 &= \mathfrak{N}^{(1)}, \\ \mathcal{U}_2 &= \text{Sym}^2(\mathfrak{N}^{(1)}) \oplus \mathfrak{N}^{(2)}, & \mathcal{U}_3 &= \text{Sym}^3(\mathfrak{N}^{(1)}) \oplus \mathfrak{N}^{(1)} \otimes \mathfrak{N}^{(2)} \oplus \mathfrak{N}^{(3)}. \end{aligned}$$

The $\mathfrak{N}^{(k)}$ are \mathfrak{k} -modules because the adjoint action of \mathfrak{k} preserves the degree. We will make these expressions more explicit in the example in Sect. 5. For the full universal enveloping algebra one has

$$\mathcal{U}(\mathfrak{N}(\mathbb{K}[u])) \cong \mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{N}_+) \quad (4.3)$$

as a tensor product of \mathbb{K} -vector spaces. In terms of the multiplication \cdot in $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$ the above decomposition is better written as

$$\mathcal{U}(\mathfrak{N}(\mathbb{K}[u])) = \mathcal{U}(\mathfrak{k}) \cdot \mathcal{U}(\mathfrak{N}_+) \quad (4.4)$$

This fact is due to the PBW-theorem applied to a suitably chosen order that is such that elements of degree 0 appear to the left in the basis that is provided by the PBW-theorem. Recall that all elements of $\mathfrak{k} = \mathfrak{N}^{(0)} \subset \mathfrak{N}(\mathbb{K}[u])$ have degree 0.

Now consider a finite-dimensional \mathfrak{k} -module \mathfrak{V}_0 which we view as a left $\mathcal{U}(\mathfrak{k})$ -module. As $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$ allows the structure of a right $\mathcal{U}(\mathfrak{k})$ -module we build the induced $\mathfrak{N}(\mathbb{K}[u])$ -module via the tensor product

$$\mathfrak{V} := \mathcal{U}(\mathfrak{N}(\mathbb{K}[u])) \otimes_{\mathcal{U}(\mathfrak{k})} \mathfrak{V}_0, \quad (4.5)$$

where the tensor product is defined with $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$ as a right $\mathcal{U}(\mathfrak{k})$ -module. Explicitly one has

$$a \otimes (x \cdot v) = ax \otimes v$$

for $v \in V$, $x \in \mathcal{U}(\mathfrak{k})$, $a \in \mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$. \mathfrak{V} , however, is viewed as a left $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$ -module, i.e. for $a, b \in \mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$, $v \in V$ one has

$$a \cdot (b \otimes v) = (a \cdot b) \otimes v.$$

Lemma 17. *The induced module \mathfrak{V} inherits the \mathbb{N} -grading*

$$\mathfrak{V} = \bigoplus_{i=0}^{\infty} \mathfrak{V}_i \quad (4.6)$$

from $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$ and is an infinite-dimensional representation of $\mathfrak{N}(\mathbb{K}[u])$. \mathfrak{V} is generated by the action of $\mathcal{U}(\mathfrak{N}_+)$ on $1 \otimes \mathfrak{V}_0$. Furthermore, a \mathbb{K} -basis of \mathfrak{V} is given by the Kronecker product of a PBW-basis of $\mathcal{U}(\mathfrak{N}_+)$ with a basis of \mathfrak{V}_0 .

Proof. The inherited gradation on \mathfrak{V} is given by assigning $\deg(u \otimes v) = \deg(u)$. It is infinite-dimensional because $\mathcal{U}(\mathfrak{N}_+)$ is. In order to determine a \mathbb{K} -basis of (4.5), consider it as a \mathfrak{k} -module and show that it is isomorphic to the \mathbb{K} -tensor product of $\mathcal{U}(\mathfrak{N}_+)$ and \mathfrak{V}_0 . Recall that \mathfrak{k} acts on \mathfrak{N}_+ via the adjoint action from $\mathfrak{N}(\mathbb{K}[u])$. Due to the above decomposition (4.4) one has for $x \in \mathcal{U}(\mathfrak{k})$, $y \in \mathcal{U}(\mathfrak{N}_+)$ and $v \in V$ that

$$xy \otimes v = (yx + [x, y]) \otimes v = y \otimes v' + y' \otimes v \quad (4.7)$$

with $y' = [x, y] \in \mathcal{U}(\mathfrak{N}_+)$ and $v' = x.v \in V$. We want to view (4.5) as a \mathbb{K} -tensor product modulo an equivalence relation. The equivalence relation on $\mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V$ induced by $\otimes_{\mathcal{U}(\mathfrak{k})}$ now is the multilinear extension of

$$x \otimes_{\mathbb{K}} y \otimes_{\mathbb{K}} v \sim 1 \otimes_{\mathbb{K}} y \otimes_{\mathbb{K}} v' + 1 \otimes_{\mathbb{K}} y' \otimes_{\mathbb{K}} v, \quad (4.8)$$

where $y' = [x, y] \in \mathcal{U}(\mathfrak{N}_+)$, $v' = x.v \in V$ and

$$\mathcal{U}(\mathfrak{N}(\mathbb{K}[u])) \otimes_{\mathcal{U}(\mathfrak{k})} V \cong \mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V / \sim$$

as \mathbb{K} -vector spaces. The original equivalence relation in $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u])) \otimes_{\mathbb{K}} V$ is

$$yx \otimes_{\mathbb{K}} v \sim y \otimes_{\mathbb{K}} xv \quad \forall x \in \mathcal{U}(\mathfrak{k})$$

but decomposing $yx = xy - [x, y]$ according to (4.4) leads to the formulation (4.8) of \sim . Equation (4.8) shows that each element $x \otimes_{\mathbb{K}} y \otimes_{\mathbb{K}} v$ is \sim -equivalent to an element of $1 \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V \cong \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V$ where the isomorphism is as \mathbb{K} -vector spaces. Since

$$1 \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V \subset \mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V$$

one deduces that the elements of $\mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{K}} \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V / \sim$ are in 1-1-correspondence with the elements of $\mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V$ and hence,

$$\mathcal{U}(\mathfrak{N}(\mathbb{K}[u])) \otimes_{\mathcal{U}(\mathfrak{k})} V \cong \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V \quad (4.9)$$

as \mathbb{K} -vector spaces. Now $\mathcal{U}(\mathfrak{k})$ acts on $\mathcal{U}(\mathfrak{N}_+) \otimes_{\mathcal{U}(\mathfrak{k})} V$ via left-multiplication but as a result of (4.7) one has that

$$xy \otimes v = [x, y] \otimes v + y \otimes xv.$$

This action is identical to the action of $\mathcal{U}(\mathfrak{k})$ on the \mathbb{K} -tensor product of \mathfrak{k} -modules $\mathcal{U}(\mathfrak{N}_+)$ and V . Thus, one finds

$$\mathcal{U}(\mathfrak{N}(\mathbb{K}[u])) \otimes_{\mathcal{U}(\mathfrak{k})} V \cong \mathcal{U}(\mathfrak{N}_+) \otimes_{\mathbb{K}} V$$

as \mathfrak{k} -modules. The Kronecker basis of this tensor product also provides a basis for the $\mathcal{U}(\mathfrak{N}(\mathbb{K}[u]))$ -module \mathfrak{V} because $\mathcal{U}(\mathfrak{N}_+) \cap \mathcal{U}(\mathfrak{k}) = \mathbb{K} \cdot 1$. □

Due to the \mathbb{N} -graded structure on $\mathfrak{N}(\mathbb{K}[u])$, \mathfrak{V} has many invariant subspaces. In particular, any

$$\mathfrak{V}^{(N)} = \bigoplus_{i=N}^{\infty} \mathfrak{V}_i \quad (4.10)$$

for $N > 0$ is a proper invariant subspace of \mathfrak{V} . The quotient representation

$$\mathfrak{V}/\mathfrak{V}^{(N)} \quad (4.11)$$

is by construction a finite-dimensional representation of $\mathfrak{N}(\mathbb{K}[u])$ for any fixed choice N . We can now prove the first part of Theorem B:

Proposition 18. *The quotient $\mathfrak{V}/\mathfrak{V}^{(N)}$ defined in (4.11) is a finite-dimensional module of $\mathfrak{N}(\mathbb{P}_N)$ and therefore, by Propositions 5 and 11, a representation of \mathfrak{k} .*

Proof. This statement follows from the fact that $\mathcal{I}_N \otimes (\mathfrak{k} \oplus \mathfrak{p}) \cap \mathfrak{N}(\mathbb{K}[u])$ acts trivially on $\mathfrak{V}/\mathfrak{V}^{(N)}$ by the \mathbb{N} -grading and therefore $\mathfrak{V}/\mathfrak{V}^{(N)}$ is a representation of the corresponding quotient. Since the quotients $\mathbb{K}[u]/\mathcal{I}_N$ and $\mathbb{K}[[u]]/\mathcal{I}_N$ are isomorphic, we deduce that $\mathfrak{V}/\mathfrak{V}^{(N)}$ is a finite-dimensional representation of $\mathfrak{N}(\mathbb{P}_N)$ defined in (3.11) and therefore can be pulled back to a representation of \mathfrak{k} by Proposition 11. \square

Remark 19. The quotient algebra acts non-trivially on $\mathfrak{V}/\mathfrak{V}^{(k)}$ and is given by all \mathfrak{k} generators of degree at most k . While all other elements of the parabolic model $\mathfrak{N}(\mathbb{K}[u])$ act trivially, we deduce from Proposition 5 that *infinitely* many generators of \mathfrak{k} act non-trivially.

