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SHAPE OPTIMIZATION FOR MONGE-AMPÈRE EQUATIONS VIA DOMAIN DERIVATIVE

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Abstract. In this note we prove that, if Ω is a smooth, strictly convex, open set in $\mathbb{R}^n$ $(n \geq 2)$ with given measure, the L^1 norm of the convex solution to the Dirichlet problem $\det D^2u = 1$ in Ω, $u = 0$ on $\partial\Omega$, is minimum whenever Ω is an ellipsoid.

1. **Introduction.** In the classical paper [13] Talenti proves that, if Ω varies in the class of bounded, open sets in $\mathbb{R}^n$ $(n \geq 2)$ with given measure, denoted by u the weak solution to the Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

the L^1 norm of u is maximum if and only if Ω is a ball. This result contains the positive answer to the famous conjecture of St. Venant that, among all the 2-dimensional open, bounded regions, the disks take the biggest torsional rigidity. Moreover, the proof relies on the classical isoperimetric inequality and the Schwarz symmetrization methods. Although symmetrization techniques have been successfully applied to prove sharp estimates of solutions to many PDE's (see, for example, [14] for a wide bibliography), they fail whenever one wants to prove a Talenti's type result for the Monge-Ampère operator. The main reason is that the Monge-Ampère operator is invariant under measure preserving affine transformations. In particular, if $\mathcal{A}$ is a measure preserving affine transformation and u is a solution to

$$\begin{cases} \det D^2 u = 1 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a convex, open set in $\mathbb{R}^n$, then $u_{\mathcal{A}}(x) = u(\mathcal{A}x)$ is a solution to (1) in $\mathcal{A}^{-1}\Omega$ and $\|u\|_{L^1(\Omega)} = \|u_{\mathcal{A}}\|_{L^1(\mathcal{A}^{-1}\Omega)}$. These considerations suggest to look for affine isoperimetric inequalities and use them to construct a suitable symmetrization method. This is what we have done in a previous paper [3] to prove, among other things, the following result.

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Theorem 1.1. *Let Ω be a smooth, strictly convex, open set in $\mathbb{R}^n$ $(n \geq 2)$ and let u be the convex solution to (1) in Ω. Then*

$$\int_\Omega |u|\, dx \geq \frac{|\Omega|^{\frac{n+2}{n}}}{(n+2)\omega_n^{\frac{2}{n}}},$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$ and equality holds if and only if Ω is an ellipsoid.

Remark 1. We explicitly observe that if Ω is an ellipsoid, then the solution to (1) is

$$u(x) = \frac{\sum_{i,j=1}^{n} A^{ij}(x_i - x_i^0)(x_j - x_j^0) - 1}{2(\det A)^{1/n}} \tag{2}$$

for some symmetric, positive definite matrix $A = (A^{ij})$ such that $\sum_{i,j=1}^{n} A^{ij}(x_i - x_i^0)(x_j - x_j^0) = 1$ describes $\partial\Omega$, being x_0 the center of Ω. In such a case the equality sign in the statement of Theorem 1.1 holds.

In this note we provide a completely different proof of Theorem 1.1; the main tools we are using are the domain derivative and the notion of affine curvature flow of a smooth convex set. The domain derivative has been extensively applied in several contexts, for instance in [1, 6] to prove a priori estimates for solutions to elliptic PDE's, in [7, 8] to face some shape optimization questions, in [4] to solve overdetermined problems. Concerning the affine curvature flow for smooth convex sets, existence and regularity results are contained in [2], where, among other things, they are used to give an alternative proof of some affine isoperimetric inequalities.

2. **Notation and preliminaries.**

In this section we collect some definitions and notation that will be useful in the sequel.

Let Ω be a convex open set in $\mathbb{R}^n$ and let $u \in C^2(\Omega)$ be a convex function. The Monge-Ampère operator $\det D^2u$ is elliptic with respect to convex functions and homogeneous of degree n. Denoted by

$$S^{ij}(D^2u) = \frac{\partial}{\partial u_{ij}}(\det D^2u)$$

the cofactor matrix of D^2u, where subscripts stand for partial differentiations, Euler identity for homogeneous functions gives

$$\det D^2u = \frac{1}{n} S^{ij}(D^2u)u_{ij},$$

where the convention over repeated indices is in force.

