On resolutions of ideals associated to subspace arrangements and the algebraic matroid of the determinantal variety

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Introduction

Motivated by providing a rigorous solution to the motion segmentation problem in computer vision, where one seeks algorithms for automatically determining the different motions in a video sequence, René Vidal in his 2003 PhD thesis [**Vid03**] formally introduced and studied the problem of clustering data points sampled from a subspace arrangement. His seminal work together with Yi Ma and Shankar Sastry led to the formation of a new subfield of machine learning known as *Generalized Principal Component Analysis (GPCA)* [**VMS16**] or *Subspace Clustering* [**Vid11**]. Here the attribute *Generalized* indicates a generalization of the classical Principal Component Analysis, a cornerstone of statistics, dating back to Legendre and Gauss, which involves modeling data with a single linear subspace. Since then, GPCA has evolved into an interdisciplinary research area in the intersection of computer science and applied mathematics. Vidal's original solution [**VMS05, VMS03**], somewhat recently revisited by the author and Vidal [**TV17, TV18**], was based on algebraic geometry and in particular on the structure of the vanishing ideal of a subspace arrangement. An instrumental property in the theory of that method is that for an arrangement of n linear subspaces $V_1, \dots, V_n$ of a finite-dimensional vector space over an infinite field k, V_i being the vanishing locus of an ideal I_i generated by linear forms, the vanishing ideal $\bigcap_{i\in[n]} I_i$ of $\bigcup_{i\in[n]} V_i$ coincides at degree n with the product ideal $J = \prod_{i\in[n]} I_i$, as long as the V_i's are transversal; here $[n] = \{1, \dots, n\}$. The method went on by computing a k-basis of J_n, the homogeneous component of J at degree n, and extracting k-bases for the I_i's by polynomial differentiation. Remarkably, the above property was proved via entirely independent motivations by Conca and Herzog, also in 2003, in a paper [**CH03**] that was to become landmark itself in commutative algebra. In that paper the authors were concerned with well-behaved classes of ideals in a polynomial ring, in the sense that the Castelnuovo-Mumford regularity of their product can be bounded from above by the sum of the individual regularities. A primary decomposition of J was described and was used in proving that the regularity of J is always equal to its minimum possible value n for any I_i, that is J always has a linear resolution. Under the transversality assumption, the equality of homogeneous components $\left(\bigcap_{i\in[n]} I_i\right)_n = J_n$ followed as a corollary of the regularity result.

It is not an exaggeration to say that the birthmark of the present thesis is the (*non-reduced*) intersection point of the two works [**Vid03, CH03**] described above. The thesis itself discusses aspects of commutative algebra, algebraic geometry and combinatorics as they relate to subspace arrangements, matrices of bounded rank and Grassmannians, these being prevalent objects in machine learning and signal processing theories and applications.

The first two chapters are concerned with combinatorial and homological properties of the ideal J mentioned above, which, via GPCA, is related to numerous applications such as motion segmentation and face clustering in computer vision, document clustering in machine learning, gene clustering in bioinformatics and identification of hybrid linear systems in control theory, as well as multiple-view geometry in computer vision. For all these, see [**VMS16**] and references therein. The mathematical contributions are as follows.

In Chapter 1, which is the joint work [**CT19**] of Conca with the author, the main highlight is an explicit description of the minimal graded free resolution of the ideal J. The resolution is supported on a polymatroid obtained from the underlying representable polymatroid by means of the so-called Dilworth truncation. Formulas for the projective dimension and Betti numbers as well as a characterization of the associated primes are given in terms of the polymatroid. Along the way it is shown that J has linear quotients. In fact, this is done for a large class of ideals J_P, where P is a certain poset ideal associated to the underlying subspace arrangement.

For Chapter 2 we let $S = k[x_1, \dots, x_r]$ be a polynomial ring over an infinite field k, and I a homogeneous ideal of S generated in degree d. The existence of an integer n_I is proved, such that for a set $\mathcal{X}$ of at least n_I general points in $\mathbb{P}^{r-1}$, the ideal $I \prod_{p \in \mathcal{X}} I(p)$ has a linear resolution, where $I(p)$ is the vanishing ideal of the point $p \in \mathbb{P}^{r-1}$. It is also proved that $n_I \leq r(\operatorname{reg}(I) - d)$, where $\operatorname{reg}(I)$ is the Castelnuovo-Mumford regularity of I. This can be seen as a generalization of the well-known fact that $I\mathfrak{m}^s$ always has a linear resolution for $s \geq \operatorname{reg}(I) - d$ where $\mathfrak{m} = (x_1, \dots, x_r)$. These results were published in [**Tsa20c**].

Chapters 3 and 4 are of a somewhat different flavor and deal with two inverse problems that occur in machine learning and signal processing. The first one, with which Chapter 3 is concerned, is the well-known low-rank matrix completion [**CR09, CT10**], where the objective is to reconstruct a low-rank matrix from a subset of its entries. Applications are abundant, ranging from recommendation systems in machine learning to quantum tomography in physics. The specific aspect that we consider here is the characterization of the minimal observation patterns, for which there are only finitely many completions of a partially observed generic matrix of the appropriate rank. The mathematical equivalent is to characterize the base sets of the corresponding algebraic matroid, and it is this latter avenue that we follow in this thesis. For an exposition of this work suitable for a general audience, the reader is referred to [**Tsa20b**]. The second problem, by which Chapter 4 is inspired, is more recent and is known under the name *unlabeled sensing* [**UHV18**] or *linear regression without correspondences* [**HSS17**]. In its simplest form, this problem amounts to solving a linear system of equations for which the right-hand-side vector is given only up to a permutation. This formulation is relevant in applications where input/output data are available but the correspondences between inputs and outputs are unknown. Applications include record linkage in machine learning, multi-target tracking and image registration in computer vision, neuron matching in neuroscience, acoustical imaging in signal processing and many others. Chapter 4 itself is concerned with a generalization of this problem, termed *homomorphic sensing*, where one allows for arbitrary linear transformations instead of permutations, and develops the theory of unique recovery. An exposition suitable for a general audience is [**TP19**], while an algebraic geometry method for linear regression without correspondences is developed in [**TPC⁺20**]. The mathematical contributions of these two chapters are as follows.

In Chapter 3 a class of base sets is presented for the algebraic matroid of the determinantal variety of $m \times n$ matrices of rank at most r over an infinite field, whose known characterizations are available only for $r = 1, 2, m-1$. It is conjectured that these bases completely characterize the matroid and the conjecture is reduced to a purely combinatorial statement, which is verified for the case $r = m-2$. Towards that end, matrix completion is interpreted from a point of view of linear sections on the Grassmannian via Plücker coordinates. A critical ingredient is a class of local coordinates on the Grassmannian induced by so-called *supports of linkage matching fields*, described by Sturmfels & Zelevinsky [**SZ93**]. As a byproduct, a conjecture of Rong, Wang & Xu [**RWX19**] is proved.

Chapter 4 introduces *homomorphic sensing*, an intersection problem in linear algebra whose solution quickly becomes commutative algebraic. Specifically, with k an infinite field, $\mathcal{T}$ a finite set of linear endomorphisms of k^m, and V a linear subspace, one is interested in conditions for which $\tau_1(v_1) = \tau_2(v_2)$ for $v_1, v_2 \in V$ and $\tau_1, \tau_2 \in \mathcal{T}$ necessarily implies $v_1 = v_2$. The main result is a dimension bound on an open locus of a determinantal scheme, under which, a general subspace V of dimension $n \leq m/2$ satisfies this property. By specializing to permutations composed by coordinate projections and computing the dimension of the corresponding open subscheme, we obtain the *unlabeled sensing* theorem of [**UHV18**].

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CHAPTER 1

Resolution of ideals associated to subspace arrangements

A subspace arrangement $\mathcal{V}$ is a finite collection $V_1, \dots, V_n$ of vector subspaces of a given vector space V over a field K. Several geometric objects can be associated to $\mathcal{V}$ and their investigation has attracted the attention of many researchers, see for example Björner [**Bjö94**], De Concini and Procesi [**DCP95**] and Björner, Peeva and Sidman [**BPS05**]. Subspace arrangements interplay as well with multigraded commutative algebra and geometric computer vision, see [**AST13**], [**Con07**], [**CS10**], and [**CDNG18**], where a subspace arrangement $\mathcal{V}$ gives rise to a multigraded ideal, called the multiview ideal.

In this chapter we consider the product J of the ideals generated by the V_i's in the polynomial ring $S = \mathrm{Sym}_K(V)$. In [**CH03**] a primary decomposition of J is presented. It is indeed a "combinatorial" decomposition since the ideals involved are powers of ideals generated by sums of the V_i's. From the primary decomposition one reads immediately that J is saturated from degree n. This is the key ingredient of the proof in [**CH03**] asserting the minimal free resolution of J is linear, i.e. the Castelnuovo-Mumford regularity of J is exactly n. In [**Der07**] Derksen proved that the Hilbert function of J is a combinatorial invariant, that is, it just depends on the rank function:

$$\mathrm{rk}_{\mathcal{V}} : 2^{[n]} \to \mathbb{N}, \qquad A \subseteq [n], \qquad \mathrm{rk}_{\mathcal{V}}(A) = \dim_K \sum_{i \in A} V_i.$$

As observed by Derksen, since the resolution is linear, this implies that the algebraic Betti numbers of J are themselves combinatorial invariants. Attached to the rank function we have a discrete polymatroid

$$\mathrm{P}(\mathcal{V}) = \{x \in \mathbb{N}^n : \sum_{i \in A} x_i \le \mathrm{rk}_{\mathcal{V}}(A) \text{ for all } A \subseteq [n]\}$$

that plays a role in the sequel.

The goal of the chapter is to describe the minimal free resolution of J and give an explicit formula for the Betti numbers and for the projective dimension. Indeed we prove that the minimal free resolution of J can be realized as a subcomplex of the tensor product of the Koszul complexes associated with generic generators of the V_i. Such a resolution is supported on the subpolymatroid

$$\mathrm{P}(\mathcal{V})^* = \left\{x \in \mathbb{N}^n : \sum_{i \in A} x_i \le \mathrm{rk}_{\mathcal{V}}(A) - 1 \text{ for all } \varnothing \ne A \subseteq [n]\right\}$$

of $\mathrm{P}(\mathcal{V})$ whose rank function $\mathrm{rk}^*_{\mathcal{V}}$ is obtained by the so-called Dilworth truncation

$$\mathrm{rk}^*_{\mathcal{V}}(A) = \min\left\{\sum_{i=1}^{p} \mathrm{rk}_{\mathcal{V}}(A_i) - p : A_1, \dots, A_p \text{ is a partition of } A\right\}.$$

It turns out that the (algebraic) Betti numbers $\beta_i(J)$ of J are given by:

$$\sum_{i\geq 0} \beta_i(J) z^i = \sum_{i\geq 0} \gamma_i(\mathcal{V})(1+z)^i$$

where $\gamma_i(\mathcal{V}) = \#\{x \in \mathrm{P}(\mathcal{V})^* : |x| = i\}$ and the projective dimension of J is given by the formula:

$$\operatorname{projdim} J = \operatorname{rk}^*_{\mathcal{V}}([n]) = \min\left\{\sum_{i=1}^{p} \operatorname{rk}_{\mathcal{V}}(A_i) - p : A_1, \dots, A_p \text{ is a partition of } [n]\right\}.$$

The formulas for the Betti numbers and the projective dimension hold over any base field K while the description of the minimal free resolution depends on the choice of generic bases (in a precise sense, see 1.1) of the V_i's whose existence is guaranteed only over an infinite base field.

Our results apply indeed to an entire family of ideals associated with the subspace arrangement that makes possible inductive arguments. As a by-product we prove that the ideal J has linear quotients.

A. Notation and basic facts

Let K be an infinite field and V a K-vector space of dimension d. Let S be the symmetric algebra of V, i.e. a polynomial ring over K of dimension d. Let $\mathcal{V} = V_1, \dots, V_n$ be a collection of non-zero K-subspaces of V. Let $d_i = \dim_K V_i$. Such a collection $\mathcal{V}$ is called a subspace arrangement of dimension $(d_1, \dots, d_n)$. For $i \in [n]$ let $\{f_{ij} : j \in [d_i]\}$ be an ordered K-basis of V_i. The arrangement of vectors

$$\{f_{ij} : i \in [n] \text{ and } j \in [d_i]\}$$

is called a collection of bases of $\mathcal{V}$. Here and in the following for $u \in \mathbb{N}$ we denote by $[u]$ the set $\{1, \dots, u\}$. As usual for $i \in [n]$ we will denote by $e_i \in \mathbb{N}^n$ the vector with zeros everywhere except a 1 at position i and for $a \in \mathbb{N}^n$ we set $|a| = a_1 + \cdots + a_n$.

For every $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ with $a_i \leq d_i$ we define a K-subspace of V by

$$W_a = \langle f_{ij} : i \in [n] \text{ and } j \in [a_i]\rangle,$$

which clearly depends on the subspace arrangement but also on the collection of bases chosen.

Assumption 1.1. *Given $\mathcal{V} = V_1, \dots, V_n$ we assume that the collection of bases $\{f_{ij}\}$ is general in the sense that for all $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ with $a_i \leq d_i$ the dimension of W_a is the largest possible.*

A collection of bases satisfying 1.1 always exists (here we use the fact that the base field is infinite). In other words, the subspace arrangement can be special with inclusions and even equalities allowed, but for each V_i we pick a general basis.

For later purposes we define two discrete polymatroids associated to the subspace arrangement $\mathcal{V} = V_1, \dots, V_n$. For general facts and terminology on polymatroids we refer the reader to the classical paper by Edmonds [**Edm70**] and to monographs [**Fuj05**] and [**Mur98**] for modern accounts. The subspace arrangement $\mathcal{V}$ gives rise to the rank function $\operatorname{rk}_{\mathcal{V}} : 2^{[n]} \to \mathbb{N}$ defined by

$$\operatorname{rk}_{\mathcal{V}}(A) = \dim_K \sum_{i \in A} V_i$$

and the associated discrete polymatroid:

$$\mathrm{P}(\mathcal{V}) = \left\{ x \in \mathbb{N}^n : \sum_{i\in A} x_i \le \mathrm{rk}_{\mathcal{V}}(A) \text{ for all } A \subseteq [n] \right\}.$$

Let us denote by $\mathrm{rk}_{\mathcal{V}} -1 : 2^{[n]} \to \mathbb{N}$ the function that takes a non-empty $A \subset [n]$ to $\mathrm{rk}_{\mathcal{V}}(A)-1$ and takes the value 0 at $\varnothing$. This function defines the discrete polytope

$$\mathrm{P}(\mathcal{V})^* = \left\{ x \in \mathbb{N}^n : \sum_{i\in A} x_i \le (\mathrm{rk}_{\mathcal{V}} -1)(A) \text{ for all } A \subseteq [n] \right\}.$$

Proposition 1.2. *The set* $\mathrm{P}(\mathcal{V})^*$ *is a discrete polymatroid with associated rank function the so-called Dilworth truncation* $\mathrm{rk}^*_{\mathcal{V}} : 2^{[n]} \to \mathbb{N}$ *of* $\mathrm{rk}_{\mathcal{V}} -1$ *defined as*

$$\mathrm{rk}^*_{\mathcal{V}}(A) = \min \left\{ \sum_{i=1}^{p} \mathrm{rk}_{\mathcal{V}}(A_i) - p : A_1, \dots, A_p \text{ is a partition of } A \right\}$$

if $A \ne \varnothing$ *and* $\mathrm{rk}^*_{\mathcal{V}}(\varnothing) = 0$.

Proof. Properties (a),(b),(c),(d) in [**Edm70**, p.12] characterize the rank functions of discrete polymatroids, see also [**HH02**]. For $\mathrm{rk}^*_{\mathcal{V}}$ only submodularity (c) is not obvious. In the language of Fujishige's monograph [**Fuj05**], the function $\mathrm{rk}_{\mathcal{V}} -1$ is intersecting-submodular, meaning that for any $A, B \subset [n]$ such that $A \cap B \ne \varnothing$, we have $(\mathrm{rk}_{\mathcal{V}} -1)(A) + (\mathrm{rk}_{\mathcal{V}} -1)(B) \ge (\mathrm{rk}_{\mathcal{V}} -1)(A\cap B) + (\mathrm{rk}_{\mathcal{V}} -1)(A \cup B)$; this follows from the submodularity of $\mathrm{rk}_{\mathcal{V}}$. According to Theorem 2.5 in [**Fuj05**], there is a unique submodular function inducing $\mathrm{P}(\mathcal{V})^*$. By Theorem 2.6 in [**Fuj05**] that function is the Dilworth truncation of $\mathrm{rk}_{\mathcal{V}} -1$, which is exactly $\mathrm{rk}^*_{\mathcal{V}}$. □

In other words, $\mathrm{rk}^*_{\mathcal{V}}$ is the unique rank function such that:

$$\mathrm{P}(\mathcal{V})^* = \left\{ x \in \mathbb{N}^n : \sum_{i\in A} x_i \le \mathrm{rk}^*_{\mathcal{V}}(A) \text{ for all } A \subseteq [n] \right\}.$$

In particular,

$$\max\{|x| : x \in \mathrm{P}(\mathcal{V})^*\} = \mathrm{rk}^*_{\mathcal{V}}([n]).$$

We collect now some simple facts about the vector spaces W_a associated to a given subspace arrangement $\mathcal{V}$ and their relations with the two polymatroids just introduced.

We have:

Lemma 1.3. *Assume that there is a nontrivial linear dependence relation among the generators of* W_a *involving one of the generators of* V_q. *Then* $V_q \subseteq W_{a-e_q}$.

Proof. For the given q let p be the largest index such that f_{qp} appears in a nontrivial linear dependence relation among the generators of W_a. This implies that $f_{qp} \in W_b$ with $b = (b_1, \dots, b_n)$ and $b_k = a_k$ for $k \ne q$ and $b_q = p-1$. But because of the choice of the f_{ij}'s this implies that $V_q \subseteq W_b \subseteq W_{a-e_q}$. □

Lemma 1.4. *Set* $T = \{i \in [n] : V_i \subseteq W_a\}$ *and* $b \in \mathbb{N}^n$ *with* $b_i = 0$ *if* $i \in T$ *and* $b_i = a_i$ *otherwise. Furthermore set* $c = a - b$. *Then*

(1) $W_a = W_b + \sum_{i\in T} V_i$,
(2) $\dim_K W_b = |b|$, *i.e. the elements* f_{ij} *with* $i \notin T$ *and* $j \le a_i$ *are linearly independent,*
(3) $W_b \cap (\sum_{i\in T} V_i) = 0$,

(4) $W_c = \sum_{i\in T} V_i$,
(5) $\dim_K W_a = \sum_{i\notin T} a_i + \mathrm{rk}_{\mathcal{V}}(T)$.

Proof. (1) is obvious. (2) follows from Lemma 1.3 and the definition of T. For (3) we set $u \in \mathbb{N}^n$ with $u_i = d_i$ if $i \in T$ and $u_i = a_i$ otherwise. Then we observe that, by (1) we have $W_a = W_u$. If, by contradiction, $W_b \cap (\sum_{i\in T} V_i)$ is non-zero then there is a non-trivial linear relation among the generators of W_u involving an element f_{ij} with $i \notin T$. Applying Lemma 1.3 we have that $V_i \subseteq W_u = W_a$, a contradiction with the definition of T. Finally (4) and (5) follow from (1)-(3). □

Proposition 1.5. *We have:*

$$\dim_K W_a = \min\left\{\sum_{i\notin T} a_i + \mathrm{rk}_{\mathcal{V}}(T) : T \subseteq [n]\right\}$$

Proof. For every $T \subseteq [n]$ we have

$$W_a \subseteq \langle f_{ij} : i \notin T \text{ and } j \leq a_i\rangle + \sum_{i\in T} V_i$$

and therefore

$$\dim_K W_a \leq \sum_{i\notin T} a_i + \mathrm{rk}_{\mathcal{V}}(T).$$

It remains to prove that at least for one subset T we have equality and this follows from Lemma 1.4 part (5). □

Corollary 1.6. *The following conditions are equivalent:*

(1) $\dim_K W_a = |a|$, *i.e. the* f_{ij}*'s with* $j \leq a_i$ *are linearly independent.*
(2) $\sum_{i\in T} a_i \leq \mathrm{rk}_{\mathcal{V}}(T)$ *for every* $T \subseteq [n]$, *i.e.* $a \in \mathrm{P}(\mathcal{V})$.

Proof. The implication (1) $\Longrightarrow$ (2) is obvious. The implication (2) $\Longrightarrow$ (1) follows from Proposition 1.5. □

Proposition 1.7. *The following conditions are equivalent:*

(1) *for every* i *one has* $V_i \nsubseteq W_a$.
(2) *for every* $\emptyset \neq T \subseteq [n]$ *one has* $\sum_{i\in T} a_i \leq \mathrm{rk}_{\mathcal{V}}(T) - 1$, *i.e.* $a \in \mathrm{P}(\mathcal{V})^*$.

Proof. (1) $\Longrightarrow$ (2): By virtue of Lemma 1.3 we know that the f_{ij}'s with $j \leq a_i$ are linearly independent. Hence for every non-empty $T \subseteq [n]$ we have

$$\sum_{i\in T} a_i = \dim_K\langle f_{ij} : i \in T \text{ and } j \leq a_i\rangle \leq \mathrm{rk}_{\mathcal{V}}(T)$$

and, if equality holds, we have $\sum_{i\in T} V_i \subseteq W_a$ contradicting the assumption.

(2) $\Longrightarrow$ (1). The assumption and Corollary 1.6 imply that the f_{ij}'s's with $j \leq a_i$ are linearly independent. By contradiction suppose that $T = \{i \in [n] : V_i \subseteq W_a\}$ is not empty. By Lemma 1.4 (5) we have

$$\dim_K W_a = \sum_{i\notin T} a_i + \mathrm{rk}_{\mathcal{V}}(T)$$

and by hypothesis $\mathrm{rk}_{\mathcal{V}}(T) > \sum_{i\in T} a_i$. It follows that $\dim_K W_a > |a|$ which is clearly a contradiction. □

B. Ideals associated to subspace arrangements and poset ideals

Given a subspace arrangement $\mathcal{V} = V_1, \ldots, V_n$ of dimension $(d_1, \ldots, d_n)$ we consider the ideal I_i of S generated by V_i and set

$$J = J_1 J_2 \cdots J_n.$$

We fix a collection of bases $f = \{f_{ij} : i \in [n] \text{ and } j \in [d_i]\}$ of $\mathcal{V}$ satisfying Assumption 1.1. On $\mathbb{N}^n$ we consider the standard poset structure defined as $a \geq b$ if $a_i \geq b_i$ for every $i \in [n]$. Indeed $(\mathbb{N}^n, \leq)$ is a distributive lattice with

$$a \wedge b = (\min(a_1, b_1), \ldots, \min(a_n, b_n))$$

and

$$a \vee b = (\max(a_1, b_1), \ldots, \max(a_n, b_n)).$$

Consider the hyper-rectangle $D = [d_1] \times \cdots \times [d_n] \subset \mathbb{N}^n$ with the induced poset structure. A poset ideal of D is a subset $P \subseteq D$ such that if $a, b \in D$ and $a \leq b \in P$ implies $a \in P$.

For every $a \in D$ we set $f_a = \prod_{i=1}^n f_{ia_i}$ and observe that $J = (f_a : a \in D)$. Furthermore for $a \in \mathbb{N}^n$ with $a_i \leq d_i$ we denote by I_a the ideal of S generated by the vector space $W_a = \langle f_{ij} : i \in [n] \text{ and } j \leq a_i \rangle$. For every poset ideal P of D we define an ideal of the polynomial ring S as follows:

$$J_P = (f_a : a \in P).$$

Clearly J_P depends on $\mathcal{V}$ but also on the collection of bases f that we consider. In particular $J = J_D$ and $J_\varnothing = \{0\}$. Let a be a maximal element of a non-empty poset ideal P. Then $Q = P \setminus \{a\}$ is itself a poset ideal. Furthermore set $b = a - (1, 1, \ldots, 1)$. With this notation our first goal is to prove:

Theorem 1.8.

(1) *J_P has a linear resolution.*
(2) *If $f_a \notin I_b$ then $J_Q : (f_a) = I_b$ and if $f_a \in I_b$ then $f_a \in J_Q$ i.e. $J_Q : (f_a) = S$.*

Proof. We prove the assertions by induction on the cardinality of P. Both assertions are obvious when P has only one element. Note that (2) actually implies (1) because we have either $J_Q = J_P$ and we conclude by induction or we have the short exact sequence

$$0 \to S/I_b(-n) \to S/J_Q \to S/J_P \to 0$$

and again we conclude by induction. So it remains to prove (2). Set $A = \{u \in D : u < a\}$. By construction $A \subseteq Q$ is a poset ideal and

$$I_b f_a \subseteq J_A \subseteq J_Q \subseteq I_b.$$

Hence

$$I_b \subseteq J_Q : f_a \subseteq I_b : f_a.$$

Since I_b is prime we have that $I_b = J_Q : f_a$ provided $f_a \notin I_b$.

It remains to prove that if $f_a \in I_b$ then actually $f_a \in J_Q$. Since I_b is prime we have that $f_{ia_i} \in I_b$ for at least one $i \in [n]$ and this implies, by the choice of the f_{ij}'s, that $V_i \subseteq W_b$. Therefore the set $T = \{i \in [n] : V_i \subseteq W_b\}$ is not empty. Up to a permutation of the coordinates we may assume that $T = \{1, \ldots, m\}$ for some $m > 0$. Set $a' = (a_1, \ldots, a_m)$, $A' = \{u' \in \mathbb{N}^m : u' < a'\}$ and $b' = (b_1, \ldots, b_m)$. We have $I_{b'} \subseteq J_{A'} : f_{a'}$ by construction and $W_{b'} = \sum_{i \in [m]} V_i$ by Lemma 1.4 (4), i.e. $I_{b'}$

is the maximal homogeneous ideal of the sub-polynomial ring S' of S generated by $\sum_{i\in[m]} V_i$. Since the generators of $J_{A'}$ and $f_{a'}$ already belong to S', we have that $f_{a'}$ is in the saturation of $J_{A'}$ in S'. Note that A' is a poset ideal of $D' = [d_1]\times\cdots\times[d_m]$ and $|A'| \le |A| < |P|$. Hence, by induction, $J_{A'}$ has a linear resolution and therefore it is saturated from degree m and on. It follows that $f_{a'} \in J_{A'}$ and then

$$f_a = f_{a'} \prod_{i=m+1}^{n} f_{ia_i} \in J_{A'}\left(\prod_{i=m+1}^{n} f_{ia_i}\right) \subseteq J_A \subseteq J_Q$$

as desired. □

Theorem 1.8 has some important corollaries. We set

$$D_{\mathcal{V}} = (1,\dots,1) + \mathrm{P}(\mathcal{V})^* = \left\{a \in D : \sum_{i\in T} a_i - |T| \le \mathrm{rk}_{\mathcal{V}}(T) - 1 \text{ for every } \varnothing \ne T \subseteq [n]\right\}.$$

Corollary 1.9. *Let P be a poset ideal of D. Set $P' = P \cap D_{\mathcal{V}}$. We have $J_P = J_{P'}$. In particular, $J = J_{D_{\mathcal{V}}}$.*

Proof. Using the notations of Theorem 1.8 we have seen that $f_a \in J_Q$ iff $f_a \in I_b$. The latter condition holds iff $V_i \subseteq I_b$ for some i and this is equivalent, in view of Proposition 1.7, to the the fact that $b \notin \mathrm{P}(\mathcal{V})^*$. In other words, if $a \in P \setminus D_{\mathcal{V}}$ then $f_a \in J_Q$, i.e. $J_P = J_Q$. Iterating the argument one obtains $J_P = J_{P'}$. □

In view of Corollary 1.9 when studying the ideal J_P we may assume $P \subseteq D_{\mathcal{V}}$.

