

Quasinormal modes of slowly-rotating black holes in dynamical Chern-Simons gravity

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(Dated: March 19, 2021)

The detection of gravitational waves from compact binary mergers by the LIGO/Virgo collaboration has, for the first time, allowed us to test relativistic gravity in its strong, dynamical and nonlinear regime, thus opening a new arena to confront general relativity (and modifications thereof) against observations. We consider a theory which modifies general relativity by introducing a scalar field coupled to a parity-violating curvature term known as dynamical Chern-Simons gravity. In this theory, spinning black holes are different from their general relativistic counterparts and can thus serve as probes to this theory. We study linear gravito-scalar perturbations of black holes in dynamical Chern-Simons gravity at leading-order in spin and (i) obtain the perturbed field equations describing the evolution of the perturbed gravitational and scalar fields, (ii) numerically solve these equations by direct integration to calculate the quasinormal mode frequencies for the dominant and higher multipoles and tabulate them, (iii) find strong evidence that these rotating black holes are linearly stable, and (iv) present general fitting functions for different multipoles for gravitational and scalar quasinormal mode frequencies in terms of spin and Chern-Simons coupling parameter. Our results can be used to validate the ringdown of small-spin remnants of numerical relativity simulations of black hole binaries in dynamical Chern-Simons gravity and pave the way towards future tests of this theory with gravitational wave ringdown observations.

I. INTRODUCTION

General Relativity (GR) has passed a plethora of experimental tests both in the Solar System [1] and in binary pulsars systems [2, 3] making it one of the most successful physical theories. These tests probe situations in which gravitational fields are either weak, as in the Solar System, or systems where the field is strong but the system is slowly-varying, as in binary pulsars. However, the observation of gravitational waves (GW) by the LIGO/Virgo collaboration offers a new arena, where the spacetime is highly dynamical and strongly curved, in which the predictions of Einstein’s theory have been once more shown to agree with observations [4]. Complementary, GW observations also allow one to constrain modifications to GR [5–7] and with more ground-based and space-based detectors in the future, these constraints will become more stringent (see e.g. [8–12]).

But why one should study modifications to GR? There are a couple observational and theoretical anomalies that GR in its simplest form (i.e., without additional “dark” components or a UV completion) fails to answer. These include the late-time acceleration of the Universe [13, 14], the anomalous galaxy rotation curves [15, 16], the matter-antimatter asymmetry of the Universe [17], and the existence of spacetime singularities, e.g. in the interior of black holes. A resolution to these anomalies may reside in a modification to Einstein’s theory that passes all current tests, yet yields deviations in other extreme regimes where the gravitational interaction is simultaneously strong, non-linear and highly dynamical. On the

theoretical side, the incompatibility of GR with quantum mechanics has prompted efforts in a variety of unified theories, including string theory and loop quantum gravity. On the phenomenological side, the observational anomalies we described have motivated various extensions to GR [18, 19], such as $f(R)$ gravity and scalar-tensor theories [20, 21], tensor-vector-scalar theories [22], massive gravity [23] and bi-gravity [24]. Whether these attempts at modifying GR have any physical implications, requires one to first derive the predictions of such theories (in a given scenario) which should be followed by a comparison of these predictions against observations.

Although the correct completion of GR is yet unknown, GWs from compact binary coalescence observations can help in constraining and excluding entire arrays of modified theories of gravity. For instance, the GWs emitted in the inspiral of black hole binaries can tell us about the presence of extra radiative degrees of freedom, which provide an extra energy sink to which orbital energy and angular momentum can be extracted from the binary and hence affecting the systems orbital evolution (see e.g. [7]). Here we concentrate on GWs emitted when the newly formed black hole relaxes towards its final equilibrium state by emitting its last dying GW notes, the so-called ringdown. The GWs emitted during the ringdown can be described by a set of quasinormal modes (QNMs) with complex-valued frequencies whose imaginary part dictate how fast the mode decays in time. In GR, the observation of two or more QNMs in the ringdown signal allows one to uniquely infer the properties of the remnant Kerr black hole, similarly to how the observation of emission lines allows one to identify chemical elements [25]. This “black hole spectroscopy” thus allows one to test the “Kerr hypothesis” [26–28] i.e., that the BHs found in Nature are

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described by the Kerr metric. In general, modified theories of gravity do not admit the Kerr metric as a solution (see e.g. [19]) and even when they do so [29–31], the presence of the modifications to GR can be probed by perturbations to the Kerr metric [32]. This makes BH spectroscopy a powerful probe into beyond-GR physics.

Here we concentrate on modifications to GR which introduce a scalar field non-minimally coupled to squared curvature scalars, known as quadratic gravity theories [33, 34]. One subset of these theories, known as dynamical Chern-Simons (dCS) gravity [35], was proposed as an explanation to the matter-antimatter asymmetry of the universe by introducing additional parity-violating gravitational interactions, challenging a fundamental pillar of GR [36, 37]. The theory is poorly constrained by Solar System experiments (see [38] for an overview), and remains unconstrained by both binary pulsars [39] and GWs [40] observations. Nonetheless, first constraints on dCS were obtained through multi-messenger neutron star observations in [41].

In dCS, nonrotating BHs are identical to their GR counterparts, but when spun a nontrivial scalar field configuration arises and whose presence affects the spacetime metric. Perturbations of spherically symmetric BHs in dCS were first studied in [42] who found the system of equation to be coupled and complicated. Later work decoupled these equations and studied them extensively [43–45]. Here extend all of these results to axisymmetric, slowly-rotating BHs in dCS gravity and study their QNM spectra and stability.

Executive summary

We study the QNM spectra of slowly-rotating BHs in dCS gravity, generalizing [43, 44] which focused on the non-rotating case. To do so, we consider as a background the BH solution obtained in [46, 47]. We analyze the most general linear perturbations to this solution, taking into consideration both gravitational and scalar perturbations, but limiting ourselves to leading order in the dimensionless spin a/M and the Chern-Simons (CS) coupling strength α/M^2 , where M is the BH’s mass. The outcome of this calculation is a pair of coupled, inhomogeneous ODEs for the axial gravitational and scalar perturbations and a single homogeneous equation for the polar gravitational perturbations. The latter is identical to that found for slowly-rotating Kerr BHs in GR [48].

With these equations in hand, we numerically calculate the QNM frequencies ω , exploring their dependence on spin and coupling strength. We find that the QNM spectra can be split into three categories: (i) the scalar-led modes, whose frequencies in the limit $\alpha/M^2 \rightarrow 0$ reduce to that of a test scalar field on a slowly-rotating Kerr BH background; (ii) the (axial) gravitational-led modes, whose frequencies in the limit $\alpha/M^2 \rightarrow 0$ reduce to that of the axial-gravitational modes in a slowly-rotating Kerr background; (iii) the (polar) gravitational modes, with

frequencies identical to that of a slowly-rotating Kerr BH in GR regardless of α . Our results show that the isospectrality existent in GR between axial and polar gravitational modes is broken due to the coupling of the former with the scalar field. This was first observed in [44] in the nonrotating limit and is shown here to persist when small rotation is added.

We found (at fixed spin a/M) that the axial gravitational modes decay fast, but oscillate slower in dCS than in GR, with the latter being more sensitive to the CS coupling. We also found that these corrections scale with α^2 due to the coupling between the scalar and axial modes. We calculated a large set of QNM frequencies (see Appendix C) which we used to obtain fitting formulas for their real and imaginary parts as function of dimensionless spin and CS coupling [cf. Eqs. (54a) and (54b)]. We also found the QNMs decay for all values of spin and CS coupling within the limits of the slow rotation and small coupling approximation we employ. This is strong evidence that slowly rotating BH solutions in dCS gravity are linearly stable against gravito-scalar perturbations.

In the rest of this paper we show how these results were obtained. In Sec. II we give a short overview of dCS gravity and present the slowly-rotating BH solutions in this theory, whose QNM frequencies we are interested in computing. In Sec. III we review some general aspects of BH perturbation theory and derive the master equations that govern linear perturbations of our background BH spacetime. In Sec. IV we explain how these equations can be integrated numerically and in Sec. V we present our numerical results. Finally, in Sec. VI we summarize our main findings and discuss some avenues for future work.

We adopt the following conventions unless stated otherwise: we work in 4-dimensions with metric signature $(-, +, +, +)$ as in [49]. Greek indices (α, β, \dots) represent spacetime indices, round brackets around indices represent symmetrization, ∂_μ represents a partial derivative and $\square = \nabla_\mu \nabla^\mu$ the d’Alembertian operator. The Einstein summation convention is used throughout and we work in geometrical units in which $G = 1 = c$.

II. DYNAMICAL CHERN-SIMONS GRAVITY

A. Basics

Let us start with a brief review of dCS gravity and establish some notation [37]. In vacuum, the theory is described by the action

$$S = S_{\text{EH}} + S_\partial + S_{\text{CS}}, \quad (1)$$

where the Einstein-Hilbert term is

$$S_{\text{EH}} = \kappa \int d^4x \sqrt{-g} R, \quad (2)$$

where $\kappa = (16\pi)^{-1}$, R is the Ricci scalar and g is the determinant of the metric g_{ab} . The action for the scalar

field is

$$S_{\vartheta} = -\frac{1}{2} \int d^4x \sqrt{-g} [g^{\mu\nu} (\nabla_{\mu} \vartheta) (\nabla_{\nu} \vartheta) + 2V(\vartheta)], \quad (3)$$

where ∇_{μ} is the covariant derivative operator compatible with the metric, while $V(\vartheta)$ is a potential for the scalar that we set to zero. The scalar field is nonminimally coupled to the Pontryagin density $*RR$ as

$$S_{\text{CS}} = \frac{\alpha}{4} \int d^4x \sqrt{-g} \vartheta *RR, \quad *RR = *R^{\mu}{}_{\nu}{}^{\kappa\delta} R^{\nu}{}_{\mu\kappa\delta}, \quad (4)$$

where α is the CS coupling constant with units of $[\text{Length}]^2$ and $*R^{\mu}{}_{\nu}{}^{\kappa\delta}$ is the dual Riemann tensor

$$*R^{\mu}{}_{\nu}{}^{\kappa\delta} = \frac{1}{2} \epsilon^{\mu}{}_{\nu\alpha\beta} R^{\alpha\beta\kappa\delta}, \quad (5)$$

and $\epsilon^{\mu}{}_{\nu\alpha\beta}$ is the Levi-Civita tensor.

The field equations are obtained by varying the action in Eq. (1) with respect to the (inverse) metric $g^{\mu\nu}$ and scalar field ϑ . Variation with respect to $g^{\mu\nu}$ gives

$$G_{\mu\nu} + \frac{\alpha}{\kappa} C_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu}^{\vartheta}, \quad (6)$$

where $G_{\mu\nu}$ is the Einstein tensor, $C_{\mu\nu}$ is the (trace-free) C-tensor

$$C_{\mu\nu} = (\nabla_{\sigma} \vartheta) \epsilon^{\sigma\delta\alpha(\mu} \nabla_{\alpha} R^{\nu)}_{\delta} + (\nabla_{\sigma} \nabla_{\delta} \vartheta) *R^{\delta(\mu\nu)\sigma}, \quad (7)$$

which contains derivatives of the scalar field, and $T_{\mu\nu}^{\vartheta}$ is the canonical scalar field stress-energy tensor

$$T_{\mu\nu}^{\vartheta} = \left[(\nabla_{\mu} \vartheta) (\nabla_{\nu} \vartheta) - \frac{1}{2} g_{\mu\nu} (\nabla^{\sigma} \vartheta) (\nabla_{\sigma} \vartheta) \right]. \quad (8)$$

Variation with respect to ϑ gives the inhomogenous wave equation

$$\square \vartheta = -\frac{\alpha}{4} *RR. \quad (9)$$

One can show that $*RR$ vanishes for static, spherically symmetric spacetimes, resulting in $\vartheta = \text{const.}$ as the only regular solutions of dCS in BH spacetimes with these symmetries [35, 42]. This is no longer the case when these symmetries are lifted as we will see next.

B. Slowly-rotating Black Holes

Slowly-rotating BH solutions in dCS are known both analytically [46, 47, 50, 51] and numerically [52]. Here we will consider the solution found in [46, 47], which was obtained by solving the field equations (6) and (8) perturbatively to linear order in spin a and to quadratic order in the coupling strength α . Following the notation of [46], the line element of this solution is

$$d\bar{s}^2 = ds_{\text{SR}}^2 + \frac{5}{4} \frac{\alpha^2}{\kappa} \frac{a}{r^4} \left(1 + \frac{12}{7} \frac{M}{r} + \frac{27}{10} \frac{M^2}{r^2} \right) \sin^2 \theta dt d\phi, \quad (10)$$

where the bar stands for ‘‘background’’ and where ds_{SR}^2 is the line element for a slowly-rotating Kerr BH, also to linear order in a ,

$$ds_{\text{SR}}^2 = -f(r) dt^2 - \frac{4Ma \sin^2 \theta}{r} dt d\phi + f(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (11)$$

and $f(r) = 1 - 2M/r$ is the Schwarzschild factor. The solution also contains a nontrivial scalar field,

$$\bar{\vartheta} = \frac{5}{8} \frac{a\alpha}{M} \frac{\cos \theta}{r^2} \left(1 + \frac{2M}{r} + \frac{18M^2}{5r^2} \right). \quad (12)$$

A far-field analysis shows that the BH has a scalar dipole charge $-(5/8)(a\alpha/M)$.