Other invariant subspaces of \mathfrak{V} can be considered by taking any vector $w \in \mathfrak{V}$ and considering the subrepresentation $\mathfrak{W} \subset \mathfrak{V}$ that it generates under the action of $\mathfrak{N}(\mathbb{K}[u])$. A natural choice would be to select some irreducible \mathfrak{k} representation \mathfrak{W}_k within one of the \mathfrak{V}_k . Clearly, the submodule \mathfrak{W} generated by \mathfrak{W}_k is a subspace of $\mathfrak{V}^{(k)}$. In general, it can be of arbitrary co-dimension in it. The representation of $\mathfrak{N}(\mathbb{K}[u])$ we are interested in then is the quotient

$$\mathfrak{V}/\mathfrak{W}. \quad (4.12)$$

If this quotient is finite-dimensional it can again be pulled back to a representation of \mathfrak{k} since it is a quotient of one of the spaces $\mathfrak{V}/\mathfrak{V}^{(N)}$ described above. Analysing (4.12) in general is more complicated than in the case (4.11). To illustrate this point, we shall analyse an explicit example for \mathfrak{e}_9 in the next Sect. 5. But first we prove the second part of Theorem B.

Proposition 20. *Recall the graded decomposition (4.6) of \mathfrak{V} and consider its formal completion*

$$\overline{\mathfrak{V}} := \{(v_i) \mid v_i \in \mathfrak{V}_i\}. \quad (4.13)$$

On $\overline{\mathfrak{V}}$ there exists a natural, faithful $\mathfrak{N}(\mathbb{K}[[u]])$ -action. Furthermore, $\overline{\mathfrak{V}}$ is the inverse limit of the modules of proposition 18 and it is spinorial as a \mathfrak{k} -representation if the \mathfrak{k} -representation \mathfrak{V}_0 that induces \mathfrak{V} is. Also, faithfulness shows that \mathfrak{k} is residually finite-dimensional, that is, for each non-trivial element $x \in \mathfrak{k}$ there exists a finite-dimensional representation on which x acts non-trivially.³

Proof. The natural action is given by

$$v \mapsto x \cdot v \quad \text{with} \quad (x \cdot v)_i = \sum_{j=0}^i x_j \cdot v_{i-j} \quad \text{for } i \in \mathbb{N} \quad (4.14)$$

for $x = (x_i) \in \mathfrak{N}(\mathbb{K}[[u]])$ and $v = (v_i) \in \overline{\mathfrak{V}}$. Since only x_i and v_{N-i} enter at degree N this action commutes with the projection to $\overline{\mathfrak{V}}/\overline{\mathfrak{V}}^{(i)}$, where $\overline{\mathfrak{V}}^{(i)}$ denotes the formal completion of $\mathfrak{V}^{(i)}$. As $\overline{\mathfrak{V}}/\overline{\mathfrak{V}}^{(i)} \cong \mathfrak{V}/\mathfrak{V}^{(i)}$, one can check that $\overline{\mathfrak{V}}$ is the inverse limit of the finite-dimensional representations $\mathfrak{V}/\mathfrak{V}^{(i)}$ by also establishing that the elements in $\overline{\mathfrak{V}}$ and the inverse limit of the $\mathfrak{V}/\mathfrak{V}^{(i)}$ are in 1-1-correspondence. The representation is faithful, because for $(x_i) \in \mathfrak{N}(\mathbb{K}[[u]])$ one can pick $(v_0, 0, 0, \dots) \in \overline{\mathfrak{V}}$

³ We believe this property to be known already for affine $\mathfrak{g}(A)$, although we were unable to find a reference. Hence, our proposition just reproves residual finiteness of \mathfrak{k} as a byproduct.

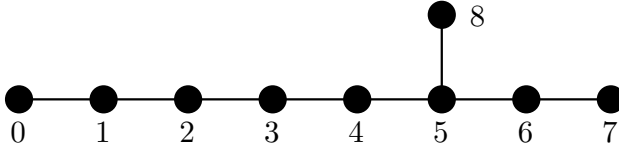


Fig. 1. Dynkin diagram of E_9 with nodes labelled

with $0 \neq v_0 \in \mathfrak{V}_0$. Since $\mathfrak{N}^{(k)}$ is part of the PBW-basis of $\mathcal{U}(\mathfrak{N}(\mathbb{K}[[u]]))$, there always exists a $k > 0$ and $0 \neq v_0$ such that $x_k \cdot v_0 \neq 0 \in \mathfrak{V}$ according to Lemma 17. The only exception is when $(x_i) \in \mathfrak{N}(\mathbb{K}[[u]])$ is such that only $x_0 \neq 0$. Even if \mathfrak{V}_0 is the trivial module, the action of this element is nontrivial because $\mathcal{U}(\mathfrak{N}_+)$ is a faithful \mathfrak{k} -module. Spinorality of \mathfrak{V}_0 implies that of \mathfrak{V} because \mathfrak{k} maps to $\mathfrak{N}^{(0)} \subset \mathfrak{N}(\mathbb{K}[[u]])$. \square

5. A Detailed Example: $\mathfrak{k}(\mathfrak{e}_9)$

In this section, we illustrate the general considerations from the previous sections in the case of $\mathfrak{k}(\mathfrak{e}_9)$ where $\mathfrak{g} = \mathfrak{e}_9 \equiv \mathfrak{e}_8^{(1)}$ denotes the non-twisted affine extension of the split real exceptional Lie algebra $\hat{\mathfrak{g}} = \mathfrak{e}_8$ (also denoted as $\mathfrak{e}_{8(8)}$). The algebra $\mathfrak{k}(\mathfrak{e}_9)$ enters in the fermionic sector of maximal supergravity in $D = 2$ dimensions where unfaithful representations have been found previously [NS] and is therefore of particular interest in physics.

5.1. *The Affine Algebra \mathfrak{e}_9 .* The Cartan decomposition of \mathfrak{e}_8 is

$$\mathfrak{e}_8 = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad \mathfrak{k} \cong \mathfrak{so}(16) \quad (5.1)$$

and \mathfrak{p} is a 128-dimensional irreducible spinor representation of $\mathfrak{so}(16)$. To explicitly parametrise the symmetric space decomposition we introduce the following adapted basis of generators

$$\begin{aligned} \mathfrak{so}(16) &= \langle X^{IJ} \mid X^{IJ} = -X^{JI}, \quad I, J = 1, \dots, 16 \rangle, \\ \mathfrak{p} &= \langle Y^A \mid A = 1, \dots, 128 \rangle, \end{aligned} \quad (5.2)$$

so that the X^{IJ} are the exterior square of the defining 16-dimensional representation of $\mathfrak{so}(16)$ and the Y^A are in an irreducible spinor representation. The conjugate spinor representation will be denoted with dotted indices A and will play a role in the construction of $\mathfrak{k}(\mathfrak{e}_9)$ representations below.

In order to write the \mathfrak{e}_8 commutation relations in the basis (5.2), we utilise $\mathfrak{so}(16)$ gamma matrices Γ_{AA}^I where AA denote the matrix components of degree one elements in the Clifford algebra of 16-dimensional Euclidean space. These are real 128-by-128 matrices satisfying the Clifford multiplication law

$$\Gamma_{AA}^I \Gamma_{BA}^J + \Gamma_{AA}^J \Gamma_{BA}^I = 2\delta^{IJ} \delta_{AB} \quad (5.3)$$

where repeated indices are summed over (a convention we shall employ throughout this section) and δ^{IJ} and δ_{AB} are the Euclidean $\mathfrak{so}(16)$ -invariant bilinear forms in the representation spaces. k -fold antisymmetric products of gamma matrices are written as

$\Gamma^{I_1 \dots I_k}$ normalised with a $1/k!$ times the signed sum over permutations of I_1, \dots, I_k , e.g.

$$\begin{aligned}\Gamma_{AB}^{I_1 I_2} &= \frac{1}{2!} \left(\Gamma_{AA}^{I_1} \Gamma_{BA}^{I_2} - \Gamma_{AA}^{I_2} \Gamma_{BA}^{I_1} \right), \\ \Gamma_{\dot{A}\dot{B}}^{I_1 I_2} &= \frac{1}{2!} \left(\Gamma_{AA}^{I_1} \Gamma_{AB}^{I_2} - \Gamma_{AA}^{I_2} \Gamma_{AB}^{I_1} \right), \\ \Gamma_{AB}^{I_1 I_2 I_3} &= \frac{1}{3!} \left(\Gamma_{AA}^{I_1} \Gamma_{BA}^{I_2} \Gamma_{BB}^{I_3} \pm 5 \text{ permutations} \right).\end{aligned}\quad (5.4)$$

For more details on the use of these gamma matrices see [NS, KNP].

The explicit commutations relations of (5.1) in the basis (5.2) are then given by

$$\left[X^{IJ}, X^{KL} \right] = \delta^{JK} X^{IL} - \delta^{IK} X^{JL} - \delta^{JL} X^{IK} + \delta^{IL} X^{JK}, \quad (5.5a)$$

$$\left[X^{IJ}, Y^A \right] = -\frac{1}{2} \Gamma_{AB}^{IJ} Y^B, \quad (5.5b)$$

$$\left[Y^A, Y^B \right] = \frac{1}{4} \Gamma_{AB}^{IJ} X^{IJ}. \quad (5.5c)$$

The first line is just the $\mathfrak{k} \cong \mathfrak{so}(16)$ Lie algebra while the others are manifestations of the Cartan decomposition's properties $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. In particular, the last relation can be viewed as a projection from $\text{Alt}^2(\mathfrak{p})$ to the factor that is isomorphic to the adjoint module of \mathfrak{k} .