Corollary 1.10. *Let $P \subseteq D_{\mathcal{V}}$ be a poset ideal. We have:*

(1) *J_P has linear quotients. More precisely, any total order on P that refines the partial order $\le$ gives rise to a total order on the generators of J_P that have linear quotients.*
(2) *We have:*
$$\sum_{j\ge 0} \beta_i(J_P) z^i = \sum_{a\in P} (1+z)^{|a|-n}.$$

Proof. (1) follows immediately from Theorem 1.8 part (2) while (2) follows from the short exact sequence used in the proof of Theorem 1.8. □

Let us single out the special case

Corollary 1.11. (1) *J is minimally generated by f_a with $a \in D_{\mathcal{V}}$.*
(2) *J has linear quotients. Indeed ordering the generators f_a with $a \in D_{\mathcal{V}}$ according to a linear extension of the partial order $\le$ gives linear quotients.*
(3) *The Betti numbers of J are given by the formula:*
$$\sum_{i\ge 0} \beta_i(J) z^i = \sum_{a\in D_{\mathcal{V}}} (1+z)^{|a|-n} = \sum_{i\ge 0} \gamma_i(\mathcal{V})(1+z)^i$$
where $\gamma_i(\mathcal{V}) = \#\{x \in \mathrm{P}(\mathcal{V})^* : |x| = i\}$.
(4) *The projective dimension* $\operatorname{projdim} J$ *of J is the rank of* $\mathrm{P}(\mathcal{V})^*$*, i.e.*
$$\operatorname{projdim} J = \min\left\{\sum_{i=1}^{p} \mathrm{rk}_{\mathcal{V}}(A_i) - p : A_1,\dots,A_p \text{ is a partition of } [n]\right\}.$$

We can go one step further and characterize the Betti numbers of J^ν for $\nu \in \mathbb{N}$ in terms of $P(\mathcal{V})^*$:

PROPOSITION 1.12. *For* $\nu \in \mathbb{N}$ *we have* $\operatorname{projdim} J^\nu = \operatorname{projdim} J$. *More precisely,* $\beta_i(J^\nu)$ *is the degree* $\operatorname{rank}^*_{\mathcal{V}}([n])$ *polynomial*

$$\beta_i(J^\nu) = \sum_{j \geq i} \binom{j}{i} \sum_{x \in P(\mathcal{V})^* : |x| = j} \binom{x_1 + \nu - 1}{\nu - 1} \cdots \binom{x_n + \nu - 1}{\nu - 1}$$

PROOF. J^ν is a product of linear ideals associated to the subspace arrangement $\mathcal{V}^\nu = (V_{ij} = V_i : i \in [n], j \in [\nu])$. Moreover, $P(\mathcal{V}^\nu)^*$ consists of those $(x_{ij} : i \in [n], j \in [\nu])$ for which $(\sum_{j \in [\nu]} x_{1j}, \dots, \sum_{j \in [\nu]} x_{nj}) \in P(\mathcal{V})^*$. Each $x = (x_i : i \in [n]) \in P(\mathcal{V})^*$ induces $\binom{x_1+\nu-1}{\nu-1} \cdots \binom{x_n+\nu-1}{\nu-1}$ elements of $P(\mathcal{V}^\nu)^*$ and each element of $P(\mathcal{V}^\nu)^*$ is associated with a unique element of $P(\mathcal{V})^*$. Then the claim follows from Corollary 1.11 (3). □

In [**Der07**] Derksen described a combinatorial procedure for the recursive computation of the Betti numbers of J in terms of the Betti numbers of the $J_T = \prod_{i \in T} I_i$, for all $T \subseteq [n]$. His proof made use of a theorem in Sidman's PhD thesis [**Sid02b**] concerning the vanishing of the homologies of a complex that involves all such products. Translating Derksen's formula into the context of the polymatroid $P(\mathcal{V})^*$ yields a recursive formula for the $\gamma_i(\mathcal{V}_T)$'s; here $\mathcal{V}_T$ denotes the subspace arrangement that involves only the subspaces indexed by T:

PROPOSITION 1.13. *For* $T \subseteq [n]$ *we have for every* $\ell = 0, \dots, \operatorname{rk}_{\mathcal{V}}(T) - 1$

$$\sum_{S \subseteq T} \sum_{j = 0, \dots, \#S} (-1)^{\#S - j} \binom{\#S}{j} \gamma_{\ell - j}(V_S) = 0$$

We close with a conjecture supported by numerical computations:

CONJECTURE 1.14. *J has linear quotients for any ordering of the f_a's.*

C. Irredundant primary decomposition and stability of associated primes

We keep the notation of the previous section. As noted earlier, the key ingredient in proving $\operatorname{reg}(J) = n$ was a description given in [**CH03**] of a (possibly redundant) primary decomposition of J, i.e.

$$J = \bigcap_{\emptyset \neq A \subseteq [n]} I_A^{\#A}$$

where for $A \subseteq [n]$ we have set $I_A = \sum_{i \in A} I_i$. Here one notes that I_A is an ideal generated by linear forms and hence prime with primary powers. The first reason why the decomposition can be redundant is that different components might have the same radical. We consider the set of the so-called flats of the polymatroid $P(\mathcal{V})$

$$F(\mathcal{V}) = \{B \subseteq [n] : \operatorname{rk}_{\mathcal{V}}(B) < \operatorname{rk}_{\mathcal{V}}(A) \text{ for all } B \subsetneq A \subseteq [n]\}$$

and observe that if $A \subseteq [n]$ and B is its closure, i.e. $B = \{i : \operatorname{rk}_{\mathcal{V}}(A) = \operatorname{rk}_{\mathcal{V}}(A \cup \{i\})\} \in F(\mathcal{V})$ then $I_A^{\#A} \supseteq I_B^{\#B}$. Hence

$$J = \bigcap_{B \in F(\mathcal{V})} I_B^{\#B}$$

is still a primary decomposition and now the radicals of the components are distinct. To get an irredundant primary decomposition it is now enough to identify for which $B \in F(\mathcal{V})$ the prime ideal I_B is associated to J.

Proposition 1.15. *For $B \in F(\mathcal{V})$ we have that the prime ideal I_B is associated to J if and only if* $\mathrm{rk}^*_{\mathcal{V}}(B) = \mathrm{rk}_{\mathcal{V}}(B) - 1$.

Proof. Set $P = I_B$. Since $B \in F(\mathcal{V})$, we have that $JS_P = \prod_{i\in B} I_i S_P$. We have that P is associated to J if and only if P is associated to $\prod_{i\in B} I_i$. So we may assume right away that $B = [n]$ and $I_{[n]}$ is the graded maximal ideal of S. By part (4) of Corollary 1.11 we have $\operatorname{projdim} J = \mathrm{rk}^*_{\mathcal{V}}([n])$ and by the Auslander-Buchsbaum formula $\operatorname{projdim} J = \mathrm{rk}_{\mathcal{V}}([n]) - 1$ if and only if $I_{[n]} \in \operatorname{Ass}(S/J)$. Hence $I_{[n]} \in \operatorname{Ass}(S/J)$ if and only if $\mathrm{rk}^*_{\mathcal{V}}([n]) = \mathrm{rk}_{\mathcal{V}}([n]) - 1$. □

Summing up we have:

Theorem 1.16. *An irredundant primary decomposition of J is given by*

$$J = \bigcap_B I_B^{\#B}$$

where B varies in the set $\{B \in F(\mathcal{V}) : \mathrm{rk}^*_{\mathcal{V}}(B) = \mathrm{rk}_{\mathcal{V}}(B) - 1\}$. In particular,

$$\operatorname{Ass}(S/J) = \{I_B : B \in F(\mathcal{V}) \text{ and } \mathrm{rk}^*_{\mathcal{V}}(B) = \mathrm{rk}_{\mathcal{V}}(B) - 1\}.$$

Proof. To obtain an irredundant primary decomposition of J it is enough to remove from the possibly redundant primary decomposition $J = \bigcap_{B\in F(\mathcal{V})} I_B^{\#B}$ the components not corresponding to associated primes. Hence by 1.15 we get the irredundant primary decomposition described in the statement. The assertion about the associated primes in then an immediate consequence. □

Corollary 1.17. *Suppose that the subspace arrangement $\mathcal{V} = V_1, \dots, V_n$ of V is linearly general, i.e. $\dim \sum_{i\in A} V_i = \min\{\sum_{i\in A} d_i, d\}$ for all $A \subseteq [n]$ where $d_i = \dim V_i$ and $d = \dim V$. Assume $n > 1$ and $d_i < d$ for all i and set $I_i = (V_i)$. We have $\Pi_{i=1}^n I_i = \cap_{i=1}^n I_i$ if and only if $d_1 + d_2 + \cdots + d_n < d + n - 1$.*

Proof. If $d_1 + d_2 + \cdots + d_n \le d$ then the assertion is obvious. So we may assume $d_1 + d_2 + \cdots + d_n > d$. In particular, $I_{[n]}$ is the maximal ideal $\mathfrak{m}$ of S and $\mathrm{rk}_{\mathcal{V}}([n]) = d$. It has been already observed in [**CH03**] that for a linearly general subspace arrangement a primary decomposition of the product ideal J is given by $J = \cap_{i=1}^n I_i \cap \mathfrak{m}^n$. Therefore we have that $J = \cap_{i=1}^n I_i$ if and only if $\mathfrak{m}$ is not associated to J. In view of the characterization given in 1.16, the latter is equivalent to $\mathrm{rk}^*_{\mathcal{V}}([n]) < \mathrm{rk}_{\mathcal{V}}([n]) - 1 = d - 1$, that is, $\sum_{i=1}^p \mathrm{rk}_{\mathcal{V}}(A_i) - p < d - 1$ for some partition $A_1, \dots, A_p$ of $[n]$. Summing up, we have to prove that the following conditions are equivalent:

(1) $d_1 + d_2 + \cdots + d_n < d + n - 1$
(2) $\sum_{i=1}^p \mathrm{rk}_{\mathcal{V}}(A_i) - p < d - 1$ for some partition $A_1, \dots, A_p$ of $[n]$.

That (1) implies (2) is clear, just take $p = n$ and $A_i = \{i\}$. Vice versa, let $A_1, \dots, A_p$ be a partition of $[n]$ such that $\sum_{i=1}^p \mathrm{rk}_{\mathcal{V}}(A_i) - p < d - 1$. If $\sum_{j\in A_v} d_j \ge d$ for some v one has $\mathrm{rk}_{\mathcal{V}}(A_v) = d$, contradicting the assumption. Hence $\sum_{j\in A_i} d_j < d$ for all i. By assumption this implies that $\mathrm{rk}_{\mathcal{V}}(A_i) = \sum_{j\in A_i} d_j$ for all i. It follows that $d_1 + d_2 + \cdots + d_n = \sum_{i=1}^p \mathrm{rk}_{\mathcal{V}}(A_i) < d - 1 + p \le d - 1 + n$ as desired. □

Now we turn our attention to the properties of the powers J^u of the ideal J with $u > 0$. Clearly J^u is associated to the subspace arrangement $\mathcal{V}^u = \{V_{ij} : (i,j) \in [n] \times [u]\}$ with $V_{ij} = V_i$ for all j. The polymatroids and rank functions associated to $\mathcal{V}^u$ are very tightly related to those of $\mathcal{V}$ as we now explain. Since $\mathcal{V}^u$

is indexed on $[n]\times[u]$ the domain of the associated rank function $\mathrm{rk}_{\mathcal{V}^u}$ is $2^{[n]\times[u]}$. Let $\pi:[n]\times[u]\to[n]$ be the projection on the first coordinate. We have:

LEMMA 1.18. *For every subset $A\subseteq[n]\times[u]$ we have*

$$\mathrm{rk}_{\mathcal{V}^u}(A)=\mathrm{rk}_{\mathcal{V}}(\pi(A))=\mathrm{rk}_{\mathcal{V}^u}(\pi^{-1}\pi(A))$$

and

$$\mathrm{rk}^*_{\mathcal{V}^u}(A)=\mathrm{rk}^*_{\mathcal{V}}(\pi(A))=\mathrm{rk}^*_{\mathcal{V}^u}(\pi^{-1}\pi(A)).$$

PROOF. For the first assertion one observes that

$$\mathrm{rk}_{\mathcal{V}^u}(A)=\dim\sum_{(i,j)\in A}V_{i,j}=\dim\sum_{i\in\pi(A)}V_i=\mathrm{rk}_{\mathcal{V}}(\pi(A)).$$

For the second, set $\nu=\mathrm{rk}^*_{\mathcal{V}^u}(A)$ and let $A_1,\dots,A_p$ be a partition of A such that $\nu=\sum_{c=1}^{p}\mathrm{rk}_{\mathcal{V}^u}(A_c)-p$. If for some i,j one has $\pi(A_i)\cap\pi(A_j)\neq\varnothing$ then let $k\in\pi(A_i)\cap\pi(A_j)$. Let $A'_1,\dots,A'_q$ be obtained from $A_1,\dots,A_p$ by replacing A_i with $A_i\cup\{(a,b)\in A_j:a=k\}$ and A_j with $\{(a,b)\in A_j:a\neq k\}$ if $\{(a,b)\in A_j:a\neq k\}\neq\varnothing$ (in this case $q=p$), or simply by removing A_j if $\{(a,b)\in A_j:a\neq k\}=\varnothing$ (and in this case $q=p-1$). One can check that the new partition satisfies $\sum_{c=1}^{q}\mathrm{rk}_{\mathcal{V}^u}(A'_c)-q\leq\nu$ and hence $\sum_{c=1}^{q}\mathrm{rk}_{\mathcal{V}^u}(A'_c)-q=\nu$. We may repeat the process until we obtain a partition $A_1,\dots,A_s$ of A such that $\nu=\sum_{c=1}^{s}\mathrm{rk}_{\mathcal{V}^u}(A_c)-s$ and $\pi(A_i)\cap\pi(A_j)=\varnothing$ for every $i\neq j$. Then $\pi(A_1),\dots,\pi(A_s)$ is a partition of $\pi(A)$ and $\mathrm{rk}^*_{\mathcal{V}^u}(A)=\nu=\sum_{c=1}^{s}\mathrm{rk}_{\mathcal{V}^u}(A_c)-s=\sum_{c=1}^{s}\mathrm{rk}_{\mathcal{V}}(\pi(A_c))-s\geq\mathrm{rk}^*_{\mathcal{V}}(\pi(A))$. Vice versa if $B_1,\dots,B_s$ is a partition on $\pi(A)$ such that $\sum_{c=1}^{s}\mathrm{rk}_{\mathcal{V}}(B_c)-s=\mathrm{rk}^*_{\mathcal{V}}(\pi(A))$ then with $A_i=A\cap\pi^{-1}(B_i)$ one gets a partition $A_1,\dots,A_s$ of A such that $\mathrm{rk}^*_{\mathcal{V}^u}(A)\leq\sum_{c=1}^{s}\mathrm{rk}_{\mathcal{V}^u}(A_c)-s=\mathrm{rk}^*_{\mathcal{V}}(\pi(A))$. □

We obtain:

THEOREM 1.19. *For every $u>0$ we have:*

(a) $\operatorname{projdim}J=\operatorname{projdim}J^u$,
(b) $\operatorname{Ass}(S/J)=\operatorname{Ass}(S/J^u)$,
(c) *an irredundant primary decomposition of J^u is obtained by raising to power u the components in the irredundant primary decomposition of J described in 1.16, i.e.*

$$J^u=\bigcap_B I_B^{u\#B}$$

where $B\in F(\mathcal{V})$ and $\mathrm{rk}^*_{\mathcal{V}}(B)=\mathrm{rk}_{\mathcal{V}}(B)-1$.

PROOF. (a) By 1.11(4) $\operatorname{projdim}J=\mathrm{rk}^*_{\mathcal{V}}([n])$ and $\operatorname{projdim}J^u=\mathrm{rk}^*_{\mathcal{V}^u}([n]\times[u])$ and by 1.18 $\mathrm{rk}^*_{\mathcal{V}}([n])=\mathrm{rk}^*_{\mathcal{V}^u}([n]\times[u])$.

Assertions (b) and (c): by 1.16 the associated primes of J^u arise form subsets $C\subseteq[n]\times[u]$ such that $\mathrm{rk}^*_{\mathcal{V}^u}(C)=\mathrm{rk}_{\mathcal{V}^u}(C)-1$ and $C\in F(\mathcal{V}^u)$, i.e. $\mathrm{rk}_{\mathcal{V}^u}(C)<\mathrm{rk}_{\mathcal{V}^u}(A)$ for all $C\subsetneq A$. The second condition together with 1.18 implies that $C=\pi^{-1}(B)$ with $B=\pi(C)$. But then, again by 1.18, $\mathrm{rk}^*_{\mathcal{V}^u}(C)=\mathrm{rk}_{\mathcal{V}^u}(C)-1$ is equivalent to $\mathrm{rk}^*_{\mathcal{V}}(B)=\mathrm{rk}_{\mathcal{V}}(B)-1$. Summing up, $F(\mathcal{V}^u)=\{\pi^{-1}(B):B\in F(\mathcal{V})\}$ and hence the associated primes of J^u are exactly the associated primes of J. The assertion concerning the primary decomposition follows immediately since $\#\pi^{-1}(B)=u\#B$. □

The established relations 1.18 among the rank functions translate immediately to the following relation involving the associated polymatroids:

Proposition 1.20. *For every u we have:*

$$P(\mathcal{V}^u)^* = \left\{(x_{ij}) \in \mathbb{N}^{[n]\times[u]} : \left(\sum_{j\in[u]} x_{1j}, \dots, \sum_{j\in[u]} x_{nj}\right) \in P(\mathcal{V})^*\right\}.$$

Since the Betti numbers can be expressed in terms of the points in $P(\mathcal{V}^u)^*$, using 1.20 one can deduce a formula for the Betti numbers of J^u that just depends on $P(\mathcal{V})^*$:

Corollary 1.21. *For every $u > 0$ and every $i \geq 0$ one has:*

$$\beta_i(J^u) = \sum_{x\in P(\mathcal{V})^*} \binom{|x|}{i} \prod_{j=1}^{n} \binom{u + x_j - 1}{x_j}$$

Remark 1.22. *As a further generalization, instead of the powers J^u of $J = I_1 I_2 \cdots I_n$ one can consider a product of powers of the I_i's, that is $I_1^{u_1} \cdots I_n^{u_n}$ with $(u_1, \dots, u_n) \in \mathbb{N}^n$ and the arguments we have presented extend immediately. Assuming $u_i > 0$ for all i one has:*

(a) *the results in 1.19 (a), (b), (c) hold with the ideal J^u replaced by $I_1^{u_1} \cdots I_n^{u_n}$ and the exponent $u\#B$ replaced by $\sum_{i\in B} u_i$.*

(b) *The polymatroid associated to the subspace arrangement $\mathcal{V}^{(u_1,\dots,u_n)} = \{V_{ij}\}$ with $V_{ij} = V_i$ for all $j \in [u_i]$ is:*

$$P(\mathcal{V}^{(u_1,\dots,u_n)})^* = \left\{(x_{ij}) \in \mathbb{N}^{[u_1]\times\cdots\times[u_n]} : \left(\sum_{j\in[u_1]} x_{1j}, \dots, \sum_{j\in[u_n]} x_{nj}\right) \in P(\mathcal{V})^*\right\}.$$

(c) *The formula for the Betti numbers is:*

$$\beta_i(I_1^{u_1} \cdots I_n^{u_n}) = \sum_{x\in P(\mathcal{V})^*} \binom{|x|}{i} \prod_{j=1}^{n} \binom{u_j + x_j - 1}{x_j}$$

The case $i = 0$ of the formula 1.22(c) deserves a special attention because of its relation with the so-called multiview variety that arises in geometric computer vision. Let us recall from [**AST13, Con07, CS10, CDNG18, Li18**] that the subspace arrangement $\mathcal{V}$ defines a multiprojective variety whose coordinate ring can be identified with the subring

$$A = K[V_1 y_1, \dots, V_n y_n]$$

of the Segre product $K[x_i y_j : i = 1, \dots, d \text{ and } j = 1, \dots n]$. The ring A is $\mathbb{Z}^n$-graded by $\deg y_j = e_j \in \mathbb{Z}^n$. Given $u = (u_1, \dots, u_n) \in \mathbb{N}^n$, the u-th homogeneous component A_u of A is $V_1^{u_1} \cdots V_n^{u_n}$ and its dimension equals to $\beta_0(I_1^{u_1} \cdots I_n^{u_n})$. We get a relatively simple and new proof of an improved version of the main result of [**Li18**]:

Theorem 1.23. *For every $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ the multigraded Hilbert function of the coordinate ring $A = K[V_1 y_1, \dots, V_n y_n]$ of the multiview variety associated with the subspace arrangement $\mathcal{V} = \{V_1, \dots, V_n\}$ is given by:*

$$\dim_K A_u = \sum_{x\in P(\mathcal{V})^*} \prod_{j=1}^{n} \binom{u_j + x_j - 1}{x_j}$$

In particular, the multidegree of A is multiplicity free and supported on the maximal elements of the polymatroid $P(\mathcal{V})^*$.

D. Resolution of J_P

For every subspace arrangement $V_1, \dots, V_n$ of dimension $(d_1, \dots, d_n)$ with a given collection of bases $f = \{f_{ij} : i \in [n] \text{ and } j \le d_i\}$ satisfying Assumption 1.1 and for every poset ideal P of $D = [d_1] \times \cdots \times [d_n]$ we have proved that the ideal J_P has a linear resolution and that the Betti numbers are combinatorial invariants. Our goal is now to describe explicitly a minimal free resolution of J_P. We start with the "generic" case.

D.I. Resolution of J_P: the generic case. Assume firstly that, for the given $(d_1, \dots, d_n)$, the V_i's are as generic as possible. That is, we assume that there is a basis $\{x_{ij} : i \in [n] \text{ and } j \in [d_i]\}$ of the ambient vector space such that V_i is generated by $\{x_{ij} : j \in [d_i]\}$. Note that the collection of bases $x = \{x_{ij} : i \in [n] \text{ and } j \in [d_i]\}$ satisfy the Assumption 1.1 and we will consider the ideals J_P with respect to x. In this case

$$S = K[x_{ij} : i \in [n] \text{ and } j \in [d_i]].$$

The corresponding ideal J is the product of transversal ideals $I_i = (x_{ij} : j \in [d_i])$ because each factor uses a different set of variables. Then the resolution of J is given by the tensor product of the resolutions of the I_i's, the (truncated) Koszul complex on the set x_{ij} with $j \in [d_i]$. More explicitly, let $\mathcal{K}^{(i)}$ be the Koszul complex on x_{ij} with $j \in [d_i]$ with the 0-th component removed and homologically shifted so that

$$\mathcal{K}^{(i)}_j = \wedge^{j+1} S^{d_i}.$$

This is sometimes called the first syzygy complex of the full Koszul complex. Denote by $e_{i1}, \dots, e_{id_i}$ the canonical basis of S^{d_i}. For every non-empty subset $A_i = \{j_1, j_2, \dots\}$ of $[d_i]$ with $j_1 < j_2 < \dots$ we have the corresponding basis element $e_{A_i} = e_{ij_1} \wedge e_{ij_2} \wedge \dots$ of $\mathcal{K}^{(i)}$ in homological degree $|A_i| - 1$. Then

$$\mathcal{K} = \mathcal{K}^{(d_1, \dots, d_n)} = \mathcal{K}^{(1)} \otimes \mathcal{K}^{(2)} \otimes \cdots \otimes \mathcal{K}^{(n)}$$

is the free resolution of $J = J_1 \cdots J_n$. An S-basis of $\mathcal{K}$ can be described as follows. Let $A = (A_1, \dots, A_n)$ with A_i a non-empty subset of $[d_i]$. Set $e_A = e_{A_1} \otimes e_{A_2} \otimes \cdots \otimes e_{A_n} \in \mathcal{K}$. Then the homological degree of e_A is $\sum_{i=1}^n |A_i| - n$ and the set of all e_A's form an S-basis of $\mathcal{K}$. The differential $\partial_{\mathcal{K}}$ of $\mathcal{K}$ can be described as follows:

$$\partial_{\mathcal{K}}(e_A) = \sum_{i \in [n], |A_i| > 1} \sum_{b \in A_i} (-1)^{\sigma(i,b)} x_{ib} \; e_{A_1} \otimes \cdots \otimes e_{A_i \setminus \{b\}} \otimes \cdots \otimes e_{A_n}$$

where

$$\sigma(i,b) = \sum_{j<i} (|A_j| - 1) + |\{c \in A_i : c < b\}|.$$

For a given poset ideal P of D we define

$$\mathcal{K}_P = \mathcal{K}_P^{(d_1, \dots, d_n)} = \oplus S e_A$$

where the sum is extended to all the e_A such that $(\max(A_1), \dots, \max(A_n)) \in P$. Clearly $\mathcal{K}_P$ is a subcomplex of $\mathcal{K}$ and $(\mathcal{K}_P)_0 = \oplus_{a \in P} S e_{1a_1} \otimes e_{2a_2} \cdots \otimes e_{na_n}$ and our goal is to prove:

THEOREM 1.24. *The complex $\mathcal{K}_P$ is a minimal free resolution of J_P.*

Augmenting the complex $\mathcal{K}_P$ with the map

$$(\mathcal{K}_P)_0 \to S$$

sending $e_{1a_1} \otimes e_{2a_2} \cdots \otimes e_{na_n}$ to $f_a = x_{1a_1} \dots x_{na_n}$ one gets a complex $\tilde{\mathcal{K}}_P$ and we will actually prove it is a resolution of S/J_P. We need the following properties that follow immediately from the definitions.

Remark 1.25.

(1) *An inclusion $P_1 \subseteq P_2$ of poset ideals of D induces an inclusion of the associated complexes $\tilde{\mathcal{K}}_{P_1} \subseteq \tilde{\mathcal{K}}_{P_2}$.*
(2) *Given two poset ideals Q_1, Q_2 of D both $Q_1 \cup Q_2$ and $Q_1 \cap Q_2$ are poset ideals and one has $\tilde{\mathcal{K}}_{Q_1} \cap \tilde{\mathcal{K}}_{Q_2} = \tilde{\mathcal{K}}_{Q_1 \cap Q_2}$ and $\tilde{\mathcal{K}}_{Q_1} + \tilde{\mathcal{K}}_{Q_2} = \tilde{\mathcal{K}}_{Q_1 \cup Q_2}$.*
(3) *Given two poset ideals Q_1, Q_2 of D one has a short exact sequence of complexes*

$$0 \to \tilde{\mathcal{K}}_{Q_1 \cap Q_2} \to \tilde{\mathcal{K}}_{Q_1} \oplus \tilde{\mathcal{K}}_{Q_2} \to \tilde{\mathcal{K}}_{Q_1 \cup Q_2} \to 0$$

where the first map sends y to (y, y) and the second sends (y, z) to $y - z$.

Later on we will also need the following assertion that is part of the folklore of the subject.

Lemma 1.26. *Let S be a positively graded ring and M a finitely generated graded S-module. Let $x_1, \dots, x_h$ be elements of degree 1 of S and set $I = (x_1, \dots, x_h)$. Denote by $\mathrm{HS}(M, z)$ the Hilbert series of M. Assume $\mathrm{HS}(M/IM, z) = \mathrm{HS}(M, z)(1-z)^h$. Then $x_1, \dots, x_h$ is an M-regular sequence.*

Proof. For $i = 0, 1, \dots, h$ we set $I_i = (x_1, \dots, x_i)$ and $N_i = M/I_iM$. Denote by T_i the kernel of multiplication by x_{i+1} on N_i. For $i < h$ we have an exact sequence:

$$0 \to T_i \to N_i(-1) \to N_i \to N_{i+1} \to 0$$

and hence

$$\mathrm{HS}(N_{i+1}, z) = \mathrm{HS}(N_i, z)(1 - z) + \mathrm{HS}(T_i, z)$$

Taking into consideration that $N_0 = M$ it follows that for every $j \geq 0$ one has

$$\mathrm{HS}(N_j, z) = \mathrm{HS}(M, z)(1 - z)^j + \sum_{i<j} \mathrm{HS}(T_i, z)(1 - z)^{j-1-i}.$$

Setting $j = h$ and using the assumption one has:

$$\sum_{i<h} \mathrm{HS}(T_i, z)(1 - z)^{h-1-i} = 0$$

Since $\mathrm{HS}(T_i, z)$ are series with non-negative terms and the least degree component of $(1 - z)^{h-1-i}$ is positive, $\mathrm{HS}(T_i, z) = 0$ for every i, that is $T_i = 0$ for every i. □

Theorem 1.27. *The complex $\tilde{\mathcal{K}}_P$ is a minimal free resolution of S/J_P.*

Proof. By construction we have that $H_0(\tilde{\mathcal{K}}_P) = S/J_P$ and hence we have to show that $H_i(\tilde{\mathcal{K}}_P) = 0$ for $i > 0$. We do it by induction on $|P|$. The case $|P| = 1$ is obvious. Let M be the set of maximal elements in P.