We see from Eq. (10) that the metric for slowly-rotating BHs in dCS gravity has an additional modification in the $(t\phi)$ -component when compared to the metric in GR given in Eq. (11). This modifies the horizon angular frequency Ω_{H} as observed by a zero angular momentum observer at the horizon location r_{H} at first order in spin and second order in α [46] as

$$\Omega_{\text{H}} = \frac{a}{4M^2} - \frac{709\alpha^2 a}{28672\kappa M^6} = \frac{a}{4M^2} \left(1 - \frac{709}{7168} \zeta \right), \quad (13)$$

where we defined

$$\zeta = \alpha^2 / (M^4 \kappa). \quad (14)$$

As we will see later, this α^2 term affects the QNMs.

III. BLACK HOLE PERTURBATION THEORY

A. Decomposition of the fundamental fields

We consider linear perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon \delta_{\mu\nu}, \quad \vartheta = \bar{\vartheta} + \epsilon \delta\vartheta, \quad (15)$$

to the background BH spacetime [Eqs. (10) and (12)], where ϵ is a bookkeeping parameter and both $\delta g_{\mu\nu}$ and $\delta\vartheta$ are functions of the coordinates of the metric.

The angular dependence of these perturbations can be described by scalar, vector and tensor spherical harmonics. The metric decomposition comes from the transformation properties of the ten components of the perturbation tensor $\delta g_{\mu\nu}$ under a rotation of the frame of origin [53]. These quantities transform as three SO(2) scalars δg_{MN} , two SO(2) vectors δg_{mN} and one SO(2) second rank tensor δg_{ab} which can be expanded into a complete basis formed by spherical harmonics of different corresponding rank. Under a parity transformation, i.e., the simultaneous shifts $\theta \rightarrow \pi - \theta$ and $\phi \rightarrow \phi + \pi$, the above mentioned metric quantities can be classified into the odd (or ‘‘axial’’) and even (or ‘‘polar’’) sectors which are multiplied by $(-1)^{\ell+1}$ and $(-1)^{\ell}$ respectively, such that

$$\delta g_{\mu\nu}(t, r, \theta, \phi) = \delta g_{\mu\nu}^{\text{odd}}(t, r, \theta, \phi) + \delta g_{\mu\nu}^{\text{even}}(t, r, \theta, \phi), \quad (16)$$

where

$$\delta g_{\mu\nu}^{\text{odd}} = \begin{pmatrix} 0 & 0 & h_0^{\ell m}(t, r) S_\theta^{\ell m}(\theta, \phi) & h_0^{\ell m}(t, r) S_\phi^{\ell m}(\theta, \phi) \\ * & 0 & h_1^{\ell m}(t, r) S_\theta^{\ell m}(\theta, \phi) & h_1^{\ell m}(t, r) S_\phi^{\ell m}(\theta, \phi) \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (17)$$

and

$$\delta g_{\mu\nu}^{\text{even}} = \begin{pmatrix} H_0^{\ell m}(t, r) Y_{\ell m}(\theta, \phi) & H_1^{\ell m}(t, r) Y_{\ell m}(\theta, \phi) & 0 & 0 \\ * & H_2^{\ell m}(t, r) Y_{\ell m}(\theta, \phi) & 0 & 0 \\ * & * & r^2 K^{\ell m}(t, r) Y_{\ell m}(\theta, \phi) & 0 \\ * & * & * & r^2 \sin^2 \theta K^{\ell m}(t, r) Y_{\ell m}(\theta, \phi) \end{pmatrix}, \quad (18)$$

where the asterisk denotes symmetric components, $Y_{\ell m}(\theta, \phi)$ are the scalar spherical harmonics, while

$$S_\theta^{\ell m}(\theta, \phi) = -\frac{1}{\sin \theta} \partial_\phi Y_{\ell m}(\theta, \phi), \quad (19)$$

$$S_\phi^{\ell m}(\theta, \phi) = \sin \theta \partial_\theta Y_{\ell m}(\theta, \phi), \quad (20)$$

and a sum over ℓ and m in the usual sense ($\ell \geq 0$ and $|m| \leq \ell$) is implicit. Equations (17) and (18) hold under the Regge-Wheeler gauge [53], which can be applied in theories with a massless graviton that support the usual two polarizations. In certain modified theories, a graviton can propagate with up to six polarizations, thereby leaving no residual gauge freedom. However, for the case of dCS gravity, GWs continue to propagate with only two polarizations (as measured at future null infinity) [54, 55], and thus, one retains enough gauge freedom to impose the Regge-Wheeler gauge. Such a decomposition separates the axial and polar perturbations with different harmonic index ℓ , i.e., for a given ℓ , we have two systems of evolution equations, one for the axial sector and one for the polar sector. These separate sets of equations completely characterize the linear response of the system.

Additional fields in the system, such as vectors and scalars, can be decomposed into spherical harmonics of the corresponding type. For scalar fields, we use scalar spherical harmonics and the perturbed scalar field reads

$$\delta\vartheta(t, r, \theta, \phi) = \frac{R_{\ell m}(r, t)}{r} Y_{\ell m}(\theta, \phi). \quad (21)$$

B. Evolution equations for perturbations of a slowly rotating black hole

Having established the background spacetime and explained how the linear perturbations can be decomposed into scalar and tensor harmonics, we can now derive the perturbed form of the field equations in dCS gravity. The procedure is as follows:

- Substitute the linear perturbations (15) into the field equations Eq. (6) and (9) and expand to linear order in ϵ .

- Expand the perturbed field equations to linear order in the spin parameter a (slow-rotation approximation) and in the coupling parameter α (small-coupling approximation). Even though the metric (10) is of order $\mathcal{O}(\alpha^2)$ we only evaluate the field equation up to linear-order in α for reasons explained later.

- Use the orthogonality properties of the spherical harmonics presented in Appendix A to eliminate the angular dependence of the functions $A^a \in \{h_0, h_1, H_0, H_1, H_2, K, R\}$, making them functions of t and r only. Moreover, assume an harmonic time-dependence in time, i.e.,

$$A_{\ell m}^a(t, r) = e^{-i\omega t} A_{\ell m}^a(r). \quad (22)$$

- The previous steps yields a system of 11 equations; 10 from the metric field equation (6) and one from the scalar field equation (9). The latter gives the evolution equation for the scalar field perturbations, whereas 3 of the metric equations reduce to the axial gravitational perturbation equation and the remaining 7 give an expression for the polar gravitational perturbation equation.
- This system of equations can then be expressed in general by an equation of the form

$$\mathcal{D}\Psi_j + V_j\Psi_j = S_j[\Psi_k, \partial_r\Psi_k], \quad (23)$$

where $j \in \{\text{R, RW, ZM}\}$, $k \in \{\{\text{R, RW, ZM}\} - j\}$ with $\{\text{R, RW, ZM}\}$ denoting the scalar, axial and polar gravitational perturbations respectively, \mathcal{D} is a second-order radial differential operator, which in tortoise coordinates (r_*) reduces to d^2/dr_*^2 , and V_j is the effective potential. The source term S_j is found to be a linear combination of the Ψ_k master functions and its first radial derivatives, e.g. when $j = \text{R}$, the master function $\Psi_{\text{R}} = R_{\ell m}$, and the source term S_{R} is a function of Ψ_{RW} and Ψ_{ZM} and their first radial derivatives.

In the next subsections, we provide the final expressions for the perturbation equations for the scalar and gravitational sectors. A derivation these equation can be found in a Mathematica notebook which will be made available upon request.

1. Scalar sector

The full equation describing the scalar field perturbation $R_{\ell m}$ is given by

$$\begin{aligned} f(r)^2 \partial_{rr} R_{\ell m} + \frac{2M}{r^2} f(r) \partial_r R_{\ell m} + [\omega^2 - V_{\text{eff}}^S(r, a)] R_{\ell m} \\ = \alpha f(r) \{ [g(r) + amh(r)] \Psi_{\ell m}^{\text{RW}} + amj(r) \partial_r \Psi_{\ell m}^{\text{RW}} \} \\ + \alpha a \{ q_{\ell m} [k_1(r) \Psi_{\ell-1, m}^{\text{ZM}} + k_2(r) \partial_r \Psi_{\ell-1, m}^{\text{ZM}}] \\ + q_{\ell+1, m} [k_3(r) \Psi_{\ell+1, m}^{\text{ZM}} + k_4(r) \partial_r \Psi_{\ell+1, m}^{\text{ZM}}] \}, \end{aligned} \quad (24)$$

where $f(r) = 1 - 2M/r$,

$$q_{\ell m} = \sqrt{\frac{\ell^2 - m^2}{4\ell^2 - 1}}, \quad (25)$$

and ∂_r denotes radial derivatives. In Eq. (24), the functions g, h, j and k_i ($i = 1, \dots, 4$) also depend on ℓ and the mass of the black hole M in addition to the radial coordinate. Their explicit forms are shown in Appendix B and in a Mathematica notebook available upon request. We also followed [48] and defined the Regge-Wheeler function $\Psi_{\ell m}^{\text{RW}}$, which to leading order in a is given by

$$\Psi_{\ell m}^{\text{RW}} = \frac{f(r)}{r} \left(1 + \frac{2mMa}{r^3\omega} \right) h_1^{\ell m}, \quad (26)$$

where $h_1^{\ell m}$ comes from Eq. (17). Similarly, we introduced the Zerilli-Moncrief function $\Psi_{\ell m}^{\text{ZM}}$, which to leading order in a is given by

$$\Psi_{\ell m}^{\text{ZM}} = (z_1/z_2) - a(z_3/z_4), \quad (27)$$

where

$$z_1 = -2iH_1^{\ell m}(r - 2M) + 2K^{\ell m}r^2\omega, \quad (28a)$$

$$z_2 = (6M + \lambda_\ell r), \quad (28b)$$

$$\begin{aligned} z_3 = -4imM\omega [H_1^{\ell m}(r - 2M) + iK^{\ell m}r^2\omega] \\ \times \{ 48M^3 - 24M^2r + \lambda_\ell r^3(\lambda_\ell + 2r^2\omega^2) \\ + 2Mr^2(\lambda_\ell^2 + 6r^2\omega^2) \}, \end{aligned} \quad (28c)$$

$$z_4 = \ell(\ell + 1)r^4z_2^3, \quad (28d)$$

with $\lambda_\ell = (\ell + 2)(\ell - 1)$, and $H_1^{\ell m}$ and $K^{\ell m}$ coming from Eq. (18).

2. Metric sector

The full equation describing the axial gravitational perturbation $\lambda_{\ell m}$ is given by

$$\begin{aligned} f(r)^2 \partial_{rr} \Psi_{\ell m}^{\text{RW}} + \frac{2M}{r^2} f(r) \partial_r \Psi_{\ell m}^{\text{RW}} + [\omega^2 - V_{\text{eff}}^A(r, a)] \Psi_{\ell m}^{\text{RW}} \\ = \alpha f(r) \{ [v(r) + amn(r)] R_{\ell m} + amp(r) \partial_r R'_{\ell m} \} \\ + a \{ q_{\ell m} [p_1(r) \Psi_{\ell-1, m}^{\text{ZM}} + p_2(r) \partial_r \Psi_{\ell-1, m}^{\text{ZM}}] \\ + q_{\ell+1, m} [p_3(r) \Psi_{\ell+1, m}^{\text{ZM}} + p_4(r) \partial_r \Psi_{\ell+1, m}^{\text{ZM}}] \}, \end{aligned} \quad (29)$$

where the functions v, n, p and p_i ($i = 1, \dots, 4$) also depend on ℓ and M in addition to the radial coordinate. Their explicit forms are given in Appendix B and in a Mathematica notebook available upon request.

Finally, the polar gravitational perturbation $Z_{\ell m}$ satisfies the equation

$$\begin{aligned} f(r)^2 \partial_{rr} \Psi_{\ell m}^{\text{ZM}} + \frac{2M}{r^2} f(r) \partial_r \Psi_{\ell m}^{\text{ZM}} + [\omega^2 - V_{\text{eff}}^P(r, a)] \Psi_{\ell m}^{\text{ZM}} \\ = \alpha a f(r) \{ q_{\ell m} (s_1(r) R_{\ell-1, m} + s_2(r) \partial_r R_{\ell-1, m}) \\ + q_{\ell+1, m} (s_3(r) R_{\ell+1, m} + s_4(r) \partial_r R_{\ell+1, m}) \} \\ + a \{ q_{\ell m} (r_1(r) \Psi_{\ell-1, m}^{\text{RW}} + r_2(r) \partial_r \Psi_{\ell-1, m}^{\text{RW}}) \\ + q_{\ell+1, m} (r_3(r) \Psi_{\ell+1, m}^{\text{RW}} + r_4(r) \partial_r \Psi_{\ell+1, m}^{\text{RW}}) \}, \end{aligned} \quad (30)$$

where the functions s_i and r_i ($i = 1, \dots, 4$) depend on r, ℓ and M . Their explicit forms are shown in Appendix B and in a Mathematica notebook available upon request.

3. Selection Rule and Propensity Rule

The perturbation equations [Eqs. (24), (29) and (30)] explicitly show that ℓ modes couple to ℓ modes and $\ell \pm 1$ modes. As in GR for slowly-rotating BHs, however, these equations possess a selection rule [56, 57]. In GR and at linear order in spin, the selection rule is that the ℓ -th axial (polar) mode couples to the $\ell \pm 1$ polar (axial) mode; at second order in spin, this simple selection rule needs to be modified [57]. Similarly, in dCS gravity and at linear order in spin, the same selection rule applies to the perturbation equations. The only modification is that in dCS gravity we have two fields with axial parity: the scalar field (encoded in $R_{\ell m}$) and the Regge-Wheeler function (encoded in $\Psi_{\ell m}^{\text{RW}}$). Thus, in dCS gravity, $R_{\ell m}$ ($\Psi_{\ell m}^{\text{RW}}$) couples to $\Psi_{\ell m}^{\text{RW}}$ ($R_{\ell m}$) and to $\Psi_{\ell \pm 1, m}^{\text{ZM}}$, while $\Psi_{\ell m}^{\text{ZM}}$ couples to both $R_{\ell \pm 1, m}$ and $\Psi_{\ell \pm 1, m}^{\text{RW}}$.