The affine algebra is then obtained according to (2.4) by introducing the loop generators

$$X_m^{IJ} = t^m \otimes X^{IJ}, \quad Y_m^A = t^m \otimes Y^A \quad (5.6)$$

for $m \in \mathbb{Z}$. The commutation relations, including the central element K and derivation d , are then given by

$$\begin{aligned}\left[X_m^{IJ}, X_n^{KL} \right] &= \delta^{JK} X_{m+n}^{IL} - \delta^{IK} X_{m+n}^{JL} - \delta^{JL} X_{m+n}^{IK} + \delta^{IL} X_{m+n}^{JK} \\ &\quad - m \left(\delta^{IK} \delta^{JL} - \delta^{JK} \delta^{IL} \right) \delta_{m, -n} K,\end{aligned}\quad (5.7)$$

$$\left[X_m^{IJ}, Y_n^A \right] = -\frac{1}{2} \Gamma_{AB}^{IJ} Y_{m+n}^B, \quad (5.8)$$

$$\left[Y_m^A, Y_n^B \right] = \frac{1}{4} \Gamma_{AB}^{IJ} X_{m+n}^{IJ} + m \delta^{AB} \delta_{m, -n} K, \quad (5.9)$$

$$\left[d, X_m^{IJ} \right] = -m X_m^{IJ}, \quad \left[d, Y_m^A \right] = -m Y_m^A, \quad (5.10)$$

and K commutes with everything.

As is well known, there is an action of the Virasoro algebra on loop algebras [K, GO]. The Virasoro algebra is a central extension of the Witt algebra generated by the operators

$$L_m = -t^{m+1} \frac{d}{dt} \quad (m \in \mathbb{Z}) \quad (5.11)$$

acting on Laurent polynomials and the non-trivial commutators with the ϵ_9 loop generators are

$$\left[L_m, X_n^{IJ} \right] = -n X_{m+n}^{IJ}, \quad \left[L_m, Y_n^A \right] = -n Y_{m+n}^A. \quad (5.12)$$

L_0 therefore acts like d . The commutators among the Virasoro generators is

$$\left[L_m, L_n \right] = (m - n) L_{m+n} + \frac{c_{\text{Vir}}}{12} m(m^2 - 1) \delta_{m, -n} K \quad (5.13)$$

with the same central element K as in the affine algebra and c_{Vir} a free coefficient. In any given highest or lowest weight representation of the affine algebra, one can find a realisation of the Virasoro algebra in the universal enveloping algebra of the loop algebra via the Sugawara construction and this fixes c_{Vir} , see [GO] for a review.

5.2. Involutory Subalgebra $\mathfrak{k}(\epsilon_9)$. The involutory subalgebra $\mathfrak{k} \equiv \mathfrak{k}(\epsilon_9)$ was defined in (2.6). We shall give explicit forms of the filtered structure and the parabolic model discussed in Sects. 2 and 3, respectively.

5.2.1. Filtered basis We recall that the central element K and the derivation d are not invariant under the involution ω and therefore not part of $\mathfrak{k}(\epsilon_9)$. The generators of $\mathfrak{k}(\epsilon_9)$ can be expressed in terms of (5.6) according to

$$\mathcal{X}_m^{IJ} = \frac{1}{2} (t^m + t^{-m}) \otimes X^{IJ} \equiv \frac{1}{2} \left(X_m^{IJ} + X_{-m}^{IJ} \right), \quad (5.14a)$$

$$\mathcal{Y}_m^A = \frac{1}{2} (t^m - t^{-m}) \otimes Y^A \equiv \frac{1}{2} \left(Y_m^A - Y_{-m}^A \right). \quad (5.14b)$$

where t is the spectral parameter and $m \in \mathbb{N}$. Note that $\mathcal{Y}_0^A \equiv 0$. Here, we have introduced a factor of $1/2$ such that $\mathcal{X}_0^{IJ} = X_0^{IJ}$ is also an $\mathfrak{so}(16)$ Lie algebra with the same normalisation.

The complete $\mathfrak{k}(\epsilon_9)$ Lie algebra in the basis (5.14) reads

$$\begin{aligned} \left[\mathcal{X}_m^{IJ}, \mathcal{X}_n^{KL} \right] &= \frac{1}{2} \delta^{JK} \left(\mathcal{X}_{m+n}^{IL} + \mathcal{X}_{|m-n|}^{IL} \right) - \frac{1}{2} \delta^{IK} \left(\mathcal{X}_{m+n}^{JL} + \mathcal{X}_{|m-n|}^{JL} \right) \\ &\quad - \frac{1}{2} \delta^{JL} \left(\mathcal{X}_{m+n}^{IK} + \mathcal{X}_{|m-n|}^{IK} \right) + \frac{1}{2} \delta^{IL} \left(\mathcal{X}_{m+n}^{JK} + \mathcal{X}_{|m-n|}^{JK} \right), \end{aligned} \quad (5.15a)$$

$$\left[\mathcal{X}_m^{IJ}, \mathcal{Y}_n^A \right] = -\frac{1}{4} \Gamma_{AB}^{IJ} \left(\mathcal{Y}_{m+n}^B - \text{sgn}(m - n) \mathcal{Y}_{|m-n|}^B \right), \quad (5.15b)$$

$$\left[\mathcal{Y}_m^A, \mathcal{Y}_n^B \right] = \frac{1}{8} \Gamma_{AB}^{IJ} \left(\mathcal{X}_{m+n}^{IJ} - \mathcal{X}_{|m-n|}^{IJ} \right), \quad (5.15c)$$

where $\text{sgn}(k)$ is the sign function with $\text{sgn}(0) = 0$.

In this $\mathfrak{so}(16)$ -covariant formulation, $\mathfrak{k}(\epsilon_9)$ can be described as the algebra generated by the \mathcal{X}_0^{IJ} , \mathcal{X}_1^{IJ} and \mathcal{Y}_1^A with the property that the \mathcal{X}_0^{IJ} form an $\mathfrak{so}(16)$ algebra and that \mathcal{X}_1^{IJ} and \mathcal{Y}_1^A transform correctly under this algebra according to (5.15). The \mathcal{X}_0^{IJ} , \mathcal{X}_1^{IJ} and \mathcal{Y}_1^A obey

$$7 \Gamma_{AB}^{IJ} \left[\mathcal{Y}_1^A, \mathcal{Y}_1^B \right] - 32 \left[\mathcal{X}_1^{IK}, \mathcal{X}_1^{KJ} \right] = -448 \mathcal{X}_0^{IJ}, \quad (5.16a)$$

$$\Gamma_{AB}^{I_1 \dots I_6} \left[\mathcal{Y}_1^A, \mathcal{Y}_1^B \right] = 0. \quad (5.16b)$$

These are the $\mathfrak{so}(16)$ -covariant forms of the Berman relation $[x_0, [x_0, x_1]] = -x_1$ in a Chevalley–Serre basis

$$x_i := e_i - f_i \quad (5.17)$$

and where we use the convention that 0 is the affine node of the e_9 Dynkin diagram that attaches to the adjoint node, labelled 1, of the e_8 Dynkin diagram, see Fig. 1. To write out the ‘affine’ Berman generator x_0 explicitly in terms of the basis (5.14), we need to make use of the $\text{SO}(8)$ decompositions (A.9) in Appendix A of [KNP]; using the notation and transformations of Appendix B of that reference there we have

$$x_0 = -\frac{1}{2} \gamma_{\alpha\beta}^1 (\mathcal{X}_1^{\alpha\beta} + \mathcal{Y}_1^{\alpha\beta}) \quad (5.18)$$

with the $\text{SO}(8)$ gamma matrices $\gamma_{\alpha\beta}^i$ (for $i = 1, \dots, 8$). The remaining Berman generators in this basis are given by

$$x_i = \frac{1}{4} \left(\gamma_{\alpha\beta}^{i,i+1} \mathcal{X}_0^{\alpha\beta} + \gamma_{\dot{\alpha}\dot{\beta}}^{i,i+1} \mathcal{X}_0^{\dot{\alpha}\dot{\beta}} \right) \quad (\text{for } i = 1, \dots, 7), \quad x_8 = -\frac{1}{2} \gamma_{\alpha\beta}^{678} \mathcal{X}_0^{\alpha\beta}. \quad (5.19)$$

The Virasoro algebra likewise can be restricted to an involutory subalgebra [JN]. The ‘maximal compact’ subalgebra of the Virasoro algebra is generated by

$$\mathcal{K}_m := L_m - L_{-m} \equiv -(t^{m+1} - t^{-m+1}) \frac{d}{dt} \quad \text{for } m \geq 1. \quad (5.20)$$

The relevant commutation relations when acting on $K(e_9)$ in the basis (5.14) read

$$\begin{aligned} [\mathcal{K}_m, \mathcal{X}_n^{IJ}] &= -n (\mathcal{X}_{m+n}^{IJ} - \mathcal{X}_{|m-n|}^{IJ}) \\ [\mathcal{K}_m, \mathcal{Y}_n^A] &= -n (\mathcal{Y}_{m+n}^A - \text{sgn}(m-n) \mathcal{Y}_{|m-n|}^A) \end{aligned} \quad (5.21)$$

and

$$[\mathcal{K}_m, \mathcal{K}_n] = (m-n) \mathcal{K}_{m+n} - \text{sign}(m-n) (m+n) \mathcal{K}_{|m-n|} \quad (5.22)$$

Note that the central term drops out here as well.

5.2.2. Parabolic model As discussed in Sects. 3 and 4, it is very useful for constructing representations of $\mathfrak{k}(e_9)$ to consider the parabolic algebras $\mathfrak{N}(\mathbb{K}[[u]])$ and $\mathfrak{N}(\mathbb{K}[u])$ defined in (3.1) and (4.1), respectively. Since polynomials have a graded product, $\mathfrak{N}(\mathbb{K}[u])$ is a graded Lie algebra. We write the variable of the polynomial as u as in Sect. 3 and it is related to the variable t in the filtered basis by $u = \frac{1-t}{1+t}$ so that expansions around $u = 0$ are expansions around $t = 1$ and vice versa.