If $|M| = 1$, say $M = \{a\}$ with $a = (a_1, \dots, a_n)$, then $P = \{b \in D : b \leq a\}$ and $J_P = \prod_{i=1}^n (x_{i1}, \dots, x_{ia_i})$. Then a resolution of S/J_P is given by the augmented complex obtained by the tensor product of the truncated Koszul complexes associated to $x_{i1}, \dots, x_{ia_i}$ which is exactly $\tilde{\mathcal{K}}_P$.

If instead $|M| > 1$, say $M = \{m_1, \dots, m_v\}$ set $Q_1 = \{b \in D : b \le m_i \text{ for some } i < v\}$ and $Q_2 = \{b \in D : b \le m_v\}$ so that $P = Q_1 \cup Q_2$. By 1.25(3) we have a short exact sequence of complexes:

$$0 \to \tilde{\mathcal{K}}_{Q_1 \cap Q_2} \to \tilde{\mathcal{K}}_{Q_1} \oplus \tilde{\mathcal{K}}_{Q_2} \to \tilde{\mathcal{K}}_P \to 0.$$

The associated long exact sequence on homology together with the fact that, by induction, we already know the statement for Q_1, Q_2 and $Q_1 \cap Q_2$, imply that $H_i(\tilde{\mathcal{K}}_P) = 0$ for $i > 1$ and that $H_1(\tilde{\mathcal{K}}_P)$ fits in the exact sequence:

$$0 \to H_1(\tilde{\mathcal{K}}_P) \to S/J_{Q_1 \cap Q_2} \to S/J_{Q_1} \oplus S/J_{Q_2} \to S/J_P \to 0$$

But $J_{Q_1 \cap Q_2} = J_{Q_1} \cap J_{Q_2}$ and $J_P = J_{Q_1} + J_{Q_2}$ because of Lemma 1.28 and then it follows that $H_1(\tilde{\mathcal{K}}_P)$ vanishes as well. □

LEMMA 1.28. *Let P_1, P_2 be poset ideals of D. Then $J_{P_1 \cap P_2} = J_{P_1} \cap J_{P_2}$ and $J_{P_1 \cup P_2} = J_{P_1} + J_{P_2}$.*

PROOF. The second assertion and the inclusion $J_{P_1 \cap P_2} \subseteq J_{P_1} \cap J_{P_2}$ are obvious. For the other inclusion, since the ideals involved are monomial ideals, the intersection $J_{P_1} \cap J_{P_2}$ is generated by $\mathrm{LCM}(f_a, f_b)$ with $a \in P_1$ and $b \in P_2$. But $f_{a \wedge b} | \mathrm{LCM}(f_a, f_b)$ and $a \wedge b \in P_1 \cap P_2$. □

D.II. Resolution of J_P: arbitrary configurations. Now let us return to the case of an arbitrary subspace arrangement $\mathcal{V} = V_1, \dots, V_n$ of dimension $(d_1, \dots, d_n)$ and fix a collection of bases $\{f_{ij}\}$ satisfying Assumption 1.1. Consider the K-algebra map:

$$T = K[x_{ij} : i \in [n] \text{ and } j \in [d_i]] \to S$$

sending x_{ij} to f_{ij} which, without loss of generality, we may assume is surjective. We consider S as a T-module via this map. We have:

THEOREM 1.29. *For every poset ideal $P \subseteq D_{\mathcal{V}}$ the complex $\tilde{\mathcal{K}}_P \otimes_T S$ is a minimal S-free resolution of S/J_P.*

PROOF. In the proof we need to distinguish the ideal J_P associated with the arbitrary subspace arrangement $V_1, \dots, V_n$ and collection of bases f with the one, that we will denote by J_P^g, associated with the generic arrangement of dimension $(d_1, \dots, d_n)$ and collection of bases x. Let U be the kernel of the map $T \to S$. By construction, U is generated by $h = \sum_{i=1}^n d_i - \dim_K \sum_{i=1}^n V_i$ linear forms and one has

$$T/J_P^g \otimes_T S = T/(J_P^g + U) = S/J_P.$$

Since by Theorem 1.27 $\tilde{\mathcal{K}}_P$ is a resolution of T/J_P^g it is enough to prove that the generators of U form a T/J_P^g-regular sequence. Note that by Corollary 1.10 T/J_P^g and S/J_P have the same Betti numbers and hence their Hilbert series differ only by the factor $(1-z)^h$. Then by Lemma 1.26 one concludes that the generators of U form a T/J_P^g-regular sequence. □

As a consequence we have that Lemma 1.28 holds for arbitrary subspace configurations:

COROLLARY 1.30. *Let P_1, P_2 be poset ideals of $D_{\mathcal{V}}$. Then $J_{P_1 \cap P_2} = J_{P_1} \cap J_{P_2}$ and $J_{P_1 \cup P_2} = J_{P_1} + J_{P_2}$.*

Proof. The second assertion and the inclusion $J_{P_1 \cap P_2} \subseteq J_{P_1} \cap J_{P_2}$ are obvious. The short exact sequence of complexes

$$0 \to \tilde{\mathcal{K}}_{P_1 \cap P_2} \otimes S \to (\tilde{\mathcal{K}}_{P_1} \otimes S) \oplus (\tilde{\mathcal{K}}_{P_2} \otimes S) \to \tilde{\mathcal{K}}_{P_1 \cup P_2} \otimes S \to 0$$

induces an exact sequence in homology that, by virtue of Theorem 1.29, yields the following short exact sequence:

$$0 \to S/J_{P_1 \cap P_2} \to S/J_{P_1} \oplus S/J_{P_2} \to S/J_{P_1} + J_{P_2} \to 0$$

that in turns implies the desired equality. □

As a special case of Theorem 1.29 we have:

Theorem 1.31. *For every subspace arrangement $\mathcal{V} = V_1, \dots, V_n$ the complex $\tilde{\mathcal{K}}_{D_\mathcal{V}} \otimes_T S$ is a minimal S-free resolution of S/J.*

Remark 1.32. *The formulas for the Betti numbers and projective dimension hold over any base field. The resolution described works provided the base field is infinite.*

E. Examples

Example 1.33. *Let k be an infinite field and consider the polynomial ring $S = k[x, y]$. Let $V_1 = V_2 = \langle x, y\rangle$ and thus $J = (x, y)^2$. Since J is generated by the 2×2 minors of a 2×3 matrix, its minimal free resolution is well known to be*

$$0 \to S(-3)^2 \xrightarrow{\phi_1} S(-2)^3 \overset{\phi_0}{\to} J \to 0$$

$$\phi_1 = \begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x \end{bmatrix}, \ \phi_0 = \begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}$$

as seen either by Hilbert-Burch or Eagon-Northcott. We show how to get this resolution by Theorem 1.29. First, we consider the generic case as in §D.I. So let $T = [x_{11}, x_{12}, x_{21}, x_{22}]$ and $V_i^g = \langle x_{i1}, x_{i2}\rangle$, $i \in [2]$. The ideal $I_i^g = (x_{i1}, x_{i2})$ of T is resolved by the truncated Koszul complex

$$0 \to T(-2) \overset{[x_{i2}\ -x_{i1}]^\intercal}{\to} T(-1)^2 \overset{[x_{i1}\ x_{i2}]}{\to} I_i^g \to 0$$

with $T(-2)$ free on $e_{i1} \wedge e_{i2}$ and $T(-1)^2$ free on e_{i1}, e_{i2}. The tensor product of these two free resolutions

$$\tilde{\mathcal{K}}:\ 0 \to T(-4) \xrightarrow{\phi_2} T(-3)^4 \xrightarrow{\phi_1} T(-2)^4 \xrightarrow{\phi_0} J_1^g J_2^g \to 0$$

$$\phi_2 = [x_{12}\ \ -x_{11}\ \ -x_{22}\ \ x_{21}]$$

$$\phi_1 = \begin{bmatrix} x_{22} & 0 & x_{12} & 0 \\ -x_{21} & 0 & 0 & x_{12} \\ 0 & x_{22} & -x_{11} & 0 \\ 0 & -x_{21} & 0 & -x_{11} \end{bmatrix}$$

$$\phi_0 = [x_{11}x_{21}\ \ x_{11}x_{22}\ \ x_{12}x_{21}\ \ x_{12}x_{22}]$$

is a minimal free resolution of $J_1^g J_2^g$. A direct computation shows that for $\mathcal{V} = V_1, V_2$ we have $D_\mathcal{V} = \{(1,1), (1,2), (2,1)\}$. To obtain the complex $\tilde{\mathcal{K}}_{D_\mathcal{V}}$ we must discard from $\tilde{\mathcal{K}}$ the generators

$$(e_{11} \wedge e_{12}) \otimes (e_{21} \wedge e_{22}),\ e_{12} \otimes (e_{21} \wedge e_{22}),\ (e_{11} \wedge e_{12}) \otimes e_{22},\ e_{12} \otimes e_{22}$$

at homological degrees 2, 1, 1, 0 respectively. By Theorem 1.24 the resulting complex

$$\tilde{\mathcal{K}}_{D_{\mathcal{V}}}: \ 0 \to T(-3)^2 \xrightarrow{\phi_1} T(-2)^3 \xrightarrow{\phi_0} J^g_{D_{\mathcal{V}}} \to 0$$

$$\phi_1 = \begin{bmatrix} x_{22} & x_{12} \\ -x_{21} & 0 \\ 0 & -x_{11} \end{bmatrix}$$

$$\phi_0 = [x_{11}x_{21} \ \ x_{11}x_{22} \ \ x_{12}x_{21}]$$

is a minimal free resolution of $J^g_{D_{\mathcal{V}}}$*.*

Now we must define a map $T \to S$ *such that for* $i = 1, 2$ *the map takes* x_{i1}, x_{i2} *to* k*-bases of* V_1, V_2 *that satisfy Assumption 1.1. One such map is*

$$x_{11} \mapsto x, \ \ x_{12} \mapsto y, \ \ x_{21} \mapsto y, \ \ x_{22} \mapsto x$$

The kernel is generated by $x_{11} - x_{22}$ *and* $x_{12} - x_{21}$ *and Theorem 1.29 together with Corollary 1.9 assert that*

$$\tilde{\mathcal{K}}_{D_{\mathcal{V}}} \otimes_T S: \ 0 \to S(-3)^2 \xrightarrow{\phi_1} S(-2)^3 \xrightarrow{\phi_0} J_{D_{\mathcal{V}}} \to 0$$

$$\phi_1 = \begin{bmatrix} x & y \\ -y & 0 \\ 0 & -x \end{bmatrix}$$

$$\phi_0 = [xy \ \ x^2 \ \ y^2]$$

is a minimal free resolution of $J_{D_{\mathcal{V}}} = (x, y)^2$*. Up to a permutation of coordinates this is the same complex as in the beginning of the example.*

Example 1.34. *With* k *infinite we let* $S = k[x, y, z]$ *and consider the subspace arrangement* $\mathcal{V}$ *given by* $V_1 = V_2 = \langle x, y\rangle, V_3 = V_4 = \langle y, z\rangle$*. As in Example 1.33 set* $T = k[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}]$ *and* $I^g_i = (x_{i1}, x_{i2})$ *for* $i \in [4]$*. Each of the* I^g_i *is resolved by a truncated Koszul complex*

$$0 \to T(-2) \overset{[x_{i2} \ -x_{i1}]^{\intercal}}{\to} T(-1)^2 \overset{[x_{i1} \ x_{i2}]}{\to} I^g_i \to 0$$

with $T(-2)$ *free on* $e_{i1} \wedge e_{i2}$ *and* $T(-1)^2$ *free on* e_{i1}, e_{i2}*. The tensor product of these four free resolutions is a free resolution of* $J^g = J^g_1 J^g_2 J^g_3 J^g_4$ *and has the form*

$$\tilde{\mathcal{K}}: \ 0 \to T(-8) \to T(-7)^8 \to T(-6)^{24} \to T(-5)^{32} \to T(-4)^{16} \to J^g \to 0$$

Let us verify those Betti numbers via the formula of part (3) in Corollary 1.11. For this we need to compute the polymatroid $P(\mathcal{V}^g)^*$*. A simple calculation shows that*

$$P(\mathcal{V}^g)^* = \{a \in \mathbb{N}^4 : a_i \leq 1\}$$

$$\gamma_0(\mathcal{V}^g) = 1, \ \gamma_1(\mathcal{V}^g) = 4, \ \gamma_2(\mathcal{V}^g) = 6, \ \gamma_3(\mathcal{V}^g) = 4, \ \gamma_4(\mathcal{V}^g) = 1$$

Applying the formula we get

$$\sum_{0 \leq i \leq 4} \beta_i(J^g) z^i = (1+z)^0 + 4(1+z) + 6(1+z)^2 + 4(1+z)^3 + (1+z)^4$$

and so indeed

$$\beta_0(J^g) = 16, \ \beta_1(J^g) = 32, \ \beta_2(J^g) = 24, \ \beta_3(J^g) = 8, \ \beta_4(J^g) = 1$$

Since the maximal elements of $D_{\mathcal{V}}$ *are*

$$(1, 2, 1, 2), \ (2, 1, 2, 1), \ (2, 1, 1, 2), \ (1, 2, 2, 1)$$

in the above complex we must discard all generators that simultaneously involve e_{12} *and* e_{22} *or* e_{32} *and* e_{42} *to obtain a minimal free resolution of* $J^g_{D_{\mathcal{V}}}$. *From this we see that there will be no components in homological degrees* 4 *and* 3 *in* $\tilde{\mathcal{K}}_{D_{\mathcal{V}}}$. *The resolution has the form*

$$\tilde{\mathcal{K}}_{D_{\mathcal{V}}} :\ 0 \to T(-6)^4 \to T(-5)^{12} \to T(-4)^9 \to J^g \to 0$$

The minimal free resolution of J *is* $\tilde{\mathcal{K}}_{D_{\mathcal{V}}} \otimes_T S$ *where* $T \to S$ *is a ring homomorphism that sends the* x_{i1}, x_{i2} *to a basis of* V_i *such that Assumption 1.1 holds true. Note that such a choice yielding monomial minimal generators for* J *in the free resolution is not possible. Instead, a valid choice is*

$$\begin{array}{ll} x_{11} \mapsto x & x_{12} \mapsto y \\ x_{21} \mapsto y + x & x_{22} \mapsto x \\ x_{31} \mapsto y + z & x_{32} \mapsto z \\ x_{41} \mapsto z & x_{42} \mapsto y + z \end{array}$$

Finally, we verify the Betti numbers by computing them from $P(\mathcal{V}^*)$ *as above. The maximal elements of* $P(\mathcal{V}^*)$ *are* $(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1)$. *Hence*

$$\gamma_0(\mathcal{V}) = 1,\ \gamma_1(\mathcal{V}) = 4,\ \gamma_2(\mathcal{V}) = 4$$

and part (3) in Corollary 1.11 gives

$$\beta_0(J) = 9,\ \beta_1(J) = 12,\ \beta_2(J) = 4$$

Example 1.35. *We close with an example showing that the infiniteness of* k *is in general necessary for Assumption 1.1, the point being that in a finite field we may not have enough linearly independent* f_{ij}*'s. Let* $k = \mathbb{Z}_2$ *and* $V_i = k^2$, $i \in [4]$. *Then we can set*

$$f_{11} = e_1 \quad f_{21} = e_2 \quad f_{31} = e_1 + e_2$$

but no matter how we choose f_{41} *we will have that* $\langle f_{i1}, f_{41} \rangle$ *will be* 1*-dimensional for some* $i < 4$. *On the other hand, there exist elements* $u \in V_i$, $v \in V_4$ *such that* $\langle u, v \rangle$ *is* 2*-dimensional.*

CHAPTER 2

Linearization of resolutions via products

Given a homogeneous ideal I in a polynomial ring $S = k[x_1, \dots, x_r]$, it is of interest to be able to quantify how *complicated* I is. One option is to consider the maximal degree among any minimal set of generators of I, which can be shown to be an invariant of I. However, this invariant does not provide any information regarding the relations between the generators of I (first syzygies of I), or the relations of these relations (second syzygies of I) and so on, which should be taken into account when measuring the *complexity* of I. Instead, this is achieved by working with the Castelnuovo-Mumford regularity of I [**Eis95**], which is the smallest integer m such that for each i the ith syzygy of I is generated in at most degree $m + i$.

The Castelnuovo-Mumford regularity of I has been proved to be a very fruitful notion, used among others, as a measure of complexity of computing a Gröbner basis for I [**BM93**]. In general, the regularity admits a doubly exponential bound $\operatorname{reg}(I) \leq (2d)^{2^{r-2}}$ [**CS05**], where d is the maximal degree at which I is generated. Moreover, in the absence of any assumptions on I, this bound is nearly sharp, as shown in [**MM82, BS88**]. Even for a homogeneous prime ideal over an algebraically closed field, efficient bounds in terms of invariants such as the codimension (height) of the ideal or the multiplicity (degree) of the quotient ring remain elusive [**EG84, MP18**].

On the other hand, products or intersections of ideals generated by linear forms have remarkable properties. Specifically, the product always has a linear resolution [**CH03**], while the regularity of the intersection is bounded by the number of factors [**DS02, DS04**]. Moreover, these results together with [**Der07**] have been important in the theoretical foundations of algebraic machine learning methods for clustering data in a subspace arrangement [**VMS03, VMS05, MYDF08, TV14, TV17, TV18**], a problem that over the past 15 years has received a lot of attention in the computer science community [**EV13, LLY$^+$13, VMS16**].

In general, it is of interest to bound the regularity of the product and intersection of any given ideals in terms of their individual regularities. However, in the absence of any further hypothesis this is a very hard problem. On the other hand,

$$\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J), \tag{1}$$

for any ideal I as soon as $\dim(S/J) \leq 1$ [**CH03**]; a generalization of the earlier result $\operatorname{reg}(J^n) \leq n \operatorname{reg}(J)$ of [**Cha97**], subsequently further generalized in [**Sid02a, Cav07, EHU06**]. More can be said for monomial ideals: In [**CMT07**] (1) was established for monomial complete intersections, in [**Cim09**] for Borel-type ideals, and in [**YCQ15**] for I Borel-type and J monomial complete intersection. Bounding the regularity of the intersection ideal is an even harder problem, partly because the generators of the intersection are not in principle available as for the product. This difficulty is not present for monomial ideals, and the work of [**Her07**] proved that

$$\operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J),$$

by constructing a free resolution of $S/(I+J)$ for any monomial ideals I, J; see [**BBC15, BCV15, BC17a, BC17b**] for other recent related results.

Inspired by [**CH03**] as well as by the fact that in computer vision subspace arrangements often appear mixed with other non-linear varieties [**RYSM10**], we study the regularity of the product of an arbitrary homogeneous ideal I generated in a single degree with ideals of general points. We prove the rather surprising fact that multiplication of I with sufficiently many such ideals, yields an ideal that has linear resolution. More precisely, for $r \geq 3$, k infinite but not necessarily algebraically closed, and $I(p)$ the vanishing ideal of a point $p \in \mathbb{P}^{r-1}$, we have:

THEOREM 2.1. *Let I be any homogeneous ideal of S, generated in degree d. Then there exists an integer $n_I \leq r(\operatorname{reg}(I) - d)$, such that for any set $\mathcal{X}$ of general points of $\mathbb{P}^{r-1}$ with $\#\mathcal{X} \geq n_I$, the ideal $I \prod_{p \in \mathcal{X}} I(p)$ has a linear resolution.*

In Theorem 2.1 the points $\mathcal{X}$ are required to be general in the sense that there must exist $\operatorname{reg}(I) - d$ disjoint subsets of $\mathcal{X}$, each containing r points, so that no such subset of r points lies in a hyperplane.

Given that $\operatorname{reg}(I)$ can be doubly exponential in r and d, Theorem 2.1 in principle requires a large number of points[1]. If on the other hand I is a general complete intersection of degree d, then $r(d-1)$ linear ideals are enough to *linearize* I:[2]

THEOREM 2.2. *Let I_{ci} be an ideal generated by $\ell \leq r$ general forms of degree d. With $\mathcal{X}$ a set of r general points of $\mathbb{P}^{r-1}$, the ideal $I_{ci}(\prod_{p \in \mathcal{X}} I(p))^{d-1}$ has a linear resolution.*

A. Generalities

For a positive integer ℓ we let $[\ell] = \{1, \dots, \ell\}$. We work over a polynomial ring $S = k[x_1, \dots, x_r]$ over an infinite field k which need not be algebraically closed, and we assume that $r \geq 3$. We assume the standard grading on S, where each x_i has degree 1, and we let $\mathfrak{m} = (x_1, \dots, x_r)$. Given a finitely generated graded S-module M and an integer ν we denote by M_ν the degree-ν component of M, which is a finite-dimensional k-vector space of dimension $\operatorname{HF}(\nu, M)$. For large enough ν this vector space dimension is given by the Hilbert polynomial of M, denoted by p_M, which is a polynomial of degree $\dim M - 1$. The *Castelnuovo-Mumford regularity* $\operatorname{reg}(M)$ of M is defined to be the smallest integer m such that every module $\operatorname{Tor}_i^S(M, k)$ vanishes at degree higher than $i + m$. This is equivalent to saying that the ith syzygy module $\operatorname{Syz}_i(M)$ of M is generated in degree at most $i+m$. Equivalently, $\operatorname{reg}(I)$ is the smallest integer m such that $\operatorname{Ext}_S^i(M, S)$ vanishes at degrees below $-m-i$ [**Eis95**], or in terms of local cohomology, the smallest m such that $\mathcal{H}_{\mathfrak{m}}^i(M)_j = 0$, $\forall i + j > m$ [**BH98**]. By definition, $\operatorname{reg}(M)$ bounds from above the maximal degree d in which M is generated. When $\operatorname{reg}(M) = d$, we say that M has a *linear resolution*, in the sense that $\operatorname{Syz}_i(M)$ is generated in degree $i + d$. The regularity is well-behaved on short exact sequences of finitely generated

[1]The question of how sharp these bounds are is the subject of current research. Preliminary investigations for small r using Macaulay2 suggest that sharper bounds might exist.

[2]The description of the non-empty Zariski open subset associated to Theorem 2.2 is more involved and is deferred to the proof the theorem.

graded S-modules: given such a sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, we have that

$$\operatorname{reg}(M) \le \max\{\operatorname{reg}(M'), \operatorname{reg}(M'')\},$$
$$\operatorname{reg}(M') \le \max\{\operatorname{reg}(M), \operatorname{reg}(M'') + 1\},$$
$$\operatorname{reg}(M'') \le \max\{\operatorname{reg}(M), \operatorname{reg}(M') - 1\}.$$

For a graded S-module N of finite length, $\operatorname{reg}(N)$ admits a simple characterization: it is the largest degree at which N is non-zero. An ideal generated by linear forms always has regularity 1, and in fact we have the following more general result.

Proposition 2.3 (Theorem 3.1 in [**CH03**]). *Let $L_1, \ldots, L_n$ be ideals of S, each generated by linear forms. Then* $\operatorname{reg}(L_1 \cdots L_n) = n$.

The proof of Proposition 2.3 relies on a more fundamental result about the primary decomposition of products of linear ideals, which we use in this chapter and state next.

Proposition 2.4 (Lemma 3.2 in [**CH03**]). *Let $L_1, \ldots, L_n$ be ideals of S, each generated by linear forms. Then a primary decomposition for $L_1 \cdots L_n$ is given by*

$$L_1 \cdots L_n = \bigcap_{\mathcal{A} \subset [n]} \Big(\sum_{i \in \mathcal{A}} L_i \Big)^{\#\mathcal{A}}. \tag{2}$$

Given an ideal I of S, the saturation I^{sat} of I is defined as $I^{\text{sat}} = \{f \in S : \mathfrak{m}^n f \subset I, \text{ for some } n\}$. In fact, I^{sat} is equal to the intersection of all the primary components of I except the one that corresponds to $\mathfrak{m}$, if any, so that I^{sat}/I has finite length. If such a component is not present, then $I = I^{\text{sat}}$ in which case I is called *saturated*. The *saturation index* $\operatorname{sat}(I)$ of I is defined as the smallest degree n such that $I_m = I^{\text{sat}}_m$, $\forall m \ge n$, and admits the following simple characterization.

Proposition 2.5 (Follows from Proposition 2.1 in [**BG06**]). *Let I be a non-saturated homogeneous ideal in S. Then* $\operatorname{sat}(I) - 1$ *is the largest degree among the elements of $I : \mathfrak{m}$ not belonging to I.*

We conclude with a very useful formula that relates $\operatorname{sat}(I)$ with $\operatorname{reg}(I)$.

Proposition 2.6 (Corollary 1.3 in [**CH03**]). *Let I be a homogeneous ideal in S and x a linear form that is a non-zero divisor of S/I^{sat}. Then* $\operatorname{reg}(I) = \max\{\operatorname{reg}(I + (x)), \operatorname{sat}(I)\}$.

B. Proof of Theorem 2.1

Let I be a homogeneous ideal of S generated in degree d. By Theorem 2.4 in [**CH03**] it is enough to prove that the product of I with n_I ideals of general points has a linear resolution. If I already has a linear resolution (in particular, if $d = 1$), then we can take $n_I = 0$. So we assume that I does not have a linear resolution (in particular, $d > 1$) and we proceed in several steps. To begin with, for $i = 1, \ldots, r$ we let L_i be the linear ideal of codimension $r - 1$ generated by all variables except x_i and we let $J = L_1 \cdots L_r$. The following lemma is used in this section for $\ell = 1$ but the more general case is needed in section C.

Lemma 2.7. *For any $\ell \ge 1$, the ideal J^ℓ is generated by all monomials of degree $r\ell$ except the ones that are divided by x_i^s for some $i \in [r]$ and $s > (r-1)\ell$.*

Proof. By Proposition 2.4 we have

$$J^\ell = (L_1 \cdots L_r)^\ell = L_1^\ell \cap \cdots \cap L_r^\ell \cap \mathfrak{m}^{r\ell}. \tag{3}$$

Since a monomial $v = x_1^{\nu_1} \cdots x_r^{\nu_r}$ of degree $r\ell$ is in L_i^ℓ if and only if $\sum_{j \neq i} \nu_j \geq \ell$, or equivalently $\nu_i \leq (r-1)\ell$, we have that the generators of J^ℓ are the monomials $v = x_1^{\nu_1} \cdots x_r^{\nu_r}$ such that $\sum_i \nu_i = r\ell$ and $\nu_i \leq (r-1)\ell$ for every $i \in [r]$. □

The next lemma is the crucial computation behind Theorem 2.1.