In addition to this selection rule, the perturbation equations also suggest a propensity rule. More specifically, we see that when $\ell = |m|$, which dominates the linear response of the system, Eq. (25) yields $q_{\ell m} = 0$, and thus the coupling of ℓ modes with $\ell - 1$ modes is suppressed. This is similar in nature to the propensity rule of atomic physics, which states that transitions involving $\ell \rightarrow \ell + 1$ are favored over those involving $\ell \rightarrow \ell - 1$ [56]. Thus, we expect that the dominant modes are coupled only to the ℓ or the $\ell + 1$ modes, after imposing the selection rule.

C. Simplification of the Perturbation Equations

Mode coupling between perturbation of different ℓ mode leads to a rich spectrum of solutions, but this paper is concerned with the QNM frequencies, which, as it turns out, are not affected by mode coupling to leading order in spin. Indeed, in GR this has been known since the 1990s, thanks to the work of Kojima and others [57–60]. Let us

then establish the same result in dCS gravity to leading order in spin and in coupling parameter. Without loss of generality, let us rewrite the perturbation equations as

$$A_{\ell m} + ma\tilde{A}_{\ell m} + a(q_{\ell}\hat{P}_{\ell-1,m} + q_{\ell+1}\hat{P}_{\ell+1,m}) = 0, \quad (31)$$

$$P_{\ell m} + ma\tilde{P}_{\ell m} + a(q_{\ell}\hat{A}_{\ell-1,m} + q_{\ell+1}\hat{A}_{\ell+1,m}) = 0. \quad (32)$$

In these equations, $A_{\ell m}$, $\tilde{A}_{\ell m}$ and $\hat{A}_{\ell\pm 1,m}$ are linear combinations of odd (axial) perturbations, which include the Regge-Wheeler function $\Psi_{\ell m}^{\text{RW}}$ and its derivatives, and also the scalar field perturbation $R_{\ell m}$ and its derivatives; we remind the reader that the scalar field perturbations are of odd (axial) parity as seen from Eq. (9). The prefactors of α^0 and α^1 have also been suppressed in the above functions for simplicity of notation. Similarly, $P_{\ell m}$, $\tilde{P}_{\ell m}$ and $\hat{P}_{\ell\pm 1,m}$ are linear combinations of polar perturbations encoded in the Zerilli-Moncrief function $\Psi_{\ell m}^{\text{ZM}}$ and its derivatives.

In GR, Kojima [58] showed using symmetry arguments for the $m = 0$ mode that the terms $\hat{P}_{\ell\pm 1,m}$ and $\hat{A}_{\ell\pm 1,m}$ in Eqs. (31) and (32) make no contribution to the QNMs. This argument was later extended to other values of m in [57] for massive vector field perturbations of the slowly-rotating Kerr metric in GR. Following [57], we now extend this argument to slowly rotating BHs in dCS gravity.

Consider a simultaneous transformation,

$$\begin{aligned} x_{\ell,m} &\rightarrow \mp x_{\ell,-m}, & y_{\ell,m} &\rightarrow \pm y_{\ell,-m}, \\ m &\rightarrow -m, & a &\rightarrow -a, \end{aligned} \quad (33)$$

where $x_{\ell m}$ and $y_{\ell m}$ represent the axial and polar perturbation variables respectively, with indices (ℓ, m) given in Eqs. (31) and (32) which remain invariant under such a transformation. The boundary conditions for QNMs of slowly rotating BHs in dCS are also invariant under such transformation. Then, in the slow rotation limit, the QNM frequencies can be expanded as

$$\omega = \omega_0 + ma\omega_1 + a\omega_2 + \mathcal{O}(a^2), \quad (34)$$

where ω_0 is the eigenfrequency of the non-rotating BH in dCS gravity, which in our case is just a Schwarzschild BH. The effective potential presented in Sec. III B is proportional to a^0 and ma , but not to a alone (see Appendix B). Hence, $\omega_2 = 0$, because the above potential would not source such a term¹.

The only terms that could source ω_2 , at least in principle, are $(\hat{P}_{\ell\pm 1,m}, \hat{A}_{\ell\pm 1,m})$. This is because the second terms in Eqs. (31) and (32) are explicitly proportional to m , and because $(\hat{A}_{\ell m}, \hat{P}_{\ell m})$ cannot be inversely

proportional to m . Whether $(\hat{P}_{\ell\pm 1,m}, \hat{A}_{\ell\pm 1,m})$ source ω_2 depends on if they are independent of m or not. However, since the second and third terms in Eqs. (31) and (32) are linear in a , both the functions $(\tilde{A}_{\ell m}, \tilde{P}_{\ell m})$ and $(\hat{P}_{\ell\pm 1,m}, \hat{A}_{\ell\pm 1,m})$ must be kept only to $\mathcal{O}(a^0)$. Therefore, they correspond to perturbations of a non-rotating BH, which is necessarily spherically symmetric, implying in particular that $(\hat{P}_{\ell\pm 1,m}, \hat{A}_{\ell\pm 1,m})$ are actually independent of m . But since ω_2 vanishes by the arguments presented above, it follows that $(\hat{P}_{\ell\pm 1,m}, \hat{A}_{\ell\pm 1,m})$ need not be included when computing the QNM spectrum.

Since the mode coupling terms can be neglected, our perturbation equations reduce to

$$\begin{aligned} f(r)^2 \partial_{rr} R_{\ell m} + \frac{2M}{r^2} f(r) \partial_r R_{\ell m} + [\omega^2 - V_{\text{eff}}^S(r, a)] R_{\ell m} \\ = \alpha f(r) \{ [g(r) + amh(r)] \Psi_{\ell m}^{\text{RW}} + amj(r) \partial_r \Psi_{\ell m}^{\text{RW}} \}, \end{aligned} \quad (35a)$$

$$\begin{aligned} f(r)^2 \partial_{rr} \Psi_{\ell m}^{\text{RW}} + \frac{2M}{r^2} f(r) \partial_r \Psi_{\ell m}^{\text{RW}} + [\omega^2 - V_{\text{eff}}^A(r, a)] \Psi_{\ell m}^{\text{RW}} \\ = \alpha f(r) \{ [v(r) + amn(r)] R_{\ell m} + amp(r) \partial_r R_{\ell m} \}, \end{aligned} \quad (35b)$$

$$\begin{aligned} f(r)^2 \partial_{rr} \Psi_{\ell m}^{\text{ZM}} + \frac{2M}{r^2} f(r) \partial_r \Psi_{\ell m}^{\text{ZM}} + [\omega^2 - V_{\text{eff}}^P(r, a)] \Psi_{\ell m}^{\text{ZM}} \\ = 0, \end{aligned} \quad (35c)$$

where the values of all these functions are the same as before and are given in Appendix B.

Equations (35) can be recast in a Schrödinger-like form by introducing tortoise coordinates r_* ,

$$r_* = r + 2M \log \left(\frac{r}{2M} - 1 \right). \quad (36)$$

However, when solving these equations, we will stick to the form given above to avoid confusion. Note in passing that the tortoise coordinate is typically used to map Schwarzschild coordinates to a horizon-penetrating (typically ingoing Eddington-Finkelstein) coordinate system, which is well adapted to imposing boundary conditions at the BH horizon. The standard transformation known in GR, however, may need to be modified by terms of $\mathcal{O}(\alpha^2 a^2)$ to transform a dCS BH to horizon-penetrating coordinates. Since we are here working to a lower order in perturbation theory, we do not need to worry about such details.

D. dCS coupling dependence of ω through a Fermi estimate

As seen from the previous section, for the calculation of the QNM frequencies, the perturbed equations take the form in Eqs. (35). In this section, we discuss the dependence of the QNM frequency ω on the dCS coupling parameter α . We see that for the polar gravitational sector Eq. (35c) is entirely independent of the CS coupling parameter, whereas Eqs. (35a) and (35b) have a non-vanishing linear-in- α source term.

¹ Another (more physical) way to see this is by considering the $a \rightarrow -a$ transformation of Eq. (33). Such a transformation would physically correspond to inverting the direction of the BH's spin angular momentum. However, the QNM frequencies should not change due to the spin orientation. Hence, in general, $\omega_2 = 0$ making Eq. (34) invariant under the symmetry in Eq. (33).

What does this imply on the dCS corrections to the QNM frequencies ω ? We can answer this question with a Fermi estimate in which we replace any radial derivative by a characteristic radius $\partial_r \rightarrow 1/\mathcal{R}$ and evaluate the equation at this characteristic radius. Doing this in the perturbation equations (35a) and (35b) gives

$$[G(\mathcal{R}) + \omega^2 - V_{\text{eff}}^S(\mathcal{R})] R_{lm} = \alpha H(\mathcal{R}) \Psi_{\text{RW}}^{lm}, \quad (37a)$$

$$[G(\mathcal{R}) + \omega^2 - V_{\text{eff}}^A(\mathcal{R})] \Psi_{\text{RW}}^{lm} = \alpha I(\mathcal{R}) R^{lm}, \quad (37b)$$

where $G(\mathcal{R}), H(\mathcal{R}), I(\mathcal{R})$ are polynomials in the radial coordinate r evaluated at $r = \mathcal{R}$, namely

$$G(\mathcal{R}) = \frac{f(\mathcal{R})^2}{r^2} + \frac{2Mf(\mathcal{R})}{\mathcal{R}^3}, \quad (38)$$

$$H(\mathcal{R}) = f(\mathcal{R}) \left[g(\mathcal{R}) + am \left(h(\mathcal{R}) + \frac{j(\mathcal{R})}{\mathcal{R}} \right) \right], \quad (39)$$

$$I(\mathcal{R}) = f(\mathcal{R}) \left[v(\mathcal{R}) + am \left(n(\mathcal{R}) + \frac{p(\mathcal{R})}{\mathcal{R}} \right) \right]. \quad (40)$$

Equations (37a) can be solved for $R_{\ell m}$ and then inserted into Eq. (37b), which renders an equation for ω since $\Psi_{\ell m}^{\text{RW}}$ cancels and the ω dependence in $H(\mathcal{R})$ and $I(\mathcal{R})$ cancels when they are multiplied together, and one finds

$$[G + \omega^2 - V_{\text{eff}}^S] [G + \omega^2 - V_{\text{eff}}^A] = \alpha^2 I H, \quad (41)$$

where we have suppressed the argument of the functions. This is a quadratic equation for ω^2 , which we can solve perturbatively in $\alpha \ll 1$ to find

$$\omega = \omega_{\text{GR}} + \zeta \delta\omega + \mathcal{O}(\alpha^3), \quad (42)$$

where, recall, $\zeta = \alpha^2/(M^4\kappa)$ and

$$\omega_{\text{GR}} = (V_{\text{eff}}^A - G)^{1/2}, \quad (43)$$

$$\delta\omega = \pm \frac{I H}{2 (V_{\text{eff}}^A - G)^{1/2} (V_{\text{eff}}^A - V_{\text{eff}}^S)}. \quad (44)$$

If we evaluate these expressions at $\mathcal{R} = 3M$, which is close to where the effective potentials V_{eff}^A and V_{eff}^S are extremized, we find for the $\ell = 2$ mode at $a = 0$ that $M\omega_{\text{GR}} = 1/3$ (≈ 0.333) and $M\delta\omega = -2/81$ (≈ -0.025), both of which are close to the real part of the correct numerical answers we will find later. The precise numerical factors, however, do not matter here. What matters is that the above Fermi estimate shows explicitly that the dCS corrections to the QNM frequencies will be of $\mathcal{O}(\alpha^2)$. If we had included terms proportional to α^2 in the perturbation equations, they would have only generated corrections of $\mathcal{O}(\alpha^3)$ in the QNM frequencies. This is why it suffices to keep only terms up to $\mathcal{O}(\alpha)$ in the perturbation equations.

IV. CALCULATION OF THE QUASINORMAL MODES

The late-time GW signal from a perturbed BH is generally dominated by a sum of exponentially damped sinusoids known as the QNMs. These correspond to the

characteristic vibrational modes of the spacetime [61] and are complex valued. The real part represents the temporal oscillations whereas the imaginary part represents an exponentially decaying temporal part of the oscillations. Using the slowly-rotating approximation for finding QNMs of BHs allows us to use well-established numerical methods for their calculation. In this section, we show the boundary conditions and the numerical integration technique that will be used to calculate the QNMs.

A. Boundary conditions

The QNMs are solutions of the inhomogeneous wave equations [Eqs. (35a) to (35c)] with appropriate boundary conditions. For the case of slowly-rotating BHs, we have two boundaries: one at spatial infinity and the other at the horizon r_{H} . The horizon of the BH described by the metric of Eq. (10) coincides with the Schwarzschild horizon (they are both located at $r = 2M$ in Schwarzschild-like coordinates) at leading order in spin.

A QNM represents waves which are of purely ingoing at the horizon r_{H} and purely outgoing at spatial infinity, which are characterized by complex frequency ω with $\text{Re}(\omega) \geq 0$. These boundary conditions are,

$$\Psi_j \propto \begin{cases} e^{-i\omega_{\text{H}} r_*}, & r \rightarrow r_{\text{H}}, \\ e^{i\omega r_*}, & r \rightarrow \infty, \end{cases} \quad (45)$$

where recall that $\Psi_j = \{R_{\ell m}, \Psi_{\ell m}^{\text{RW}}, \Psi_{\ell m}^{\text{ZM}}\}$, r_* is the tortoise coordinate and $\omega_{\text{H}} = \omega - m\Omega_{\text{H}}$, with Ω_{H} , given by Eq. (13), is the horizon angular frequency for the BH under consideration.