A direct computation yields that $(S^{1j}(D^2u), \dots, S^{nj}(D^2u))$ is divergence free, i.e.

$$(S^{ij}(D^2u))_i = 0;$$

hence $\det D^2u$ can be written in divergence form as follows

$$\det D^2u = \frac{1}{n}(S^{ij}(D^2u)u_j)_i. \tag{3}$$

Moreover, if Ω is of class C^2_+ (i.e. Ω is a nonempty, compact, convex set whose boundary is of class C^2 with nonvanishing Gaussian curvature $k_{\partial\Omega}$), the following pointwise identity holds (see for instance [10])

$$k_{\{u=h\}} = \left.\frac{S^{ij}(D^2u)u_iu_j}{|Du|^{n+1}}\right|_{\{u=h\}}, \quad h \in \mathbb{R}. \tag{4}$$

We also recall the following affine isoperimetric inequality, known as Petty inequality, stating that (see, for example, [9, 11])

$$\int_{\partial\Omega} k_{\partial\Omega}^{\frac{1}{n+1}} \, d\mathcal{H}^{n-1} \le n\omega_n^{\frac{2}{n+1}} |\Omega|^{\frac{n-1}{n+1}}, \tag{5}$$

equality holding if and only if Ω is an ellipsoid. The integral on the left hand side of (5) is known as affine surface area and $k_{\partial\Omega}^{\frac{1}{n+1}}$ is known as affine curvature.

Now we recall a result proved in [2] concerning affine curvature flow.

Lemma 2.1. *Let $\varphi_0 : \mathbb{S}^{n-1} \to \mathbb{R}^n$ be a smooth (C^∞) strictly convex embedding of the unit sphere in $\mathbb{R}^n$; then there exists a unique $t_E > 0$ and a unique*

$$\varphi(z,t) \in C^\infty(\mathbb{S}^{n-1} \times [0, t_E[; \mathbb{R}^n)$$

such that, for all $0 \le t < t_E$, $\varphi(\cdot,t) : \mathbb{S}^{n-1} \to \Gamma_t \subset \mathbb{R}^n$ is a smooth, closed surface, uniformly convex (i.e. with strictly positive Gaussian curvature) for $t > 0$, and for all $(z,t) \in \mathbb{S}^{n-1} \times [0, t_E[$ φ is a solution to the following partial differential equation

$$\frac{\partial \varphi}{\partial t}(z,t) = -(k_{\Gamma_t}(\varphi(z,t)))^{\frac{1}{n+1}} \nu_{\Gamma_t}(\varphi(z,t)),$$

where $k_{\Gamma_t}(x)$ and $\nu_{\Gamma_t}(x)$ are, respectively, the Gaussian curvature and the outer normal of Γ_t at the point $x \in \Gamma_t$ and $\varphi(z,0) = \varphi_0(z)$.

Moreover

i) Γ_t converges to a point as $t \nearrow t_E$,
ii) after rescaling about the final point to make the enclosed volume constant, Γ_t converges in C^∞ to an ellipsoid.

The affine curvature flow of a convex surface is a flow where each point of the surface moves in the direction of the inner normal with velocity equal to the affine curvature of the surface itself. The previous lemma states that, for any initial smooth, convex, closed surface, it is possible to find a unique one parameter family of solutions to the affine curvature flow. Such a family is smooth and shrinks to a point by approaching an ellipsoidal shape.