The definition of the basis generators of $\mathfrak{N}(\mathbb{K}[u])$ is then explicitly

$$\mathcal{A}_{2m}^{IJ} := u^{2m} \otimes X^{IJ}, \quad \mathcal{S}_{2m+1}^A := u^{2m+1} \otimes Y^A \quad \text{for } m \in \mathbb{N}. \quad (5.23)$$

The Lie algebra of these generators is graded and given by

$$\left[\mathcal{A}_{2m}^{IJ}, \mathcal{A}_{2n}^{KL} \right] = \frac{1}{2} \delta^{JK} \mathcal{A}_{2(m+n)}^{IL} - \frac{1}{2} \delta^{IK} \mathcal{A}_{2(m+n)}^{JL}$$

$$-\frac{1}{2}\delta^{JL}\mathcal{A}_{2(m+n)}^{IK} + \frac{1}{2}\delta^{IL}\mathcal{A}_{2(m+n)}^{JK}, \quad (5.24a)$$

$$\left[\mathcal{A}_{2m}^{IJ}, \mathcal{S}_{2n+1}^{KL}\right] = -\frac{1}{4}\Gamma_{AB}^{IJ}\mathcal{S}_{2(m+n)+1}^B, \quad (5.24b)$$

$$\left[\mathcal{S}_{2m+1}^A, \mathcal{S}_{2n+1}^B\right] = \frac{1}{8}\Gamma_{AB}^{IJ}\mathcal{A}_{2(m+n+1)}^{IJ}. \quad (5.24c)$$

The generators of $\mathfrak{N}(\mathbb{K}[[u]])$ also include infinite linear combinations of (5.23) since $\mathfrak{N}(\mathbb{K}[[u]])$ is constructed using power series rather than polynomials.

The maps (3.9a) and (3.9b) from the filtered to the parabolic bases now read

$$\rho_{\pm}(\mathcal{X}_n^{IJ}) = (\pm 1)^n \frac{1}{2} \sum_{k \geq 0} a_{2k}^{(n)} \mathcal{A}_{2k}^{IJ}, \quad \rho_{\pm}(\mathcal{Y}_n^A) = (\pm 1)^n \frac{1}{2} \sum_{k \geq 0} a_{2k+1}^{(n)} \mathcal{S}_{2k+1}^A, \quad (5.25)$$

where the factors of $\frac{1}{2}$ are due to our definition (5.14). More specifically, the images of the first few generators of $\mathfrak{k}(\mathfrak{e}_9)$ according to Proposition 5 are given in the above basis by

$$\begin{aligned} \rho_+(\mathcal{X}_0^{IJ}) &= \mathcal{A}_0^{IJ}, \\ \rho_+(\mathcal{X}_1^{IJ}) &= \mathcal{A}_0^{IJ} + 2 \sum_{k \geq 1} \mathcal{A}_{2k}^{IJ}, \\ \rho_+(\mathcal{X}_2^{IJ}) &= \mathcal{A}_0^{IJ} + 8 \sum_{k \geq 1} k \mathcal{A}_{2k}^{IJ}, \\ \rho_+(\mathcal{X}_3^{IJ}) &= \mathcal{A}_0^{IJ} + 2 \sum_{k \geq 1} (1 + 8k^2) \mathcal{A}_{2k}^{IJ}, \\ \rho_+(\mathcal{X}_4^{IJ}) &= \mathcal{A}_0^{IJ} + \frac{32}{3} \sum_{k \geq 1} (k + 2k^3) \mathcal{A}_{2k}^{IJ}, \\ \rho_+(\mathcal{X}_5^{IJ}) &= \mathcal{A}_0^{IJ} + \frac{2}{3} \sum_{k \geq 1} (3 + 40k^2 + 32k^4) \mathcal{A}_{2k}^{IJ} \end{aligned} \quad (5.26)$$

for the \mathcal{X}_m^{IJ} . For the \mathcal{Y}_m^A the relations read

$$\begin{aligned} \rho_+(\mathcal{Y}_1^A) &= -2 \sum_{k \geq 0} \mathcal{S}_{2k+1}^A, \\ \rho_+(\mathcal{Y}_2^A) &= -4 \sum_{k \geq 0} (2k + 1) \mathcal{S}_{2k+1}^A, \\ \rho_+(\mathcal{Y}_3^A) &= -2 \sum_{k \geq 0} (8k^2 + 8k + 3) \mathcal{S}_{2k+1}^A, \\ \rho_+(\mathcal{Y}_4^A) &= -\frac{8}{3} \sum_{k \geq 0} (3 + 10k + 12k^2 + 8k^3) \mathcal{S}_{2k+1}^A, \\ \rho_+(\mathcal{Y}_5^A) &= -\frac{2}{3} \sum_{k \geq 0} (15 + 56k + 88k^2 + 64k^3 + 32k^4) \mathcal{S}_{2k+1}^A. \end{aligned} \quad (5.27)$$

We note that, unlike (5.16), there is no known presentation of $\mathfrak{N}(\mathbb{K}[[u]])$ as a finitely generated algebra, say by \mathcal{A}_0^{IJ} and \mathcal{S}_1^A , with a finite number of Berman-like relations. Using the algebra (5.24) we find the following Berman-type relations for all $k \geq 1$

$$7 \Gamma_{AB}^{IJ} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1+k_2=k-1}} \left[\mathcal{S}_{2k_1+1}^A, \mathcal{S}_{2k_2+1}^B \right] - 32 \sum_{\substack{k_1, k_2 \geq 1 \\ k_1+k_2=k}} \left[\mathcal{A}_{2k_1}^{IK}, \mathcal{A}_{2k_2}^{KJ} \right] = 448 \mathcal{A}_{2k}^{IJ}, \quad (5.28a)$$

$$\Gamma_{AB}^{I_1 \dots I_6} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1+k_2=k-1}} \left[\mathcal{S}_{2k_1+1}^A, \mathcal{S}_{2k_2+1}^B \right] = 0, \quad (5.28b)$$

where we have evaluated the commutators involving \mathcal{A}_0 in the first line.

The maximal compact Virasoro generators \mathcal{K}_m introduced in (5.20) can also be expressed using the variable u . In particular,

$$\mathcal{K}_1 = L_1 - L_{-1} = (1 - t^2) \frac{d}{dt} = 2u \frac{d}{du} \quad (5.29)$$

generates an $\text{SO}(1,1)$ group, and acts as a counting operator on the basis of the parabolic model (5.23):

$$\left[\mathcal{K}_1, \mathcal{A}_{2k}^{IJ} \right] = 4k \mathcal{A}_{2k}^{IJ}, \quad \left[\mathcal{K}_1, \mathcal{S}_{2k+1}^A \right] = 2(2k+1) \mathcal{S}_{2k+1}^A \quad (5.30)$$

More generally the operators \mathcal{K}_m admit the following realisation as differential operators

$$\mathcal{K}_m = \frac{1}{2} (\pm 1)^m \frac{1}{(1-u^2)^{m-1}} \left[(1-u)^{2m} - (1+u)^{2m} \right] \frac{d}{du} \quad (5.31)$$

Proceeding as before we find for instance

$$\left[\mathcal{K}_2, \mathcal{A}_{2m}^{IJ} \right] = 8m \left(\mathcal{A}_{2m}^{IJ} + 2\mathcal{A}_{2m+2}^{IJ} + 2\mathcal{A}_{2m+4}^{IJ} + \dots \right) \quad (5.32)$$

so for $m \geq 2$ these operators mix all levels.

5.3. Representations. Representations of $\mathfrak{k}(\mathfrak{e}_9)$ can be constructed via the technique described in Sect. 4. This means that we construct a basis of the universal enveloping algebra of $\mathfrak{N}_+ \subset \mathfrak{N}$ (given by all generators in (5.23) with degree greater than 0) and let this act on a $\mathfrak{k} \cong \mathfrak{so}(16)$ representation \mathfrak{V}_0 as the initial vector space. We shall exploit that everything is $\mathfrak{so}(16)$ covariant and graded.

5.3.1. Universal enveloping algebra basis The basis (4.3) of $\mathcal{U}(\mathfrak{N}_+)$ at the first few levels becomes

$$\ell = 0 : 1 \quad (5.33a)$$

$$\ell = 1 : \mathcal{S}_1^A \quad (5.33b)$$

$$\ell = 2 : \mathcal{S}_1^{(A} \mathcal{S}_1^{B)}, \mathcal{A}_2^\alpha \quad (5.33c)$$

$$\ell = 3 : \mathcal{S}_1^{(A} \mathcal{S}_1^B \mathcal{S}_1^{C)}, \mathcal{S}_1^A \mathcal{A}_2^\alpha, \mathcal{S}_3^A \quad (5.33d)$$

$$\ell = 4 : \mathcal{S}_1^{(A} \mathcal{S}_1^B \mathcal{S}_1^C \mathcal{S}_1^{D)}, \mathcal{S}_1^{(A} \mathcal{S}_1^B) \mathcal{A}_2^\alpha, \mathcal{S}_1^B \mathcal{S}_3^A, \mathcal{A}_2^{(\alpha} \mathcal{A}_2^{\beta)}, \mathcal{A}_4^\alpha \quad (5.33e)$$

where we now use $\alpha \equiv [IJ]$ for $I < J$ to denote an adjoint index of $\mathfrak{so}(16)$. The symmetrisations (\dots) for similar generators on the same level are necessary to implement the ordering in accordance with the PBW theorem.