B. Evaluation of the QNMs: direct integration

To compute the QNMs, we use the direct integration method. In this method, we integrate the equations twice, once from finite distance outside the horizon towards spatial infinity, and once from a finite distance far from the horizon but inwards towards the event horizon. The integrations are started using the boundary conditions presented above, with a given choice of ω . We then compare the two numerical solutions at a matching point r_m that is somewhere between the horizon and spatial infinity, to check whether the master functions and their radial derivatives are continuous at r_m . Typically, this is not the case, so we then iterate this process over various values of ω until one finds a choice of the complex frequency that leads to continuous and differentiable solutions. In practice, this can be done by finding the value of ω for which the Wronskian W of the two solutions vanishes at r_m [62].

However, as simple as this approach appears, it is numerically difficult to implement due to inherent numerical instabilities. If one chooses the wrong value for ω then the trial integrations will tend to diverge as one approaches spatial infinity (for the outward solution) or as

one approaches the BH horizon (for the inward solution). Also, since numerically we cannot start the integrations exactly at the horizon or at spatial infinity, the boundary conditions are not sufficiently accurate in general.

To improve numerical stability, one can improve the boundary conditions by finding analytic asymptotic solutions to the perturbation equations about spatial infinity and the horizon. More specifically, we can write [63],

$$\Psi_j \propto \begin{cases} e^{-i\omega r_*} \sum_{k=0}^{\infty} (\gamma_k)_j (r - r_H)^k, & r \rightarrow r_H, \\ e^{i\omega r_*} \sum_{k=0}^{\infty} (\eta_k)_j r^{-k}, & r \rightarrow \infty, \end{cases} \quad (46)$$

where the coefficients γ_k and η_k can be determined order by order in a series expansion of the perturbation equations around spatial infinity or the horizon. These coefficients can all be written entirely in terms of $(\gamma_0)_j$ and $(\eta_0)_j$, but they are long and not illuminating, so we will not present them in the paper, and instead they will be made available through a Mathematica notebook upon request. We will use these boundary conditions for our numerical integrations.

We will here compute the QNMs adapting the procedure in [64] to our case. We begin by constructing two square matrices W_o and W_e , which is four dimensional for the axial case due to the coupling between $\Psi_{\ell m}^{RW}$ and $R_{\ell m}$, and two dimensional for the polar case due to the lack of coupling. The columns of $W_{o,e}$ are independent solutions of the perturbation equations, so

$$W_o = \begin{pmatrix} {}_H\Psi_{\ell m}^{RW(1)} & {}_I\Psi_{\ell m}^{RW(1)} & {}_H\Psi_{\ell m}^{RW(2)} & {}_I\Psi_{\ell m}^{RW(2)} \\ \partial_{r_H}\Psi_{\ell m}^{RW(1)} & \partial_{r_I}\Psi_{\ell m}^{RW(1)} & \partial_{r_H}\Psi_{\ell m}^{RW(2)} & \partial_{r_I}\Psi_{\ell m}^{RW(2)} \\ {}_H R_{\ell m}^{(1)} & {}_I R_{\ell m}^{(1)} & {}_H R_{\ell m}^{(2)} & {}_I R_{\ell m}^{(2)} \\ \partial_{r_H}R_{\ell m}^{(1)} & \partial_{r_I}R_{\ell m}^{(1)} & \partial_{r_H}R_{\ell m}^{(2)} & \partial_{r_I}R_{\ell m}^{(2)} \end{pmatrix}, \quad (47)$$

$$W_e = \begin{pmatrix} {}_H\Psi_{\ell m}^{ZM} & {}_I\Psi_{\ell m}^{ZM} \\ \partial_{r_H}\Psi_{\ell m}^{ZM} & \partial_{r_I}\Psi_{\ell m}^{ZM} \end{pmatrix}, \quad (48)$$

where, the pre-subscript to the perturbation function ${}_{H,I}\Psi$ denote whether the solutions is obtained by integration from the horizon to r_m , or from spatial infinity to r_m , while the superscripts $\Psi^{(1),(2)}$ denote two solutions evaluated with different initial conditions at the boundaries.

In principle, any set of independent solutions will do for the calculation of these $W_{o,e}$ matrices, but in this paper we make the following choices. For the even sector, we choose the solution to be that obtained by integrating the perturbation equations with the boundary conditions in Eq. (46) and $[(\gamma_0)_{ZM}, (\eta_0)_{ZM}] = (1, 1)$. For the odd sector, we choose the two solutions to be those obtained by integrating with the boundary conditions in Eq. (46) and with $[(\gamma_0)_{RW}, (\eta_0)_{RW}, (\gamma_0)_R, (\eta_0)_R] = (1, 1, 0, 0)$ or with $[(\gamma_0)_{RW}, (\eta_0)_{RW}, (\gamma_0)_R, (\eta_0)_R] = (0, 0, 1, 1)$. In general, these solutions are linearly independent, unless ω is the

correct QNM frequency, in which case

$$\det(W)|_{r=r_m} = 0. \quad (49)$$

We can use a root-finding algorithm to find the ω such that the Wronskian vanishes at the matching point.

Using this method, we have calculated the QNMs of a slowly-rotating BH in dCS gravity. In practice, all numerical integrations that start at the horizon are initiated at $r_{\text{initial},r_+} = (2 + 10^{-4})M$, while those that start at spatial infinity are initiated at $r_{\text{initial},i_0} = 60M$, with the matching always performed at $r_m = 30M$. We have checked the numerical stability of the QNM frequencies against changes in the values of r_{initial,r_+} , r_{initial,i_0} and r_m . All numerical integrations are done with the NDSolve package of Mathematica, with accuracy and precision set to 10 digits.

V. NUMERICAL RESULTS

In this section, we present our numerical results for the QNM frequencies. For clarity, we have divided this section into three parts. First, we take the GR limit (i.e., $\alpha/M^2 = 0$) of our equation and discuss the associated QNMs of a slowly-rotating Kerr BH with the metric expanded to leading-order in spin. We compare these results with the exact Kerr QNMs obtained using Leaver's continued fraction method [65]. This preliminary step will give us an estimate of where the slow-rotation approximation breaks down and how the numerical errors due to this approximation compare against the modifications to the QNM frequencies due to the CS coupling. Second, we consider a nonzero CS coupling α/M^2 and study in detail how the QNMs behave as functions of this coupling and of the BH spin. Third, we construct fitting formulas for the real and imaginary parts of the QNMs, valid within the errors associated to the slow-rotation approximation.

In the main body of this paper, we will show numerical results for the fundamental $n = 0$, $\ell = m = 2$ frequencies since these are dominant for tensorial perturbations. We leave our results for the QNM frequencies for the fundamental mode with $\ell = 2$, $\ell = 3$ and $\ell = 4$ modes, and all m modes to Appendix C. To aid in the presentation of these numerical results, we will work with dimensionless parameters by rescaling a , α and ω as

$$a \rightarrow a/M, \quad \alpha \rightarrow \alpha/M^2, \quad \text{and} \quad \omega \rightarrow \omega M. \quad (50)$$

In our numerical calculations, we work in code units, in which $M = 1$, thus making the code quantities a , α and ω dimensionless.

A. Slow rotation: GR

The calculation of the QNMs of a slow-rotating Kerr BH in GR were presented in [48], but to our knowledge

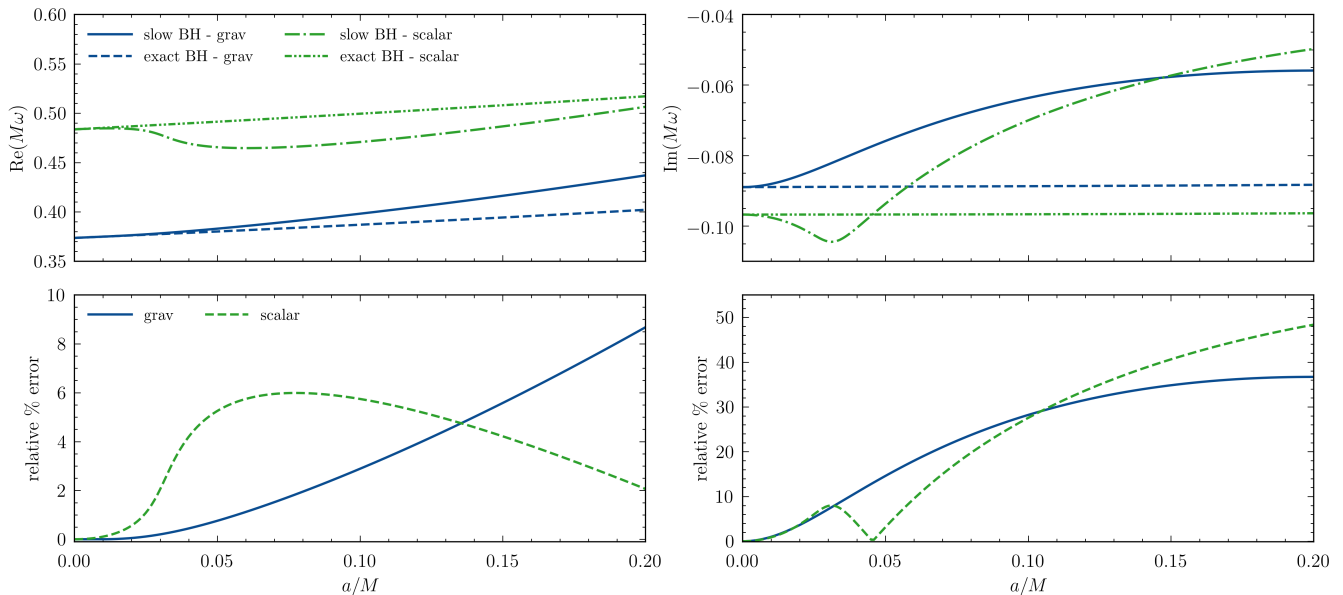


FIG. 1. Comparison between the fundamental dominant $\ell = m = 2$ gravitational and scalar QNMs calculated using the exact Kerr metric and its expansion to leading-order in a . The dependence of the real (imaginary) part of the frequency on the spin a is shown in the left (right) top panel. The left (right) bottom panel shows the relative error defined by Eq. (52). The colors distinguish the scalar (green) and gravitational (blue) QNMs. Unsurprisingly, we see that the relative percent error $\delta\omega$ increases with spin a/M , doing so faster for the imaginary part of the frequency, although the absolute error is $\mathcal{O}(a^2)$. The legends shown in the left panels apply to the corresponding right panels as well.

a comparison between these results and those obtained using the exact Kerr metric has not appeared in the literature. See [66, 67] for complementary studies. This comparison is important for our purposes for two reasons. First, it will tell us up to what values of a we can trust our slow-rotation approximation. Second, it will tell us whether the errors due to this approximation are degenerate with modifications to the GR QNMs introduced by the CS coupling. We refer the interested reader to [68] for a similar analysis, but in a different context.

In the GR limit ($\alpha/M^2 = 0$) the perturbation equations [Eqs. (24), (29) and (30)] decouple and each of them reduces to equations of the form

$$f(r)^2 y''_{\ell m} + (2/r^2) f(r) y'_{\ell m} + [\omega^2 + v_{\text{eff}}(r, a)] y_{\ell m} = 0, \quad (51)$$

for the field variables and effective potential pairs $\{y_{\ell m}, v_{\text{eff}}\} = \{R_{\ell m}, V_{\text{eff}}^S\}$, $\{\Psi_{\ell m}^{\text{RW}}, V_{\text{eff}}^A\}$ and $\{\Psi_{\ell m}^{\text{ZM}}, V_{\text{eff}}^P\}$. For the gravitational perturbations these equations agree with those in [48], whereas for the scalar perturbation they agree with the small- a limit of that in [69]. Moreover, in this limit, the axial and the polar gravitational equation become isospectral.

As a benchmark for our numerical code, we calculated the gravitational and scalar QNMs of a Schwarzschild BH by taking the nonrotating limit ($a = 0$). We find excellent agreement with the well-known result summarized e.g. in [70, 71]. Next, we computed the QNMs associated with Eq. (51) and compared them against the QNMs obtained using the exact Kerr metric (i.e., with-

out performing any small- a expansion) by means of the continuous fraction method [65] and tabulated in [70, 71]

Figure 1 presents the results of this exercise, focusing on the fundamental mode with $\ell = m = 2$. The top panels show a comparison between the behavior of the real (left) and the imaginary (right) parts of the QNM frequencies for the slowly-rotating Kerr metric (solid) and the exact Kerr metric (dashed) as a function of the spin parameter a for both the scalar (green curves) and the gravitational modes (blue curves). The bottom panels show the relative percent error due to the slow rotation approximation, which we define via

$$\delta(\text{Im } \omega) = |1 - (\text{Im } \omega_{\text{SR}})/(\text{Im } \omega_{\text{K}})| \times 100, \quad (52a)$$

$$\delta(\text{Re } \omega) = |1 - (\text{Re } \omega_{\text{SR}})/(\text{Re } \omega_{\text{K}})| \times 100, \quad (52b)$$

where, ω_{K} are the QNM frequencies calculated from the exact Kerr metric [65, 70], whereas ω_{SR} are the QNM frequencies we calculated for the Kerr metric expanded to linear order in a/M .