3. **Proof of Theorem 1.1.** Let us consider a one parameter family of transformations $\Phi(x,t)$ satisfying, for some $\delta > 0$, the following conditions:

(a) $\Phi(\cdot,t)$ and $\Phi^{-1}(\cdot,t)$ belong to $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ for all $0 \le t < \delta$,
(b) the mappings $t \to \Phi(x,\cdot)$ and $t \to \Phi^{-1}(x,\cdot)$ belong to $C^\infty([0,\delta[)$ for all $x \in \mathbb{R}^n$,
(c) $\Omega(t) = \Phi(\Omega,t)$ is a one parameter family of strictly convex and C^∞ domains such that $\Omega(0) = \Omega$.

We then look at the corresponding family $u(x,t)$ of solutions to

$$\begin{cases} \det D^2u(x,t) = 1 & \text{in} \quad \Omega(t) \\ u(x,t) = 0 & \text{on} \quad \partial\Omega(t). \end{cases} \tag{6}$$

Classical regularity theory for the Monge-Ampère equation ensures that for all $t \in [0, \delta[$ $u(x,t) \in C^\infty(\overline{\Omega}(t))$, see for instance [5]. Moreover, arguing as in [8, Chapter

5], there exists at least some positive $\varepsilon < \delta$ such that the function $u(x,t)$ belongs to $C^\infty([0,\varepsilon[; C^\infty(\overline{\Omega}(t)))$.

For the reader convenience, in what follows we will denote by $u_i = \dfrac{\partial u(x,t)}{\partial x_i}$, $u_{ij} = \dfrac{\partial^2 u(x,t)}{\partial x_i \partial x_j}$, while the subscript t will denote $\dfrac{\partial}{\partial t}$.

Differentiating with respect to t the equation in (6) we get

$$\begin{aligned} 0 &= \frac{d}{dt}(\det D^2 u) = \frac{\partial}{\partial u_{ij}}(\det D^2 u) u_{ijt} \\ &= S^{ij}(D^2u)u_{ijt} = S^{ij}(D^2u)u_{ijt} + (S^{ij}(D^2u))_j u_{it} = (S^{ij}(D^2u)u_{it})_j \end{aligned}$$

that is

$$(S^{ij}(D^2u)u_{it})_j = 0 \qquad \text{in } \Omega(t). \tag{7}$$

Since $\Omega(t)$ is, at any time, the zero-level set of $u(x,t)$, if $x \in \partial\Omega$ then $\Phi(x,t) \in \partial\Omega(t)$ and we have

$$u(\Phi(x,t),t) = 0 \qquad \text{for all } 0 < t < \varepsilon. \tag{8}$$

The boundary point $\Phi(x,t)$ of $\Omega(t)$ moves with velocity $\frac{\partial}{\partial t}\Phi(x,t)$. Differentiating (8) with respect to t, the projection of such a velocity along the direction of the inner unit normal $-\nu$ is equal to $\dfrac{u_t}{|Du|}$.

If $f(x,t)$ is a smooth function and $J(t) = \displaystyle\int_{\Omega(t)} f(x,t)\,dx$, the classical Hadamard formula gives (see, for instance, [8, 12])

$$J'(t) = \int_{\Omega(t)} f_t(x,t)\,dx - \int_{\partial\Omega(t)} f(x,t)\,\frac{u_t}{|Du|}\,d\mathcal{H}^{n-1}.$$

Therefore,

$$\text{if } G(t) = \int_{\Omega(t)} |u|\,dx, \quad \text{then } G'(t) = -\int_{\Omega(t)} u_t\,dx \tag{9}$$

and

$$\text{if } H(t) = |\Omega(t)|, \quad \text{then } H'(t) = -\int_{\partial\Omega(t)} \frac{u_t}{|Du|}\,d\mathcal{H}^{n-1}. \tag{10}$$