In terms of $\mathfrak{so}(16)$ representations the first levels \mathcal{U}_ℓ of $\mathcal{U}(\mathfrak{N}_+)$ are⁴

$$\ell = 0 : \quad \mathbf{1} \quad (5.34a)$$

$$\ell = 1 : \quad \mathbf{128}_s \quad (5.34b)$$

$$\ell = 2 : \quad \left(\mathbf{1} \oplus \mathbf{1820} \oplus \mathbf{6435}_+ \right) \oplus \mathbf{120} \quad (5.34c)$$

$$\begin{aligned} \ell = 3 : \quad & \left(\mathbf{128}_s \oplus \mathbf{13312}_s \oplus \mathbf{161280}_s \oplus \mathbf{183040}_s \right) \\ & \oplus \left(\mathbf{128}_s \oplus \mathbf{1920}_c \oplus \mathbf{13312}_s \right) \oplus \mathbf{128}_s \end{aligned} \quad (5.34d)$$

Here, we have labelled the $\mathfrak{so}(16)$ representations by their real dimensions. A translation to highest weight labels can be found in ‘‘Appendix C’’.

The module \mathfrak{V} as in (4.5) built from an $\mathfrak{k} \cong \mathfrak{so}(16)$ representation \mathfrak{V}_0 is graded as

$$\mathfrak{V} = \bigoplus_{\ell=0}^{\infty} \mathfrak{V}_\ell, \quad \mathfrak{V}_\ell := \mathcal{U}_\ell \otimes \mathfrak{V}_0$$

and each level decomposes further as a $\mathfrak{so}(16)$ -representation. For instance, in the case of the $\mathfrak{V}_0 = \mathbf{16}$ we obtain

$$\mathfrak{V}_0 = \mathbf{16}, \quad (5.35a)$$

$$\mathfrak{V}_1 = \mathbf{128}_c \oplus \mathbf{1920}_s, \quad (5.35b)$$

$$\mathfrak{V}_2 = \left(\mathbf{16} \oplus \mathbf{560} \oplus \mathbf{4368} \oplus \mathbf{11440} \oplus \mathbf{24192} \oplus \mathbf{91520}_+ \right) \oplus \left(\mathbf{16} \oplus \mathbf{560} \oplus \mathbf{1344} \right), \quad (5.35c)$$

$$\begin{aligned} \mathfrak{V}_3 = & \left(\mathbf{128}_c \oplus 2 \times \mathbf{1920}_s \oplus \mathbf{13312}_c \oplus 2 \times \mathbf{56320}_s \oplus \mathbf{141440}_s \oplus \mathbf{161280}_c \right. \\ & \left. \oplus \mathbf{326144}_s \oplus \mathbf{439296}_c \oplus \mathbf{2036736}_s \oplus \mathbf{2489344}_s \right) \\ & \oplus \left(2 \times \mathbf{128}_c \oplus 3 \times \mathbf{1920}_s \oplus \mathbf{13312}_c \oplus \mathbf{15360}_c \oplus \mathbf{56320}_s \oplus \mathbf{141440}_s \right) \\ & \oplus \left(\mathbf{128}_c \oplus \mathbf{1920}_s \right). \end{aligned} \quad (5.35d)$$

5.3.2. A quotient example As is clear from (5.35), the module grows very rapidly and it is desirable to find smaller $\mathfrak{k}(\mathfrak{e}_9)$ representations by identifying invariant subspaces and taking quotients.

The simplest quotient example is to consider $\mathfrak{W}_{s=1/2} = \mathfrak{V}^{(1)}$ in the notation (4.10) to be given by all spaces of degree greater than zero, then we obtain as $\mathfrak{k}(\mathfrak{e}_9)$ representation simply $\mathfrak{V}/\mathfrak{W}_{s=1/2} \cong \mathfrak{V}_0 \cong \mathbf{16}$ which is nothing but the irreducible spin-1/2 representation appearing in supergravity [NS].

A non-trivial example can be obtained by looking at the construction (4.12) that uses the invariant subspaces \mathfrak{W} generated by an $\mathfrak{so}(16)$ representation \mathfrak{W}_k sitting at a given

⁴ The parentheses are used to group the representations according to the different words in the induced representation module list after decomposing into $\mathfrak{so}(16)$.

level. As the example we take $\mathfrak{W}_1 = \mathbf{1920}_s$ within (5.35). In order to describe this, we shall need a more explicit parametrisation of the elements of the module's homogeneous parts \mathfrak{W}_ℓ for $0 \leq \ell \leq 2$. At level $\ell = 0$ we need an element of the 16-dimensional defining representation of $\mathfrak{so}(16)$ that we write as φ_0^I .

The elements of $\mathfrak{W}_1 \cong \mathbf{128}_s \otimes \mathbf{16}$ are of the form $\mathcal{S}_1^A \varphi_0^I$ and decompose into a conjugate spinor $\mathbf{128}_c$ and a traceless vector-spinor $\mathbf{1920}_s$

$$\mathcal{S}_1^A \varphi^I = \Gamma_{AA}^I \chi_1^{\dot{A}} + \chi_1^{IA}$$

according to $\mathfrak{W}_1 = \mathbf{128}_c \oplus \mathbf{1920}_s$. Note that the occurrence of $\mathbf{128}_c$ in the above tensor product is tied to the existence of a suitable Γ -matrix Γ_{AA}^I . The condition that projects onto the $\mathbf{1920}_s$ is

$$\chi_1^{IA} = \mathcal{S}_1^A \varphi_0^I - \frac{1}{16} (\Gamma^I \Gamma^J)_{AB} \mathcal{S}_1^B \varphi_0^J$$

and the quotient we want to consider is the one where this component and all vectors generated from it are set to zero. In other words, all elements of the quotient module $\mathfrak{W}/\mathfrak{W}$ satisfy

$$\mathcal{S}_1^B \varphi_0^I = \frac{1}{16} (\Gamma^I \Gamma^J)_{BC} \mathcal{S}_1^C \varphi_0^J \quad (5.36)$$

and any relations obtained from it by acting with $\mathfrak{k}(\mathfrak{e}_9)$.

To find what this imposes on \mathfrak{W}_2 as given in (5.35c) we parametrise all its elements as

$$\mathcal{S}_1^A \mathcal{S}_1^B \varphi_0^I = \delta^{AB} \varphi_2^I + \Gamma_{AB}^{J_1 J_2} \varphi_2^{J_1 J_2; I} + \Gamma_{AB}^{J_1 \dots J_4} \varphi_2^{J_1 \dots J_4; I} + \Gamma_{AB}^{J_1 \dots J_8} \varphi_2^{J_1 \dots J_8; I}, \quad (5.37)$$

This formula follows from the $\mathfrak{so}(16)$ tensor product $\mathbf{128}_s \otimes \mathbf{128}_s$ that is relevant for the word $\mathcal{S}_1^A \mathcal{S}_1^B$ and that we decompose into its symmetric and anti-symmetric parts

$$\begin{aligned} \text{Sym}^2(\mathbf{128}_s) &= \mathbf{1} \oplus \mathbf{1820} \oplus \mathbf{6435}_+, \\ \text{Alt}^2(\mathbf{128}_s) &= \mathbf{120} \oplus \mathbf{8008}. \end{aligned}$$

The first line consists of a scalar, a four-form and a self-dual eight-form of $\mathfrak{so}(16)$, while the second line represents a two-form and a six-form. These intertwiners from $\mathbf{128}_s \otimes \mathbf{128}_s$ to p -forms are given by the Γ -matrices $\Gamma_{AB}^{I_1 \dots I_p}$. As the anti-symmetric product $\mathcal{S}_1^A \mathcal{S}_1^B$ is proportional to the commutator (5.24c) that does not contain a six-form, the ansatz (5.37) does not contain a term in $\Gamma_{AB}^{I_1 \dots I_6}$. The final $\mathfrak{so}(16)$ tensor product on the left-hand side of (5.37) is then to multiply $\mathbf{128}_s \otimes \mathbf{128}_s$ (written as p -forms) with $\mathbf{16}$ which is the representation of φ_0^I . These tensor products are written using a semi-colon, so that for instance

$$\varphi_2^{J_1 J_2; I} \in \mathbf{120} \otimes \mathbf{16} = \mathbf{16} \oplus \mathbf{560} \oplus \mathbf{1344}.$$