Lemma 2.8. *Let L_i be the ideal generated by all variables except x_i and $J = L_1 \cdots L_r$. Let I be any homogeneous ideal generated in degree d. Then*

$$\operatorname{sat}(IJ) \leq \max\{r + d, \operatorname{reg}(I) + 2\}.$$

Proof. If IJ is saturated the statement is trivially true, so suppose that $IJ \subsetneq (IJ)^{\text{sat}}$. Let u be an element of maximal degree s among the elements of $IJ : \mathfrak{m}$ that do not belong to IJ. Then $\operatorname{sat}(IJ) = s + 1$ by Proposition 2.5. If $s < r + d$, we are done; so suppose $s \geq r + d$. First, suppose that $u \notin I$. Since $(IJ)^{\text{sat}} \subset I^{\text{sat}}$, we have that $u \in I^{\text{sat}} \setminus I$. Then Propositions 2.5 and 2.6 give that $s \leq \operatorname{sat}(I) - 1 \leq \operatorname{reg}(I) - 1$, by which we are done. Thus, suppose that $u \in I$, by which we can write

$$u = p_1 f_1 + \cdots + p_n f_n,$$

where $f_1, \ldots, f_n$ are minimal generators of I, each of degree d, and p_i are homogeneous polynomials of degree $s - d$. For each $j \in [n]$ we decompose each p_j as $p_j = \bar{p}_j + \tilde{p}_j$, where

$$\bar{p}_j = c_j^{(1)} x_1^{s-d} + \cdots + c_j^{(r)} x_r^{s-d}, \quad c_j^{(i)} \in k,$$

and $\tilde{p}_j$ is supported only by monomials each divisible by at least two variables. Since by hypothesis $s - d \geq r$, this implies that $\tilde{p}_j \in J$. Since $f_j \tilde{p}_j \in IJ$ and $u \notin IJ$, we may replace u by $u - \sum_{j=1}^n f_j \tilde{p}_j$ and assume that

$$\begin{aligned} u &= \bar{p}_1 f_1 + \cdots + \bar{p}_n f_n \\ &= \left(c_1^{(1)} x_1^{s-d} + \cdots + c_1^{(r)} x_r^{s-d}\right) f_1 + \cdots + \left(c_n^{(1)} x_1^{s-d} + \cdots + c_n^{(r)} x_r^{s-d}\right) f_n \\ &= x_1^{s-d}\left(c_1^{(1)} f_1 + \cdots + c_n^{(1)} f_n\right) + \cdots + x_r^{s-d}\left(c_1^{(r)} f_1 + \cdots + c_n^{(r)} f_n\right). \end{aligned} \tag{4}$$

Since u is non-zero, not all $c_j^{(i)}$ are equal to zero, and after a possible re-indexing of the variables x_i, we may assume that there exists a minimal integer $1 \leq r' \leq r$ such that $c_j^{(i)} = 0$, $\forall i > r'$, $\forall j \in [n]$, while for $i \leq r'$ at least one of the coefficients $c_1^{(i)}, \ldots, c_n^{(i)}$ is non-zero. Since $f_1, \ldots, f_n$ are minimal generators of I, this implies that $c_1^{(i)} f_1 + \cdots + c_n^{(i)} f_n$ is non-zero for every $i \leq r'$. Since $u \in IJ : \mathfrak{m}$, we have that $x_i u \in IJ$, $\forall i \in [r]$, which in particular implies that $0 \neq x_i^{s-d+1}\left(c_1^{(i)} f_1 + \cdots + c_n^{(i)} f_n\right) \in IJ$ for every $i \leq r'$. Hence, there exist polynomials $h_1, \ldots, h_n$ in J of degree $s - d + 1$, such that

$$0 \neq x_i^{s-d+1}\left(c_1^{(i)} f_1 + \cdots + c_n^{(i)} f_n\right) = h_1 f_1 + \cdots + h_n f_n, \quad \forall i \leq r'. \tag{5}$$

Consider the exact sequence $0 \longrightarrow \operatorname{Syz}_1(I) \longrightarrow \bigoplus_{j=1}^n S(-d) \longrightarrow I \longrightarrow 0$ of graded morphisms (i.e., each arrow has degree zero), where the second arrow sends the generator of each direct summand to a generator of I. Then equation (5) says that

$$Z_{1i} = \left(c_1^{(i)} x_i^{s-d+1} - h_1, \ldots, c_n^{(i)} x_i^{s-d+1} - h_n\right), \quad i \in [r']$$

is a non-zero element of $\mathrm{Syz}_1(I)$ of (shifted) degree $(s-d+1)+d=s+1$. Let

$$(q_{11},\dots,q_{1n}),\dots,(q_{\ell 1},\dots,q_{\ell n})$$

be minimal homogeneous generators of $\mathrm{Syz}_1(I)$ of (shifted) degrees $t_1,\dots,t_\ell$ respectively, i.e., $\deg(q_{\alpha\beta})=t_\alpha-d$, and set $t=\max_{\alpha\in[\ell]}\{t_\alpha\}$.

If $t\geq s+1$, then by definition of $\mathrm{reg}(I)$ we must have that the degree of Z_{1i} must be bounded from above by $\mathrm{reg}(I)+1$, i.e., $s+1\leq \mathrm{reg}(I)+1$, and so $\mathrm{sat}(IJ)\leq \mathrm{reg}(I)+1$, by which we are done.

So suppose that $t\leq s$, in which case we can write

$$\begin{aligned} Z_{1i} &= \left(c_1^{(i)}x_i^{s-d+1}-h_1,\dots,c_n^{(i)}x_i^{s-d+1}-h_n\right)\\ &= v_1\,(q_{11},\dots,q_{1n})+\cdots+v_\ell\,(q_{\ell 1},\dots,q_{\ell n}), \end{aligned} \tag{6}$$

where each v_α is a homogeneous polynomial of degree $s+1-t_\alpha>0$.

If $t=s$, then by the definition of $\mathrm{reg}(I)$ we must have that $s\leq\mathrm{reg}(I)+1$, from which we have $\mathrm{sat}(I)\leq\mathrm{reg}(I)+2$, and we are done.

Hence, suppose that $t\leq s-1$; we will show that in this case we arrive at a contradiction. To begin with, we isolate the jth coordinate of equation (6):

$$c_j^{(i)}x_i^{s-d+1}-h_j=v_1q_{1j}+\cdots+v_\ell q_{\ell j}. \tag{7}$$

For every $\alpha\in[\ell]$, we can write $v_\alpha=b_\alpha x_i^{s+1-t_\alpha}+\tilde{v}_\alpha$, where $b_\alpha\in k$, and $\tilde{v}_\alpha\in L_i$. Substituting in (7) and reordering terms we have

$$\begin{aligned} &x_i^{s+1-t}\left(c_j^{(i)}x_i^{t-d}-b_1x_i^{t-t_1}q_{1j}-\cdots-b_\ell x_i^{t-t_\ell}q_{\ell j}\right)\\ &=h_j+\tilde{v}_1q_{1j}+\cdots+\tilde{v}_\ell q_{\ell j}. \end{aligned} \tag{8}$$

This last equation shows that the polynomial $h_j+\tilde{v}_1q_{1j}+\cdots+\tilde{v}_\ell q_{\ell j}$ is divisible by x_i^{s+1-t}, by which we can write

$$h_j+\tilde{v}_1q_{1j}+\cdots+\tilde{v}_\ell q_{\ell j}=x_i^{s+1-t}\xi_j, \tag{9}$$

where ξ_j is either the zero polynomial or homogeneous of degree $t-d$. Since $h_j,\tilde{v}_\alpha\in L_i$, we necessarily have that $\xi_j\in L_i$. Combining (8) and (9) we get

$$c_j^{(i)}x_i^{t-d}-\xi_j=b_1x_i^{t-t_1}q_{1j}+\cdots+b_\ell x_i^{t-t_\ell}q_{\ell j}. \tag{10}$$

Since (10) is true for all $j\in[n]$, we equivalently have

$$\begin{aligned} &\left(c_1^{(i)}x_i^{t-d}-\xi_1,\dots,c_n^{(i)}x_i^{t-d}-\xi_n\right)\\ &=b_1x_i^{t-t_1}\,(q_{11},\dots,q_{1n})+\cdots+b_\ell x_i^{t-t_\ell}\,(q_{\ell 1},\dots,q_{\ell n}). \end{aligned}$$

This implies that $\left(c_1^{(i)}x_i^{t-d}-\xi_1,\dots,c_n^{(i)}x_i^{t-d}-\xi_n\right)\in\mathrm{Syz}_1(I)$, and so

$$x_i^{t-d}\left(c_1^{(i)}f_1+\cdots+c_n^{(i)}f_n\right)=\xi_1f_1+\cdots+\xi_nf_n.$$

Now, if $\xi_j\neq 0$, then $\deg(\xi_j)=t-d$ and every $x_i^{s-t}\xi_j$ is a polynomial of degree $s-d\geq r$. Moreover, since $\xi_j\in L_i$, every supporting monomial of $x_i^{s-t}\xi_j$ is divisible by at least two variables, and so $x_i^{s-t}\xi_j\in J$. Consequently, multiplying this last equation with x_i^{s-t} we see that

$$x_i^{s-d}\left(c_1^{(i)}f_1+\cdots+c_n^{(i)}f_n\right)\in IJ.$$

Since this last equation is true for any $i \in [r']$, by (4) and the definition of r' we have that $u \in IJ$, in contradiction to the hypothesis that $u \notin IJ$. Consequently, the hypothesis that $t < s$ is not a valid one, and the proof is concluded. ☐

PROPOSITION 2.9. *Let L_i be the ideal generated by all variables except x_i and set $J = L_1 \cdots L_r$. Then for I any homogeneous ideal generated in degree d we have*

$$\operatorname{reg}(IJ) \le \max\{r + d, \operatorname{reg}(I) + 2\}.$$

PROOF. Because the underlying field k is assumed infinite, the set of regular elements on S/I^{sat} is a non-empty open set of k^r and similarly for $S/(IJ)^{\text{sat}}$. Since k^r is irreducible, the intersection of these two open sets is non-empty, hence a linear form $\mu = c_1x_1 + \cdots + c_rx_r$, $c_i \in k$, that is regular on both $S/(I)^{\text{sat}}$ and $S/(IJ)^{\text{sat}}$ exists. Then by Proposition 2.6 we have that

$$\operatorname{reg}(I) = \max\{\operatorname{reg}(I + (\mu)), \operatorname{sat}(I)\}, \tag{11}$$

$$\operatorname{reg}(IJ) = \max\{\operatorname{reg}(IJ + (\mu)), \operatorname{sat}(IJ)\}. \tag{12}$$

We first bound from above $\operatorname{reg}(IJ + (\mu))$. Towards that end, suppose without loss of generality that $c_r \neq 0$, and let $S' = k[x_1, \ldots, x_{r-1}]$, $\mathfrak{m}' = (x_1, \ldots, x_{r-1})S'$, and I' the ideal of S' generated by the generators of I with x_r substituted with $-c_r^{-1}(c_1x_1 + \cdots + c_{r-1}x_{r-1})$. Then $S/(IJ + (\mu)) \cong S'/I'(\mathfrak{m}')^r$, and so

$$\operatorname{reg}(IJ + (\mu)) = \operatorname{reg}(I'(\mathfrak{m}')^r). \tag{13}$$

Since I' is a homogeneous ideal of S' generated in degree d, we have that the ideal $(\mathfrak{m}')^{\operatorname{reg}(I')-d}I'$ has a linear resolution and regularity $\operatorname{reg}(I')$. If $r \le \operatorname{reg}(I') - d$, then $\operatorname{reg}((\mathfrak{m}')^rI') = \operatorname{reg}(I')$, and otherwise $\operatorname{reg}((\mathfrak{m}')^rI') = \operatorname{reg}(I') + r - (\operatorname{reg}(I') - d) = r + d$. Consequently,

$$\operatorname{reg}(I'(\mathfrak{m}')^r) = \max\{r + d, \operatorname{reg}(I')\}. \tag{14}$$

On the other hand,

$$\operatorname{reg}(S'/I') = \operatorname{reg}(S/(I + (\mu))) \overset{(11)}{\le} \operatorname{reg}(S/I),$$

from which we conclude that $\operatorname{reg}(I') \le \operatorname{reg}(I)$. Thus (13) and (14) give

$$\operatorname{reg}(IJ + (\mu)) \le \max\{r + d, \operatorname{reg}(I)\}.$$

Combining this with Lemma 2.8 into (12) concludes the proof. ☐

COROLLARY 2.10. *Let L_i be the ideal generated by all variables except x_i and set $J = L_1 \cdots L_r$. Then for I any homogeneous ideal generated in degree d we have*

$$\operatorname{reg}(IJ) \le \operatorname{reg}(I) + \operatorname{reg}(J) - 1.$$

PROOF. First, note that $\operatorname{reg}(J) = r$ by Proposition 2.3. Next, since I is assumed to not have a linear resolution, $\operatorname{reg}(I) > d$. By Proposition 2.9 we have that $\operatorname{reg}(IJ) \le \max\{r + d, \operatorname{reg}(I) + 2\}$. Now, $r + d = (d + 1) + (r - 1) \le \operatorname{reg}(I) + \operatorname{reg}(J) - 1$. Also, $\operatorname{reg}(I) + 2 \le \operatorname{reg}(I) + (r - 1) = \operatorname{reg}(I) + \operatorname{reg}(J) - 1$. ☐

We now complete the proof of Theorem 2.1 as follows. First, recall that there is a $1 - 1$ correspondence between points p of $\mathbb{P}^{r-1}$ and ideals of S of dimension 1 that are generated by linear forms. Next, let $I(p_1), \ldots, I(p_r)$ be r such linear ideals defined by r points $p_1, \ldots, p_r$ of $\mathbb{P}^{r-1}$. We call these ideals *general* if the points $p_1, \ldots, p_r$ do not lie in any hyperplane of $\mathbb{P}^{r-1}$. In such a case, there exists a change of coordinates that maps each p_i to e_i, the latter having zeros everywhere except the ith coordinate. This change of coordinates induces a ring isomorphism $\phi : S \longrightarrow S$ such that $\phi(I(p_i)) = L_i := I(e_i)$ and $\operatorname{reg}(\phi(\mathcal{Q})) = \operatorname{reg}(\mathcal{Q})$ for any ideal $\mathcal{Q}$. Now let Λ be a collection of $r(\operatorname{reg}(I) - d)$ linear ideals of dimension 1, such that Λ admits a partition into $\operatorname{reg}(I) - d$ subsets $\Lambda_1, \dots, \Lambda_{\operatorname{reg}(I)-d}$, each consisting of r general ideals. For $\alpha \in [\operatorname{reg}(I) - d]$ denote by J_α the product of all r ideals in Λ_α. Moreover, let $\phi_\alpha : S \longrightarrow S$ be the ring isomorphism that takes all ideals of Λ_α to $L_1, \dots, L_r$. Then by Corollary 2.10 we have

$$\operatorname{reg}(IJ_1) = \operatorname{reg}(\phi_1(I)J) \le \operatorname{reg}(\phi_1(I)) + \operatorname{reg}(J) - 1 = \operatorname{reg}(I) + \operatorname{reg}(J_1) - 1.$$

Similarly,

$$\begin{aligned}\operatorname{reg}(IJ_1J_2) &= \operatorname{reg}(\phi_2(IJ_1)J) \le \operatorname{reg}(\phi_2(IJ_1)) + \operatorname{reg}(J) - 1 \\ &= \operatorname{reg}(IJ_1) + \operatorname{reg}(J_2) - 1 \le \operatorname{reg}(I) + \operatorname{reg}(J_1) + \operatorname{reg}(J_2) - 2.\end{aligned}$$

Repeating this for $\alpha = 3, \dots, \operatorname{reg}(I) - d$, we arrive at

$$\operatorname{reg}(IJ_1 \cdots J_{\operatorname{reg}(I)-d}) \le \operatorname{reg}(J_1) + \cdots + \operatorname{reg}(J_{\operatorname{reg}(I)-d}) + d = r(\operatorname{reg}(I) - d) + d.$$

But now $r(\operatorname{reg}(I) - d) + d$ is precisely the degree of the generators of $IJ_1 \cdots J_{\operatorname{reg}(I)-d}$, which means that the latter has a linear resolution.

C. Proof of Theorem 2.2

We proceed in several steps starting with a lemma, which is interesting on its own.

LEMMA 2.11. *Let $L_1, \dots, L_n$ be linear ideals of S, and I a homogeneous ideal generated in a single degree $d \ge 2$. Set $J = L_1 \cdots L_n$. If*

(i) $(I + J)_n = S_n$,
(ii) $(I \cap J)_{n+d} = (IJ)_{n+d}$, *and*
(iii) $n \ge \operatorname{reg}(I) - d$,

then $\operatorname{reg}(IJ) = n + d$, *i.e., IJ has a linear resolution.*

PROOF. By Proposition 2.3 $\operatorname{reg}(S/J) = n-1$. By hypothesis (i) $\operatorname{reg}(S/(I+J)) \le n - 1$. Then by the exact sequence

$$0 \to \frac{S}{I \cap J} \to \frac{S}{I} \oplus \frac{S}{J} \to \frac{S}{I + J} \to 0, \tag{15}$$

we get that

$$\begin{aligned}\operatorname{reg}(S/I \cap J) &\le \max\{\operatorname{reg}(S/I), \operatorname{reg}(S/J), \operatorname{reg}(S/(I+J)) + 1\} \\ &\le \max\{\operatorname{reg} I - 1, n - 1, n\} \\ &\overset{(iii)}{\le} \max\{n + d - 1, n - 1, n\} \overset{d \ge 2}{=} n + d - 1,\end{aligned} \tag{16}$$

i.e., $\operatorname{reg}(I \cap J) \le n + d$. This implies that $I \cap J$ is generated at most in degree $n + d$, which together with hypothesis (ii) gives $\operatorname{reg}(I \cap J / IJ) \le n + d - 1$. Hence, the exact sequence

$$0 \to \frac{I \cap J}{IJ} \to \frac{S}{IJ} \to \frac{S}{I \cap J} \to 0,$$

together with (16) gives

$$\operatorname{reg}(S/IJ) \le \max\{n + d - 1, n + d - 1\} = n + d - 1,$$

i.e., $\operatorname{reg}(IJ) \le n + d$. But IJ is generated in degree $n + d$ and so $\operatorname{reg}(IJ) = n + d$. $\square$

Remark 2.12. *Let* $\alpha = \operatorname{reg}(I)$. *Then* $\mathfrak{m}^{\alpha-d}I$ *has a linear resolution. Notice that this is a special case of Lemma 2.11. Indeed* $n = \alpha - d = \operatorname{reg} I - d$ *so that condition* (iii) *is satisfied. Moreover,* $(\mathfrak{m}^{\alpha-d} + I)_n = (\mathfrak{m}^{\alpha-d} + I)_{\alpha-d} = (\mathfrak{m}^{\alpha-d})_{\alpha-d} = S_{\alpha-d} = S_n$, *and so condition* (i) *is satisfied. Finally, if* $\nu \geq n + d = \alpha$, *write* $\nu = \alpha + \ell$. *Then* $(\mathfrak{m}^{\alpha-d}I)_\nu = (\mathfrak{m}^{\alpha-d}I)_{\alpha+\ell} = I_{\alpha+\ell} = (\mathfrak{m}^{\alpha+\ell} \cap I)_{\alpha+\ell} = (\mathfrak{m}^{\alpha-d} \cap I)_{\alpha+\ell} = (\mathfrak{m}^{\alpha-d} \cap I)_\nu$, *and so condition* (ii) *is satisfied.*

Remark 2.13. *The conditions of Lemma 2.11 are not necessary. For example for* $r = 3, L_1 = (x_1, x_2), L_2 = (x_1, x_3), L_3 = (x_2, x_3), I = (x_2^2, x_3^2)$, $L_1L_2L_3I$ *has a linear resolution, but condition* (i) *is not true.*

In what follows, for $i \in [r]$ we let L_i be the linear ideal generated by all variables except x_i and $J = L_1 \cdots L_r$. Let $s = \operatorname{height}(I_{\mathrm{ci}}) - 1$ and assume throughout that $d \geq 2$. In the next four lemmas we show that for the particular complete intersection ideal $I = (x_1^d, \ldots, x_s^d, x_{s+1}^d + \cdots + x_r^d)$, the ideal IJ^{d-1} satisfies the conditions of Lemma 2.11 and thus has a linear resolution.

Lemma 2.14. *The ideal* $J^{d-1} + I$ *agrees with* S *at degree* $r(d-1)$.

Proof. By Lemma 2.7 it is enough to show that $J^{d-1} + I$ contains all monomials of degree $r(d-1)$ of the form $x_i^\nu v$, where $\nu \geq (r-1)(d-1)+1$, for every $i \in [r]$, with v not divisible by x_i and of degree at most $d-2$. If $i \leq s$, this is true because in that case $x_i^d \in I$; so suppose that $i \geq s+1$. Without loss of generality we need to show that $x_{s+1}^\nu v \in J^{d-1} + I$. But this follows by noting that $x_{s+1}^{\nu-d} v (x_{s+1}^d + \cdots + x_r^d) \in I$, and every monomial of the form $x_{s+1}^{\nu-d} x_j^d v$ with $j > s+1$ lies in J^{d-1}, since the exponent of every variable in that monomial is less or equal than $(r-1)(d-1)$. □

Lemma 2.15. *Let* $u \notin J^{d-1}$ *be a monomial of degree* $r(d-1)$ *such that* $x_1^d u \in J^{d-1}$. *Then there exists some* $i > 1$, *integer* $\nu_i \geq (r-1)(d-1)+1$, *and monomial* v *of degree at most* $d-2$ *not divisible by* x_i, *such that* $x_1^d u = x_i^d u'$, *with* $u' = x_1^d x_i^{\nu_i - d} v \in J^{d-1}$.

Proof. By Lemma 2.7 there exists some $i \in [r]$ such that $u = x_i^{\nu_i} v$, with $\nu_i \geq (r-1)(d-1)+1$, and v not divisible by x_i and of degree at most $d-2$. If $i = 1$, then the exponent of x_1 in $x_1^d u$ is at least $d + (r-1)(d-1) + 1 = r(d-1) + 2$. Now, the hypothesis $x_1^d u \in J^{d-1}$ means that we can write $x_1^d u = wu''$, for some $u'' \in J^{d-1}$ of degree $r(d-1)$ and some w of degree d. Then by Lemma 2.7 the exponent of x_1 in u'' is at most $(r-1)(d-1)$ and since the exponent of x_1 in w is at most d, we have that the exponent of x_1 in $x_1^d u$ is at most $d + (r-1)(d-1) = r(d-1)+1$, which is a contradiction. Hence, $i > 1$, and without loss of generality we can take $i = 2$, i.e., $u = x_2^{\nu_2} v$, where $\nu_2 \geq (r-1)(d-1)+1$, v is not divisible by x_2 and $\deg(v) \leq d-2$.

Now $d \leq (r-1)(d-1)+1$, and so x_2^d divides u, and we can write $x_1^d u = x_2^d (x_1^d u / x_2^d) = x_2^d (x_1^d x_2^{\nu_2 - d} v)$. Let us show that $x_1^d x_2^{\nu_2-d} v \in J^{d-1}$. This will follow from Lemma 2.7 if we show that the exponent of every variable in $x_1^d x_2^{\nu_2-d} v$ does not exceed $(r-1)(d-1)$. For x_2 this exponent is at most $r(d-1) - d = r(d-1) - (d-1) - 1 = (r-1)(d-1) - 1 < (r-1)(d-1)$. Since the degree of v is at most $d-2$, and $d-2 < (r-1)(d-1)$, the exponent of x_i, for $i > 2$ in $x_1^d x_2^{\nu_2-d} v$ is strictly less than $(r-1)(d-1)$. Finally, the exponent of x_1 in $x_1^d x_2^{\nu_2-d} v$ is at most $d + (d-2) = 2(d-1) \leq (r-1)(d-1)$, since $r \geq 3$. □

Lemma 2.16. *The ideal* $I \cap J^{d-1}$ *agrees with the ideal* IJ^{d-1} *at degree* $r(d-1)+d$.

Proof. Let p be a polynomial of degree $r(d-1)+d$ that lies in $I \cap J^{d-1}$. Since $p \in I$, we can write

$$p = x_1^d p_1 + \cdots + x_s^d p_s + (x_{s+1}^d + \cdots + x_r^d)q,$$

where $p_1, \ldots, p_s, q$ are polynomials of degree $r(d-1)$. We will show that p is in IJ^{d-1}. Towards that end, we can without loss of generality assume that every monomial in the support of $p_1, \ldots, p_s, q$ does not lie in J^{d-1}. We may also assume without loss of generality that for every monomial u_j in the support of p_j, $j \in [s]$, the monomial $x_j^d u_j$ is in the support of the polynomial $x_1^d p_1 + \cdots + x_s^d p_s$.

We now show that every such $x_j^d u_j$ must lie in the support of p. For if not, then we must have that $x_j^d u_j = x_{j'}^d u$, for some $j' \geq s+1$ and u monomial in the support of q. Without loss of generality we can take $j = 1$ and $j' = s+1$, i.e., $x_1^d u_1 = x_{s+1}^d u$. Now, recalling that by hypothesis $u_1 \notin J^{d-1}$, Lemma 2.7 gives that $u_1 = x_{j''}^{\nu_{j''}} v$, for some $j'' \in [r]$, $r(d-1) \geq \nu_{j''} \geq (r-1)(d-1)+1$ and v not divisible by $x_{j''}$ and of degree at most $d-2$. Thus, $x_1^d x_{j''}^{\nu_{j''}} v = x_{s+1}^d u$ and by degree considerations we see that j'' must be equal to $s+1$. Hence, $u = x_1^d x_{s+1}^{\nu_{s+1}-d} v$. But then Lemma 2.7 gives that $u \in J^{d-1}$, which is a contradiction.

Next, we show that for every $j \in [s]$ and u_j monomial in the support of p_j, we have that $x_j^d u_j \in J^{d-1}I$. Without loss of generality we can take $j = 1$. By what we have already established $x_1^d u_1$ is in the support of p, and so $p \in J^{d-1} \cap I$ implies that $x_1^d u_1 \in J^{d-1}$, because J^{d-1} is a monomial ideal. Since by hypothesis $u_1 \notin J^{d-1}$, by Lemma 2.15 there exists $i > 1$, integer $r(d-1) \geq \nu_i \geq (r-1)(d-1)+1$, and monomial v of degree at most $d-2$ not divisible by x_i, such that $x_1^d u_1 = x_i^d u_1'$, with $u_1' = x_1^d x_i^{\nu_2 - d} v \in J^{d-1}$. If $i \leq s$, we are done, since then $x_i^d u_1' \in IJ^{d-1}$. So suppose that $i > s$ and without loss of generality take $i = s+1$. Thus $x_1^d u_1 = x_{s+1}^d (x_1^d x_{s+1}^{\mu_{s+1}} v)$, with $x_1^d x_{s+1}^{\mu_{s+1}} v \in J^{d-1}$, $(r-1)(d-1)-1 \geq \mu_{s+1} \geq (r-2)(d-1)$, and v of degree at most $d-2$ and not divisible by x_{s+1}. Then

$$\begin{aligned} x_1^d u_1 &= x_{s+1}^d (x_1^d x_{s+1}^{\mu_{s+1}} v) \\ &= -x_1^d (x_{s+2}^d + \cdots + x_r^d) x_{s+1}^{\mu_{s+1}} v + (x_{s+1}^d + \cdots + x_r^d) x_1^d x_{s+1}^{\mu_{s+1}} v, \end{aligned}$$

from which we see that $x_1^d u_1 \in IJ^{d-1}$, since by degree considerations $x_j^d x_{s+1}^{\mu_{s+1}} v \in J^{d-1}$ for $j = 1, s+2, \ldots, r$.

