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Probabilistic hesitant fuzzy multiple attribute decision-making based on regret theory for the evaluation of venture capital projects

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ABSTRACT
The selection of venture capital investment projects is one of the most important decision-making activities for venture capitalists. Due to the complexity of investment market and the limited cognition of people, most of the venture capital investment decision problems are highly uncertain and the venture capitalists are often bounded rational under uncertainty. To address such problems, this article presents an approach based on regret theory to probabilistic hesitant fuzzy multiple attribute decision-making. Firstly, when the information on the occurrence probabilities of all the elements in the probabilistic hesitant fuzzy element (P.H.F.E.) is unknown or partially known, two different mathematical programming models based on water-filling theory and the maximum entropy principle are provided to handle these complex situations. Secondly, to capture the psychological behaviours of venture capitalists, the regret theory is utilised to solve the problem of selection of venture capital investment projects. Finally, comparative analysis with the existing approaches is conducted to demonstrate the feasibility and applicability of the proposed method.

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1. Introduction

Through decades of sustained development, China's capital market is growing fast and has begun to take shape. It attracts a lot of investors to participate and invest their money in different fields, such as the stock market (Shen & Tzeng, 2015), the property market (Heidi, 2009), the electronics industry (Lin, Chen, & Ting, 2011), the automobile industry (Buckley, Clegg, Ping, Siler, & Giorgioni, 2007) and so on. Venture capital is an important driving force for promoting economic development, and the investment project selection is an important issue that relates to the survival of enterprises. How to select an appropriate investment project for investors will be a

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challenging problem. Therefore, making the right investment decision is one of the most important issues faced by investors. To solve the investment project selection problems, some decision methods were presented. Wang, Wang, and Wang (2018) put forward a method with interval neutrosophic probability and applied it to stock selection problems. Wu, Kou, Peng, and Ergu (2012) proposed an approach based on improved A.H.P. for evaluating investment risk. To select the optimal investment market, Zeng and Xiao (2016) presented an intuitionistic fuzzy ordered weighted averaging weighted averaging (O.W.A.W.A.) distance T.O.P.S.I.S. method. Considering that the capital market is characterised by uncertainty, risk and fuzziness, some different decision-making methods with fuzzy information has been researched deeply. Zhang, Du, and Tian (2018) put forward a method based on regret theory for dealing with risky multiple attribute decision-making problems. Liu, Jin, Zhang, Su, and Wang (2011) presented a risk decision-making method based on prospect theory under uncertain linguistic environment. Based on fuzzy and rough set theory, Renigier-Biłozor, Janowski, and d'Amato (2019) proposed an automated valuation model for real estate market. In these methods, the decision-making process tends to be uncertain and ambiguous as it involves complexity of human cognitive thinking (Liu et al., 2019a, 2019b; Zeng, Peng, Baležentis, & Streimikiene, 2019). Therefore, it is hard for decision-makers to provide precise assessments in the assessment process (Chi, Yeh, & Lai, 2011; Gao, 2018; Lu, Tang, Wei, Wei, & Wei, 2019).

As pointed by Dadras, Momeni, and Majd (2008), uncertainty is widely found in the complex realities (Wang, Gao, Wei, & Wei, 2019; Wu, Liu, Wang, & Zhang, 2019; Wu, Wang & Gao, 2019). To model the uncertainty, Torra (2010) proposed the concept of hesitant fuzzy set (H.F.S.), which is an extension of fuzzy set and can be considered as an effective tool for handling the uncertainty and fuzziness in the uncertain data (Liu, Wang, & Hetzler, 2017, 2018a, 2018b). With the in-depth research, a significant drawback with H.F.S. appears, namely, the loss of information. To overcome this drawback, Zhu and Xu (2018) proposed the concept of probabilistic hesitant fuzzy set (P.H.F.S.), which incorporate distribution information in H.F.S. P.H.F.S. depicts not only the hesitancy of decision-makers when they are irresolute for one thing or another, but also the hesitant distribution information (Li & Wang, 2018; Wu, Liu, et al., 2019). Afterwards, Zhang, Xu, and He (2017) defined the operations over P.H.F.S.s and presented an improved P.H.F.S. that can incorporate incomplete evaluation information. Li and Wang (2017) extended the Q.U.A.L.I.F.L.E.X. method to accommodate probabilistic hesitant fuzzy environment and applied the proposed method to the green supplier selection. Gao, Xu, and Liao (2017) presented a dynamic reference point method with probabilistic hesitant fuzzy information for emergency response. Xu and Zhou (2017) proposed the concept of P.H.F.E., which is often taken as the unit of P.H.F.S., and several probabilistic hesitant fuzzy aggregation operators were put forward to fuse probabilistic hesitant fuzzy information. Wu, Jin, and Xu (2018) provided a novel consensus reaching process for probabilistic hesitant fuzzy group decision-making and applied the proposed method to evaluate the strategic positions of energy channels. Thus it can be seen that P.H.F.S. has aroused widespread interest of scholars and has been applied to many areas.