We can extract several conclusions from this figure. First, the relative error introduced by the slow-rotation approximation is larger for the imaginary part of the frequency than for the real part. For instance, when $a/M = 0.1$, the relative error is approximately eight times larger on the imaginary part than on the real part for both gravitational and scalar-led modes. In addition, we also find that the absolute error $|\omega_{\text{SR}} - \omega_{\text{K}}|$ is of $\mathcal{O}(a^2)$ for both the real and imaginary parts of the QNM frequencies. This naturally follows from the fact that the

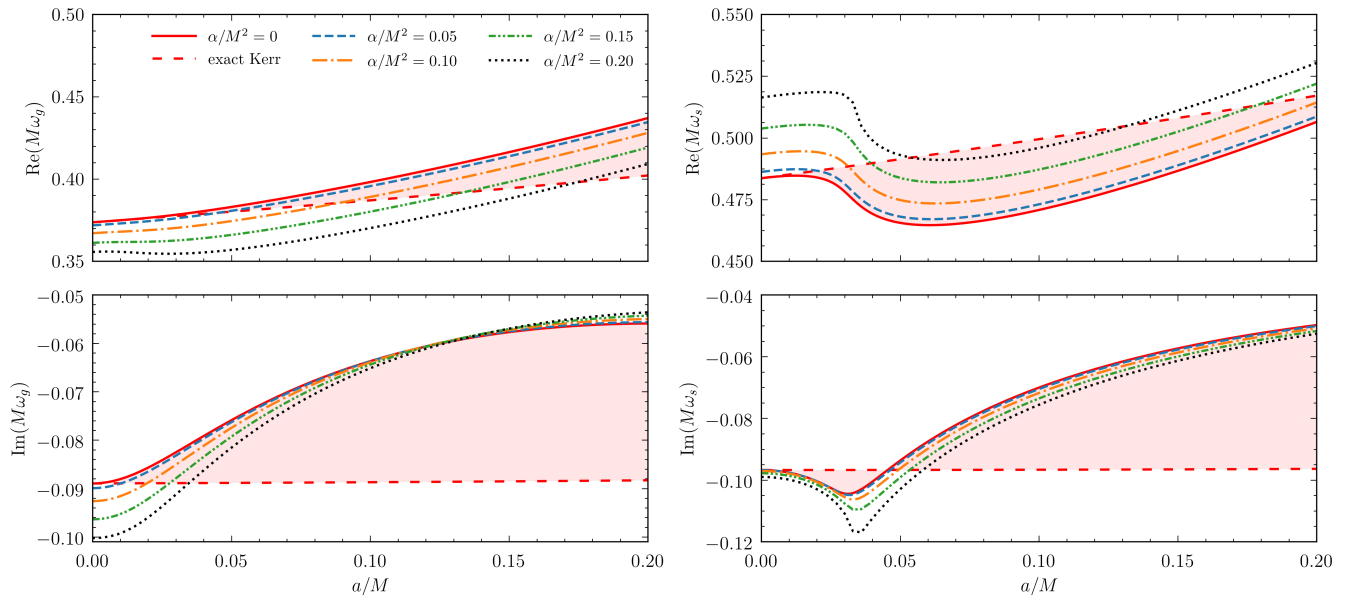


FIG. 2. Real (Imaginary) parts of the QNM frequency for the $n = 0$, $\ell = m = 2$ mode as a function of the spin parameter for slowly rotating BHs in dCS gravity with different CS couplings α/M^2 are shown in the top (bottom) panels. The individual QNM frequencies increase for both the gravitational- and scalar-led modes with spin parameter a/M whereas with increase in the CS coupling α/M^2 , the frequencies are found to decrease with the exception of $\text{Re}(\omega_s)$. The imaginary part is larger in magnitude with increasing α , suggesting that modes are more strongly damped in dCS gravity. The shaded red region shows the region of error introduced due to the slow rotation approximation and the results can be accurately estimated only for an average spin parameter value of up to 0.0375. The curve for the imaginary part of the exact Kerr metric QNM frequency varies slowly with a/M in the range shown. The legends shown apply to all panels.

error should indeed be of $\mathcal{O}(a^2)$, since we have evaluated ω to leading order in the spin parameter. The results show that the approximation introduces errors smaller than 10% when $a \lesssim 0.2$ for the real part and $a \lesssim 0.04$ for the imaginary part.

B. Slow rotation: dCS

In this section, we present how the QNM frequencies discussed in the previous section are modified by the presence of a nonzero CS coupling α/M^2 for the dominant $n = 0$, $\ell = m = 2$ mode. Tabulated values for the QNM frequencies for the fundamental mode with $\ell = 2$, $\ell = 3$, and $\ell = 4$ can be found in Appendix C.

As we discussed in Sec. III B 3, our calculations are valid to linear order in spin a/M , but second-order in the CS coupling α/M^2 . Moreover, we recall that the polar sector [governed by Eq. (35c)] does not depend on α/M^2 , and therefore, the polar gravitational QNMs are identical to those of GR. This extends to linear order in spin, the result found in [44]. Hence, we focus on the coupled system of Eqs. (35b) and (35a) alone and use the numerical procedure described in Sec. IV to calculate the QNMs. Our results for the fundamental modes are summarized in Figs. 2 and 3.

In Fig. 2 we show the dependence of the real and

imaginary values of gravitational- and scalar-led QNMs as function of spin parameter a/M for $\alpha/M^2 = \{0, 0.05, 0.10, 0.15, 0.20\}$. We also show QNMs of an exact Kerr BH for comparison, and shaded the region between those and our $\alpha = 0$ curve, to give a sense of the error due to the small-rotation approximation. In Fig. 3 we complement this analysis, by showing the dependence of the QNMs on α/M^2 for $a/M = \{0, 0.05, 0.10, 0.15, 0.20\}$. These two figures can be thought of as showing the slow-rotation corrections to the non-rotating BH QNM frequencies in dCS and the dCS modifications to the GR QNM frequencies respectively.

As seen from Fig. 2 and 3, the behavior of the real and the imaginary part for gravitational-led and the scalar-led sectors is distinct. Therefore, we present individual analysis for each.

- **Real gravitational-led QNM** (top left panel of Figs. 2 and 3). The $\text{Re}(\omega_g)$ increases with BH spin just as in GR (see Sec. V A) for constant CS coupling, while it remains mostly constant with a slight decrease as the CS coupling increases for constant spin.
- **Imaginary gravitational-led QNM** (bottom left panel of Figs. 2 and 3).
- **Real scalar-led QNM** (top right panel of Figs. 2 and 3). The $\text{Re}(\omega_s)$ initially shows a sinuous be-

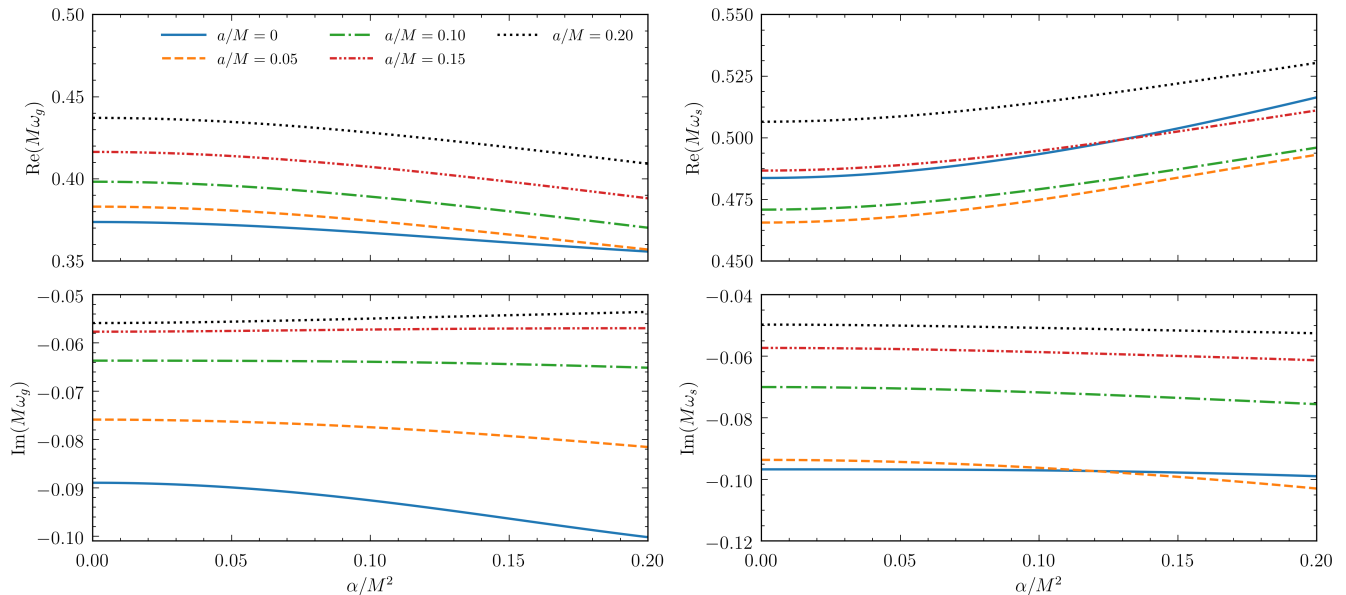


FIG. 3. Real (top) and imaginary (bottom) parts of ω as a function of the CS coupling for slowly rotating BH in dCS gravity for different values of the spin parameter a (given in legend) are shown above. The deviation introduced on a non-rotating BH in dCS gravity on introduction of spin can be seen by a strong shift in gradient when spin is introduced. $\text{Re}(\omega)$ decreases as the CS coupling gets stronger for gravitational-led mode whereas it increases for the scalar-led mode. The inferences made in Fig. 2 are supported here as well. For higher spins, it is seen that change in decay rate is small as the CS coupling is increased for $\text{Im}(\omega)$ for both sectors which can be an artifact of the slow rotation approximation as discussed previously. It is evident from the above panels that for a particular value of spin, there is a quadratic order dependence on CS coupling for the QNM frequencies supporting our estimate in Sec. III D. The legends shown apply to all panels.

havior as we increase spin and hold the CS coupling constant up to $a = 0.05$, and then it increases with BH spin. On the other hand, the $\text{Re}(\omega_s)$ increases monotonically with the CS coupling while keeping the BH spin constant.

- **Imaginary scalar-led QNM** (bottom right panel of Figs. 2 and 3). The $\text{Im}(\omega_s)$ initially decreases with BH spin for constant CS coupling, reaches a minimum and then increases again. On the other hand, the $\text{Im}(\omega_s)$ remains essentially constant as we change the CS coupling for constant spin.

The modes decay for all values of spin parameter and CS coupling within the limits of our approximation, giving strong evidence that slowly-rotating BHs in dCS gravity are linearly stable against gravito-scalar perturbations, extending the results of [44, 45] to small spins.

In interpreting the figures, especially Fig. 2, one must be careful to take into account the approximate nature of our results. The numerical calculations we have performed are only valid in the slow-rotation approximation (to leading order in the BH spin and second order in the CS coupling). This is why we included the red shaded regions in Fig. 2, which quantify approximately the error in the slow-rotation approximation (see also Sec. V A). When the spin is large enough that this error becomes comparable to the dCS correction, then our numerical results should not be trusted any longer. This occurs

roughly at $a/M \gtrsim 0.2$ for $\text{Re}(\omega_{g,s})$, $a/M \gtrsim 0.05$ for $\text{Im}(\omega_g)$ and $a/M \gtrsim 0.07$ for $\text{Im}(\omega_s)$. Therefore, to be conservative, henceforth we restrict our attention to the regime $a/M \leq 0.0375$, where the errors introduced by the slow-rotation approximation are very small and the dCS corrections we have calculated are meaningful.

Our results allow us to estimate the magnitude of the dCS deviations with respect to the GR QNMs. As an example, let us use saturate the constraint $\alpha^{1/2} \leq 8.5$ km (at 90% confidence) obtained in [41], and consider the smallest remnant BH mass observed so far ($M \approx 18 M_\odot$), the product of the event GW170608 [72]. Combining this value of M with the maximum value of α allowed from [41], we find $\alpha/M^2 \approx 0.1$. We can now use the data presented in Tables III and IV (see also Fig. 2) to find that the maximum deviations from the GR QNMs is about 2% (2%) for the oscillation frequency and about 9% (6%) for the decay rate for the fundamental dominant $\ell = m = 2$ gravitational- (scalar-) led modes and a BH with spin $a/M = 0.0375$ (i.e., at the upper limit of our slow-rotation approximation).

It is important to stress that a GW detector responds only to gravitational degrees of freedom that propagate from the source to the detector (taken to be at spatial infinity), which in dCS gravity was shown to consist of the standard “plus” and “cross” transverse-traceless polarization modes, just as in GR [55]. This means that in practice only deviations to the gravitational-led modes

can be used to test dCS gravity, since these are the modes that affect the detector. To constrain the modest deviations predicted here one would require a signal with very high signal-to-noise ratio, in addition to detection and characterization of at least two ringdown modes (to break degeneracies between mass, spin and dCS coupling). One should bear in mind however, that the small deviations found here may be a consequence of the small-spin approximation. For BHs in dCS gravity, the larger the spin, the larger the deformations away from the Kerr metric become, which in turn may reflect on larger deviations in the QNM frequencies. This implies that tests of dCS gravity through BH spectroscopy would benefit from an extension of our work to larger spin values.

C. Fitting formulas for the QNMs

We have calculated a large catalog of QNM frequencies from which we can construct ready-to-use fitting formulas for the fundamental mode with $\ell = m = 2$, as well as other multipoles. In general, we use the fitting functions

$$M\omega_g(M, a) = a_g + b_g (1 - a/M)^{c_g} , \quad (53a)$$

$$M\omega_s(M, a) = a_s + \sum_{j=1}^3 d_j (a/M)^j , \quad (53b)$$

where (a_g, b_g, c_g) and (a_s, b_s, c_s) are fitting coefficients for the gravitational-led and the scalar-led sectors of the QNMs, respectively. Table I presents the numerical values of these coefficients for the fundamental dominant $n = 0, \ell = 2, m = 2$ mode, and selected values of the CS coupling α/M^2 .