As a consequence, we are reduced to showing that if $p = (x_{s+1}^d + \cdots + x_r^d)q$ is in $I \cap J^{d-1}$, then $p \in IJ^{d-1}$, where q is of degree $r(d-1)$ and every monomial in its support lies outside J^{d-1}. By an argument similar as before, we have that for every monomial u in the support of q all monomials $x_{s+1}^d u, \ldots, x_r^d u$ are in the support of p, and as a consequence $x_{s+1}^d u, \ldots, x_r^d u \in J^{d-1}$. Then by Lemma 2.15 we have that $u = x_j^{\nu_j} v$, where $j \in [s]$, $\nu_j \geq (r-1)(d-1)+1$ and v is of degree at most $d-2$ and not divisible by x_j. Without loss of generality we can take $j = 1$. But then

$$(x_{s+1}^d + \cdots + x_r^d)u = x_1^d (x_{s+1}^d x_1^{\nu_1 - d} v + \cdots + x_r^d x_1^{\nu_1 - d} v) \in J^{d-1} I.$$

□

Lemma 2.17. *The ideal $J^{d-1}I$ has a linear resolution.*

Proof. Lemma 2.14 shows that condition $i)$ of Lemma 2.11 is satisfied. Lemma 2.16 shows that condition $ii)$ of Lemma 2.11 is satisfied. Moreover, $x_{s+1}^d + \cdots + x_r^d$ is $S/(x_1^d, \ldots, x_s^d)$-regular, since the only associated prime of $S/(x_1^d, \ldots, x_s^d)$ is

$(x_1, \ldots, x_s)$. Hence, I is generated by a regular sequence and so $\mathrm{reg}(I) = (s+1)d - s$. This, together with the fact that the number of linear ideals in the product J^{d-1} is $r(d-1)$ shows that condition *iii*) of Lemma 2.11 is also true. □

We now complete the proof of Theorem 2.2 as follows. Let $I(p_1), \ldots, I(p_r)$ be ideals of points $p_1, \ldots, p_r \in \mathbb{P}^{r-1}$, set $J_{\mathrm{fp}} = \big(I(p_1), \ldots, I(p_r)\big)^{d-1}$ and let I_{ci} be an ideal[3] generated by ℓ forms of degree d. Using J_{fp} as the product of linear forms in Lemma 2.11, we see that condition (i) is satisfied on an open set $\mathcal{U}_1$ (possibly empty) of the space parametrizing the generators of $I(p_1), \ldots, I(p_r), I_{\mathrm{ci}}$. From the exact sequence (15) we have that on $\mathcal{U}_1$

$$\mathrm{HF}(\nu, S/(I_{\mathrm{ci}} \cap J_{\mathrm{fp}})) = \mathrm{HF}(\nu, S/I_{\mathrm{ci}}) + \mathrm{HF}(\nu, S/J_{\mathrm{fp}}), \ \ \forall \nu \geq r(d-1).$$

On an open set $\mathcal{U}_2$ the ideals $I(p_1), \ldots, I(p_r)$ are distinct and so on $\mathcal{U}_2$ we have that $\mathrm{HF}(\nu, J_{\mathrm{fp}}) = \mathrm{HF}(\nu, J^{d-1})$ for every ν. On an open set $\mathcal{U}_3$ the generators of I_{ci} form a regular sequence and condition (*iii*) of Lemma 2.11 is true. Moreover, on $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ and for $\nu = r(d-1) + d$, $\mathrm{HF}(\nu, I_{\mathrm{ci}} \cap J_{\mathrm{fp}})$ is a constant[4] c, because on that open set

$$\mathrm{HF}(\nu, S/(I_{\mathrm{ci}} \cap J_{\mathrm{fp}})) = \mathrm{HF}(\nu, S/I) + \mathrm{HF}(\nu, S/J^{d-1}). \tag{17}$$

Now, condition $\mathrm{HF}(\nu, I_{\mathrm{ci}} J_{\mathrm{fp}}) = c$ is true on an open set $\mathcal{U}_4$, and so on the open set $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4$ all conditions of Lemma 2.11 are true. Finally, $\mathcal{U}$ is non-empty because by Lemmas 2.14 and 2.16 it contains the choice $L_1, \ldots, L_r, I$.

D. Examples

EXAMPLE 2.18. *We revisit Example 2.1 in* [**CH03**]. *Let* $S = k[a,b,c,d]$ *and consider two monomial ideals* $J = (a^2b, abc, bcd, cd^2)$ *and* $I = (b,c)$. *As mentioned in* [**CH03**] $\mathrm{reg}(J) = 3$ *while* $\mathrm{reg}(IJ) = 5$, *with already a non-linear syzygy present among the generators of* IJ. *According to Theorem 2.1 multiplication of* IJ *with the product of* $4 = 4(5-4) = r(\mathrm{reg}(IJ) - d)$ *ideals of general points will yield an ideal with linear resolution. On the other hand, a symbolic computation shows that the ideal of one general point is enough. Moreover, for special points we have that*

$$\mathrm{reg}\big(IJK\big) = 6, \quad K = (a,b,c)$$
$$\mathrm{reg}\big(IJK\big) = 5, \quad K = (a,b,d)$$

EXAMPLE 2.19. *In* $S = k[a,b,c,d]$ *let* I *be an ideal generated by* 3 *general forms of degree* 2. *Then* $\mathrm{reg}(I) = 4$. *As per Theorem 2.1 a linear resolution is achieved upon multiplication by* $r(\mathrm{reg}(I) - d) = 4(4-2) = 8$ *ideals of general points. Instead, Theorem 2.2 guarantees a linear resolution after multiplication by four ideals of general points. On the other hand, a computation shows that three points are enough.*

[3]The subscripts *fp* and *ci* stand for *fat points* and *complete intersection* respectively.

[4]$\mathrm{HF}(\nu, S/I)$ can be computed from the Hilbert series $(1-t^d)^\ell/(1-t)^n$ of S/I, while Lemma 2.7 can be used to show that $\mathrm{HF}(\nu, S/J^{d-1}) = p_{S/J^{d-1}}(\nu) = r\binom{d+r-3}{r-1}$.

CHAPTER 3

On the algebraic matroid of the determinantal variety

With k an infinite field and $[s] = \{1, \dots, s\}$ for any positive integer s, we let $k[Z] = k\big[z_{ij} : (i,j) \in [m] \times [n]\big]$ be a polynomial ring in the z_{ij}'s and $I_{r+1}(Z)$ the determinantal ideal generated by all $(r+1) \times (r+1)$ minors of the matrix $Z = \big(z_{ij} : (i,j) \in [m] \times [n]\big)$. With Ω a subset of $[m] \times [n]$ and $k[Z_\Omega] = k[z_{ij} : (i,j) \in \Omega]$, the images in $k[Z]/I_{r+1}(Z)$ of the z_{ij}'s with $(i,j) \in \Omega$ are algebraically independent over k if and only if $k[Z_\Omega] \cap I_{r+1}(Z) = 0$. Since $I_{r+1}(Z)$ is a prime ideal [**BV88**], the set of all such algebraically independent subsets of z_{ij}'s forms an algebraic matroid [**RST20**]. The rank of that matroid coincides with the Krull dimension of $k[Z]/I_{r+1}(Z)$ which is $r(m+n-r)$, i.e. every base set of the matroid has cardinality $r(m+n-r)$. A $Z_\Omega = \{z_{ij} : (i,j) \in \Omega\}$ with $\#\Omega = r(m+n-r)$ is a base if and only if the projection morphism $\pi_\Omega : \mathrm{M}(r, m \times n) \to \mathbb{A}^\Omega$ has finite fibers over a Zariski dense open subset on the source. Here $\mathrm{M}(r, m \times n) = \mathrm{Spec}(k[Z]/I_{r+1}(Z))$, $\mathbb{A}^\Omega = \mathrm{Spec}(k[Z_\Omega])$ and π_Ω is induced by the ring homomorphism $k[Z_\Omega] \to k[Z]/I_{r+1}(Z)$ where $z_{ij} \mapsto z_{ij} + I_{r+1}(Z)$ for $(i,j) \in \Omega$. Whenever no confusion arises, we will identify Z_Ω with Ω.

By elimination theory, the ideal $k[Z_\Omega] \cap I_{r+1}(Z)$ of $k[Z_\Omega]$ is generated by the elements of a Gröbner basis of $I_{r+1}(Z)$ that lie in $k[Z_\Omega]$, with the underlying term order being lexicographic and the variables Z_Ω the least significant. This gives an immediate characterization of the base sets for the extreme rank values $r = 1$ and $r = m-1$, since for these cases a description of a universal Gröbner basis is available. Indeed, for $r = 1$ the independent work of Sturmfels and Villarreal implies the existence of a universal Gröbner basis supported on the cycles of the complete bipartite graph $K_{m,n}$, e.g., see Theorem 3.1 in [**Con07**]. This makes the base sets of $\mathrm{M}(1, m \times n)$ those Ω's for which the corresponding bipartite graph is a tree; see [**SC10**] for an argument based on rigidity theory. On the other hand, Bernstein & Zelevinsky [**BZ93**] proved that for $r = m-1$ the maximal minors form a universal Gröbner basis; a result that was later generalized in [**Boo12**], [**CDNG15**], [**CDNG20**]. This makes the base sets of $\mathrm{M}(m-1, m \times n)$ those Z_Ω's that consist of collections of $m-1$ rows of Z together with any $m-1$ elements from the remaining row[1]. However, for $1 < r < m-1$ it is known that the $(r+1)$-minors do not in general form a universal Gröbner basis. Instead, the $(r+1)$-minors are a Gröbner basis for $I_{r+1}(Z)$ under any diagonal or anti-diagonal term order, e.g. see Theorem 5.4 in [**BC03**] and [**Stu90**]. As noted by Kalkbrener & Sturmfels [**KS95**] this yields a class of base sets for any r: with the partial order $z_{ij} \le z_{i'j'}$ if $i \le i'$ and $j \le j'$, an Ω is a base set of $\mathrm{M}(r, m \times n)$ if it does not contain an antichain

[1]A simpler justification for this characterization is also possible.

of cardinality $r+1$. Already though for $r=1$ it is easy to find examples of base sets that do not satisfy this condition. For $r=2$ D.I. Bernstein [**Ber17**] overcame these difficulties by using the tropicalization of the Grassmannian $\mathrm{Gr}(2,m)$ [**SS04**] and a connection with the completion of tree metrics to characterize the bases of $\mathrm{M}(2,m\times n)$ as those bipartite graphs for which an acyclic orientation exists with no alternating trails. This approach though appears to be intractable for $r\geq 3$.

Characterizing the matroid of $\mathrm{M}(r,m\times n)$ is also of great importance in the machine learning problem of low-rank matrix completion. There, one is interested in knowing a minimal number of precise locations in a matrix of rank r, that if observed, lead to a finite number of possible rank-r completions, i.e. to a finite fiber $\pi_\Omega^{-1}\big(\pi_\Omega(X)\big)$, e.g., see [**KT12**], [**KTT15**]; see also [**BBS20**] for a variation where the minimal completion rank is sought.

We call a set $\Phi=\bigcup_{j\in[m-r]}\phi_j\times\{j\}\subset[m]\times[m-r]$ an (r,m)-SLMF (Support of a Linkage Matching Field), if Φ satisfies the following conditions:

$$\#\phi_j=r+1,\ j\in[m-r] \quad \text{and} \quad \#\bigcup_{j\in\mathcal{J}}\phi_j\geq\#\mathcal{J}+r,\ \ \mathcal{J}\subseteq[m-r]. \tag{18}$$

SLMF's arise as the supports of the vertices of the Newton polytope of the product of maximal minors of an $m\times(m-r)$ matrix of variables. These were introduced by Sturmfels & Zelevinsky [**SZ93**] in their effort to establish the aforementioned universal Gröbner basis property of maximal minors, and have recently found applications in tropical geometry, e.g., see [**FR15**], [**LS20**]. Here we need a generalization of the notion of SLMF. Let us write $\Omega=\bigcup_{j\in[n]}\omega_j\times\{j\}$ for ω_j's subsets of $[m]$ and $\Omega_\mathcal{J}=\bigcup_{j\in\mathcal{J}}\omega_j\times\{j\}$ for $\mathcal{J}\subset[n]$.

Definition 3.1. *For $\mathcal{J}\subset[n]$ and ν a positive integer we call $\Omega_\mathcal{J}$ a relaxed (ν,r,m)-SLMF if $\sum_{j\in\mathcal{J}}\max\big\{\#\omega_j\cap\mathcal{I}-r,0\big\}\leq\nu(\#\mathcal{I}-r)$ for every $\mathcal{I}\subset[m]$ with $\#\mathcal{I}\geq r+1$, and equality for $\mathcal{I}=[m]$. Note that an (r,m)-SLMF is always a relaxed $(1,r,m)$-SLMF.*

In [**SZ93**] Sturmfels & Zelevinsky showed that a family of local coordinates on $\mathrm{Gr}(r,m)$ already known by Gelfand, Graev & Retakh [**GGR90**] from an analytic point of view, could be seen as induced by SLMF's. This connection is one of the key ingredients for the main result of this chapter:

Theorem 3.2. *If $\#\Omega=r(m+n-r)$ and there is a partition $[n]=\bigcup_{\ell\in[r]}\mathcal{J}_\ell$ with $\Omega_{\mathcal{J}_\ell}$ a relaxed $(1,r,m)$-SLMF for $\ell\in[r]$, then Ω is a base set of the algebraic matroid of* $\mathrm{M}(r,m\times n)$.

Another key ingredient in the proof of Theorem 3.2 is a novel interpretation of matrix completion in terms of linear sections on the Grassmannian $\mathrm{Gr}(r,m)$ via the use of Plücker coordinates. A natural consequence of this view is:

Proposition 3.3. *If Ω is a base set of the algebraic matroid of* $\mathrm{M}(r,m\times n)$*, then Ω is a relaxed (r,r,m)-SLMF.*

Therefore, a complete characterization of the algebraic matroid of $\mathrm{M}(r,m\times n)$ will be achieved once the following purely combinatorial conjecture is proved.

Conjecture 3.4. *Suppose $\#\Omega=r(m+n-r)$ and without loss of generality that each vertex in the bipartite graph associated with Ω has degree at least $r+1$ (Lemma 3.16). Then Ω is a relaxed (r,r,m)-SLMF if and only if there is a partition $[n]=\bigcup_{\ell\in[r]}\mathcal{J}_\ell$ with $\Omega_{\mathcal{J}_\ell}$ a relaxed $(1,r,m)$-SLMF for every $\ell\in[r]$.*

REMARK 3.5. *Conjecture 3.4 is trivially true for $r = 1$, which shows that the notion of a relaxed $(1,1,m)$-SLMF coincides with the notion of a tree. This fact is also easy to prove directly. The conjecture is immediate for $r = m-1$ and easy for $r = m-2$, while a proof for other values of r remains elusive. We discuss this further in §C. Finally, for $r = 2$ a comparison with D. I. Bernstein's characterization mentioned above is also not available.*

The following is a useful consequence of Theorem 3.2:

COROLLARY 3.6. *Suppose Ω satisfies the hypothesis of Theorem 3.2. Then there is a Zariski dense open set $U_\Omega \subset \mathrm{M}(r, m\times n)$ with $\pi_\Omega^{-1}\big(\pi_\Omega(X)\big)$ finite for every $X \in U_\Omega$.*

Among all bases of the matroid of $\mathrm{M}(r, m\times n)$ it is interesting to characterize those for which the fiber $\pi_\Omega^{-1}\big(\pi_\Omega(X)\big)$ contains a single element. Recently, for $k = \mathbb{R}, \mathbb{C}$, replacing π_Ω by $X \overset{\pi_F}{\longmapsto} \big(f_i(X) : i \in [N]\big)$ for an arbitrary collection $F = \big(f_i \in \mathrm{Hom}_k\big(k^{m\times n}, k\big) : i \in [N]\big)$ Rong, Wang & Xu [**RWX19**] proved that for general F the map π_F is injective on a dense subset of the rank-r matrices as long as $N > \dim \mathrm{M}(r, m\times n)$. They further conjectured the existence of special F's with $N = \dim \mathrm{M}(r, m\times n)$ that allow the same conclusion. Our next result settles their conjecture in the affirmative. For this, we note that if $\Omega_{\mathcal{J}}$ is a relaxed $(1,r,m)$-SLMF and denoting by Ω_j the set of subsets of ω_j of cardinality $r+1$, then there exist $\phi_{j'} \in \bigcup_{j\in\mathcal{J}} \Omega_j$ for $j' \in [m-r]$ such that $\Phi = \bigcup_{j'\in[m-r]} \phi_{j'} \times \{j'\}$ is an (r,m)-SLMF (Lemma 3.17). We say that $\Omega_{\mathcal{J}}$ induces the SLMF Φ.

PROPOSITION 3.7. *In addition to the hypothesis of Theorem 3.2 suppose that each $\Omega_{\mathcal{J}_\ell}$ induces the same (r,m)-SLMF $\Phi = \Phi_\ell$ for every $\ell \in [r]$. Then there exists a Zariski dense open set $U_\Omega \subset \mathrm{M}(r, m\times n)$ such that π_Ω is injective at the k-valued points of U_Ω. Moreover, any (r,m)-SLMF Φ induces such an Ω.*

A. Preliminaries

A.I. Local coordinates on $\mathrm{Gr}(r,m)$ induced by SLMF's. We recall the beautiful relationship between SLMF's and local coordinates on $\mathrm{Gr}(r,m)$ described in [**SZ93**], here presented more generally over an infinite field k.

Let $S \in \mathrm{Gr}(r,m)$ be a k-valued point and $S^\perp \in \mathrm{Gr}(m-r,m)$ the orthogonal complement of S. That is, if s_ℓ, $\ell \in [r]$ is a basis for S then $S^\perp$ is the vanishing locus of the linear forms induced by the s_ℓ's. Working with the standard basis of k^m the canonical isomorphism $\mathrm{Gr}(r-m,m) \to \mathrm{Gr}(r,m)$ sends the Plücker coordinate $[\psi]_{S^\perp}$ to $\sigma(\psi, [m]\smallsetminus\psi)\big[[m]\smallsetminus\psi\big]_S$, where ψ is any subset of $[m]$ of cardinality $m-r$ and $\sigma(\psi,[m]\smallsetminus\psi)$ is -1 raised to the number of elements $(a,b) \in \psi\times([m]\smallsetminus\psi)$ with $a > b$[2]. Let $A \in k^{m\times(m-r)}$ contain a basis of $S^\perp$ in its columns. Let $\Phi = \bigcup_{j\in[m-r]} \phi_j \times \{j\}$ be an (r,m)-SLMF. For $j \in [m-r]$ denote by H_j the k-subspace of k^{m-r} spanned by these rows of A indexed by $[m]\smallsetminus\phi_j$. The locus V_Φ of $\mathrm{Gr}(r,m)$ where the H_j's have codimension 1 and $\bigcap_{j\in[m-r]} H_j = 0$ is open. Suppose $S \in V_\Phi$. Then there is an automorphism of k^{m-r} that takes H_j to the hyperplane with normal vector e_j, the latter having zeros everywhere and a 1 at position j. Changing the basis we see that S can also be represented by some $\tilde{A} \in k^{m\times(m-r)}$ which is sparse with support on Φ. Let $\mathfrak{m}_{jj'}$ be the minor of A corresponding to row indices in $[m]\smallsetminus\phi_j$ and

[2]E.g., see §1.6 in Bruns & Herzog [**BH98**].

column indices $[m-r] \smallsetminus \{j'\}$ and set $M = (\mathfrak{m}_{jj'} : j, j' \in [m-r])$. Use respective notations $\tilde{\mathfrak{m}}_{jj'}$ and $\tilde{M} = (\tilde{\mathfrak{m}}_{jj'} : j, j' \in [m-r])$ for $\tilde{A}$. By definition of V_Φ all $\tilde{\mathfrak{m}}_{jj}$'s are non-zero thus, viewed as an element of $\mathbb{P}^{m-1}$, the j-th column $\tilde{a}_j$ of $\tilde{A}$ satisfies

$$\tilde{a}_{\phi_{ij}j} = (-1)^{i-1}\big[\phi_j \smallsetminus \{\phi_{ij}\}\big]_S \text{ for } i \in [r+1] \quad \text{and} \quad \tilde{a}_{ij} = 0 \text{ for } i \notin \phi_j$$

where ϕ_{ij} is the i-th element of ϕ_j. Next consider the rational maps $\gamma_{\phi_j} : \mathrm{Gr}(r,m) \dashrightarrow \mathbb{P}^r$ and $\gamma_\Phi : \mathrm{Gr}(r,m) \dashrightarrow \prod_{j \in [m-r]} \mathbb{P}^r$ given by

$$S \xmapsto{\gamma_{\phi_j}} \big((-1)^{i-1}[\phi_j \smallsetminus \{\phi_{ij}\}]_S : i \in [r+1]\big)$$
$$S \xmapsto{\gamma_\Phi} \big(\gamma_{\phi_j}(S) : j \in [m-r]\big)$$

Proposition 3.8 (Sturmfels & Zelevinsky [**SZ93**]). *The rational map γ_Φ is an open embedding on V_Φ. In particular, for k-valued $S \in V_\Phi$ the columns of $\tilde{A}|_S$ contain a basis for $S^\perp$, where $\tilde{A}|_S$ denotes the evaluation of $\tilde{A}$ at S interpreted as an element of $k^{m\times(m-r)}$.*

Let $T = k\big[[\psi] : \psi \subset [m], \#\psi = r\big]$ be a polynomial ring generated by variables $[\psi]$'s associated with the Plücker embedding of $\mathrm{Gr}(r,m)$, i.e. $\mathrm{Gr}(r,m) = \mathrm{Proj}(T/\mathfrak{p})$ with $\mathfrak{p}$ the ideal generated by the Plücker relations. By computing the normal vectors of the H_j's in terms of the $\mathfrak{m}_{jj'}$'s it follows that $S \in V_\Phi$ if and only if $\det(M) \neq 0$. Since $\tilde{M} = MC$ where C is an invertible matrix[3] we see that V_Φ is defined by the non-vanishing of the polynomial $\prod_{j\in[m-r]} \tilde{\mathfrak{m}}_{jj}$. This gives the equation of this hypersurface in Plücker coordinates [4]:

$$p_\Phi = \det\Big(\big[\phi_\alpha \smallsetminus \{\beta\}\big] : \alpha \in [m-r] \smallsetminus 1,\, \beta \in [m] \smallsetminus \phi_1\Big) \in T \tag{19}$$

A.II. Fibers of morphisms and dominance. For convenience we recall as needed the upper semicontinuity of the fiber dimension:

Proposition 3.9 (Exercise II.3.22 in Hartshorne [**Har77**]). *Let $g : Y \to W$ be a dominant morphism of integral schemes of finite type over a field k. Then for any $y \in Y$ we have that $\dim g^{-1}\big(g(y)\big) \geq \dim Y - \dim W$ with equality on a dense open set of Y.*

B. Proofs

We begin with some preparations. For $\omega \subseteq [m]$ define the coordinate projection $\pi_\omega : k^m \to k^{\#\omega}$ by $(\xi_i)_{i\in[m]} \mapsto (\xi_i)_{i\in\omega}$. For $B \in k^{m\times r}$ let $\pi_\omega(B) \in k^{\#\omega\times r}$ the matrix obtained by applying π_ω to the columns of B. For $j \in [n]$ we let Ω_j be the set of all subsets of ω_j of cardinality $r+1$. A k-valued point of $\mathrm{Gr}(r,m)$ is an r-dimensional linear subspace of k^m.

Lemma 3.10. *Let $S \in \mathrm{Gr}(r,m)$ be a k-valued point, $\omega \subseteq [m]$ with $\#\omega \geq r$ and suppose that $\dim \pi_\omega(S) = r$. Suppose that for some $x \in k^m$ we have $\pi_\omega(x) \in \pi_\omega(S)$. Then there exists unique $y \in S$ such that $\pi_\omega(y) = \pi_\omega(x)$.*

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[3]This follows from the functoriality of $\wedge^{m-r-1}$.

[4]If $\beta \notin \phi_\alpha$ then $\big[\phi_\alpha \smallsetminus \{\beta\}\big] = 0$. Only the sign may change if one replaces 1 by any $j \in [m-r]$ in (19).

Proof. Let $B \in k^{m\times r}$ be a basis for S. By hypothesis $\pi_\omega(B) \in k^{\#\omega\times r}$ is a basis of $\pi_\omega(S)$. Then there is a unique $c \in k^r$ such that $\pi_\omega(x) = \pi_\omega(B)c$. Define $y = Bc$, clearly $\pi_\omega(x) = \pi_\omega(y)$. Suppose $\pi_\omega(y) = \pi_\omega(y')$ for some other $y' \in S$. There is a unique $c' \in k^r$ such that $y' = Bc'$. On the other hand, the equation $\pi_\omega(y) = \pi_\omega(y')$ implies that $\pi_\omega(B)(c - c') = 0$. But $\pi_\omega(B)$ has rank r and so $c = c'$. □

Lemma 3.11. *Let $S \in \mathrm{Gr}(r,m)$ be a k-valued point, $x \in k^m$ and $\omega, \omega' \subseteq [m]$ with $\pi_\omega(x) \in \pi_\omega(S)$ and $\pi_{\omega'}(x) \in \pi_{\omega'}(S)$. If $\dim \pi_{\omega\cap\omega'}(S) = r$ then $\pi_{\omega\cup\omega'}(x) \in \pi_{\omega\cup\omega'}(S)$.*

Proof. There exist $y, y' \in S$ such that $\pi_\omega(x) = \pi_\omega(y)$ and $\pi_{\omega'}(x) = \pi_{\omega'}(y')$. This implies that $\pi_{\omega\cap\omega'}(x) = \pi_{\omega\cap\omega'}(y) = \pi_{\omega\cap\omega'}(y')$. Lemma 3.10 gives $y = y'$. □

Lemma 3.12. *Let $\phi = \{i_1 < \cdots < i_{r+1}\} \subseteq [m]$, let $x \in k^m$ and $S \in \mathrm{Gr}(r,m)$ a k-valued point with $\dim \pi_\phi(S) = r$. Then $\pi_\phi(x) \in \pi_\phi(S)$ if and only if*

$$\sum_{\alpha\in[r+1]} (-1)^{\alpha-1}\, x_{i_\alpha}\, [\phi \setminus \{i_\alpha\}]_S = 0$$

Proof. With $B \in k^{m\times r}$ a basis for S and any $\alpha \in [r+1]$ we identify $[\phi \setminus \{i_\alpha\}]_S$ with $\det\left(\pi_{\phi\setminus\{i_\alpha\}}(B)\right)$. Applying Laplace expansion on the first column of the matrix $[\pi_\phi(x)\ \pi_\phi(B)] \in k^{(r+1)\times(r+1)}$ shows that $\det\left([\pi_\phi(x)\ \pi_\phi(B)]\right) = 0$ is equivalent to the formula in the statement. Since $\pi_\phi(B)$ has rank r, $\det\left([\pi_\phi(x)\ \pi_\phi(B)]\right) = 0$ is equivalent to $\pi_\phi(x) \in \pi_\phi(S)$. □

Lemma 3.13. *Let $\Phi = \bigcup_{j\in[m-r]} \phi_j \times \{j\}$ be an (r,m)-SLMF and let $S \in V_\Phi$ be a k-valued point. Then $\dim \pi_{\phi_j}(S) = r$ for every $j \in [m-r]$.*

Proof. Since $S \in V_\Phi$ Proposition 3.8 gives that $\tilde{A}|_S$ has full column rank. On the other hand, $\dim \pi_{\phi_j}(S) < r$ if and only if all Plücker coordinates $[\phi_j \setminus \{\phi_{ij}\}]_S$ are zero, where ϕ_{ij} denotes the i-th element of ϕ_j. But in that case the j-th column of $\tilde{A}|_S$ would be zero by definition of $\tilde{A}$. □

Lemma 3.14. *Let $\Phi = \bigcup_{j\in[m-r]} \phi_j \times \{j\}$ be an (r,m)-SLMF and let $S \in V_\Phi$ be a k-valued point. If $\pi_{\phi_j}(x) \in \pi_{\phi_j}(S)$ for every $j \in [m-r]$, then $x \in S$.*

Proof. By Lemma 3.13 $\dim \pi_{\phi_j}(S) = r$ for every $j \in [m-r]$. Then Lemma 3.12 implies that the relation $\pi_{\phi_j}(x) \in \pi_{\phi_j}(S)$ is equivalent to $\pi_{\phi_j}(x)$ being orthogonal to the j-th column of $\tilde{A}|_S$. Since this is true for every $j \in [m-r]$ and since the columns of $\tilde{A}|_S$ form a basis for $S^\perp$ this implies that $x \in S$. □

Lemma 3.15. *Let $k \hookrightarrow K$ be a field extension. Then the algebraic matroid of T/I coincides with the algebraic matroid of $T/I \otimes_k K$.*

Proof. A set of z_{ij}'s with $(i,j) \in \Omega$ form an independent set in the matroid of T/I if and only if the ring homomorphism $k[z_{ij} : (i,j) \in \Omega] \to T/I$ is injective. Since K is a faithfully flat k-module, this is equivalent to the injectivity of $K[z_{ij} : (i,j) \in \Omega] \to T/I \otimes_k K$. □