Acting on (5.36) with \mathcal{S}_1^A then leads to the relation

$$\begin{aligned} &\delta_{AB} \varphi_2^I + \Gamma_{AB}^{J_1 J_2} \varphi_2^{J_1 J_2; I} + \Gamma_{AB}^{J_1 \dots J_4} \varphi_2^{J_1 \dots J_4; I} + \Gamma_{AB}^{J_1 \dots J_8} \varphi_2^{J_1 \dots J_8; I} \\ &\stackrel{!}{=} \frac{1}{16} (\Gamma^I \Gamma^K)_{BC} \left(\delta^{AC} \varphi_2^K - \Gamma_{CA}^{J_1 J_2} \varphi_2^{J_1 J_2; K} + \Gamma_{CA}^{J_1 \dots J_4} \varphi_2^{J_1 \dots J_4; K} + \Gamma_{CA}^{J_1 \dots J_8} \varphi_2^{J_1 \dots J_8; K} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{16} \delta_{AB} \varphi_2^I - \frac{1}{16} \Gamma_{AB}^{IJ} \varphi_2^J + \frac{1}{16} \Gamma_{AB}^{J_1 J_2} \varphi_2^{J_1 J_2; I} - \frac{1}{16} \Gamma_{AB}^{I J_1 J_2 J_3} \varphi_2^{J_1 J_2; J_3} \\
 &+ \frac{1}{8} \Gamma_{AB}^{J_1 J_2} \varphi_2^{I J_1; J_2} - \frac{1}{8} \Gamma_{AB}^{IJ} \varphi_2^{JK; K} \\
 &+ \frac{1}{8} \delta_{AB} \varphi_2^{IJ; J} + \frac{1}{16} \Gamma_{AB}^{J_1 \dots J_4} \varphi_2^{J_1 \dots J_4; I} - \frac{1}{16} \Gamma_{AB}^{I J_1 \dots J_5} \varphi_2^{J_1 \dots J_4; J_5} - \frac{1}{4} \Gamma_{AB}^{I J_1 \dots J_3} \varphi_2^{J_1 \dots J_3 K; K} \\
 &+ \frac{1}{4} \Gamma_{AB}^{J_1 \dots J_4} \varphi_2^{I J_1 \dots J_3; J_4} + \frac{3}{4} \Gamma_{AB}^{J_1 J_2} \varphi_2^{I J_1 J_2 K; K} + \frac{1}{16} \Gamma_{AB}^{J_1 \dots J_8} \varphi_2^{J_1 \dots J_8; I} - \frac{1}{16} \Gamma_{AB}^{I J_1 \dots J_9} \varphi_2^{J_1 \dots J_8; J_9} \\
 &- \frac{1}{2} \Gamma_{AB}^{I J_1 \dots J_7} \varphi_2^{J_1 \dots J_7 K; K} - \frac{1}{2} \Gamma_{AB}^{J_1 \dots J_8} \varphi_2^{I J_1 \dots J_7; J_8} + \frac{7}{2} \Gamma_{AB}^{J_1 \dots J_6} \varphi_2^{I J_1 \dots J_6 K; K}. \tag{5.38}
 \end{aligned}$$

Projecting this onto the various irreducible pieces in (5.35c) leads to the conditions

$$\begin{aligned}
 \varphi_2^{IJ; J} &= \frac{15}{2} \varphi_2^I && \text{(relation between the two } \mathbf{16}) \\
 \varphi_2^{I_1 I_2 I_3 J; J} &= \frac{13}{12} \varphi_2^{[I_1 I_2; I_3]} && \text{(relation between the two } \mathbf{560}) \tag{5.39}
 \end{aligned}$$

and the fact that all other irreducible components must vanish. Therefore, at level $\ell = 2$, the quotient is given by only

$$\mathfrak{W}_2 / \mathfrak{W}_2 \cong \mathbf{16} \oplus \mathbf{560}, \tag{5.40}$$

a comparatively small subspace of (5.35c). Already at the next level the above computation becomes almost unfeasible. If one formally subtracts the $\mathfrak{so}(16)$ -decompositions of $\mathcal{U}_{\ell+1} \otimes \mathfrak{W}_0$ and $\mathcal{U}_\ell \otimes \mathfrak{W}_1$ the result indicates that only $\mathbf{128}_c$ survives at level three, and that there are only two $\mathbf{16}$ s at level four, after which the procedure terminates. In summary, the above computation shows

$$\begin{aligned}
 \mathfrak{W}_0 / \mathfrak{W}_0 &\cong \mathfrak{W}_0 \cong \mathbf{16}, \\
 \mathfrak{W}_1 / \mathfrak{W}_1 &\cong \mathbf{128}_c, \\
 \mathfrak{W}_2 / \mathfrak{W}_2 &\cong \mathbf{16} \oplus \mathbf{560}, \tag{5.41}
 \end{aligned}$$

and we conjecture

$$\begin{aligned}
 \mathfrak{W}_3 / \mathfrak{W}_3 &\cong \mathbf{128}_c, \\
 \mathfrak{W}_4 / \mathfrak{W}_4 &\cong 2 \times \mathbf{16}, \\
 \mathfrak{W}_\ell / \mathfrak{W}_\ell &\cong 0 \quad \forall \ell \geq 5. \tag{5.42}
 \end{aligned}$$

This is related to the analogue of the spin- $\frac{5}{2}$ representation studied in [KN5]. The spin- $\frac{3}{2}$ representation of supergravity [NS, KNP] can also be obtained from this construction by taking a further quotient. More precisely, one quotients by all \mathfrak{W}_ℓ with $\ell > 2$ and also by the $\mathbf{560}$ representation in (5.40). The remaining $\mathfrak{so}(16)$ representations are $\mathbf{16} \oplus \mathbf{128} \oplus \mathbf{16}$ that form one chiral half of the supergravity spin- $\frac{3}{2}$ fields.

Remark 21. As is evident from the analysis above, determining the quotient $\mathfrak{W} / \mathfrak{W}$ can become intricate quickly since the precise structure of the submodule \mathfrak{W} is hard to analyse. In the case of complex simple Lie algebras a similar problem arises when constructing irreducible highest weight representations as quotients of Verma modules by the maximal proper submodule. In that case, there is a description of the quotient in terms of the Weyl character formula. A similar technique for representations of \mathfrak{k} is not known to the best of our knowledge.

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A. $\mathfrak{k}(\mathfrak{e}_9)(\mathbb{C})$ as the Quotient of a GIM-Algebra

Generalised intersection algebras, GIM-algebras in short for Generalised Intersection Matrix, are constructed similarly to Kac–Moody algebras. One starts from a so-called *generalised intersection matrix* A where one replaces the condition $A_{ij} \leq 0$ for $i \neq j$ by $A_{ij} \leq 0 \Leftrightarrow A_{ji} \leq 0$ and $A_{ij} > 0 \Leftrightarrow A_{ji} > 0$. GIMs can be visualised quite neatly by drawing solid lines for $A_{ij} < 0$ and dotted edges for $A_{ij} > 0$. In [B], Berman explores a connection between his involutory subalgebras and GIM-algebras. As it turns out due to the work of Slodowy, GIM-algebras fall into two classes. The first class consists of those GIM-algebras which are in fact isomorphic to a Kac–Moody-algebra and the second class are those which are isomorphic to an involutory subalgebra of a Kac–Moody-algebra according to Berman’s construction. This isomorphism relies on doubling the number of vertices of the respective GIM-diagram and the involution that is used involves a diagram automorphism of the new diagram which is of Kac–Moody-type. So even though our \mathfrak{k} is an involutory subalgebra it is not of the type that is directly isomorphic to a GIM-algebra. For $\mathfrak{k}(E_9)(\mathbb{C})$ it turns out that it is a quotient of a GIM-algebra, although we do not know the precise structure of the defining ideal. In particular we do not know whether it is generated by (A.15)–(A.19) or whether one needs additional relations. We will collect the most essential definitions and results here (see the work of Slodowy [S] for more details) and state our result in Proposition 27.

Definition 22. Let I be a finite index set and \mathfrak{h} a \mathbb{C} -vector space of dimension r . Let $\Delta^\vee := \{h_i \mid i \in I\} \subset \mathfrak{h}$ and $\Delta := \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$. Then $(\mathfrak{h}, \Delta^\vee, \Delta)$ is called a \mathbb{C} -root basis, its reductive rank is defined to be equal to r , whereas its semi-simple rank is defined to be $|I|$. Associate a matrix A to $(\mathfrak{h}, \Delta^\vee, \Delta)$, called its structure matrix, by setting

$$A_{ij} := \alpha_j(h_i) \quad \forall i, j \in I. \tag{A.1}$$

A root basis is called free if both Δ^\vee and Δ are linearly independent.

Definition 23. Let $A \in \mathbb{Z}^{\ell \times \ell}$ such that

$$(i) \quad A_{ii} = 2 \quad \forall i = 1, \dots, \ell$$

- (ii) $A_{ij} < 0 \Leftrightarrow A_{ji} < 0 \forall i \neq j$
 (iii) $A_{ij} > 0 \Leftrightarrow A_{ji} > 0 \forall i \neq j,$

then A is called a generalised intersection matrix (GIM). A GIM A is called symmetrisable if there exist $D, B \in \mathbb{Q}^{\ell \times \ell}$ such that D is diagonal and B is symmetric and it holds $A = DB$. A root basis $(\mathfrak{h}, \Delta^\vee, \Delta)$ whose structure matrix is a generalised intersection matrix is called a GIM-root basis.

Definition 24. Let $(\mathfrak{h}, \Delta^\vee, \Delta)$ be a GIM-root basis with structure matrix A . Then $\underline{\Delta}(A)$ is a coloured, weighted graph with vertices Δ with:

1. Two vertices i and j are connected by a dotted edge if $A_{ij} = \alpha_j(h_i) > 0$.
2. Two vertices i and j are connected by a solid edge if $A_{ij} = \alpha_j(h_i) < 0$.
3. There is no edge between two vertices i and j if $A_{ij} = 0$.
4. The edges (i, j) are weighted by the weight m_{ij} according to the following table

$\alpha_i(h_j) \cdot \alpha_j(h_i)$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

The weights $m_{ij} = 2, 3$ are not set apart graphically in the diagram's visualisation. In the case of $m_{ij} = 4, 6$ one draws an arrow towards the node j if $|A_{ji}| = |\alpha_i(h_j)| > |\alpha_j(h_i)| = |A_{ij}|$.