The fitting coefficients show a quadratic dependence on α , in agreement with the arguments presented in Sec. III D. Thus, we can fit (a_g, b_g, c_g) and (a_s, b_s, c_s) as functions of $\zeta = \alpha^2/(M^4\kappa)$, recasting the fitting functions (53) to their final form, given by

$$M\omega_g(M, a, \alpha) = f_1 + f_2\kappa\zeta + (f_3 + f_4\kappa\zeta) \times (1 - a/M)^{f_5 + f_6\kappa\zeta} , \quad (54a)$$

$$M\omega_s(M, a, \alpha) = g_1 + g_2\kappa\zeta + \sum_{j=1}^3 \left(g_3^{(j)} + g_4^{(j)}\kappa\zeta \right)^2 (a/M)^j , \quad (54b)$$

These fitting coefficients vary for different values of n , ℓ , and m . For the fundamental dominant mode $n = 0$ and $\ell = m = 2$, the fitting coefficients are presented in Table II.

The average (maximum) relative error of these fit to our data is 0.4% (0.6%) and 0.7% (0.9%) for the $\text{Re}(\omega_g)$ and $\text{Im}(\omega_g)$ respectively, and 0.4% (0.6%) and 1.6% (1.86%) for the $\text{Re}(\omega_s)$ and $\text{Im}(\omega_s)$ respectively in the regime $a/M \leq 0.0375$ and $\zeta \leq 0.04/\kappa M^4$. We can fit the QNM frequencies of other $\ell = 2$ multipoles and other higher multipoles with the same functional form as that

presented above. The numerical values of these fitting functions and their average errors are tabulated in Appendix C.

VI. DISCUSSION

We here investigated the quasinormal modes excited in dCS gravity by a perturbed and slowly-rotating black hole. We began by finding the perturbation equations that describe the evolution of scalar and metric perturbations. These triply-coupled set of differential equations generalize the slowly-rotating versions of the Regge-Wheeler and the Zerilli-Moncrief master equations to the case of dCS gravity. Using symmetry arguments, we showed that not all terms in these equations contribute to the QNM frequencies, thereby simplifying our set of equations into two sets: two coupled equations for the axial and the scalar sectors, and an homogeneous equation describing the polar sector; the later was found to be identical to the GR case for a slowly-rotating Kerr BH.

We then solved these equations to calculate the QNM frequencies for slowly rotating BHs in dCS gravity. We found that the dCS corrections to the QNM frequencies scale with the square of the CS coupling. We also found that, in general, slowly-rotating BHs in dCS gravity have a decay time that is mostly independent of the CS coupling, so these BHs return to their stationary configuration only slightly more rapidly than in GR. Nonetheless, their (real) frequency of oscillation is indeed affected, decreasing when the CS coupling increases for fixed spin. Finally, we constructed fitting functions for the real and imaginary parts of the gravitational and scalar QNM frequencies as a function of the BH mass, spin and CS coupling for the fundamental ($n = 0$) mode and all $\ell \leq 4$ harmonics.

The work presented here allows for many extensions along different directions. First, as it should be clear from our paper, we have here only considered perturbations to leading-order in the spin parameter. A natural extension of our work then would be to go to higher order in spin, which should produce more accurate results for BHs that are not as slowly spinning. Extending this calculation to second order, however, will be very difficult because of mode coupling between the odd- and even-parity sectors, just as in the GR case [57].

Another interesting extension of our work would be to develop a continued fraction approach for the calculation of the QNM frequencies in dCS gravity. Such a method was introduced by Leaver long ago, but its extension to modified gravity theories is not obvious [44]. One could therefore use dCS gravity as a toy problem to extend such methods, and then compare these continued fraction results to the numerical results found in this paper.

The results that we presented in this paper can also serve as a way to verify numerical simulation of BH binaries that result in slowly-rotating BH remnants, such as the head-on collisions performed in [73]. A natural

| α/M^2 | a_g^r | b_g^r | c_g^r | a_g^i | b_g^i | c_g^i |
|--------------|---------|---------|---------|---------|---------|---------|
| 0.00 | 1.1849 | -0.8130 | 0.2959 | 0.3212 | -0.4112 | 0.6545 |
| 0.05 | 1.1846 | -0.8147 | 0.2876 | 0.3052 | -0.3962 | 0.7062 |
| 0.10 | 1.1823 | -0.8175 | 0.2648 | 0.2874 | -0.3811 | 0.8085 |
| 0.15 | 1.1706 | -0.8122 | 0.2297 | 0.2976 | -0.3952 | 0.8746 |
| 0.20 | 1.1328 | -0.7806 | 0.1826 | 0.3281 | -0.4299 | 0.8885 |

| α | a_s^i | d_1^i | d_2^i | d_3^i | a_s^i | d_1^i | d_2^i | d_3^i |
|----------|---------|---------|---------|---------|----------|---------|---------|---------|
| 0.00 | 0.4856 | -0.1033 | -9.2285 | 89.779 | -0.09494 | -0.7367 | 20.275 | -104.25 |
| 0.05 | 0.4881 | -0.0723 | -10.029 | 94.521 | -0.09488 | -0.7494 | 20.220 | -102.88 |
| 0.10 | 0.4950 | -0.0032 | -12.166 | 107.49 | -0.09489 | -0.7902 | 20.26 | -100.28 |
| 0.15 | 0.5054 | 0.0935 | -15.289 | 127.69 | -0.09514 | -0.8788 | 21.141 | -101.53 |
| 0.20 | 0.5185 | 0.1758 | -19.848 | 162.64 | -0.09359 | -1.2206 | 27.165 | -131.13 |

TABLE I. Fitting coefficients for Eqs. (53a) (top) and Eq. (53b) (bottom) with fixed CS coupling parameter, to model both the real (superscript r) and imaginary (superscript i) QNM frequencies describing the gravitational (top) and scalar (bottom) sectors.

| | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | % error |
|-----------------------|--------|--------|---------|---------|---------|--------|---------|
| $\text{Re}(\omega_g)$ | 0.2203 | 0.1506 | -1.6129 | -0.0522 | -0.0387 | 9.9922 | 0.4 |
| $\text{Im}(\omega_g)$ | 0.3030 | 0.3347 | -0.3933 | -0.6296 | 0.7026 | 5.5940 | 0.7 |

| | g_1 | g_2 | $g_3^{(1)}$ | $g_4^{(1)}$ | $g_3^{(2)}$ | $g_4^{(2)}$ | $g_3^{(3)}$ | $g_4^{(3)}$ | % error |
|-----------------------|---------|--------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| $\text{Re}(\omega_s)$ | 0.4862 | 0.8206 | -0.0845 | 6.9267 | -9.3642 | -263.20 | 89.418 | 1800.5 | 0.4 |
| $\text{Im}(\omega_s)$ | -0.0951 | 0.0287 | -0.6995 | -11.712 | 19.336 | 165.06 | -98.517 | -633.21 | 1.6 |

TABLE II. Fitting coefficients for Eqs. (54a) (top) and (54b) (bottom) for the $n = 0, \ell = 2, m = 2$ mode, which allow us to approximate both the real and the imaginary parts of the QNM frequencies as a function of both the spin and the dCS coupling.

and important extension of our work, would be to extend the validity of our calculations to larger values of spin. Currently, the only way to find the QNM frequencies of not-slowly-spinning BHs in dCS is through numerical relativity simulations of BH mergers which are computationally expensive. Another interesting path for future research would then be to find a modified Teukolsky equation for dCS gravity. Such a task, however, may not be possible given that the BHs of dCS gravity are Petrov type I and not Petrov type D [50], as assumed in the work of Teukolsky [74, 75]

Another interesting calculation would be to map our theory-specific numerical results to the theory-agnostic QNM parametrization introduced in [66] (see also [76]). Our numerical results could also be used to quantify the error in theory-agnostic QNMs calculations due to the geometrical optics approximation, as done in [77–79].

Yet another possible avenue for future work is to apply

the tools developed here to study the oscillation spectra of rotating neutron stars in dCS gravity [39, 80]. Such an analysis could have applications to GW asteroseismology [81, 82].

Finally, these results technically allow for the construction of ringdown templates that could be used by the LIGO-Virgo-Kagra collaboration to place constraints on dCS gravity. This would of course only be possible for ringdown signals produced by slowly-rotating BH remnants, which in turn only occurs when the inspiraling binary components have the right spin magnitude and orientation prior to merger. Whether the remnant is spinning or not, however, cannot currently be determined accurately enough because the signals detected so far do not have sufficiently high signal-to-noise ratio [83], and thus the posteriors on the spin are very wide [84, 85]. Moreover, such ringdown tests would require the unambiguous detection of more than one QNM mode [86–88]

As the signal-to-noise ratio increases, it may be possible to carry out such a test, and in the meantime, it would be highly desirable to extend our results to more rapidly rotating BH backgrounds.

ACKNOWLEDGEMENTS

We thank Takahiro Tanaka and Helvi Witek for discussions. Some of our algebraic work used the package xAct [89] for Mathematica. P.K.W. and N.Y. are supported by NSF grants No. PHY-1759615, PHY-1949838 and NASA ATP Grant No. 17-ATP17-0225, No. NNX16AB98G and No. 80NSSC17M0041.

Appendix A: Spherical Harmonics - Orthogonality properties

In this appendix, we are providing some useful orthogonality relations for scalar, vector and tensor spherical harmonics. The scalar spherical harmonics satisfy a fundamental identity,

$$Y_{,\theta\theta}^{\ell m} + \cot\theta Y_{,\theta}^{\ell m} + \frac{1}{\sin^2\theta} Y_{,\phi\phi}^{\ell m} = -\ell(\ell+1)Y^{\ell m}. \quad (\text{A1})$$

They also satisfy the orthogonality relation given by

$$\langle Y^{\ell m}, Y^{\ell' m'} \rangle = \delta^{\ell\ell'} \delta^{mm'}. \quad (\text{A2})$$

Appendix B: Coefficients of the perturbation equations

In this Appendix, we list the explicit forms of the coefficients appearing in the perturbation equations presented in Sec. III B. These coefficients have prefactors of a combination of α , a and m which have already been shown in the expression for the perturbation equations. However, it is worth noting that there is no direct correspondence between these coefficients. The potentials are given by:

$$V_{\text{eff}}^S = \left(1 - \frac{2M}{r}\right) \left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} \right] + am\omega \frac{4M}{r^3}, \quad (\text{B1})$$

$$V_{\text{eff}}^A = \left(1 - \frac{2M}{r}\right) \left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3} + \frac{am}{\omega} \frac{24M(3r-7M)}{\ell(\ell+1)r^6} \right] + am\omega \frac{4M}{r^3}, \quad (\text{B2})$$

$$V_{\text{eff}}^P = \left(1 - \frac{2M}{r}\right) \left[\frac{2M}{r^3} + \frac{1}{3}(\ell-1)(\ell+2) \left(\frac{1}{r^2} + \frac{2(\ell-1)(\ell+2)(\ell^2+\ell+1)}{(6M+(\ell^2+\ell-2)r)^2} \right) \right. \\ \left. + \frac{4amM}{r^8\ell(\ell+1)[(\ell^2+\ell-2)r+6M]^4\omega} \sum_{i=0}^7 \xi_i(r, \ell, \omega) M^i r^{7-i} \right]. \quad (\text{B3})$$

where the functions ξ_i are the same as those found for slowly rotating Kerr BHs in GR [48]. Moreover, we also provide these functions in a Mathematica notebook that can be made available upon request. The other functions in

Vector spherical harmonics hold the following orthogonality relations,

$$\langle Y_a^{\ell m}, Y_a^{\ell' m'} \rangle = \langle S_a^{\ell m}, S_a^{\ell' m'} \rangle = (\ell-1)(\ell+2) + 2, \quad (\text{A3})$$

where the polar and axial vector harmonics have been defined as

$$Y_a^{\ell m} = (Y_{,\theta}^{\ell m}, Y_{,\phi}^{\ell m}), \\ S_a^{\ell m} = \left(-\frac{Y_{,\theta}^{\ell m}}{\sin\theta}, \sin\theta Y_{,\theta}^{\ell m} \right). \quad (\text{A4})$$

Finally, the tensor spherical harmonics are given by

$$\langle Z_{ab}^{\ell m}, Z_{ab}^{\ell' m'} \rangle = \langle S_{ab}^{\ell m}, S_{ab}^{\ell' m'} \rangle \\ = 2\ell(\ell-1)(\ell+1)(\ell+2)\delta^{\ell\ell'} \delta^{mm'}. \quad (\text{A5})$$

There are additional orthogonality relations which can be found in [48]. The additional relations that we have used in this work are

$$\cos\theta Y^{\ell m} = q_{\ell+1,m} Y^{\ell+1,m} + q_{\ell,m} Y^{\ell-1,m}, \\ \sin\theta Y_{,\theta}^{\ell m} = q_{\ell+1,m} \ell Y^{\ell+1,m} - q_{\ell,m} (\ell+1) Y^{\ell-1,m} \\ A_{\ell' m'} \langle Y^{\ell m}, \sin\theta Y_{,\theta}^{\ell' m'} \rangle = (\ell-1)q_{\ell m} A_{\ell-1,m} \\ - (\ell+2)q_{\ell+1,m} A_{\ell+1,m}, \quad (\text{A6})$$

where $q_{\ell m}$ is defined in Eq. (25). $A_{\ell m}$ is the operator defined to separate the angular dependence of the linearized field equations within the slow-rotation approximation.

the perturbed field equations that contribute to the QNM frequencies [See Eqs. (35)] are given by

$$g(r) = \frac{6i(\ell - 1)\ell(\ell + 1)(l + 2)M}{r^5\omega}, \quad (\text{B4a})$$

$$h(r) = -\frac{i(r^4\omega^2(12(2\ell(\ell + 1) - 1)M^2 + 15Mr + 5r^2) + 144M^3(2M - r))}{2Mr^9\omega^2}, \quad (\text{B4b})$$

$$j(r) = \frac{72iM^2(r - 2M)}{r^8\omega^2}, \quad (\text{B4c})$$

and

$$v(r) = -\frac{6iM\omega}{\kappa r^5}, \quad (\text{B5a})$$

$$n(r) = \frac{i(-4224M^4 + 3306M^3r + 48M^2r^2(r^2\omega^2 - 15) + 5Mr^3 + 15r^4)}{4\kappa l(l + 1)Mr^9}, \quad (\text{B5b})$$

$$p(r) = \frac{12iM(12M - 5r)(2M - r)}{\kappa l(\ell + 1)r^8}. \quad (\text{B5c})$$

All the other functions such as k_i, p_i, s_i and r_i ($i = 1, \dots, 4$) are rather lengthy and non-illuminating and hence not provided here. Instead these can be found in a Mathematica notebook which can be made available upon request.