Lemma 3.16. *Insertion or deletion of ω_j's with $\#\omega_j = r$ do not affect the property of Ω of being a base set.*

Proof. Suppose $Z_\Omega = \{z_{ij} : (i,j) \in \Omega\}$ is algebraically dependent mod $I_{r+1}(Z)$. We consider the lexicographic term order on $k[Z]$ with $z_{11} > z_{21} > \cdots > z_{m1} > z_{12} > z_{22} > \cdots > z_{m2} > z_{13} > \cdots > z_{m-1,n} > z_{mn}$. This is a diagonal term order in the sense that the leading term of every minor is the product of the variables on the main diagonal. With respect to that order the $(r+1)$-minors of Z form a Gröbner basis (Theorem 5.4 in [**BC03**]). Since $I_{r+1}(Z) = I_{r+1}(P_1 Z P_2)$ for any permutations P_1, P_2 [**BV88**], we may assume that $\omega_1 = \{m-r+1, \dots, m\}$. Write $Z = [z\ Z']$ where z is the first column of Z and set $Z'_\Omega = \{z_{ij} : (i,j) \in \Omega,\ j > 1\}$. By elimination theory the elimination ideal $k[z_{m-r+1,1}, \dots, z_{m1}, Z'] \cap I_{r+1}(Z)$ is generated by the $(r+1)$-minors of Z'. Hence $k[Z_\Omega] \cap I_{r+1}(Z) \subset k[z_{m-r+1,1}, \dots, z_{m1}, Z'] \cap I_{r+1}(Z) \subset I_{r+1}(Z')$. Thus Z_Ω is algebraically dependent mod $I_{r+1}(Z')$. But $k[Z]/I_{r+1}(Z') \cong k[Z']/I_{r+1}(Z')[z]$. Hence Z'_Ω is algebraically dependent mod $I_{r+1}(Z')$. The converse direction is clear by definition. □

Lemma 3.17. *Suppose $\Omega_{\mathcal{J}} = \bigcup_{j \in \mathcal{J}} \omega_j \times \{j\}$ is a relaxed $(1,r,m)$-SLMF for some $\mathcal{J} \subset [n]$. Denote by Ω_j the set of subsets of ω_j of cardinality $r+1$. Then there exist $\phi_{j'} \in \bigcup_{j \in \mathcal{J}} \Omega_j$ for $j' \in [m-r]$ such that $\Phi = \bigcup_{j' \in [m-r]} \phi_{j'} \times \{j'\}$ is an (r,m)-SLMF.*

Proof. For each $j \in \mathcal{J}$ fix any $\omega'_j \subset \omega_j$ with $\#\omega'_j = r$ and for every $\kappa \in \omega_j \smallsetminus \omega'_j$ define $\phi_{j,\kappa} = \omega'_j \cup \{\kappa\}$. Setting $\mathcal{I} = [m]$ in the definition of relaxed $(1,r,m)$-SLMF gives $\sum_{j \in \mathcal{J}} \max\{\#\omega_j \smallsetminus \omega'_j - r, 0\} = m - r$ so that in total we have $m-r$ $\phi_{j,\kappa}$'s and thus we can order them as $\phi_1, \dots, \phi_{m-r}$. Then $\Phi = \bigcup_{j' \in [m-r]} \phi_{j'} \times \{j'\}$ is an (r,m)-SLMF. □

B.I. Proof of Theorem 3.2. In view of Lemma 3.15 we may assume that k is algebraically closed. In view of Lemma 3.16 we may assume that $\#\omega_j \ge r+1$ for every $j \in [n]$. By Lemma 3.17 for every $\ell \in [r]$ there are $\phi_j^\ell \in \bigcup_{j' \in \mathcal{J}_\ell} \Omega_{j'}$, $j \in [m-r]$ such that $\Phi_\ell = \bigcup_{j \in [m-r]} \phi_j^\ell \times \{j\}$ is an (r,m)-SLMF. For a closed point $X \in \mathrm{M}(r, m \times n)$ and S the column-space of X the condition $S \in \bigcap_{\ell \in [r]} V_{\Phi_\ell}$ is true on an open set of $\mathrm{M}(r, m \times n)$ which can be described as follows. Let $p = \prod_{\ell \in [r]} p_{\Phi_\ell}$ where p_{Φ_ℓ} is given by (19). For any $\psi \subseteq [n]$ with $\#\psi = r$ replace every $[\phi_\alpha^\ell \smallsetminus \{\beta\}]$ in p by the $r \times r$ minor of Z with row indices $\phi_\alpha^\ell \smallsetminus \{\beta\}$ and column indices ψ to obtain a polynomial $p_\psi \in k[Z]$. Varying ψ gives the open set $U_\Omega = \bigcup_{\psi \subseteq [n], \#\psi = r} \mathrm{Spec}(k[Z]/I_{r+1}(Z))_{\bar{p}_\psi}$ of $\mathrm{M}(r, m \times n)$, where $(k[Z]/I_{r+1}(Z))_{\bar{p}_\psi}$ is the localization of $k[Z]/I_{r+1}(Z)$ at the multiplicatively closed set $\{1, \bar{p}_\psi, \bar{p}_\psi^2, \dots\}$, with $\bar{p}_\psi$ the class of p_ψ in $k[Z]/I_{r+1}(Z)$. Then $S \in \bigcap_{\ell \in [r]} V_{\Phi_\ell}$ if and only if $X \in U_\Omega$. To see that U_Ω is non-empty, first note that $\bigcap_{\ell \in [r]} V_{\Phi_\ell}$ is the intersection of finitely many dense open sets and thus non-empty. Let $S \in \bigcap_{\ell \in [r]} V_{\Phi_\ell}$ be any closed point and s_ℓ, $\ell \in [r]$ a k-basis for S. Define $X \in \mathrm{M}(r, m \times n)$ by setting $x_j = s_\ell$ whenever $j \in \mathcal{J}_\ell$. Then $p_\psi(X) \neq 0$ for any ψ that contains exactly one index from each $\mathcal{J}_\ell$, i.e. $X \in U_\Omega$.

Let $\pi'_\Omega : U_\Omega \to \mathbb{A}^\Omega$ be the restriction of π_Ω to U_Ω. Let $X' \in \pi_\Omega'^{-1}(\pi'_\Omega(X))$ be a closed point in the fiber over the X defined above. Let S' be the column-space of X'. Then by construction $\pi_{\phi_j^\ell}(s_\ell) \in \pi_{\phi_j^\ell}(S')$ for every $j \in [m-r]$ and every $\ell \in [r]$. Since $X' \in U_\Omega$ we have $S' \in \bigcap_{\ell \in [r]} V_{\Phi_\ell}$ so that by Lemma 3.14 we must have that $s_\ell \in S'$ for every $\ell \in [r]$. But then $S' = S$. By Lemma 3.13 $\dim \pi_{\omega_j}(S') = \dim \pi_{\omega_j}(S) = r$ for every $j \in [n]$, and so Lemma 3.10 gives $X' = X$. Since X is the only closed point of the fiber, and since the fiber is a Jacobson space [**Sta20**], it is equal to the closure of X, i.e., $\pi_\Omega'^{-1}(\pi'_\Omega(X)) = \{X\}$ as a topological space and thus $\pi_\Omega'^{-1}(\pi'_\Omega(X))$ is a zero-dimensional scheme.

Now π_Ω is a morphism of integral schemes. Since an open subscheme of an integral subscheme is integral, we have that π'_Ω is also a morphism of integral schemes. Since $\dim U_\Omega = \dim \mathbb{A}^\Omega = \dim \mathrm{M}(r, m \times n)$, if π'_Ω were not dominant then the minimum fiber dimension of π'_Ω would be positive (Proposition 3.9). But as we just saw $\dim \pi_\Omega'^{-1}\big(\pi'_\Omega(X)\big) = 0$ and so π'_Ω must be dominant. But then π_Ω must also be dominant. Since a ring homomorphism $A \to B$ of integral domains is injective if and only if the corresponding morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is dominant, we have that $k[Z_\Omega] \to k[Z]/I_{r+1}(Z)$ is injective, i.e. Ω is a base set of the algebraic matroid of $k[Z]/I_{r+1}(Z)$.

B.II. Proof of Proposition 3.3. By Lemma 3.15 we may assume that k is algebraically closed. Suppose Ω is a base set of $\mathrm{M}(r, m \times n)$. Then there is a non-empty open set $U_\Omega \subset \mathrm{M}(r, m \times n)$ such that the fiber $\pi_\Omega^{-1}(\pi_\Omega(X))$ is a zero-dimensional scheme for any $X \in U_\Omega$. Fix an $X = [x_1 \cdots x_n] \in U_\Omega$ whose column-space has dimension r and preserves that dimension upon projection onto any r coordinates. Denote by x_{ij} the i-th coordinate of x_j. For any $j \in [n]$ fix a $\psi_j \subset \omega_j$ with $\#\psi_j = r$ and for every $\kappa \in \omega_j \setminus \psi_j$ let $\phi_{j,\kappa} = \psi_j \cup \{\kappa\} = \{i_1, \dots, i_{r+1}\} \in \Omega_j$. With this we define a linear form in Plücker coordinates

$$l_{j,\kappa} = \sum_{\alpha \in [r+1]} (-1)^{\alpha-1} x_{i_\alpha j} \left[\phi_{j,\kappa} \setminus \{i_\alpha\}\right] \in T$$

Let L_Ω be the ideal of T generated by $l_{j,\kappa}$ for all j's and κ's. Let p be the product of all Plücker coordinates, $\bar{p}$ its class in $T/\mathfrak{p}$ and $(T/\mathfrak{p})_{(\bar{p})}$ the homogeneous localization of $T/\mathfrak{p}$ at the multiplicatively closed set $\{1, \bar{p}, \bar{p}^2, \dots\}$. In view of Lemmas 3.11 and 3.12 every closed point of $\mathrm{Proj}\big((T/\mathfrak{p} + L_\Omega)_{(\bar{p})}\big)$ is an r-dimensional linear subspace S of k^m for which $\pi_{\omega_j}(x_j) \in \pi_{\omega_j}(S)$ for every $j \in [n]$. Thus by Lemma 3.10 every such S gives a unique closed point $X' \in \pi_\Omega^{-1}(\pi_\Omega(X))$, i.e. a completion of $\pi_\Omega(X)$. If $\mathrm{rank}(X') = r$ the column-space of X' is necessarily equal to S. Since $\pi_\Omega^{-1}(\pi_\Omega(X))$ is a finite set, there are only finitely many closed points S in $\mathrm{Gr}(r, m)$ that give rank-r completions of $\pi_\Omega(X)$. In fact, the locus $V \subset \mathrm{Proj}\big((T/\mathfrak{p} + L_\Omega)_{(\bar{p})}\big)$ where every closed point gives a rank-r completion is non-empty and open. Let $L_{\Omega,1}$ be the k-vector space of linear forms in L_Ω. By Krull's height theorem $\dim V \geq \dim \mathrm{Gr}(r, m) - \dim_k L_{\Omega,1}$. By the finiteness of $\pi_\Omega^{-1}(\pi_\Omega(X))$ we must have $\dim V = 0$ and so $\dim_k L_{\Omega,1} \geq r(m-r)$. That is, there must be at least $r(m-r)$ linearly independent $l_{j,\kappa}$'s. On the other hand, since necessarily $\#\omega_j \geq r$ and since $\sum_{j\in[n]} \#\omega_j = r(m+n-r)$, there are exactly $\sum_{j \in [n]} \max\{\#\omega_j - r, 0\} = r(m-r)$ $l_{j,\kappa}$'s. This is the condition in the definition of an (r, r, m)-SLMF obtained for $\mathcal{I} = [m]$ and it here implies that all $l_{j,\kappa}$'s must be linearly independent. Note that this must be true for any choice of the ψ_j's.

Suppose there is some $\mathcal{I} \subsetneq [m]$ for which $\sum_{j\in[n]} \max\{\#\omega_j \cap \mathcal{I} - r, 0\} > r(\#\mathcal{I} - r)$. Let us write $\sum_{j \in [n]} \max\{\#\omega_j \cap \mathcal{I} - r, 0\} = \sum_{j \in \mathcal{J}} (\#\omega_j \cap \mathcal{I} - r)$ where the terms for $j \in \mathcal{J}$ are those that have a non-zero contribution. Now for every $j \in \mathcal{J}$ choose ψ_j to lie in $\mathcal{I} \cap \omega_j$. Then the inequality above says that there are more than $r(\#\mathcal{I} - r)$ $l_{j,\kappa}$'s contributed by the ω_j's indexed by $j \in \mathcal{J}$, and they must be linearly independent by what we said above. On the other hand, these $l_{j,k}$'s are linear forms in Plücker coordinates that are supported inside $\mathcal{I}$ and thus the maximal number of linearly independent such forms can not exceed the dimension of the corresponding Grassmannian, which is $r(\#\mathcal{I} - r)$. This contradiction shows that Ω must be a relaxed (r, r, m)-SLMF.

B.III. Proof of Corollary 3.6. By Theorem 3.2 Ω is a base set of the matroid of T/I. Thus the ring homomorphism of integral domains $T_\Omega \to T/I$ is injective and so the induced morphism $\pi_\Omega : \mathrm{Spec}(T/I) \to \mathrm{Spec}(T_\Omega)$ is dominant. Then the claim follows from Proposition 3.9.

B.IV. Proof of Proposition 3.7. We prove the first statement for $\bar{k}$ the algebraic closure of k. Set $\mathrm{M}_{\bar{k}}(r, m\times n) = \mathrm{Spec}(k[Z]/I_{r+1}(Z)\otimes_k \bar{k})$ and $\mathrm{Gr}_{\bar{k}}(r,m) = \mathrm{Gr}(r,m)\times_k \mathrm{Spec}(\bar{k}) = \mathrm{Proj}(T/\mathfrak{p})\times_k \mathrm{Spec}(\bar{k})$. Write $\Phi_\ell = \Phi = \bigcup_{j\in[m-r]} \phi_j \times \{j\}$ for every ℓ. Then for every $\alpha \in [m-r]$ there is a subset $\mathcal{L}_\alpha \subseteq [n]$ of cardinality r such that $\phi_\alpha \subseteq \omega_j$, $\forall j \in \mathcal{L}_\alpha$. For a closed point $X = [x_1 \cdots x_n] \in \mathrm{M}_{\bar{k}}(r, m\times n)$ denote by $\mathfrak{c}(X)$ the column-space of X. Call $U_{\Omega,\bar{k}}$ the non-empty open set of $\mathrm{M}_{\bar{k}}(r, m\times n)$ on which $\mathfrak{c}(X)$ lies in $V_{\Phi,\bar{k}} \subseteq \mathrm{Gr}_{\bar{k}}(r,m)$, none of the Plücker coordinates of $\mathfrak{c}(X)$ vanishes and the $\{x_j : j \in \mathcal{L}_\alpha\}$ are linearly independent for every $\alpha \in [m-r]$. Since $\mathrm{Span}(x_j : j\in\mathcal{L}_\alpha)$ is the same as $\mathfrak{c}(X)$ so will be their projections under π_{ϕ_α}. Proposition 3.8 asserts that the data $\pi_{\phi_\alpha}\big(\mathfrak{c}(X)\big)$, $\alpha\in[m-r]$ uniquely determine $\mathfrak{c}(X)$ on $V_{\Phi,\bar{k}}$. Since $\#\omega_j \geq r$, $\forall j\in[n]$ and $\pi_{\omega_j}\big(\mathfrak{c}(X)\big)$ does not drop dimension, Lemma 3.10 gives that the data $\mathfrak{c}(X)$, $\pi_\Omega(X)$ uniquely determine X. Hence, the following data uniquely determine X for any closed $X\in U_{\Omega,\bar{k}}$:

$$\pi_{\phi_\alpha}\big(\mathrm{Span}(x_j : j\in\mathcal{L}_\alpha)\big),\ \alpha\in[m-r] \quad \text{and} \quad \pi_\Omega(X)$$

We have proved that the restriction of $\pi_{\Omega,\bar{k}}$ on the dense open set $U_{\Omega,\bar{k}} \subseteq \mathrm{M}_{\bar{k}}(r, m\times n)$ is injective at closed points. Now note that the defining polynomials of $U_{\Omega,\bar{k}}$ do not depend on the field $\bar{k}$. Since $U_{\Omega,\bar{k}}$ is non-empty not all of these polynomials are zero in $T/I\otimes_k \bar{k}$. But then not all of them will be zero in T/I. Hence, they also define a non-empty open set U_Ω of $\mathrm{M}(r, m\times n)$. This U_Ω must be dense because $\mathrm{M}(r, m\times n)$ is an integral scheme and thus it is irreducible. Then the injectivity at k-valued points of the restriction of π_Ω on U_Ω is inherited from the injectivity at closed points of $\pi_{\Omega,\bar{k}}$ restricted on $U_{\Omega,\bar{k}}$.

We now prove the second claim of the statement. Let $\Phi \subseteq [m]\times[m-r]$ be any (r,m)-SLMF. We prove the existence of an $\Omega\subseteq[m]\times[n]$ such that 1) $\#\Omega = \dim \mathrm{M}(r, m\times n)$, 2) $\#\omega_j \geq r$ and 3) for every $\alpha\in[m-r]$ there is a subset $\mathcal{L}_\alpha \subseteq [n]$ of cardinality r with $\phi_\alpha\subseteq\omega_j$, $\forall j\in\mathcal{L}_\alpha$. We argue by induction on n. For $n=r$ take $\Omega = [m]\times[n]$. Suppose $n>r$. By induction there is an $\Omega' \subseteq [m]\times[n-1]$ with the required as above properties. Then take $\Omega = \Omega' \cup \big([r]\times\{n\}\big)$.

C. On Conjecture 3.4

We prove the conjecture for the following extreme rank values:

Proposition 3.18. *Conjecture 3.4 is true for* $r = 1, m-2, m-1$.

Proof. Only the *only if* part needs proving. For $r=1$ there is nothing to prove. Recall that the conjecture is stated under the hypothesis that $\#\omega_j \geq r+1$ for every $j\in[n]$. We may also assume $m\leq n$ without loss of generality. Thus when $r = m-1$ we have $\omega_j = [m]$ for every $j\in[n]$. Since $\dim \mathrm{M}(m-1, m\times n) = (m-1)(n+1)$ and this value must be equal to mn we necessarily have $n = m-1$. Taking $\mathcal{J}_\ell = \{\ell\}$ for every $\ell\in[n]=[m-1]$ gives the required partition and proves the conjecture for $r = m-1$.

When $r=m-2$ each ω_j is either equal to $[m]$ or has cardinality $m-1$. Without loss of generality we assume that $\omega_j = [m]$ for $j = n-\alpha+1,\dots,n$ and $\#\omega_j = m-1$ for $j\in[\alpha]$, for some non-negative integer α. A counting argument as before shows that $n = 2m-4-\alpha$. We construct the partition $[n] = \bigcup_{\ell\in[m-2]} \mathcal{J}_\ell$ as follows. For $\ell = m-1-\alpha,\dots,m-2$ set $\mathcal{J}_\ell = \{\ell\}$. The rest of the $\mathcal{J}_\ell$'s for $\ell\in[m-2-\alpha]$ will contain two elements and we show how to get them. For $j\in[n-\alpha] = [2m-4-2\alpha]$ we re-order the ω_j's such that equal ω_j's are placed consecutively. Then we assign cyclically these ω_j's to $m-2-\alpha$ ordered cells $\mathcal{J}_1,\dots,\mathcal{J}_{m-2-\alpha}$, by placing ω_j to $\mathcal{J}_{\max\{j \bmod m-1-\alpha,1\}}$. We claim that each $\mathcal{J}_\ell$ is a relaxed $(1,m-2,m)$-SLMF. For $\ell > m-2-\alpha$ this is clear. So suppose that $\mathcal{J}_\kappa = \{\omega_i,\omega_{i+m-2-\alpha}\}$ is not a relaxed $(1,m-2,m)$-SLMF for some $\kappa\in[m-2-\alpha]$ and some $i\in[n-\alpha]$. The only way this can happen is if $\omega_i=\omega_{i+m-2-\alpha}$. In that case for $\mathcal{I}=\omega_i$ we have

$$\sum_{j\in[n]}\big(\#\omega_j\cap\mathcal{I}-(m-2)\big)=\sum_{j\in[n-\alpha]}\big(\#\omega_j\cap\mathcal{I}-(m-2)\big)+\sum_{j\geq n-\alpha+1}\big(\#\omega_j\cap\mathcal{I}-(m-2)\big)$$
$$\geq\big((m-2-\alpha)+1\big)+\alpha=m-2+1>r(\#\mathcal{I}-r)=m-2$$

which violates the hypothesis of relaxed $(m-2,m-2,m)$-SLMF on Ω. □

On the other hand, it is not clear how to generalize the clustering algorithm described in the proof for $r=m-2$, the difficulty being determining the ordering of the ω_j's. To get a better feeling for the statement of the conjecture we consider it in the boundary case where $n=r(m-r)$ and $\#\omega_j=r+1$ for every $j\in[n]$. By Hall's marriage theorem $\Phi=\bigcup_{j\in[m-r]}\phi_j\times\{j\}\subset[m]\times[m-r]$ is an (r,m)-SLMF if and only if there exists a perfect matching in the bipartite subgraph of Φ induced by removing any r indices from $[m]$. In turn, this is equivalent to saying that for any $\mathcal{I}\subset[m]$ with $\#\mathcal{I}=m-r$ the ϕ_j's have a system of distinct representatives in $\mathcal{I}$, in the sense that we can write $\mathcal{I}=\{i_1,\dots,i_{m-r}\}$ with $i_j\in\phi_j$ for every $j\in[m-r]$. In this terminology Conjecture 3.4 becomes equivalent to:

Conjecture 3.19. *Suppose that $n=r(m-r)$ and $\#\omega_j=r+1$ for every $j\in[n]$. Then for any $\mathcal{I}\subset[m]$ with $\#\mathcal{I}=m-r$ there exists a partition $[n]=\bigcup_{\ell\in[r]}\mathcal{J}_\ell$ (in general depending on $\mathcal{I}$) with $\#\mathcal{J}_\ell=m-r$ such that every $\{\omega_j : j\in\mathcal{J}_\ell\}$ has a system of distinct representatives in $\mathcal{I}$, if and only if there exists a partition $[n]=\bigcup_{\ell\in[r]}\mathcal{J}_\ell$ such that for any $\mathcal{I}\subset[m]$ with $\#\mathcal{I}=m-r$ every $\{\omega_j : j\in\mathcal{J}_\ell\}$ has a system of distinct representatives in $\mathcal{I}$.*

D. Examples

Example 3.20. *Let $r=2, m=6$ and $\Phi=\bigcup_{j\in[4]}\phi_j\times\{j\}\subset[m]\times[m-r]$:*

$$\Phi=\{2,4,6\}\times\{1\}\ \cup\ \{1,2,4\}\times\{2\}\ \cup\ \{1,2,5\}\times\{3\}\ \cup\ \{1,3,5\}\times\{4\}$$

and its representation by its indicator matrix:

$$\begin{bmatrix}0&1&1&1\\1&1&1&0\\0&0&0&1\\1&1&0&0\\0&0&1&1\\1&0&0&0\end{bmatrix}$$

This is a $(2,6)$-SLMF since it satisfies condition (18). It defines an open set V_Φ in $\mathrm{Gr}(2,6)$ *on which the rational map* $\mathrm{Gr}(2,6)\to\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$ *given by*

$$S\mapsto\left(\begin{bmatrix}[46]_S\\-[26]_S\\[24]_S\end{bmatrix},\begin{bmatrix}[24]_S\\-[14]_S\\[12]_S\end{bmatrix},\begin{bmatrix}[25]_S\\-[15]_S\\[12]_S\end{bmatrix},\begin{bmatrix}[35]_S\\-[15]_S\\[13]_S\end{bmatrix}\right)$$

is injective. These 4 elements of $\mathbb{P}^2$ are precisely the normal vectors of the 4 planes in k^3 that one gets by projecting a general 2-dimensional subspace S in k^6 onto the 3 coordinates indicated by each of the ϕ_j's.

For each $S \in V_\Phi$ there is a unique up to a scaling of its columns 6×4 matrix with the same sparsity pattern as Φ whose column-space is $S^\perp$:

$$\begin{bmatrix} 0 & [24]_S & [25]_S & [35]_S \\ [46]_S & -[14]_S & -[15]_S & 0 \\ 0 & 0 & 0 & -[15]_S \\ -[26]_S & [12]_S & 0 & 0 \\ 0 & 0 & [12]_S & [13]_S \\ [24]_S & 0 & 0 & 0 \end{bmatrix}$$

The polynomial that defines the complement of V_Φ is

$$p_\Phi = \det\left(\begin{bmatrix} [24]_S & [25]_S & [35]_S \\ 0 & 0 & -[15]_S \\ 0 & [12]_S & [13]_S \end{bmatrix}\right) = [12]_S[24]_S[15]_S$$

The following two examples illustrate Theorem 3.2.

EXAMPLE 3.21. *Let $r = 2$, $m = 6$, $n = 5$ and consider the following $\Omega \subset [6] \times [5]$ with $\#\Omega = 18 = \dim \mathrm{M}(2, 6 \times 5)$ represented by its indicator matrix:*

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Consider the partition $[5] = \mathcal{T}_1 \cup \mathcal{T}_2$ with $\mathcal{T}_1 = \{1, 2\}$ and $\mathcal{T}_2 = \{3, 4, 5\}$. Now take

$$\Phi_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Φ_1, Φ_2 are $(2, 6)$-SLMF's since they satisfy (18). Φ_1 is associated with the first 2 columns of Ω ($\mathcal{T}_1$), while Φ_2 with the last 2 columns of Ω ($\mathcal{T}_2$). A computation with `Macaulay2` suggests that $\pi_\Omega^{-1}(\pi_\Omega(X))$ consists only of X, for general X.

EXAMPLE 3.22. *Let $r = 2, m = 6, n = 8$ and Ω with $\#\Omega = 24 = \dim \mathscr{M}_k(2, 6 \times 8)$*

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

With $\mathcal{T}_1 = \{1, 2, 3, 4\}, \mathcal{T}_2 = \{5, 6, 7, 8\}$, Φ_1, Φ_2 are the leftmost and rightmost respectively blocks of Ω and both satisfy (18). A computation with `Macaulay2` suggests that $\pi_\Omega^{-1}(\pi_\Omega(X))$ consists of 2 points over a non-algebraically closed field k and 4 points otherwise.

CHAPTER 4

Determinantal conditions for homomorphic sensing

In a fascinating line of research in signal processing termed *unlabeled sensing*, it has been recently established that uniquely recovering a signal from shuffled and subsampled measurements is possible as long as the number of measurements is at least twice the intrinsic dimension of the signal [**UHV18**], while the source generating the signal is sufficiently exciting. In abstract terms, this says that if V is a general[1] n-dimensional linear subspace of $\mathbb{R}^m$, for some $m \geq 2n$, π_1, π_2 permutations on the m coordinates of $\mathbb{R}^m$ and ρ_1, ρ_2 coordinate projections viewed as endomorphisms, then $\rho_1\pi_1(v_1) = \rho_2\pi_2(v_2)$ implies $v_1 = v_2$ whenever $v_1, v_2 \in V$, provided that each ρ_i preserves at least $2n$ coordinates. A similar phenomenon has been identified in real phase retrieval [**LS18, HLS18**]. In both cases the proofs involve lengthy combinatorial arguments which show that certain determinants do not vanish. In this chapter we provide an abstract justification for this phenomenon, that may very well go under the name *homomorphic sensing*.