Moreover, from (3), (4), divergence theorem and (7) we can deduce

$$\begin{aligned} \int_{\Omega(t)} u_t\,dx &= \frac{1}{n}\int_{\Omega(t)} (S^{ij}(D^2u)u_i)_j u_t\,dx \\ &= \frac{1}{n}\int_{\partial\Omega(t)} k|Du|^n u_t\,d\mathcal{H}^{n-1} - \frac{1}{n}\int_{\Omega(t)} S^{ij}(D^2u)u_i u_{jt}\,dx \\ &= \frac{1}{n}\int_{\partial\Omega(t)} k|Du|^n u_t\,d\mathcal{H}^{n-1} - \frac{1}{n}\int_{\partial\Omega(t)} S^{ij}(D^2u)\nu_i u u_{jt}\,dx \\ &\quad + \int_{\Omega(t)} \frac{1}{n}(S^{ij}(D^2u)u_{jt})_i u\,dx \\ &= \frac{1}{n}\int_{\partial\Omega(t)} k|Du|^n u_t\,d\mathcal{H}^{n-1}, \end{aligned}$$

where $k = k_{\partial\Omega(t)}$ and then (9) can be rewritten as follows

$$G'(t) = -\frac{1}{n}\int_{\partial\Omega(t)} k|Du|^n u_t\,d\mathcal{H}^{n-1}. \tag{11}$$

Setting

$$F(t) = \frac{\displaystyle\int_{\Omega(t)} |u(t)|\,dx}{|\Omega(t)|^{\frac{n+2}{n}}},$$

from the regularity of all the involved quantities we deduce the smoothness of F and from (10)–(11) that

$$\begin{aligned} F'(t) &= \frac{n+2}{n}|\Omega(t)|^{-\frac{2(n+1)}{n}}\left(\int_{\partial\Omega(t)} \frac{u_t}{|Du|}\,d\mathcal{H}^{n-1}\right)\left(\int_{\Omega(t)}|u|\,dx\right) \\ &\quad -\frac{1}{n}|\Omega(t)|^{-\frac{n+2}{n}}\int_{\partial\Omega(t)} k|Du|^n u_t\,d\mathcal{H}^{n-1}. \end{aligned}$$

Now, the main idea of our proof consists in choosing the family Φ satisfying (a), (b), (c) and such that the boundary of the domain Ω evolves by affine curvature flow. Actually there are infinitely many families satisfying these properties but there is a natural way to define such a family through the so-called *speed method* (see [12, §2.9]). Without entering into the details, if V is a smooth vector field in $\mathbb{R}^n$, we can construct a family of transformations $\Phi(x,t)$ by solving the following ODE

$$\frac{d}{dt}\Phi(x,t) = V(\Phi(x,t))$$

with the initial condition $\Phi(x,0) = x \in \Omega$. The vector field V is the velocity field of the transformation and its regularity immediately implies the regularity of Φ. Let us now consider the affine curvature flow of the boundary of the strictly convex domain Ω. We can choose $V(x)$ to be the velocity of the shrinking surface when passing through x. The smoothness of the flow (see Lemma 2.1) ensures that $V(x)$ is smooth in the whole Ω but at the point where the surface shrinks. We can modify $V(x)$ in an arbitrarily small neighborhood of such a point in order to make it smooth and extend it outside Ω in any reasonable smooth way. Eventually, there exists $\delta > 0$ such that

$$\Phi(\varphi_0(z),t) = \varphi(z,t) \qquad \text{for } (z,t) \in \mathbb{S}^{n-1}\times[0,\delta[.$$

Now, if $x \in \partial\Omega(t)$ and $k(x,t)$ is the Gauss curvature of $\partial\Omega(t)$ at the point x, we have

$$\frac{u_t}{|Du|}(x,t) = k(x,t)^{\frac{1}{n+1}}.$$

Thus

$$\begin{aligned} F'(t) &= \frac{n+2}{n}|\Omega(t)|^{-\frac{2(n+1)}{n}}\left(\int_{\partial\Omega(t)} k^{\frac{1}{n+1}}\,d\mathcal{H}^{n-1}\right)\left(\int_{\Omega(t)}|u|\,dx\right) \\ &\quad -\frac{1}{n}|\Omega(t)|^{-\frac{n+2}{n}}\int_{\partial\Omega(t)} k^{\frac{n+2}{n+1}}|Du|^{n+1}\,d\mathcal{H}^{n-1}. \end{aligned} \tag{12}$$