Appendix C: QNM frequencies of different multipoles and fitting coefficients

In this appendix, we have tabulated the values of QNM frequencies for all multipoles of $\ell = 2$ and $\ell = 3$. $\ell = 4$ values have been compiled into a data file available upon request. In the following tables, we have shown the values for the real and imaginary parts of the QNM frequencies for both gravitational- and scalar-led modes.

We then proceed to present the numerical values for the fitting functions shown in Eqs. (54) for both the gravitational and scalar QNM frequencies. We have also calculated the average percent error in these fits and presented these in our tables below.

| a/M | α/M^2 | $m = 2$ | $m = 1$ | $m = 0$ | $m = -1$ | $m = -2$ |
|-------|--------------|--------------|--------------|--------------|--------------|--------------|
| | 0.0 | .3737, .0889 | .3737, .0889 | .3737, .0889 | .3737, .0889 | .3737, .0889 |
| 0.0 | 0.1 | .3671, .0926 | .3671, .0926 | .3671, .0926 | .3671, .0926 | .3671, .0926 |
| | 0.2 | .3557, .1002 | .3557, .1002 | .3557, .1002 | .3557, .1002 | .3557, .1002 |
| | 0.0 | .3749, .0881 | .3743, .0887 | .3737, .0889 | .3730, .0887 | .3723, .0881 |
| 0.01 | 0.1 | .3679, .0916 | .3676, .0923 | .3671, .0926 | .3664, .0923 | .3656, .0916 |
| | 0.2 | .3557, .0993 | .3559, .1000 | .3557, .1002 | .3549, .0999 | .3536, .0991 |
| | 0.0 | .3764, .0857 | .3749, .0881 | .3737, .0889 | .3723, .0881 | .3709, .0859 |
| 0.02 | 0.1 | .3689, .0888 | .3679, .0916 | .3671, .0926 | .3656, .0916 | .3637, .0890 |
| | 0.2 | .3548, .0959 | .3553, .0993 | .3557, .1002 | .3536, .0991 | .3501, .0961 |
| | 0.0 | .3783, .0825 | .3756, .0870 | .3737, .0889 | .3716, .0871 | .3693, .0829 |
| 0.03 | 0.1 | .3702, .0851 | .3684, .0904 | .3671, .0926 | .3647, .0904 | .3617, .0856 |
| | 0.2 | .3546, .0912 | .3553, .0979 | .3557, .1002 | .3519, .0978 | .3466, .0919 |
| | 0.0 | .3805, .0857 | .3764, .0857 | .3737, .0889 | .3708, .0859 | .3677, .0796 |
| 0.04 | 0.1 | .3721, .0812 | .3689, .0887 | .3671, .0926 | .3637, .0890 | .3597, .0819 |
| | 0.2 | .3554, .0862 | .3548, .0959 | .3557, .1002 | .3501, .0961 | .3434, .0874 |

TABLE III. QNM frequencies for axial gravitational-led sector with $n = 0$, $\ell = 2$ for slowly rotating BHs in dCS gravity. The format used is $M(\text{Re}(\omega), -\text{Im}(\omega))$. To save space, the leading zeros have been omitted.

| a/M | α/M^2 | $m = -2$ | $m = -1$ | $m = 0$ | $m = 1$ | $m = 2$ |
|-------|--------------|--------------|--------------|--------------|--------------|--------------|
| 0.0 | 0.0 | .4836, .0967 | .4836, .0967 | .4836, .0967 | .4836, .0967 | .4836, .0967 |
| | 0.1 | .4933, .0971 | .4933, .0971 | .4933, .0971 | .4933, .0971 | .4933, .0971 |
| | 0.2 | .5163, .0989 | .5163, .0989 | .5163, .0989 | .5163, .0989 | .5163, .0989 |
| 0.01 | 0.0 | .4817, .0976 | .4828, .0969 | .4836, .0967 | .4843, .0969 | .4846, .0967 |
| | 0.1 | .4914, .0979 | .4925, .0973 | .4933, .0971 | .4940, .0929 | .4944, .0976 |
| | 0.2 | .5147, .0999 | .5156, .0992 | .5163, .0989 | .5171, .0992 | .5177, .0998 |
| 0.02 | 0.0 | .4787, .1007 | .4817, .0976 | .4836, .0967 | .4846, .0976 | .4839, .1004 |
| | 0.1 | .4887, .1008 | .4914, .0979 | .4933, .0971 | .4944, .0979 | .4942, .1006 |
| | 0.2 | .5129, .1028 | .5147, .0998 | .5163, .0989 | .5177, .0998 | .5185, .1029 |
| 0.03 | 0.0 | .4719, .1085 | .4803, .0988 | .4836, .0967 | .4846, .0987 | .4782, .1044 |
| | 0.1 | .4839, .1082 | .4902, .0991 | .4933, .0971 | .4946, .0990 | .4897, .1056 |
| | 0.2 | .5121, .1101 | .5138, .1010 | .5163, .0989 | .5182, .1011 | .5174, .1108 |
| 0.04 | 0.0 | .4520, .1581 | .4787, .1007 | .4836, .0967 | .4839, .1004 | .4692, .1005 |
| | 0.1 | .4739, .1372 | .4887, .1008 | .4933, .0971 | .4942, .1006 | .4794, .1033 |
| | 0.2 | .5232, .1178 | .5129, .1029 | .5163, .0989 | .5185, .1029 | .4989, .1124 |

TABLE IV. Same as Table III but for the scalar-led sector with $n = 0$, $\ell = 2$.

| a/M | α/M^2 | $m = -3$ | $m = -2$ | $m = -1$ | $m = 0$ | $m = 1$ | $m = 2$ | $m = 3$ |
|-------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 0.0 | 0.0 | .5994, .0927 | .5994, .0927 | .5994, .0927 | .5994, .0927 | .5994, .0927 | .5994, .0927 | .5994, .0927 |
| | 0.1 | .5791, .0964 | .5791, .0938 | .5791, .0964 | .5791, .0964 | .5791, .0964 | .5791, .0964 | .5791, .0964 |
| | 0.2 | .5499, .1019 | .5499, .1019 | .5499, .1019 | .5499, .1019 | .5499, .1019 | .5499, .1019 | .5499, .1019 |
| 0.01 | 0.0 | .5970, .0911 | .5979, .0919 | .5987, .0925 | .5994, .0927 | .6001, .0925 | .6007, .0919 | .6013, .0911 |
| | 0.1 | .5764, .0946 | .5775, .0955 | .5784, .0962 | .5791, .0964 | .5796, .0962 | .5799, .0955 | .5803, .0945 |
| | 0.2 | .5464, .1000 | .5479, .1011 | .5491, .1017 | .5499, .1019 | .5502, .1017 | .5501, .1011 | .5498, .0999 |
| 0.02 | 0.0 | .5941, .0874 | .5961, .0900 | .5979, .0919 | .5994, .0927 | .6007, .0919 | .6019, .0899 | .6035, .0871 |
| | 0.1 | .5728, .0902 | .5752, .0933 | .5775, .0955 | .5791, .0964 | .5799, .0955 | .5806, .0931 | .5815, .0897 |
| | 0.2 | .5411, .0951 | .5446, .0986 | .5479, .1010 | .5499, .1019 | .5501, .1010 | .5494, .0983 | .5491, .0943 |
| 0.03 | 0.0 | .5912, .0829 | .5942, .0874 | .5971, .0911 | .5994, .0927 | .6013, .0911 | .6034, .0871 | .6064, .0825 |
| | 0.1 | .5693, .0852 | .5728, .0902 | .5764, .0945 | .5791, .0964 | .5803, .0944 | .5816, .0897 | .5839, .0844 |
| | 0.2 | .5363, .0891 | .5411, .0951 | .5464, .1000 | .5499, .1019 | .5497, .0999 | .5491, .0943 | .5503, .0877 |
| 0.04 | 0.0 | .5883, .0787 | .5922, .0844 | .5961, .0900 | .5994, .0927 | .6019, .0899 | .6053, .0841 | .6099, .0782 |
| | 0.1 | .5661, .0804 | .5705, .0869 | .5753, .0933 | .5791, .0964 | .5806, .0931 | .5830, .0861 | .5871, .0795 |
| | 0.2 | .5323, .0836 | .5378, .0911 | .5446, .0987 | .5499, .1019 | .5494, .0983 | .5497, .0899 | .5527, .0819 |

TABLE V. Same as Table III but for the axial gravitational-led sector with $n = 0$, $\ell = 3$.

| a/M | α/M^2 | $m = -3$ | $m = -2$ | $m = -1$ | $m = 0$ | $m = 1$ | $m = 2$ | $m = 3$ |
|-------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 0.0 | 0.0 | .6753, .0965 | .6753, .0965 | .6753, .0965 | .6753, .0965 | .6753, .0965 | .6753, .0965 | .6753, .0965 |
| | 0.1 | .7018, .0972 | .7018, .0972 | .7018, .0972 | .7018, .0972 | .7018, .0972 | .7018, .0972 | .7018, .0972 |
| | 0.2 | .7500, .1009 | .7500, .1009 | .7500, .1009 | .7500, .1009 | .7500, .1009 | .7500, .1009 | .7500, .1009 |
| 0.01 | 0.0 | .6714, .0974 | .6731, .0969 | .6744, .0966 | .6753, .0965 | .6759, .0966 | .6761, .0968 | .6758, .0971 |
| | 0.1 | .6981, .0983 | .6997, .0977 | .7008, .0973 | .7018, .0972 | .7025, .0973 | .7028, .0977 | .7027, .0982 |
| | 0.2 | .7475, .1027 | .7483, .1017 | .7492, .1011 | .7500, .1009 | .7516, .1017 | .7516, .1017 | .7522, .1028 |
| 0.02 | 0.0 | .6630, .0989 | .6693, .0979 | .6731, .0969 | .6753, .0965 | .6761, .0968 | .6750, .0974 | .6720, .0969 |
| | 0.1 | .6904, .1013 | .6962, .0991 | .6997, .0977 | .7018, .0972 | .7028, .0976 | .7022, .0988 | .6994, .0997 |
| | 0.2 | .7462, .1103 | .7467, .1043 | .7483, .1017 | .7500, .1009 | .7516, .1017 | .7528, .1045 | .7534, .1112 |
| 0.03 | 0.0 | .6515, .0961 | .6630, .0989 | .6715, .0973 | .6753, .0965 | .6758, .0972 | .6720, .0969 | .6679, .0919 |
| | 0.1 | .6771, .1005 | .6905, .1014 | .6981, .0983 | .7018, .0972 | .7027, .0982 | .6993, .0997 | .6936, .0959 |
| | 0.2 | .7099, .1187 | .7463, .1104 | .7475, .1027 | .7500, .1009 | .7522, .1028 | .7534, .1112 | .7305, .1126 |
| 0.04 | 0.0 | .6428, .0903 | .6551, .0976 | .6693, .0979 | .6753, .0965 | .6750, .0974 | .6689, .0939 | .6671, .0855 |
| | 0.1 | .6669, .0945 | .6816, .1018 | .6962, .0991 | .7018, .0972 | .7022, .0988 | .6952, .0978 | .6916, .0893 |
| | 0.2 | .6988, .1064 | .7156, .1252 | .7467, .1043 | .7500, .1009 | .7528, .1045 | .7347, .1180 | .7261, .1009 |

TABLE VI. Same as Table III but for the scalar-led sector with $n = 0$, $\ell = 3$.

| m | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | % error |
|-----|--------|---------|---------|----------|-------------|-------------|---------|
| 2 | 0.2203 | 0.1506 | -1.6129 | -0.0522 | -0.0387 | 9.9922 | 0.4 |
| 1 | 0.8345 | -2.8014 | -0.4627 | 2.3553 | 0.1969 | -4.4606 | 0.2 |
| 0 | 0.6863 | -0.2178 | -0.3137 | -0.2304 | ≈ 0 | ≈ 0 | 0.02 |
| -1 | 0.7097 | 3.4830 | -0.3370 | -3.9195 | -0.2124 | -2.2966 | 0.3 |
| -2 | 0.7138 | 13.952 | -0.3409 | -14.4265 | -0.4487 | 5.1861 | 0.3 |

| m | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | % error |
|-----|--------|---------|---------|---------|-------------|-------------|---------|
| 2 | 0.3030 | 0.3347 | -0.3933 | -0.6296 | 0.7026 | 5.5940 | 0.7 |
| 1 | 0.5822 | 2.6540 | -0.6729 | -2.9587 | 0.2034 | 1.1492 | 0.5 |
| 0 | 0.5091 | -0.4353 | -0.5984 | 0.1529 | ≈ 0 | ≈ 0 | 0.04 |
| -1 | 0.6169 | 2.2458 | -0.7078 | -2.5435 | 0.1862 | 0.9471 | 0.6 |
| -2 | 0.4171 | -3.4719 | -0.5079 | 3.1802 | 0.5551 | 9.7328 | 0.6 |