Let k be an infinite field and τ_1, τ_2 endomorphisms of k^m. Let ρ be a linear projection onto $\mathrm{im}(\tau_2)$, that is ρ is an idempotent endomorphism of k^m with $\mathrm{im}(\rho) = \mathrm{im}(\tau_2)$. Let $R, T_1, T_2 \in k^{m \times m}$ be matrix representations of ρ, τ_1, τ_2 on the canonical basis of k^m. Let $k[x] = k[x_1, \dots, x_m]$ be a polynomial ring and $I_{\rho\tau_1,\tau_2}$ the ideal generated by all 2-minors of the $m \times 2$ matrix $[RT_1x \ T_2x]$ with $x = x_1, \dots, x_n$ arranged as column vector. Consider the closed subscheme $Y_{\rho\tau_1,\tau_2} = \mathrm{Spec}(k[x]/I_{\rho\tau_1,\tau_2})$ of $\mathbb{A}_k^m = \mathrm{Spec}(k[x])$. Its k-valued points correspond to w's in k^m for which $\rho\tau_1(w), \tau_2(w)$ are linearly dependent. For a k-subspace $V \subseteq k^m$ denote by $\overline{V} = \mathrm{Spec}(k[x]/I_V)$ the closed subscheme of $\mathbb{A}_k^m$ corresponding to V, where I_V is the vanishing ideal of V. The key object is the locally closed subscheme

$$U_{\rho\tau_1,\tau_2} = Y_{\rho\tau_1,\tau_2} \setminus \left(\overline{\ker(\rho\tau_1 - \tau_2)} \cup \overline{\ker(\rho\tau_1)} \cup \overline{\ker(\tau_2)}\right)$$

Let $\mathrm{Gr}(n, m)$ be the Grassmannian of n-dimensional k-subspaces of k^m, identified by the image of the Plücker embedding with an irreducible projective variety. Our main result is:

THEOREM 4.1. *For $n \leq m/2$ suppose $\dim U_{\rho\tau_1,\tau_2} \leq m - n$, $\dim_k \mathrm{im}(\tau_2) \geq 2n$ and $\dim_k \mathrm{im}(\tau_1) \geq n$. Then there is an open dense set $\mathcal{U} \subseteq \mathrm{Gr}(n, m)$ such that for $V \in \mathcal{U}$ and $v_1, v_2 \in V$ we have $\tau_1(v_1) = \tau_2(v_2)$ only if $v_1 = v_2$.*

By a coordinate projection ρ we mean an endomorphism of k^m which preserves the values of $\mathrm{rank}(\rho)$ coordinates and sets the rest to zero.

[1]The attribute *general* is used in the algebraic geometry sense, to indicate that the claimed property is true for every V on a dense open set of the Grassmannian.

Theorem 4.2. *Let π_1, π_2 be permutations on the m coordinates of k^m and ρ_1, ρ_2 coordinate projections. Then* $\dim U_{\rho_2\rho_1\pi_1,\rho_2\pi_2} \le m - \lfloor \mathrm{rank}(\rho_2)/2 \rfloor$.

Using Theorems 4.1-4.2 we obtain a generalization of the main theorem of [**UHV18**]. The generalization consists in allowing one of the projections to preserve at least n coordinates (and not $2n$ for both projections) as well as considering sign changes. We call $\rho : k^m \to k^m$ a signed coordinate projection, if it is the composition of a coordinate projection with a map represented by a diagonal matrix with ± 1 on the diagonal.

Corollary 4.3. *Let $\mathscr{P}_m$ be the group of permutations on the m coordinates of k^m, and $\mathscr{R}_n, \mathscr{R}_{2n}, \mathscr{S}_n, \mathscr{S}_{2n}$ the set of all coordinate projections ($\mathscr{R}_n$, $\mathscr{R}_{2n}$) and signed coordinate projections ($\mathscr{S}_n$, $\mathscr{S}_{2n}$) of k^m, which preserve at least n and $2n$ coordinates respectively, for some $n \le m/2$. Then the following is true for a general n-dimensional subspace V: if $\rho_1\pi_1(v_1) = \rho_2\pi_2(v_2)$ for $v_1, v_2 \in V$ with $\rho_1 \in \mathscr{S}_n, \rho_2 \in \mathscr{S}_{2n}, \pi_1, \pi_2 \in \mathscr{P}_m$, then $v_1 = v_2$ or $v_1 = -v_2$. Moreover, if $\rho_1 \in \mathscr{R}_n$ and $\rho_2 \in \mathscr{R}_{2n}$, then $v_1 = v_2$.*

Finally, using just linear algebra, a much simpler argument gives a version of Theorem 4.1 for general points:

Proposition 4.4. *Suppose τ_1, τ_2 have rank at least $n+1$ and are not scalar multiples of each other. Then for a general n-dimensional linear subspace V of k^m and v a general point in V, we have $\tau_1(v) = \tau_2(v')$ with $v' \in V$ only if $v' = v$.*

A. Proof of Theorem 4.1

For a positive integer s set $[s] = \{1, \dots, s\}$ and $[0] = 0$. We first consider a special case where one of the endomorphisms is the identity id. Let τ be the other endomorphism with $T \in k^{m \times m}$ its matrix representation on the canonical basis of k^m. Denote by I_τ the ideal of $k[x]$ generated by the 2-minors of the $2 \times m$ matrix $[Tx \ x]$. The k-valued points of the closed subscheme $Y_\tau = \mathrm{Spec}(k[x]/I_\tau)$ form the union of the eigenspaces of the endomorphism τ corresponding to eigenvalues that lie in k. Set $U_\tau = Y_\tau \setminus \overline{\ker(\tau - \mathrm{id})}$ the open subscheme of Y_τ with the locus associated to eigenvalue 1 removed. We have:

Proposition 4.5. *Suppose that* $\dim U_\tau \le m - n$ *for some n with $m \ge 2n$. Then there is a dense open set $\mathcal{U} \subset \mathrm{Gr}(n, m)$ such that for every $V \in \mathcal{U}$ and $v_1, v_2 \in V$ we have $\tau(v_1) = v_2$ only if $v_1 = v_2$.*

Proof. See §A.I. □

Denote by $k[x]_1$ the k-vector space of degree-1 homogeneous polynomials in $k[x]$. Write $\overline{\ker(\rho\tau_1 - \tau_2)} = \mathrm{Spec}(k[x]/J)$ where J is generated by linear forms $p_\alpha \in k[x]_1$, $\alpha \in [\mathrm{codim} \ker(\rho\tau_1 - \tau_2)]$. Similarly, let q_β's and r_γ's be linear forms generating the vanishing ideals of $\ker(\rho\tau_1)$ and $\ker(\tau_2)$ respectively. Set $h_{\alpha\beta\gamma} = p_\alpha q_\beta r_\gamma$. Then

$$U_{\rho\tau_1,\tau_2} = \bigcup_{\alpha,\beta,\gamma} \mathrm{Spec}\left(k[x]/I_{\rho\tau_1,\tau_2}\right)_{h_{\alpha\beta\gamma}}$$

where $\left(k[x]/I_{\rho\tau_1,\tau_2}\right)_{h_{\alpha\beta\gamma}}$ is the localization of $k[x]/I_{\rho\tau_1,\tau_2}$ at the multiplicatively closed set $\{1, h_{\alpha\beta\gamma}, h^2_{\alpha\beta\gamma}, \dots\}$.

Set $\ell = \dim_k \mathrm{im}(\tau_2)$. There is a dense open set $\mathcal{U}_1 \subseteq \mathrm{Gr}(\ell, m)$ such that $H \cap \ker(\tau_2) = 0$ for every $H \in \mathcal{U}_1$. For any such H we have that $\tau_2|_H$ establishes an isomorphism between H and $\mathrm{im}(\tau_2)$. Let $(\tau_2|_H)^{-1} : \mathrm{im}(\tau_2) \to H$ be the inverse map. Consider the endomorphism of H given by $\tau_H = (\tau_2|_H)^{-1}\rho\tau_1|_H$. Fixing a basis $B_{\mathrm{im}(\tau_2)} \in k^{m\times\ell}$ of $\mathrm{im}(\tau_2)$ we let R' be the $k^{\ell\times m}$ matrix that sends a vector $\xi \in k^m$ to the coefficients of the representation of $\rho(\xi)$ on the basis $B_{\mathrm{im}(\tau_2)}$ and note that $R'R = R'$. Fix a basis $B_H \in k^{m\times\ell}$ of H. Then $\tau_2|_H$ is represented by the invertible matrix $T_{2,H} = R'T_2B_H \in k^{\ell\times\ell}$ and τ_H by $T_H = T_{2,H}^{-1}R'T_1B_H \in k^{\ell\times\ell}$.

Let $k[z] = k[z_1,\dots,z_\ell]$ be a polynomial ring of dimension ℓ and consider the surjective ring homomorphism $\psi : k[x] \to k[z]$ that takes x to B_Hz. The kernel of ψ is the vanishing ideal I_H of H, so that ψ induces a ring isomorphism $k[x]/I_H \cong k[z]$. The further ring isomorphism $k[x]/I_H + (p_\alpha)_\alpha \cong k[z]/(\psi(p_\alpha))_\alpha$ corresponds geometrically to the identification $\ker(\rho\tau_1 - \tau_2)\cap H \cong \ker(\tau_H - \mathrm{id})$. That is, the $\psi(p_\alpha)$'s generate the vanishing ideal of $\ker(\tau_H - \mathrm{id})$. Similarly, the $\psi(q_\beta)$'s generate the vanishing ideal of $\ker(\tau_H)$, while the $\psi(r_\gamma)$'s generate the irrelevant ideal $(z_1,\dots,z_\ell)$. Now define I_{τ_H} to be the ideal of $k[z]$ generated by all 2×2 determinants of the $\ell\times 2$ matrix $[T_Hz\ z]$. We have:

Lemma 4.6. $I_{\tau_H} = \psi(I_{\rho\tau_1,\tau_2})$.

Proof. Since $\tau_2(\tau_2|_H)^{-1}\rho = \rho$ we have $\psi([RT_1x\ T_2x]) = T_2B_H[T_Hz\ z]$. Recall that if C is a $2\times\ell$ row-submatrix of T_2B_H then

$$\det(C[T_Hz\ z]) = \sum_{\mathcal{J}\subset[\ell],\#\mathcal{J}=2} \det(C_{\mathcal{J}})\det({}_{\mathcal{J}}[T_Hz\ z])$$

where $C_{\mathcal{J}}$ and ${}_{\mathcal{J}}[T_Hz\ z]$ denote column and row 2×2 submatrices repsectively, indexed by $\mathcal{J}$. This shows that the ideal of 2×2 determinants of $\psi([RT_1x\ T_2x])$ is contained in the ideal of 2×2 determinants of $[T_Hz\ z]$. For the reverse inclusion, note that T_2B_H has rank ℓ and so there is an invertible row-submatrix A of T_2B_H of size $\ell\times\ell$. It is enough to prove that the ideal of 2×2 determinants of $A[T_Hz\ z]$ coincides with that of $[T_Hz\ z]$. The matrix A induces a k-automorphism $f : (k[z])^\ell \to (k[z])^\ell$ given by $u \mapsto Au$. This further induces a k-linear map of exterior powers $f^{(2)} : \wedge^2(k[z])^\ell \to \wedge^2(k[z])^\ell$ by taking $u\wedge v$ to $Au\wedge Av$. Note that $u\wedge v$ is the vector of 2×2 determinants of the matrix $[u\ v]$. Similarly A^{-1} induces a k-linear map $g^{(2)} : \wedge^2(k[z])^\ell \to \wedge^2(k[z])^\ell$. Since $f^{(2)}, g^{(2)}$ are inverses, the vectors of 2×2 determinants of $A[T_Hz\ z]$ and $[T_Hz\ z]$ can be obtained from each other via matrix multiplication over k, thus they generate the same ideal. ☐

Lemma 4.6 gives the ring isomorphism $\psi' : k[x]/I_{\rho\tau_1,\tau_2} + I_H \cong k[z]/I_{\tau_H}$. Together with the definition of U_{τ_H} this gives

$$\begin{aligned} U_{\tau_H} \setminus \overline{\ker(\tau_H)} &= \mathrm{Spec}(k[z]/I_{\tau_H}) \setminus \overline{\ker(\tau_H - \mathrm{id})} \cup \overline{\ker(\tau_H)} \\ &\cong \bigcup_{\alpha,\beta} \mathrm{Spec}\left(k[x]/I_{\rho\tau_1,\tau_2} + I_H\right)_{p_\alpha q_\beta} \end{aligned}$$

Let $\mathrm{Gr}(c, k[x]_1)$ be the Grassmannian of k-subspaces W of $k[x]_1$ of dimension c. The following is a folklore fact in commutative algebra.

Lemma 4.7. *Let I be a homogeneous ideal of $k[x]$. Then there exists a dense open set $\mathcal{U}^* \subseteq \mathrm{Gr}(c, k[x]_1)$ such that*

$$\dim\left(k[x]/I + (W)\right) = \max\{\dim\left(k[x]/I\right) - c, 0\}$$

for every $W \in \mathcal{U}^*$, with (W) the ideal generated by W.

Let $\mathscr{P}$ be the minimal set of homogeneous prime ideals of $k[x]$ such that $\sqrt{I_{\rho\tau_1,\tau_2}} = \bigcap_{P\in\mathscr{P}} P$. Let $\mathscr{P}_{\alpha,\beta}$ be the subset of $\mathscr{P}$ consisting of those P's that do not contain $p_\alpha q_\beta$ for some α, β. Each such P corresponds to an irreducible component of Spec $\big(k[x]/I_{\rho\tau_1,\tau_2}\big)_{p_\alpha q_\beta}$. For $P \in \mathscr{P}_{\alpha,\beta}$ Lemma 4.7 with $c = m-\ell$ and $I = P$ gives a dense open set $\mathcal{U}_P^* \subseteq \mathrm{Gr}(m-\ell, k[x]_1)$ on which $\dim\big(k[x]/P + (W)\big) = \max\{\dim\big(k[x]/P\big) - m + \ell, 0\}$ for every $W \in \mathcal{U}_P^*$. Set $\mathcal{U}_{\alpha,\beta}^* = \bigcap_{P\in\mathscr{P}_{\alpha,\beta}} \mathcal{U}_P^*$. For every $W \in \mathcal{U}_{\alpha,\beta}^*$ we have $\dim\big(k[x]/I_{\rho\tau_1,\tau_2} + (W)\big)_{p_\alpha q_\beta} = \max\big\{\dim\big(k[x]/I_{\rho\tau_1,\tau_2}\big)_{p_\alpha q_\beta} - m + \ell, 0\big\}$. By hypothesis $\dim U_{\rho\tau_1,\tau_2} \le m-n$ so $\dim\big(k[x]/I_{\rho\tau_1,\tau_2}\big)_{p_\alpha q_\beta} \le m-n$. With $\mathcal{U}^* = \bigcap_{\alpha,\beta} \mathcal{U}_{\alpha,\beta}^*$, for every $W \in \mathcal{U}^*$ we have that $\dim\big(k[x]/I_{\rho\tau_1,\tau_2} + (W)\big)_{p_\alpha q_\beta} \le \ell - n$ for every α, β. Now under the isomorphism $\mathrm{Gr}(m-\ell, k[x]_1) \cong \mathrm{Gr}(\ell, m)$ the open set $\mathcal{U}^*$ gives an open set $\mathcal{U}_2 \subset \mathrm{Gr}(\ell, m)$ such that $H \in \mathcal{U}_2$ if and only if $I_H \in \mathcal{U}^*$. We conclude that $\dim U_{\tau_H} \smallsetminus \overline{\ker(\tau_H)} \le \ell - n$ for every $H \in \mathcal{U}_2$.

The locus of H's in $\mathcal{U}_1 \cap \mathcal{U}_2$ for which i) $\dim_k \ker(\tau_H)$ is minimal, ii) $\dim_k E_{\tau_H,1} = \ell - \mathrm{rank}(R'T_1B_H - R'T_2B_H)$ is minimal, iii) a unique basis B_H exists with the top $\ell \times \ell$ block the identity matrix, is also open and non-empty; call it $\mathcal{U}_3$. For every $H \in \mathcal{U}_3$ the above mentioned unique representation B_H of H establishes a k-vector space isomorphism $H \cong k^\ell$ by sending the jth column of B_H to the jth canonical vector of k^ℓ. This further establishes an isomorphism of projective varieties $\gamma_H : \mathrm{Gr}(n, H) \xrightarrow{\sim} \mathrm{Gr}(n, \ell)$. By the definition of $\mathcal{U}_3$ $\dim_k \ker(\tau_H)$ is constant for every $H \in \mathcal{U}_3$, call that value α. If $\alpha \le n$ then τ_H satisfies the hypothesis of Proposition 4.5 for every $H \in \mathcal{U}_3$. Hence there is a dense open set $\mathcal{U}_H \subseteq \mathrm{Gr}(n, H)$ such that for every $V \in \mathcal{U}_H$ and $v_1, v_2 \in V$ we have $\tau_H(v_1) = v_2$ only if $v_1 = v_2$. If on the other hand $\alpha > n$, it is easy to see that there is another dense open set that we also call $\mathcal{U}_H \subseteq \mathrm{Gr}(n, H)$, such that for every $V \in \mathcal{U}_H$ and $v_1, v_2 \in V$ the equality $\tau_H(v_1) = v_2$ implies $v_1 = v_2 = 0$.

We now show that the incidence correspondence $V \subset H$ with $H \in \mathcal{U}_4$ and $V \in \mathcal{U}_H$ contains a non-empty open set of the flag variety $\mathcal{F}(n, \ell, m)$, the latter defined as the closed subset of $\mathrm{Gr}(n, m) \times \mathrm{Gr}(\ell, m)$ cut out by the relation $V \in \mathrm{Gr}(n, H)$. Towards that end, it is enough to show that the equations that define $\mathcal{U}_H$ are polynomials in the Plücker coordinates of V via γ_H with rational coefficients in B_H. Denote by $k(B_H)$ the field of fractions of the polynomial ring $k[B_H]$ with the free entries of B_H viewed as variables. The parametrization of $\mathcal{U}_H$ by H depends on the two numbers $\alpha = \dim_k \ker(\tau_H)$ and $\beta = \dim_k E_{\tau_H,1}$. Both these dimensions are constant for every $H \in \mathcal{U}_3$ and there are three possibilities for the structure of $\mathcal{U}_H$ determined by the cases i) $\alpha \le n$, $\beta \le m-n$, ii) $\alpha \le n$, $\beta > m-n$, iii) $\alpha > n$. We only discuss i) and ii). For case i) the last part of the proof of Proposition 4.5 shows that $\mathcal{U}_H$ is determined via γ_H by the condition $\mathrm{rank}[T_H A \ \ A] = 2n$, where $A \in k^{\ell\times n}$ is any basis of $\gamma_H(V)$. This amounts to the non-simultaneous vanishing of certain quadratic equations in the Plücker coordinates of $\gamma_H(V)$ with coefficients in $k(B_H)$. For case ii) we note that the number β is equal to the $k(B_H)$-vector space dimension of the right nullspace of the matrix $T_H - I$, where I is the identity matrix of size ℓ. By Gauss-Jordan elimination over $k(B_H)$ we compute a $k(B_H)$-basis $\mathfrak{s}_1, \dots, \mathfrak{s}_\beta \in k(B_H)^\ell$ for that nullspace. We extend this sequence by adding vectors $s_1, \dots, s_{\ell-\beta} \in k^\ell$ such that the matrix $S = [\mathfrak{s}_1 \cdots \mathfrak{s}_\beta \ \ s_1 \cdots s_{\ell-\beta}] \in k(B_H)^{\ell\times\ell}$ is invertible over $k(B_H)$. The last part of the proof of Proposition 4.5 shows that now $\mathcal{U}_H$ is determined via γ_H as the $\gamma_H(V)$'s with basis $A \in k^{\ell\times n}$ for which $\det(S_{[n]}^{-1} A) \ne 0$, where $S_{[n]}^{-1}$ is the top

$n \times m$ block of S^{-1}. This is a linear equation in the Plücker coordinates of $\gamma_H(V)$ with rational coefficients in B_H.

We have a non-empty open set $\mathscr{O} \subset \mathcal{F}(n, \ell, m)$ such that for every $(V, H) \in \mathscr{O}$ we have that V satisfies the property of interest: if $\tau_1(v_1) = \tau_2(v_2)$ for $v_1, v_2 \in V$ then $\tau_H(v_1) = v_2$ and thus necessarily $v_1 = v_2$. The equations that define $\mathscr{O}$ also define a non-empty open subscheme $\overline{\mathscr{O}}$ of the flag scheme $\overline{\mathcal{F}}(n, \ell, m)$, where the overline notation indicates scheme structure. Now, since both $\overline{\mathcal{F}}(n, \ell, m)$ and $\overline{\mathrm{Gr}}(n, m)$ are irreducible, the image of $\overline{\mathscr{O}}$ under the canonical projection $\overline{\mathcal{F}}(n, \ell, m) \to \overline{\mathrm{Gr}}(n, m)$ is dense. By Chevalley's theorem that image is constructible and thus it contains a non-empty open set $\overline{\mathcal{U}}_5$ whose k-valued points satisfy our property of interest. It remains to show how to get the open set $\mathcal{U} \subset \mathrm{Gr}(n, m)$ of the theorem. $\mathrm{Gr}(n, m), \overline{\mathrm{Gr}}(n, m)$ are locally isomorphic to the affine space of dimension $n(m - n)$. Let $\overline{\mathcal{U}}_6$ be the open set of $\overline{\mathrm{Gr}}(n, m)$ where some Plücker coordinate does not vanish. With Y an $n \times (m - n)$ matrix of indeterminates, $\overline{\mathcal{U}}_6$ is isomorphic to $\mathbb{A}^{n(m-n)} = \operatorname{Spec} k[Y]$. The non-vanishing of the same Plücker coordinate in $\mathrm{Gr}(n, m)$ gives an open set $\mathcal{U}_7 \subset \mathrm{Gr}(n, m)$, which is isomorphic to $k^{n(m-n)}$. Replacing $\overline{\mathcal{U}}_5$ by its intersection with $\overline{\mathcal{U}}_6$, we may assume that it lies in $\mathbb{A}^{n(m-n)}$. As $\overline{\mathcal{U}}_5$ is covered by basic affine open sets, we may further assume that $\overline{\mathcal{U}}_5 = \operatorname{Spec}(k[Y])_p$ for some non-zero polynomial $p \in k[Y]$. Our open set $\mathcal{U}$ is the non-vanishing locus of p in $\mathcal{U}_7$, which is non-empty by the infinity of k.

A.I. Proof of Proposition 4.5. We recall some notions from linear algebra following [**Rom08**]. For simplicity we write τv instead of $\tau(v)$. We say that a k-subspace C of k^m is τ-cyclic if it admits a basis of the form $v, \tau v, \tau^2 v, \ldots, \tau^{d-1} v$ for some $v \in k^m$ with $d = \dim_k C$. Let y be a transcendental element over k. Then k^m admits a $k[y]$-module structure under the action $p(y) \in k[y] \mapsto p(\tau) \in \operatorname{Hom}_k(k^m, k^m)$. Let $m_\tau(y)$ be the monic minimal polynomial of τ and let $m_\tau(y) = p_1^{\ell_1}(y) \cdots p_s^{\ell_s}(y)$ be its unique factorization into powers of irreducible polynomials $p_i(y) \in k[y]$. Then k^m admits a primary cyclic decomposition as a $k[y]$-module into the direct sum of τ-cyclic subspaces on which the minimal polynomial of τ is a power of one of the $p_i(y)$'s. Now τ admits an eigenvalue $\lambda \in k$ if and only if $y - \lambda$ divides $m_\tau(y)$, that is if and only if one of the $p_i(y)$'s is equal to $y - \lambda$. Let C be a τ-cyclic subspace as above in the primary decomposition with minimal polynomial of the form $(y - \lambda)^e$. Then $w_i = (\tau - \lambda)^{d-i} v$, $i \in [d]$ is a basis of C with $\tau w_1 = \lambda w_1$ and $\tau w_i = \lambda w_i + w_{i-1}$, $i = 2, \ldots, d$. We call this basis a Jordan basis and the matrix representation of $\tau|_C$ on that basis is a Jordan block

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \in k^{d \times d}$$

Thus the geometric multiplicity of the eigenvalue $\lambda \in k$ is the number of τ-cyclic subspaces in the primary decomposition of k^m for which $m_{\tau|_C}(y) = (y - \lambda)^d$ for some $d \geq 1$. Now τ induces an endomorphism of $\bar{k}^m$ in a natural way, which we also call τ. With $\lambda_i, i \in [s]$ the eigenvalues of τ over $\bar{k}$ we have that $\bar{k}^m$ admits a decomposition $\bar{k}^m = \bigoplus_{t,i} C_{t,\lambda_i}$ into τ-cyclic $\bar{k}$-subspaces with C_{t,λ_i} corresponding to eigenvalue λ_i. That is each $\mathcal{C}_{t,\lambda_i}$ admits a Jordan basis $w_1, \ldots, w_{d_{ti}}$ such that $\tau w_1 = \lambda_i w_1$ and $\tau w_j = \tau_i w_j + w_{j-1}, \ \forall j = 2, \ldots, d_{ti}$. We denote by $E_{\tau,\lambda}$ the eigenspace of τ associated to eigenvalue λ. We note that if $\lambda \in k$ then $\dim_k E_{\tau,\lambda} = \dim_{\bar{k}} E_{\tau,\lambda}$. Finally, with $K = k, \bar{k}$ we denote by $\mathrm{Gr}_K(n,m)$ the set of all n-dimensional K-subspaces of K^m.

We prove the proposition in several stages, starting with the boundary situation described in the next lemma.

LEMMA 4.8. *Suppose $m = 2n$ and $\dim_{\bar{k}} E_{\tau,\lambda} = n$ for some $\lambda \in \bar{k}$. Then there exists a $V \in \mathrm{Gr}_{\bar{k}}(n,m)$ such that $\bar{k}^m = V \oplus \tau(V)$.*

PROOF. Let $\lambda_i, i \in [s]$ be the spectrum of τ over $\bar{k}$ and suppose that $\lambda_1 = \lambda$ is the said eigenvalue. Then in the decomposition above of $\bar{k}^{2n}$ there are exactly n subspaces $C_{t,\lambda_1}, t \in [n]$ associated to λ_1 each of them contributing a single eigenvector. With $w_{t,1}, \ldots, w_{t,d_t}$ a Jordan basis for C_{t,λ_1} that eigenvector is $w_{t,1}$ and we set $v_t = w_{t,1}$ for $t \in [n]$. We produce n linearly independent vectors $u_t, t \in [n]$ to be taken as a basis for the claimed subspace V, by summing pairwise the v_t's with the remaining Jordan basis vectors across all C_{t,λ_i}'s in a manner prescribed below.

First, suppose that all $\mathcal{C}_{t,\lambda_1}$'s are 1-dimensional. Then $\mathcal{C}_{1,\lambda_2}$ is a non-trivial subspace with Jordan basis say $w_1, \ldots, w_d$, for some $d \geq 1$. We construct the first d basis vectors $u_1, \ldots, u_d$ for V as $u_j = v_j + w_j,\ j \in [d]$. A forward induction on the relations

$$\tau u_1 = \lambda_1 v_1 + \lambda_2 w_1;\ \ \tau u_j = \lambda_1 v_j + \lambda_2 w_j + w_{j-1},\ \ j = 2, \ldots, d,$$

together with $\lambda_1 \neq \lambda_2$, gives

$$\mathrm{Span}(u_1, \tau u_1, \ldots, u_d, \tau u_d) = \left(\oplus_{t \in [d]} \mathcal{C}_{t,\lambda_1}\right) \oplus \mathcal{C}_{1,\lambda_2}$$

If $d = n$ we are done, otherwise either $\mathcal{C}_{2,\lambda_2}$ or $\mathcal{C}_{1,\lambda_3}$ is a non-trivial subspace and we inductively repeat the argument above until all $\mathcal{C}_{t,\lambda_i}$'s are exhausted.