The last equation is valid whenever $t < \varepsilon$, since, in principle, the function $u(x,t)$ exists smooth on a time interval $[0,\varepsilon[$. However, denoted by t_E the extinction time of the affine curvature flow for the domain Ω, if $\varepsilon < t_E$, the regularity of the affine curvature flow yields immediately that $\lim_{t\nearrow\varepsilon} F(t) = F(\varepsilon)$. We can therefore repeat all the steps performed from the beginning of this paragraph using the uniformly convex and smooth set $\Omega(\varepsilon)$ as initial domain. By standard arguments we extend the validity of equation (12) on the whole interval $[0,t_E[$.

Since the function $F(t)$ is invariant under affine transformations of $\mathbb{R}^n$ in itself, Lemma 2.1 - ii) yields

$$\lim_{t\to t_E} F(t) = \frac{\omega_n^{-\frac{2}{n}}}{n+2} \equiv F_E,$$

where with F_E we denote the value of F when Ω is a ball.

In the following we shall prove that $F(0) \geq F_E$.
By using (3), (4), divergence theorem, Hölder inequality and Petty inequality (5) we get

$$\begin{aligned} n|\Omega(t)| &= \int_{\partial\Omega(t)} k|Du|^n \, d\mathcal{H}^{n-1} \qquad (13)\\ &\leq \left(\int_{\partial\Omega(t)} k^{\frac{n+2}{n+1}}|Du|^{n+1}\, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} \left(\int_{\partial\Omega(t)} k^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \\ &\leq \left(\int_{\partial\Omega(t)} k^{\frac{n+2}{n+1}}|Du|^{n+1}\, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} \left(n\omega_n^{\frac{2}{n+1}}|\Omega(t)|^{\frac{n-1}{n+1}}\right)^{\frac{1}{n+1}}. \end{aligned}$$

Together (12) and (13) imply

$$\begin{aligned} F'(t) &\leq (n+2)\omega_n^{\frac{2}{n+1}}|\Omega(t)|^{-\frac{n^2+5n+2}{n(n+1)}}\left(\int_{\Omega(t)}|u|\,dx\right) - \omega_n^{-\frac{2}{n(n+1)}}|\Omega(t)|^{-\frac{2}{n+1}} \qquad (14)\\ &= (n+2)\omega_n^{\frac{2}{n+1}}|\Omega(t)|^{-\frac{2}{(n+1)}}\left(|\Omega(t)|^{-\frac{n+2}{n}}\int_{\Omega(t)}|u|\,dx - \frac{\omega_n^{-\frac{2}{n}}}{n+2}\right) \\ &= (n+2)\omega_n^{\frac{2}{n+1}}|\Omega(t)|^{-\frac{2}{(n+1)}}\left(F(t)-F_E\right). \end{aligned}$$

We distinguish two cases:

Case 1. $F'(0) \geq 0$;
Case 2. $F'(0) < 0$.

In Case 1, from (14) it immediately follows that $F(0) \geq F_E$. Moreover if $F(0) = F_E$ then equality holds in (14) and (13) for $t = 0$. But in (14) and (13) we just used Petty and Hölder inequalities. The equality in Petty inequality enforces Ω to be an ellipsoid. Thereafter u has the form (2) which also explains the equality sign in the Hölder inequality. In Case 2, if $F'(t) < 0$ for every $t \in]0, t_E[$, then $F(0) > \lim_{t\to t_E} F(t) = F_E$; otherwise, there exists $\bar{t} \in]0, t_E[$ such that $F'(t) < 0$ if $0 \leq t < \bar{t}$ and $F'(\bar{t}) = 0$. By using again (14) we get $F(0) > F(\bar{t}) \geq F_E$.

Finally we have proved that

$$|\Omega|^{-\frac{n+2}{n}}\int_\Omega |u|\,dx \geq \frac{\omega_n^{-\frac{2}{n}}}{n+2}$$

with equality if and only if Ω is an ellipsoid.

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