Definition 25. Let $(\mathfrak{h}, \Delta^\vee, \Delta)$ be a GIM-root basis with structure matrix A and let \mathfrak{f} be the free Lie algebra over \mathbb{C} generated by \mathfrak{h} and elements $e_\alpha, e_{-\alpha}$ for $\alpha \in \Delta$. Let \mathfrak{J} be the ideal in \mathfrak{f} generated by the relations (identify $h_{-\alpha} \equiv -h_\alpha$ for $\alpha \in -\Delta$)

$$[h, h'] = 0, \quad [h, e_\alpha] = \alpha(h)e_\alpha \quad \forall h, h' \in \mathfrak{h}, \alpha \in \pm\Delta,$$

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad \text{ad}(e_\alpha)^{\max(1, 1-\beta(h_\alpha))}(e_\beta) = 0 \quad \forall \alpha, \beta \in \pm\Delta \text{ and } \alpha \neq -\beta.$$

Set $\mathfrak{g} := \mathfrak{f}/\mathfrak{J}$ then \mathfrak{g} is called the GIM-Lie algebra to $(\mathfrak{h}, \Delta^\vee, \Delta)$. Note that every GIM A has a free realisation $(\mathfrak{h}, \Delta^\vee, \Delta)$ that is unique up to isomorphism. In this sense one can associate a GIM-Lie algebra $\text{gim}(A)$ to a structure matrix A or equivalently a GIM-diagram $\underline{\Delta}(A)$.

If one spells out the above relations for $A_{ij} < 0$, one obtains with $e_i = e_{\alpha_i}, f_i = e_{-\alpha_i}$ the familiar Serre-relations

$$\text{ad}(e_i)^{1-A_{ij}}(e_j) = 0 = \text{ad}(f_i)^{1-A_{ij}}(f_j), \quad [e_i, f_j] = 0 = [e_j, f_i]$$

but for $A_{ij} > 0$ one obtains

$$\text{ad}(e_i)^{1+A_{ij}}(f_j) = 0 = \text{ad}(f_i)^{1+A_{ij}}(e_j), \quad [e_i, e_j] = 0 = [f_i, f_j].$$

One knows on abstract grounds that the complexification $\mathfrak{k}(A_8)(\mathbb{C})$ of the canonical subalgebra $\mathfrak{k}(A_8) < \mathfrak{k}(E_9)$ is isomorphic to $B_4(\mathbb{C}) \cong \mathfrak{so}(9, \mathbb{C})$. Let us spell out the relationship between the description of $\mathfrak{k}(A_8)(\mathbb{C})$ via Berman generators

$$x_i = e_i - f_i \tag{A.2}$$

and the usual description of $\mathfrak{g}(B_4)(\mathbb{C})$ in terms of a Chevalley–Serre basis. Let $i_1, i_2, \dots, i_k \in \{1, \dots, 9\}$ and set

$$x_{\alpha_{i_1} + \dots + \alpha_{i_k}} := [x_{i_1}, [x_{i_2}, [\dots, [x_{i_{k-1}}, x_{i_k}]]]]]. \quad (\text{A.3})$$

Note that the order in the sum $\alpha_{i_1} + \dots + \alpha_{i_k}$ matters. For $i < j$ define roots $\beta_{i,j}^{(1)}, \dots, \beta_{i,j}^{(4)} \in \Delta(A_8) \subset \Delta(E_9)$ by

$$\beta_{i,j}^{(1)} = \alpha_{2i} + \dots + \alpha_{2j-1}, \quad \beta_{i,j}^{(2)} = \alpha_{2i} + \dots + \alpha_{2j-2} \quad (\text{A.4})$$

$$\beta_{i,j}^{(3)} = \alpha_{2i-1} + \dots + \alpha_{2j-1}, \quad \beta_{i,j}^{(4)} = \alpha_{2i-1} + \dots + \alpha_{2j-2}. \quad (\text{A.5})$$

Now set

$$e_{\varepsilon_1 L_i + \varepsilon_2 L_j} := \frac{i}{2} \cdot \left(x_{\beta_{i,j}^{(1)}} - i \varepsilon_2 x_{\beta_{i,j}^{(2)}} - i \varepsilon_1 x_{\beta_{i,j}^{(3)}} - \varepsilon_1 \varepsilon_2 x_{\beta_{i,j}^{(4)}} \right) \quad \forall i < j \in \{1, 2, 3, 4\}, \quad (\text{A.6})$$

$$H_j := -i x_{2j-1}, \quad \text{for } j = 1, \dots, 4, \quad (\text{A.7})$$

$$e_{\pm L_j} := i \cdot (x_{\alpha_{2j} + \dots + \alpha_8} \mp i x_{\alpha_{2j-1} + \dots + \alpha_8}). \quad (\text{A.8})$$

Proposition 26. Consider the abelian subalgebra $\mathfrak{h}_{B_4} := \text{span}_{\mathbb{C}} \{H_1, \dots, H_4\}$ together with the linear functionals $L_i : \mathfrak{h}_{B_4}^* \rightarrow \mathbb{C}$ defined via $L_i(H_j) = \delta_{ij}$. Then with the above definitions (A.6), (A.7) and (A.8) one has

$$[h, e_{\varepsilon_1 L_i + \varepsilon_2 L_j}] = (\varepsilon_1 L_i + \varepsilon_2 L_j)(h) e_{\varepsilon_1 L_i + \varepsilon_2 L_j}, \quad [h, e_{\pm L_j}] = \pm L_j(h) e_{\pm L_j} \quad \forall h \in \mathfrak{h}_{B_4}$$

Thus, (A.6) and (A.8) provide a root space decomposition of $\mathfrak{k}(A_8)(\mathbb{C}) \cong B_4(\mathbb{C})$ with respect to the Cartan subalgebra \mathfrak{h}_{B_4} spanned by (A.7). A corresponding Chevalley–Serre basis is given by

$$\begin{aligned} e_i &:= e_{L_i - L_{i+1}}, \quad f_i := e_{-L_i + L_{i+1}}, \quad h_i := H_i - H_{i+1} \quad \forall i = 1, 2, 3, \\ e_4 &= e_{+L_4}, \quad f_4 := e_{-L_4}, \quad h_4 := 2H_4. \end{aligned}$$

Proof. One verifies that (A.6) and (A.8) are eigenvectors under the adjoint action of \mathfrak{h}_{B_4} with the correct eigenvalues by direct computation. This suffices for a root space decomposition because one knows abstractly that this exhausts $\mathfrak{k}(A_8)(\mathbb{C})$ as its isomorphism type is known. Checking the Chevalley–Serre basis is a matter of fixing suitable prefactors. \square

Now set

$$x_{\pm} := i(x_9 \mp i[x_3, x_9]) \quad (\text{A.9})$$

then one has

$$[x_+, x_-] = 2H_2, \quad [H_2, x_{\pm}] = \pm x_{\pm}, \quad [H_i, x_{\pm}] = 0 \quad \forall i \neq 2, \quad (\text{A.10})$$

and the Slodowy-type relations

$$[x_+, y] = 0 \quad \forall y \in \{f_1, e_2\} \cup \{e_3, e_4, f_3, f_4\}, \quad (\text{A.11})$$

$$\text{ad}(x_+)^3(y) = 0 = \text{ad}(y)^2(x_+) \quad \forall y \in \{e_1, f_2\} \cup \{e_3, e_4, f_3, f_4\}, \quad (\text{A.12})$$

$$[x_-, y] = 0 \quad \forall y \in \{e_1, f_2\} \cup \{e_3, e_4, f_3, f_4\}, \quad (\text{A.13})$$

$$\text{ad}(x_-)^3(y) = 0 = \text{ad}(y)^2(x_-) \quad \forall y \in \{f_1, e_2\} \cup \{e_3, e_4, f_3, f_4\}, \quad (\text{A.14})$$



Fig. 2. The diagrams associated to B_4 and B_4^\diamond with labelling of nodes

as well as additional relations that hold in $\mathfrak{k}(e_9)(\mathbb{C})$

$$[x_+, e_{\varepsilon L_1 - L_2}] = x_{\alpha_2 + \alpha_3 + \alpha_9} - i\varepsilon x_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_9} = [x_-, e_{\varepsilon L_1 + L_2}] \quad (\text{A.15})$$

$$\text{ad}(x_+)^2(e_{\varepsilon L_1 - L_2}) = 2e_{\varepsilon L_1 + L_2}, \quad \text{ad}(x_-)^2(e_{\varepsilon L_1 + L_2}) = 2e_{\varepsilon L_1 - L_2} \quad (\text{A.16})$$

$$[x_+, e_{+L_2 + \varepsilon L_3}] = 0 = [x_-, e_{-L_2 + \varepsilon L_3}] \quad (\text{A.17})$$

$$[x_+, e_{-L_2 + \varepsilon L_3}] = -\varepsilon x_{\alpha_9 + \alpha_3 + \alpha_4} - i x_{\alpha_9 + \alpha_3 + \alpha_4 + \alpha_5} = -[x_-, e_{L_2 + \varepsilon L_3}] \quad (\text{A.18})$$

$$\text{ad}(x_+)^2(e_{-L_2 + \varepsilon L_3}) = -2e_{L_2 + \varepsilon L_3}, \quad \text{ad}(x_-)^2(e_{L_2 + \varepsilon L_3}) = -2e_{-L_2 + \varepsilon L_3}. \quad (\text{A.19})$$

Consider the Cartan matrix of B_4 and a GIM which we call B_4^\diamond that extends it:

$$B_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}, \quad B_4^\diamond = \begin{pmatrix} 2 & -2 & 2 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

The nontrivial Serre relations (for the e_i only) of B_4 spell out to be

$$\begin{aligned} \text{ad}(e_i)^2(e_{i+1}) &= 0 = \text{ad}(e_{i+1})^2(e_i) \quad \forall i = 1, 2 \\ \text{ad}(e_3)^2(e_4) &= 0 = \text{ad}(e_4)^3(e_3). \end{aligned}$$

Denote the associated GIM-algebra to B_4^\diamond over \mathbb{C} by $\text{gim}(B_4^\diamond)(\mathbb{C})$. The diagrams of B_4 and B_4^\diamond are given in Fig. 2.