Next, suppose that not all C_{t,λ_1}'s are 1-dimensional. We may assume that there exists integer $0 \leq r < n$ such that $\dim C_{t,\lambda_1} = 1$ for every $t \leq r$ and $\dim C_{t,\lambda_1} = d_j > 1$ for every $t > r$. If $r = 0$, then each C_{t,λ_1} is necessarily 2-dimensional and τ has only one eigenvalue λ_1. Letting $w_{1,t}, w_{2,t}$ be the Jordan basis for C_{t,λ_1}, we define $u_t = w_{2,t},\ \forall t \in [n]$. Clearly, $\mathrm{Span}(u_t, \tau u_t) = \mathrm{Span}(w_{1,t}, w_{2,t})$, in which case $\mathrm{Span}\left(\{u_t, \tau u_t\}_{t \in [n]}\right) = \oplus_{t \in [n]} C_{t,\lambda_1} = \bar{k}^{2n}$. So suppose $1 \leq r < n$. Let $w_1, \ldots, w_{d_{r+1}}$ be a Jordan basis for C_{r+1,λ_1}. Since

$$2(n-r-1) \leq \dim \oplus_{t=r+2}^{n} C_{t,\lambda_1} \leq \mathrm{codim} \oplus_{t=1}^{r+1} C_{t,\lambda_1} = 2n - r - d_{r+1},$$

we must have

$$d_{r+1} - 2 \leq r$$

Recall that $w_1 = v_{r+1}$ and define $u_1 = v_{r+1} + w_{d_{r+1}}$ and $u_j = v_{j-1} + w_j$ for $j = 2, \ldots, d_{r+1} - 1$. Noting that $\{w_j : j \in [d_{r+1} - 1]\} = \{\tau u_j - \lambda u_j : j \in [d_{r+1} - 1]\}$, we have

$$\mathrm{Span}\left(\{u_j, \tau u_j\}_{j=1}^{d_{r+1}-1}\right) = \left(\oplus_{t=1}^{d_{r+1}-2} C_{t,\lambda_1}\right) \oplus C_{r+1,\lambda_1}$$

If $r = n-1$, we have found a $(d_n - 1)$-dimensional subspace $V' \coloneqq \mathrm{Span}(u_j : j \in [d_{r+1} - 1])$ such that $V' + \tau(V') = \oplus_{t=1}^{n} C_{t,\lambda_1}$. Otherwise if $r < n-1$, C_{r+2,λ_1} is a nontrivial subspace of dimension $d_{r+2} \geq 2$, which must satisfy $r + d_{r+1} + d_{r+2} + 2(n-r-2) \leq 2n$ or

$$d_{r+2} - 2 \leq r - (d_{r+1} - 2)$$

Letting $w_1, \ldots, w_{d_{r+2}}$ be a Jordan basis for C_{r+2,λ_1} and recalling the convention $v_{r+2} = w_1$, we define $u_{d_{r+1}}, \ldots, u_{d_{r+1}+d_{r+2}-2}$ as

$$u_{d_{r+1}} = v_{r+2} + w_{d_{r+2}},\ \ u_{d_{r+1}-1+j} = v_{d_{r+1}-3+j} + w_j,\ \ \forall j = 2, \ldots, d_{r+2} - 1$$

Then one verifies that

$$\operatorname{Span}\left(\{u_{d_{r+1}-1+j},\tau u_{d_{r+1}-1+j}\}_{j=1}^{d_{r+2}-1}\right)=\left(\oplus_{t=1}^{d_{r+2}-2}C_{d_{r+1}-2+t,\lambda_1}\right)\oplus C_{r+2,\lambda_1}$$

and in particular

$$\operatorname{Span}\left(\{u_j,\tau u_j\}_{j=1}^{d_{r+1}+d_{r+2}-2}\right)=\left(\oplus_{t=1}^{d_{r+1}+d_{r+2}-4}C_{t,\lambda_1}\right)\oplus\left(\oplus_{t\in[2]}C_{r+t,\lambda_1}\right)$$

Continuing inductively like this we exhaust all C_{t,λ_1}'s that have dimension greater than 1 and obtain

$$V'=\operatorname{Span}\left(\left\{u_j\,:\,j=1,\ldots,\sum_{j\in[n-r]}(d_{r+j}-1)\right\}\right)$$

$$V'+\tau(V')=\left(\oplus_{t\in\left[\sum_{j=1}^{n-r}(d_{r+j}-2)\right]}C_{t,\lambda_1}\right)\oplus\left(\oplus_{t\in[n-r]}C_{r+t,\lambda_1}\right)$$

with $\sum_{j=1}^{n-r}(d_{r+j}-2)\le r$. If equality is achieved then $\dim V'=n$ and we can take $V=V'$; note that in that case $s=1$. Otherwise, $\dim\oplus_{t;i>1}C_{t,\lambda_i}=r-\sum_{j=1}^{n-r}(d_{r+j}-2)=:\alpha$ and this is precisely the number of 1-dimensional C_{t,λ_1}'s that have not been used so far. Letting $\xi_1,\ldots,\xi_\alpha$ be the union of all Jordan bases of all C_{t,λ_i}'s for $i>1$, we define the remaining α basis vectors of V as $u_{n-\alpha+j}=v_{r-\alpha+j}+\xi_j$, $j\in[\alpha]$, and since

$$\operatorname{Span}\left(\{u_{n-\alpha+j},\tau(u_{n-\alpha+j})\}_{j=1}^{\alpha}\right)=\left(\oplus_{j=1}^{\alpha}C_{r-\alpha+j,\lambda_1}\right)\oplus\left(\oplus_{t;i>1}C_{t,\lambda_i}\right)$$

the proof is complete. □

We now use Lemma 4.8 to get a stronger statement for eigenspace dimensions less than or equal to half of the ambient dimension.

LEMMA 4.9. *Suppose $m=2n$ and $\dim_{\bar{k}}E_{\tau,\lambda}\le n$ for every $\lambda\in\bar{k}$. Then there exists a $V\in\operatorname{Gr}_{\bar{k}}(n,m)$ such that $\bar{k}^m=V\oplus\tau(V)$.*

PROOF. Let $\lambda_i, i\in[s]$ be the eigenvalues of τ over $\bar{k}$ and proceed by induction on n. For $n=1$ we have $s\le 2$ and $\dim E_{\tau,\lambda_i}=1$, whence the claim follows from Lemma 4.8. So let $n>1$. If $\dim_{\bar{k}}E_{\tau,\lambda_i}=n$ for some i, then we are done by Lemma 4.8. Hence suppose throughout that $\dim_{\bar{k}}E_{\tau,\lambda_i}<n$, $\forall i\in[s]$. Since the induction hypothesis applied on any $2(n-1)$-dimensional τ-invariant subspace S furnishes an $(n-1)$-dimensional subspace $V'\subset S$ such that $V'\oplus\tau(V')=S$, our strategy is to suitably select S so that for a 2-dimensional complement T there is a vector $u\in T$ such that $\operatorname{Span}(u,\tau u)=T$. Then we can take $V=V'+\operatorname{Span}(u)$.

If there are two 1-dimensional subspaces C_{1,λ_1}, C_{1,λ_2} spanned by v_1,v_2 respectively, we let $S=\oplus_{(t,i)\neq(1,1),(1,2)}C_{t,\lambda_i}$ and $u=v_1+v_2$. So suppose that there is at most one eigenvalue, say λ_1, that possibly contributes 1-dimensional subspaces C_{t,λ_1}'s. In that case, there exist t',i' such that $d:=\dim_{\bar{k}}C_{t',\lambda_{i'}}>1$. Let $w_1,\ldots,w_d$ be a Jordan basis for $\mathcal{C}_{t',\lambda_{i'}}$. Define the τ-invariant subspace $\tilde{C}_{t,\lambda_i}=\operatorname{Span}(w_1,\ldots,w_{d-2})$, taken to be zero if $d=2$. Then we let $S=\left(\oplus_{(t,i)\neq(t',i')}C_{t,\lambda_i}\right)\oplus\tilde{C}_{t,\lambda_i}$ and $u=w_d$. □

We take one step further by allowing $m\ge 2n$.

LEMMA 4.10. *Suppose $m\ge 2n$ and $\dim_{\bar{k}}E_{\tau,\lambda}\le m-n$ for every $\lambda\in\bar{k}$. Then there exists a $V\in\operatorname{Gr}_{\bar{k}}(n,m)$ such that $\dim_{\bar{k}}V\oplus\tau(V)=2n$.*

Proof. The strategy is to find a $2n$-dimensional τ-invariant subspace $S \subset \bar{k}^m$ for which $\dim_{\bar{k}} E_{\tau|_S,\lambda_i} \le n$; then the claim will follow from Lemma 4.9. We obtain S by suitably truncating the C_{t,λ_i}'s. Set $\mu = \max_{i\in[s]} \dim_{\bar{k}} E_{\tau,\lambda_i}$. If $\mu = 1$ then τ has m distinct eigenvalues and we may take $S = \oplus_{i\in[2n]} C_{i,\lambda_i}$. Suppose that $1 < \mu \le n$. Set $c = m - 2n$. If there is some $C_{t',\lambda_{i'}}$ with $d = \dim_{\bar{k}} C_{t',\lambda_{i'}} \ge c$, let $w_1, \dots, w_d$ be a Jordan basis for $C_{t',\lambda_{i'}}$ and take $S = \left(\oplus_{(t,i)\neq(t',i')} C_{t,\lambda_i} \right) \oplus \operatorname{Span}(w_1, \dots, w_{d-c})$. Otherwise, let $\beta > 1$ be the smallest number of subspaces $C_{t_1,\lambda_{i_1}}, \dots, C_{t_\beta,\lambda_{i_\beta}}$ for which $\dim_{\bar{k}} \oplus_{j\in[\beta]} C_{t_j,\lambda_{i_j}} = c + \ell$ for some $\ell \ge 0$. Then by the minimality of β we must have that $\dim_{\bar{k}} C_{t_1,\lambda_{i_1}} \ge \ell$. Now replace $C_{t_1,\lambda_{i_1}}$ by an ℓ-dimensional τ-invariant subspace $\tilde{C}_{t_1,\lambda_{i_1}}$ obtained as the span of the first ℓ vectors of a Jordan basis of $C_{t_1,\lambda_{i_1}}$ and take $S = \left(\oplus_{(t,i)\neq(t_j,\lambda_{i_j}),\, j\in[\beta]} C_{t,\lambda_i} \right) \oplus \tilde{C}_{t_1,\lambda_{i_1}}$.

Next, suppose that $\mu > n$ and we may assume that $\dim_{\bar{k}} E_{\tau,\lambda_1} = \mu = n + c_1$ with $0 < c_1 \le c$. We first treat the case $c_1 = c$. In such a case $\dim_{\bar{k}} E_{\tau,\lambda_i} \le n$ for any $i > 1$. Let r be the number of 1-dimensional C_{t,λ_1}'s, say $C_{1,\lambda_1}, \dots, C_{r,\lambda_1}$. Then we must have that

$$r + 2(n + c - r) \le 2n + c \Leftrightarrow c \le r$$

and we can take $S = \left(\oplus_{t=c+1}^{n+c} C_{t,\lambda_1} \right) \oplus \left(\oplus_{t;i>1} C_{t,\lambda_i} \right)$. Next, suppose that $c_1 < c$. If $\dim_{\bar{k}} C_{t,\lambda_i} = 1$ for every t, i, then there are $n + c - c_1$ 1-dimensional C_{t,λ_i}'s associated to eigenvalues other than λ_1. In that case we can take S to be the sum of n subspaces associated to λ_1 and any other subspaces associated to eigenvalues different than λ_1. If on the other hand $\dim_{\bar{k}} C_{t',\lambda_{i'}} > 1$ for some t', i', then we replace $\bar{k}^m$ by U_1, the latter being the sum of all C_{t,λ_i}'s with the exception that $C_{t',\lambda_{i'}}$ has been replaced by a $\tilde{C}_{t',\lambda_{i'}} \subset C_{t',\lambda_{i'}}$ of dimension one less which we rename to $C_{t',\lambda_{i'}}$. Notice that this replacement does not change μ. If $c - 1 = c_1$ or all C_{t,λ_i}'s in the decomposition of U_1 are 1-dimensional, we are done by proceeding as above. If on the other hand $c - 1 > c_1$ and there is a $C_{t',\lambda_{i'}}$ of dimension larger than one, then replace U_1 by U_2, where the latter is the sum of all C_{t,λ_i}'s except the said $C_{t',\lambda_{i'}}$, which is replaced as above by a $C_{t',\lambda_{i'}}$ of dimension one less. Continuing inductively furnishes S. □

We are now in a position to complete the proof of Proposition 4.5. Suppose first that $\dim_{\bar{k}} E_{\tau,1} \le m - n$. Then for $V \in \operatorname{Gr}_k(n,m)$ we have $\dim_k(V + \tau(V)) \le 2n$ with equality on an open set $\mathcal{U}_1 \subset \operatorname{Gr}_k(n,m)$. With $A \in k^{m\times n}$ a basis of V this open set is implicitly defined by the non-vanishing of some $2n \times 2n$ minor of the $m \times 2n$ matrix $[A \; TA]$. These minors are polynomials in A with coefficients over k. By Lemma 4.10 there exists a non-zero evaluation for one of these polynomials at some point $A^* \in \bar{k}^{m\times n}$ so that $\mathcal{U}_1$ is non-empty. Set $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ where $\mathcal{U}_2$ is the non-empty open set of V's that do not intersect the kernel of τ. Then for every $V \in \mathcal{U}$ we have $V \cap \tau(V) = 0$ so that the equality $\tau(v_1) = v_2$ implies $v_1 = v_2 = 0$.

Next, suppose that $\dim_{\bar{k}} E_{\tau,1} \ge n$. Then also $\dim_k E_{\tau,1} \ge n$. Thus in the primary cyclic decomposition of k^m as a $k[y]$-module there are at least n τ-cyclic subspaces associated to eigenvalue 1. Pick a basis of k^m that consists of the union of bases of each and every of the τ-cyclic subspaces choosing a Jordan basis whenever eigenvalue 1 is involved. Stack this basis in the columns of a matrix S and write $T = SJS^{-1}$ where J is the matrix representation of τ on that basis. Note that J is block diagonal with at least n Jordan blocks present and associated to eigenvalue 1. Hence there are indices $\mathfrak{I} = \{i_1, \dots, i_n\}$ for which the i_j-th row of J is the canonical vector e_{i_j} of all zeros except a 1 at position i_j. Let $S_{\mathfrak{I}}^{-1}$ be the row-submatrix of S^{-1}

made out of the rows indexed by $\mathfrak{I}$. Now let $\mathcal{U} \subset \mathrm{Gr}_k(n,m)$ be the non-empty open set of V's for which there is a basis $A \in k^{m\times n}$ such that the matrix $S_{\mathfrak{I}}^{-1} A \in k^{n\times n}$ is invertible. Let $V \in \mathcal{U}$ and suppose that $\tau(v_1) = v_2$ with $v_1, v_2 \in V$. Let A be a basis of V and write $v_i = A\xi_i$. Then the relation $SJS^{-1}v_1 = v_2$ implies $S_{\mathfrak{I}}^{-1}A\xi_1 = S_{\mathfrak{I}}^{-1}A\xi_2$ and so $\xi_1 = \xi_2$.

B. Proof of Theorem 4.2

We first need two lemmas.

LEMMA 4.11. *Let Π be an $\ell \times \ell$ permutation matrix consisting of a single cycle, and let Σ be an $\ell \times \ell$ diagonal matrix with its diagonal entries taking values in $\{1, -1\}$. Let $\mathcal{Q}$ be the ideal generated by the 2-minors of the matrix $[\Sigma\Pi z \ z]$ over the ℓ-dimensional polynomial ring $k[z] = k[z_1, \dots, z_\ell]$. Then* $\mathrm{height}(\mathcal{Q}) = \ell - 1$.

PROOF. Note that the height of $\mathcal{Q}$ is the same as the height of the extension of $\mathcal{Q}$ in $\bar{k}[z]$, so we may assume that $k = \bar{k}$. Let $Y \subset \bar{k}^\ell$ be the vanishing locus of $\mathcal{Q}$. Clearly $v \in Y$ if and only if v is an eigenvector of $\Sigma\Pi$. Hence Y is the union of the eigenspaces of $\Sigma\Pi$, the latter being the irreducible components of Y. With $\sigma_i \in \{1, -1\}$ the i-th diagonal element of Σ, the eigenvalues of $\Sigma\Pi$ are the ℓ distinct roots of the equation $x^\ell = \sigma_1 \cdots \sigma_\ell$. Hence $\Sigma\Pi$ is diagonalizable with ℓ distinct eigenvalues, i.e., each eigenspace has dimension 1. Thus Y has pure dimension $1 = \dim Y = \dim \bar{k}[z]/I_{\Sigma\Pi}$ whence $\mathrm{height}(\mathcal{Q}) = 1$. □

LEMMA 4.12. *Let Π be an $m \times m$ permutation matrix consisting of c cycles and Σ an $m \times m$ diagonal matrix with diagonal entries taking values in $\{1, -1\}$. For every $i \in [c]$ let $\mathcal{I}_i \subset [m]$ be the indices that are cycled by cycle i. Let $\bar{\mathcal{I}} \subset [m]$ such that $\#\bar{\mathcal{I}} \geq 2$ and $\mathcal{I}_i \notin \bar{\mathcal{I}}$ for every $i \in [c]$. Let $\mathcal{Q}$ be the ideal generated by the 2-minors of the row-submatrix Φ of $[x \ \Sigma\Pi x]$ indexed by $\bar{\mathcal{I}}$. Viewing $\mathcal{Q}$ as an ideal of the polynomial ring over k in the indeterminates that appear in Φ, we have that* $\mathrm{height}(\mathcal{Q}) = \#\bar{\mathcal{I}} - 1$.

PROOF. Let $\Phi = [x \ \Sigma\Pi x]_{\bar{\mathcal{I}}}$ be the said submatrix. Let $r \in [c]$ be such that $\bar{\mathcal{I}} \cap \mathcal{I}_r \neq \varnothing$. Since $\mathcal{I}_r \notin \bar{\mathcal{I}}$, we can partition $\bar{\mathcal{I}} \cap \mathcal{I}_r$ into subsets $\bar{\mathcal{I}}_{rj}$ for $j \in [s_r]$ for some s_r, such that each $\Phi_{rj} = [x \ \Sigma\Pi x]_{\bar{\mathcal{I}}_{rj}}$ has up to a permutation of the rows the form

$$\Phi_{rj} = \begin{bmatrix} x_\alpha & \sigma_\beta x_\beta \\ x_{\alpha+1} & \sigma_\alpha x_\alpha \\ \vdots & \vdots \\ x_{\alpha+\ell-2} & \sigma_{\alpha+\ell-3} x_{\alpha+\ell-3} \\ x_\gamma & \sigma_{\alpha+\ell-2} x_{\alpha+\ell-2} \end{bmatrix}$$

Here $\sigma_i \in \{1, -1\}$ and $x_\alpha, \dots, x_{\alpha+\ell-2}, x_\beta, x_\gamma$ are distinct variables appearing only in Φ_{rj}. Note that there is a total of $s = \sum_{\bar{\mathcal{I}}_r \neq \varnothing} s_r$ blocks Φ_{rj} and a total of $\#\bar{\mathcal{I}} + s$ distinct indeterminates appearing in Φ. Let T be the general determinantal ring over k of 2-minors of a $\#\bar{\mathcal{I}} \times 2$ matrix of variables. Then it is very well known that T is Cohen-Macaulay of dimension $\#\bar{\mathcal{I}} + 1$ [**BV88**]. Taking quotient with $\#\bar{\mathcal{I}} - s$ suitable linear forms we obtain the quotient ring associated to $\mathcal{Q}$. Taking quotient with extra s linear forms we can obtain the quotient ring of an ideal of the form appearing in Lemma 4.11. Then as per Lemma 4.11 this is 1-dimensional so that the total sequence of $\#\bar{\mathcal{I}}$ linear forms is a T-regular sequence. □

Remark 4.13. *Ignoring the sign matrix Σ, a geometric view of the proof of Lemma 4.12 is the following. When $k = \bar{k}$ the ideal $\mathcal{Q}$ corresponds to a rational normal scroll of dimension $s+1$. Then we take a sequence of s hyperplane sections of that scroll, each time getting a new scroll of dimension one less until the scroll degenerates to the union of eigenspaces of a cyclic permutation. See* [**CF17**] *for a complete classification of rational normal scrolls that arise as hyperplane sections of rational normal scrolls, see also* [**CJ97**] *for the free resolution of ideals of 2-minors of a matrix of linear forms with two columns.*

It is enough to bound as claimed the dimension of $U_{\rho_2\rho_1\pi,\rho_2}$ where π is some permutation. Since the dimension of locally finite type k-schemes is preserved under any field extension (exercise 11.2.J in [**Vak17**]) we may assume that $k = \bar{k}$. Let R_1, R_2, Π be matrix representations of ρ_1, ρ_2, π on the canonical basis of k^m. For a closed point $v \in U_{\rho_2\rho_1\pi,\rho_2}$ we have $R_2 R_1 \Pi v = \lambda R_2 v$ for some $\lambda \neq 0, 1$. For $i = 1, 2$, let $I_i \subset [m]$ be the indices that correspond to $\operatorname{im}(R_i)$, and similarly K_i the indices that correspond to $\ker(R_i)$. If $i \in I_2 \cap K_1$, then it is clear that v_i must be zero, because $\lambda \neq 0$. If $\pi(i) \in I_2 \cap K_1$, then we must also have $v_{\pi(i)} = 0$ for the same reason. If $\pi(i) \in I_2 \cap I_1$, then again $v_{\pi(i)} = 0$ because we already have $v_i = 0$ and $\lambda \neq 0$. This *domino effect* either forces v to be zero in the entire orbit of i, or until an index j in the orbit of i is reached such that $\pi(j) \in K_2 \cap K_1$. Let $I_{\text{domino}} \subset I_2$ be the coordinates of v that are forced to zero by the union of the domino effects for every $i \in I_2 \cap K_1$. Clearly $I_2 \setminus I_{\text{domino}} \subset I_2 \cap I_1$. Let $i \in I_2 \setminus I_{\text{domino}}$; if it so happens that $\pi(i) = i$, then we must have that $v_i = 0$ because $\lambda \neq 1$. Consequently the coordinates of v that correspond to fixed points of π and lie in $I_2 \setminus I_{\text{domino}}$ must be zero. Letting $I_{\text{fixed}} \subset I_2 \setminus I_{\text{domino}}$ be the set containing these indices, v must lie in the linear variety defined by the vanishing of the coordinates indexed by $I_{\text{domino}} \cup I_{\text{fixed}}$.

Next, let $\bar{\pi}_1, \dots, \bar{\pi}_{c'}$ be all the $c' \geq 0$ cycles of π of length at least two that lie entirely in $I_2 \setminus (I_{\text{domino}} \cup I_{\text{fixed}})$. Let $C_i \subset [m]$ be the indices cycled by $\bar{\pi}_i$. Since $\lambda \neq 0$, it is clear that v_{C_i} must be an eigenvector of $\bar{\pi}_i$, and so by Lemma 4.11 v_{C_i} must lie in a codimension-$(\#C_i - 1)$ variety. Adding codimensions over $i \in [c']$, and letting $I_{\text{cycles}} = \bigcup_{i\in[c']} C_i$, we get that $v_{I_{\text{cycles}}}$ must lie in a variety of codimension $\sum_{i\in[c']}(\#C_i - 1)$. Moreover, we may assume that the set $I_{\text{incomplete}} = I_2 \setminus (I_{\text{domino}} \cup I_{\text{fixed}} \cup I_{\text{cycles}})$ does not contain any complete cycles, and if $I_{\text{incomplete}} \neq \varnothing$ Lemma 4.12 gives that $v_{I_{\text{incomplete}}}$ must lie in a codimension-$(\#I_{\text{incomplete}} - 1)$ variety.

Let $\mathcal{Y}_{\text{domino}}, \mathcal{Y}_{\text{fixed}}, \mathcal{Y}_{\text{cycles}}, \mathcal{Y}_{\text{incomplete}}$ be the varieties defined by the vanishing of the coordinates in I_{domino}, the vanishing of the coordinates in I_{fixed}, as well as the vanishing of the 2-minors of the matrix $[x\ \Pi x]$ indexed by I_{cycles} and $I_{\text{incomplete}}$ respectively. Noting that these varieties are all associated with disjoint polynomial rings and that $\#I_{\text{domino}} + \#I_{\text{fixed}} + \#I_{\text{cycles}} + \#I_{\text{incomplete}} = \#I_2$, the above analysis gives that v must lie in a variety $\mathcal{Y} = \mathcal{Y}_{\text{domino}} \times \mathcal{Y}_{\text{fixed}} \times \mathcal{Y}_{\text{cycles}} \times \mathcal{Y}_{\text{incomplete}}$ so that

$$\begin{aligned}\operatorname{codim}\mathcal{Y} &\geq \#I_{\text{domino}} + \#I_{\text{fixed}} + \sum_{i\in[c']}(\#C_i - 1) + \max\{\#I_{\text{incomplete}} - 1, 0\}\\ &= \#I_2 - c' - \#I_{\text{incomplete}} + \max\{\#I_{\text{incomplete}} - 1, 0\}.\end{aligned}$$

If $I_{\text{incomplete}} = \varnothing$, then $\operatorname{codim}\mathcal{Y} \geq \#I_2 - c'$. Since $c' \leq \#I_2/2$, we have that $\operatorname{codim}\mathcal{Y} \geq \#I_2/2 \geq \lfloor \#I_2/2 \rfloor$. If on the other hand $I_{\text{incomplete}} \neq \varnothing$, then $c' \leq \lfloor(\#I_2 - 1)/2\rfloor$, so that $\operatorname{codim}\mathcal{Y} \geq \#I_2 - \lfloor(\#I_2 - 1)/2\rfloor - 1 \geq \lfloor \#I_2/2 \rfloor$, with the last inequality separately verified for $\#I_2$ odd or even.

C. Proof of Corollary 4.3

If $\rho_1 \in \mathscr{R}_n$ and $\rho_2 \in \mathscr{R}_{2n}$, then the claim is a direct corollary of Theorems 4.1 and 4.2. Otherwise, a similar set of arguments as in the proof of Theorem 4.2 establishes that $\dim \mathcal{U}^{\pm}_{\rho_2\rho_1\pi_1,\rho_2\pi_2} \leq m - \lfloor \operatorname{rank}(\rho_2)/2 \rfloor$, where now $\mathcal{U}^{\pm}_{\rho_2\rho_1\pi_1,\rho_2\pi_2} = \mathcal{U}_{\rho_2\rho_1\pi_1,\rho_2\pi_2} \setminus \ker(\rho\tau_1 + \tau_2)$. Moreover, an identical argument as in the end of the proof of Proposition 4.5 shows that we can adjust that proposition as follows: "Suppose $\dim_{\bar{k}} E_{\tau,\lambda} \leq m - n$ for every $\lambda \neq 1, -1$. Then for a general n-dimensional subspace V and $v_1, v_2 \in V$ we have $\tau(v_1) = v_2$ only if $v_1 = v_2$ or $v_1 = -v_2$." Combining everything together establishes the claim.

D. Proof of Proposition 4.4

Let $A \in k^{m \times n}$ be a basis of V. If $\tau_1(v_1) = \tau_2(v_2)$ then $\tau_2(v_2) \in \tau_1(V)$ and so $\operatorname{rank}([T_1 A \ T_2 A\xi]) \leq n$ for $\xi \in k^n$ with $v_2 = A\xi$. We show that for general V, ξ this can not happen unless $\tau_1 = \tau_2$, in which case $v_1 - v_2 \in \ker(\tau_1)$ and so $v_1 = v_2$. Suppose $\tau_1 \neq \tau_2$. We show the existence of A, ξ such that $\operatorname{rank}([T_1 A \ T_2 A\xi]) = n + 1$. Since $\tau_1 \neq \lambda\tau_2$ for all $\lambda \in k$, there exists some $v \in k^m$ such that $\tau_1(v), \tau_2(v)$ are linearly independent. Let $W = \operatorname{Span}\big(\tau_1(v), \tau_2(v)\big)$. Since $\operatorname{rank}(\tau_1) \geq n + 1$, any complement C of $W \cap \operatorname{im}(\tau_1)$ in $\operatorname{im}(\tau_1)$ has dimension at least $n - 1$. Let C_1 be a subspace of C of dimension $n - 1$. Let V_1 be a subspace of $\tau_1^{-1}(C_1)$ of dimension $n - 1$ such that $C_1 = \tau_1(V_1)$. Then for $V = V_1 + \operatorname{Span}(v)$ we have $\dim\big(\tau_1(V) + \tau_2(v)\big) = n + 1$.

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