TABLE VII. Same as Table II but for the gravitational-led sector and the $n = 0$, $\ell = 2$ mode for real (top) and imaginary (bottom) parts.

| m | g_1 | g_2 | $g_3^{(1)}$ | $g_4^{(1)}$ | $g_3^{(2)}$ | $g_4^{(2)}$ | $g_3^{(3)}$ | $g_4^{(3)}$ | % error |
|-----|--------|--------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| 2 | 0.4862 | 0.8206 | -0.0845 | 6.9267 | -9.3642 | -263.20 | 89.418 | 1800.5 | 0.4 |
| 1 | 0.4835 | 0.8015 | 0.3211 | 1.7772 | -9.4529 | 30.909 | 44.952 | -645.52 | 0.2 |
| 0 | 0.4844 | 0.8170 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | 0.04 |
| -1 | 0.4828 | 0.8024 | 0.1929 | 4.4256 | -7.6129 | -78.963 | 17.130 | 198.32 | 0.2 |
| -2 | 0.4863 | 0.8048 | -0.1903 | 7.1129 | -17.734 | -247.98 | 136.48 | 1599.8 | 0.5 |

| m | g_1 | g_2 | $g_3^{(1)}$ | $g_4^{(1)}$ | $g_3^{(2)}$ | $g_4^{(2)}$ | $g_3^{(3)}$ | $g_4^{(3)}$ | % error |
|-----|---------|---------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| 2 | -0.0951 | 0.0287 | -0.6995 | -11.712 | 19.336 | 165.06 | -98.517 | -633.21 | 1.6 |
| 1 | -0.0965 | -0.0866 | 0.0426 | 8.2154 | -6.0266 | -280.41 | 59.516 | 1833.71 | 0.6 |
| 0 | -0.0966 | -0.0563 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | 0.12 |
| -1 | -0.0977 | -0.0882 | 0.2878 | 9.9186 | -14.027 | -320.07 | 110.37 | 2028.0 | 1.1 |
| -2 | -0.0921 | -0.0710 | -1.0426 | -0.5249 | 23.221 | -75.864 | -112.92 | 754.87 | 3.5 |

TABLE VIII. Same as Table II but for the scalar-led sector and the $n = 0$, $\ell = 2$ mode.

| m | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | % error |
|-----|--------|---------|---------|----------|---------|---------|---------|
| 3 | 1.5249 | -1.1738 | -0.9328 | -0.1019 | 0.3993 | -2.3318 | 0.7 |
| 2 | 1.3566 | -2.5989 | -0.7629 | 1.3451 | 0.2718 | -2.5372 | 0.7 |
| 1 | 0.9705 | -4.9430 | -0.3756 | 3.7294 | 0.2014 | -4.5916 | 0.7 |
| 0 | 0.6223 | -1.6403 | -0.0270 | 0.4245 | -0.0123 | 0.2384 | 0.6 |
| -1 | 0.3803 | -3.5492 | 0.2152 | 2.3453 | 0.4325 | 1.7505 | 0.6 |
| -2 | 0.2432 | 0.1693 | 0.3525 | -1.3988 | 0.5483 | 9.4537 | 0.6 |
| -3 | 0.7138 | 13.952 | -0.3409 | -14.4265 | -0.4487 | 5.1861 | 0.3 |

| m | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | % error |
|-----|--------|---------|---------|---------|-------------|-------------|---------|
| 3 | 0.2432 | 2.2068 | -0.3361 | -2.4234 | 0.09602 | 0.5338 | 1.5 |
| 2 | 0.3956 | -2.6348 | -0.4902 | 2.3966 | 0.5228 | 8.2091 | 0.6 |
| 1 | 0.5619 | 2.1895 | -0.6567 | -2.4346 | 0.1878 | 0.9899 | 0.6 |
| 0 | 0.5025 | -0.3303 | -0.5959 | 0.1026 | ≈ 0 | ≈ 0 | 0.7 |
| -1 | 0.5543 | 1.6632 | -0.6489 | -1.9032 | 0.1804 | 0.7948 | 0.6 |
| -2 | 0.5511 | -1.0319 | -0.6458 | 0.7961 | 0.3890 | 3.3348 | 0.6 |
| -3 | 0.4171 | -3.4719 | -0.5079 | 3.1802 | 0.5551 | 9.7328 | 0.6 |

TABLE IX. Same as Table II but for the axial gravitational-led sector and the $n = 0$, $\ell = 3$ mode.

| m | g_1 | g_2 | $g_3^{(1)}$ | $g_4^{(1)}$ | $g_3^{(2)}$ | $g_4^{(2)}$ | $g_3^{(3)}$ | $g_4^{(3)}$ | % error |
|-----|--------|--------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| 3 | 0.6810 | 1.9293 | -0.4136 | -6.7595 | 5.7508 | -117.81 | -4.1052 | 1342.8 | 1.1 |
| 2 | 0.6803 | 1.8542 | -0.0832 | 9.7784 | -2.9483 | -455.07 | 36.346 | 3157.8 | 1.0 |
| 1 | 0.6792 | 1.8169 | 0.1311 | 4.5758 | -4.6926 | -10.148 | 27.212 | -746.03 | 1.2 |
| 0 | 0.6794 | 1.8455 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | 0.6 |
| -1 | 0.6794 | 1.7904 | -0.0676 | 10.367 | -2.5888 | -192.47 | 7.0098 | 427.89 | 0.8 |
| -2 | 0.6781 | 2.1636 | -0.2646 | 10.817 | -8.0549 | -454.10 | 59.607 | 3026.3 | 1.2 |
| -3 | 0.4862 | 0.8206 | -0.0845 | 6.9267 | -9.3642 | -263.20 | 89.418 | 1800.5 | 1.1 |

| m | g_1 | g_2 | $g_3^{(1)}$ | $g_4^{(1)}$ | $g_3^{(2)}$ | $g_4^{(2)}$ | $g_3^{(3)}$ | $g_4^{(3)}$ | % error |
|-----|---------|---------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| 3 | -0.0969 | -0.0685 | -0.1104 | -22.949 | 12.603 | 470.43 | -81.121 | -2514.6 | 1.2 |
| 2 | -0.0959 | -0.0365 | -0.2456 | -18.061 | 10.076 | 246.64 | -52.797 | -833.94 | 0.7 |
| 1 | -0.0959 | -0.1657 | -0.0963 | 14.916 | 1.7665 | -565.79 | -1.4423 | 3877.6 | 0.6 |
| 0 | -0.0963 | -0.1136 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | 0.5 |
| -1 | -0.0959 | -0.1859 | -0.0569 | 19.115 | 0.0976 | -671.08 | 6.8510 | 4477.2 | 0.7 |
| -2 | -0.0955 | -0.0333 | -0.3242 | -15.748 | 9.3870 | 174.35 | -43.658 | -390.57 | 1.0 |
| -3 | -0.0921 | -0.0710 | -1.0426 | -0.5249 | 23.221 | -75.864 | -112.92 | 754.87 | 3.5 |

TABLE X. Same as Table II but for the scalar-led sector and the $n = 0$, $\ell = 3$ mode.

| m | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | % error |
|-----|--------|---------|---------|---------|--------|---------|---------|
| 4 | 1.4148 | -2.2063 | -0.6157 | 0.2412 | 0.6881 | -5.7401 | 1.0 |
| 3 | 1.3409 | -4.8111 | -0.5413 | 2.8538 | 0.5322 | -4.0873 | 0.9 |
| 2 | 1.2119 | -11.366 | -0.4079 | 8.7904 | 0.3892 | 7.8567 | 0.9 |
| 1 | 1.0583 | -4.3496 | -0.2542 | 1.7776 | 0.2768 | -3.5423 | 0.9 |
| 0 | 0.8803 | -0.6367 | -0.0801 | -1.3170 | 0.0029 | -0.0519 | 1.0 |
| -1 | 0.5955 | -2.8643 | 0.2086 | 0.2942 | 0.3538 | 0.9393 | 1.0 |
| -2 | 0.4814 | -3.5410 | 0.3228 | 0.9744 | 0.4931 | 1.8531 | 0.9 |
| -3 | 0.3863 | -3.4811 | 0.4179 | 0.9127 | 0.5969 | 2.3454 | 1.2 |
| -4 | 0.2986 | -3.1758 | 0.5058 | 0.6033 | 0.6828 | 2.5986 | 1.1 |

| m | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | % error |
|-----|--------|---------|---------|---------|-------------|-------------|---------|
| 4 | 0.6589 | 1.6619 | -0.7550 | -1.8657 | 0.6338 | 3.5775 | 0.7 |
| 3 | 0.5833 | 1.0937 | -0.6795 | -1.3019 | 0.5223 | 3.6721 | 0.8 |
| 2 | 0.4918 | 1.9436 | -0.5875 | -2.1497 | 0.3491 | 2.0597 | 0.8 |
| 1 | 0.4567 | -0.0252 | -0.5516 | -0.2324 | 0.1167 | 1.8467 | 0.3 |
| 0 | 0.4363 | 1.9714 | -0.5312 | -2.1693 | ≈ 0 | ≈ 0 | 0.6 |
| -1 | 0.4563 | -0.0649 | -0.5512 | -0.1916 | 0.1131 | 1.7912 | 0.2 |
| -2 | 0.4975 | 0.5585 | -0.5929 | -0.8199 | 0.3355 | 4.0209 | 0.5 |
| -3 | 0.5837 | 0.9136 | -0.6796 | -1.1736 | 0.5095 | 4.8685 | 0.6 |
| -4 | 0.6507 | 2.8767 | -0.7467 | -3.1286 | 0.6398 | 3.5226 | 0.6 |

TABLE XI. Same as Table II but for the axial gravitational-led sector and the $n = 0$, $\ell = 4$ mode.

| m | g_1 | g_2 | $g_3^{(1)}$ | $g_4^{(1)}$ | $g_3^{(2)}$ | $g_4^{(2)}$ | $g_3^{(3)}$ | $g_4^{(3)}$ | % error |
|-----|--------|--------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| 4 | 0.8766 | 2.8304 | -0.0512 | 48.851 | -2.3909 | -4532.0 | 135.72 | 72648.7 | 0.8 |
| 3 | 0.8766 | 2.7934 | 0.1344 | 30.684 | -10.106 | -2661.6 | 168.70 | 38390.9 | 0.9 |
| 2 | 0.8466 | 2.8004 | 0.0270 | 24.357 | 1.0001 | -1609.1 | -49.353 | 19316.2 | 0.9 |
| 1 | 0.8725 | 3.4584 | 0.0775 | 0.1535 | -1.9811 | -57.895 | 5.7601 | -875.64 | 0.7 |
| 0 | 0.8765 | 2.8162 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | 0.7 |
| -1 | 0.8725 | 3.4583 | -0.0751 | -0.4851 | -1.8263 | 48.603 | 1.3107 | -402.55 | 0.6 |
| -2 | 0.8724 | 3.4658 | -0.1056 | -7.6281 | -11.936 | 888.34 | 127.43 | -20825.8 | 0.8 |
| -3 | 0.8725 | 3.4222 | -0.2294 | 23.032 | -20.461 | -1404.8 | 290.40 | 10821.0 | 0.7 |
| -4 | 0.8727 | 3.4217 | -0.4490 | 38.818 | -22.413 | -3473.7 | 363.66 | 51302.7 | 0.9 |

| m | g_1 | g_2 | $g_3^{(1)}$ | $g_4^{(1)}$ | $g_3^{(2)}$ | $g_4^{(2)}$ | $g_3^{(3)}$ | $g_4^{(3)}$ | % error |
|-----|---------|---------|-------------|-------------|-------------|-------------|-------------|-------------|---------|
| 4 | -0.0959 | -0.1349 | -0.2066 | -20.430 | 34.189 | -631.29 | -452.52 | 25358.0 | 1.2 |
| 3 | -0.0965 | -0.0762 | -0.0866 | -5.6460 | 25.433 | -2889.76 | -395.26 | 70752.7 | 3.1 |
| 2 | -0.0959 | -0.1978 | -0.5895 | 94.929 | 50.994 | -8517.6 | -791.63 | 142661.0 | 3.3 |
| 1 | -0.0962 | -0.1475 | 0.0016 | -0.1050 | -0.0674 | -60.637 | 13.666 | -563.08 | 0.5 |
| 0 | -0.0962 | -0.1454 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | ≈ 0 | 0.1 |
| -1 | -0.0962 | -0.1474 | 0.0023 | -0.0904 | -0.0975 | -52.641 | 7.6109 | -574.55 | 0.1 |
| -2 | -0.0955 | -0.0333 | -0.3242 | -15.748 | 9.3870 | 174.35 | -43.658 | -390.57 | 1.0 |
| -3 | -0.0961 | -0.1566 | -0.1221 | 15.374 | 12.253 | -2584.3 | -87.559 | 48317.0 | 1.5 |
| -4 | -0.0960 | -0.1045 | -0.1786 | -19.049 | 23.801 | -798.65 | -260.28 | 26756.7 | 1.4 |

TABLE XII. Same as Table II but for the scalar-led sector and the $n = 0$, $\ell = 4$ mode.

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