Proposition 27. Denote the Chevalley generators of $\text{gim}(B_4^\diamond)(\mathbb{C})$ by $E_0, \dots, E_4, F_0, \dots, F_4$ and $H_{\gamma_0}, \dots, H_{\gamma_4}$. There exists a surjective homomorphism of Lie algebras $\phi : \text{gim}(B_4^\diamond)(\mathbb{C}) \rightarrow \mathfrak{k}(E_9)(\mathbb{C})$ that is given on the Level of generators via

$$\begin{aligned} \phi(E_0) &= x_+, \quad \phi(F_0) = x_-, \quad \phi(H_{\gamma_0}) = 2H_2 = 2h_2 + 2h_3 + h_4 \\ \phi(E_i) &= e_i, \quad \phi(F_i) = f_i, \quad \phi(H_{\gamma_i}) = h_i \quad \forall i = 1, 2, 3, 4. \end{aligned}$$

Proof. One verifies that the defining relations between the generators from definition (25) are satisfied. The B_4 -relations are unproblematic and towards the relations

$$\text{ad}(E_0)^3(E_1) = 0 = \text{ad}(E_1)^2(E_0), \quad \text{ad}(E_0)^3(F_2) = 0 = \text{ad}(F_2)^2(E_0)$$

we refer to Eqs. (A.11)–(A.14). One also has to check that $H_{\gamma_0} \mapsto 2H_2$ satisfies all necessary identities which is the case. Thus, ϕ is a homomorphism of Lie-algebras.

Surjectivity follows from the fact that all Berman generators of $\mathfrak{k}(E_9)(\mathbb{C})$ can be recovered from the image of the generators of $\mathfrak{gim}(B_4^\diamond)(\mathbb{C})$. For x_1, \dots, x_8 this is a basis transformation within $B_4(\mathbb{C})$ and for x_9 one notes that (A.9) implies

$$x_+ + x_- = 2ix_9.$$

□

GIM-algebras are graded with respect to their root system although their root system does not split into positive and negative roots. Now consider an element $[E_0, [E_0, E_1]] - 2E_{L_1+L_2}$ with $E_{L_1+L_2} \propto [E_1, [E_2, [E_4, [E_4, E_3]]]]$. This element is nonzero because the two summands lie in different root spaces and are nonzero themselves. But because of (A.16) it is equal to 0 in the image. This implies that $\mathfrak{k}(E_9)(\mathbb{C})$ is a (nontrivial) quotient of $\mathfrak{gim}(B_4^\diamond)(\mathbb{C})$.

In conclusion, representations of $\mathfrak{gim}(B_4^\diamond)(\mathbb{C})$ could potentially be useful to find representations of $\mathfrak{k}(E_9)(\mathbb{C})$ if it is possible to check whether or not a given representation factors through the projection of Proposition 27. Conversely, our results from Sects. 3 and 4 provide representations of $\mathfrak{gim}(B_4^\diamond)(\mathbb{C})$ both of finite and infinite dimension.

B. The Hilbert Space Completion $\widehat{\mathfrak{k}}$ is Not a Lie Algebra

Because the restriction of the standard bilinear form is positive definite we can complete \mathfrak{k} to a Hilbert space $\widehat{\mathfrak{k}}$. Here we show by means of a simple explicit example that this Hilbert space completion is not compatible with the Lie algebra structure. Define

$$J_N(\omega) := \frac{1}{2}\omega^{IJ} \sum_{n=1}^N \frac{1}{n^{1/2+\varepsilon}} \mathcal{X}_n^{IJ} \quad (\text{B.1})$$

The positive definite bilinear form can be normalised such that (for $m, n \geq 0$)

$$\langle \mathcal{X}_m^{IJ} | \mathcal{X}_n^{KL} \rangle = \delta_{mn} \delta_{KL}^{IJ} \quad (\text{B.2})$$

The induced norm is distinguished by its invariance, which implies that the right-hand side is independent of m and n . Consequently,

$$\|J_N\|^2 = C_0 \sum_{n=1}^N \frac{1}{n^{1+2\varepsilon}} \quad (\text{B.3})$$

where $C_0 = C_0(\omega)$ is an irrelevant strictly positive constant. For $\varepsilon > 0$ this sum converges, and therefore the limit $J_\infty(\omega) \equiv \lim_{N \rightarrow \infty} J_N(\omega)$ belongs to the Hilbert space $\widehat{\mathfrak{k}}$. We next compute the commutator

$$\left[J_N(\omega_1), J_N(\omega_2) \right] = \frac{1}{2} [\omega_1, \omega_2]^{IJ} \sum_{n=1}^N f_n \mathcal{X}_n^{IJ} \quad (\text{B.4})$$

with

$$f_n = \sum_{m=1}^{n-1} \frac{1}{m^{1/2+\varepsilon}} \frac{1}{(n-m)^{1/2+\varepsilon}} + \sum_{m=1}^N \frac{1}{m^{1/2+\varepsilon}} \frac{1}{(n+m)^{1/2+\varepsilon}} \quad (\text{B.5})$$

Estimating the first sum on the right-hand side as ($n > 1$)

$$\sum_{m=1}^{n-1} \frac{1}{m^{1/2+\varepsilon}} \frac{1}{(n-m)^{1/2+\varepsilon}} > \frac{n-1}{n^{1+2\varepsilon}} \tag{B.6}$$

it is easy to see that

$$f_n > \frac{C_1}{n^{2\varepsilon}} \tag{B.7}$$

with another irrelevant strictly positive constant C_1 . For $\varepsilon < \frac{1}{4}$ the sum $\sum_{n=1}^{\infty} |f_n|^2$ diverges, whence (B.4) does not converge in the limit $N \rightarrow \infty$. In other words, although $J_{\infty}(\omega_1)$ and $J_{\infty}(\omega_2)$ separately do belong to $\widehat{\mathfrak{k}}$, their commutator does not exist as an element of $\widehat{\mathfrak{k}}$. Hence $\widehat{\mathfrak{k}}$ is not even a Lie algebra, and *a fortiori* also not a Hilbert Lie algebra, which would in addition require $\|[x, y]\| < C_2 \|x\| \|y\|$ for all $x, y \in \widehat{\mathfrak{k}}$.

We stress that the failure of $\widehat{\mathfrak{k}}$ to be a Lie algebra depends on the norm used for the completion which in the analysis above was the standard invariant bilinear form. Other norms are possible and the corresponding completions of \mathfrak{k} can be Lie algebras, and even Hilbert Lie algebras. However, in those cases the norm is not invariant.

Although $\widehat{\mathfrak{k}}$ is not a Lie algebra, an interesting open question is whether one can still make sense of the commutator *as a distribution*, by considering the commutator of two elements belonging to a dense subspace of $\widehat{\mathfrak{k}}$. This would fit with earlier observations in [KNP].

C. $\mathfrak{so}(16)$ Representations

The translation of dimensions of $\mathfrak{so}(16)$ representations to the labels of the highest weight in the conventions of the LiE software [vLCL] are

1	\leftrightarrow	[0,0,0,0,0,0,0]	56320_s	\leftrightarrow	[0,0,1,0,0,0,0,1]
16	\leftrightarrow	[1,0,0,0,0,0,0]	60060	\leftrightarrow	[1,0,0,0,1,0,0,0]
120	\leftrightarrow	[0,1,0,0,0,0,0]	91520₊	\leftrightarrow	[1,0,0,0,0,0,0,2]
128_s	\leftrightarrow	[0,0,0,0,0,0,1]	112320	\leftrightarrow	[1,0,0,0,0,1,0,0]
128_c	\leftrightarrow	[0,0,0,0,0,1,0]	141372	\leftrightarrow	[0,1,0,1,0,0,0,0]
560	\leftrightarrow	[0,0,1,0,0,0,0]	141440_s	\leftrightarrow	[1,1,0,0,0,0,0,1]
1344	\leftrightarrow	[1,1,0,0,0,0,0]	161280_s	\leftrightarrow	[0,0,0,1,0,0,0,1]
1820	\leftrightarrow	[0,0,0,1,0,0,0]	161280_c	\leftrightarrow	[0,0,0,1,0,0,1,0]
1920_s	\leftrightarrow	[1,0,0,0,0,0,1]	162162	\leftrightarrow	[1,0,0,0,0,0,1,1]
4368	\leftrightarrow	[0,0,0,0,1,0,0]	183040_s	\leftrightarrow	[0,0,0,0,0,0,0,3]
6435₊	\leftrightarrow	[0,0,0,0,0,0,2]	326144_s	\leftrightarrow	[0,0,0,0,1,0,0,1]
7020	\leftrightarrow	[1,0,1,0,0,0,0]	439296_c	\leftrightarrow	[0,0,0,0,0,0,1,2]
8008	\leftrightarrow	[0,0,0,0,0,1,0,0]	465920_c	\leftrightarrow	[0,0,0,0,0,0,1,1,0]
11440	\leftrightarrow	[0,0,0,0,0,0,1,1]	595595₊	\leftrightarrow	[0,1,0,0,0,0,0,2]
13312_s	\leftrightarrow	[0,1,0,0,0,0,0,1]	670208_s	\leftrightarrow	[1,0,1,0,0,0,0,1]
13312_c	\leftrightarrow	[0,1,0,0,0,0,1,0]	2036736_s	\leftrightarrow	[1,0,0,1,0,0,0,1]
15360_c	\leftrightarrow	[2,0,0,0,0,0,1,0]	2489344_s	\leftrightarrow	[1,0,0,0,0,0,0,3]
24192	\leftrightarrow	[1,0,0,1,0,0,0,0]	6223360_s	\leftrightarrow	[1,0,0,0,0,1,0,1]

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