In the above-mentioned researches, the exact values are given to depict the occurrence probabilities of elements in the P.H.F.E. However, the probabilities of the elements in the P.H.F.E. are hard to determine through subjective evaluation of a decision-maker (Zhou & Xu, 2018). Zhou and Xu (2018) and Li and Wang (2018) proposed an approach to calculate the probabilities of the elements in the P.H.F.E. based on probabilistic hesitant fuzzy preference relations (P.H.F.P.R.s), respectively. In fact, the decision-makers sometimes cannot provide their judgements by pairwise comparison of alternatives, namely, P.H.F.P.R.s, instead they give their assessed values for attributes directly, namely, P.H.F.E. Therefore, how to objectively determine the probabilities of the elements in the P.H.F.E. is an urgent problem to be solved, which is also one of the keys of this article.

In addition, the behavioural experiments show that decision-makers are often bounded rational under uncertainty and risk (Camerer, 1998; Kahneman & Tversky, 1979; Tversky & Kahneman, 1992). Therefore, the psychological behaviours of decision-makers should be considered in the decision process. Especially, in the field of investment decision, the noteworthy features of venture capital are uncertainty and high-risk (Ruhnka & Young, 1991), and venture capitalists are usually bounded rational rather than complete rational when making decisions (Tian, Xu, & Fujita, 2018a, Tian, Xu, Gu, & Herrera-Viedma, 2018b; Zhang et al., 2018). Tian, Xu, and Fujita (2018a) proposed an approach based on prospect theory and probabilistic hesitant fuzzy preferences to study the sequential decision-making of the venture capitalists. Zhang et al. (2018) have extended the T.O.D.I.M. method, which is based on prospect theory, to probabilistic hesitant fuzzy environment for the evaluation of venture capital projects. In these studies, prospect theory is adopted to solve the investment decision problems of venture capital under probabilistic hesitant fuzzy environment. However, prospect theory fails to explain many aspects of decision-making (Nwogugu, 2006). According to neurobiology, Nwogugu (2006) proved that the natural mental process of human beings would bring about decision patterns, which are different from those implied in prospect theory. Moreover, Nagarajan and Shechter (2014) also demonstrated that the consistent empirical findings would not be explained by prospect theory. Therefore, it is necessary to develop a more realistic decision model. To depict intuitive judgements more consistently, Loomes and Sugden (1982) and Bell (1982) put forward regret theory independently, namely rejoice and regret factors were introduced into the utility values. Then, Quiggin (1994) extended it to a more general form. To date, regret theory has been applied to many fields (Liu, Wang, & Zhang, 2018b; Zhang, Zhu, Liu, & Chen, 2016; Zhang, Du, et al.2018; Zhou, Wang, & Zhang, 2017).

In this article, we expand the application of regret theory and extend it to accommodate probabilistic hesitant fuzzy environment. Then a novel approach to evaluate the venture capital projects is presented. The main novelties of this article can be summarised as follows:

1. Approaches to determine the occurrence probabilities of all the possible elements in the P.H.F.E. are presented. As is stated above, the probabilities of the elements are a key component of the P.H.F.E. It is hard to identify the probabilities of

elements in the P.H.F.E. through subjective evaluation of decision-makers, especially when the information on occurrence probabilities of elements in the P.H.F.E. is partially known. Therefore, approaches for calculating the probabilities of the elements in the P.H.F.E. are proposed whatever the information on the occurrence probabilities of the elements in the P.H.F.E. is unknown or partially known.

2. The regret theory is extended to accommodate probabilistic hesitant fuzzy environment. As mentioned above, decision-makers are often bounded rational under uncertainty and risk. Decisions are often correlated with behaviours, and thus the psychological behaviours of decision-makers should be integrated into decision analysis. In this article, the regret theory is introduced into decision-making system under probabilistic hesitant fuzzy environment. Then, a novel approach to probabilistic hesitant fuzzy multiple attribute decision-making is proposed.
3. An investigation on the selection of venture capital investment project is conducted. Venture capital plays a critical role in supporting innovation activities, and the investment decision phase has an effect on venture capital performance (Cheng, Gu, & Xu, 2018). It can be considered as a multiple attribute decision-making process that needs to consider both the bounded rationality of venture capitalists and uncertain decision environment for venture capital investment project. Therefore, in this article, a practical example for selecting the promising venture capital project is given, and comparative analysis are conducted to demonstrate the superiority of the proposed method.

The remainder of this article is organised as follows. Some basic concepts are provided in Section 2. In Section 3, we describe the probabilistic hesitant fuzzy multiple attribute decision-making problems. With the aid of the maximum entropy principle and water-filling theory, we present two different mathematical programming models to determine the occurrence probabilities of the elements in the P.H.F.E., and then an approach to probability hesitant fuzzy multiple attribute decision-making based on regret theory is proposed. Section 4 provides a real case on investment decision for venture capital and the comparisons with other methods are also conducted. Conclusion remarks are offered in Section 5.

2. Preliminaries

2.1. Probabilistic hesitant fuzzy sets

As an enhanced version of H.F.S. (Torra, 2010), P.H.F.S. can not only be used to handle the situation when decision-makers are hesitant among several evaluation values to express their perception, but also can assign different probabilities to the assessed values. Therefore, it has a wider application range. In this section, some concepts related to P.H.F.S. are introduced.

Definition 1 (Zhu & Xu, 2018). Let X be a reference set. A P.H.F.S. on X is defined as

$$H_p = \{\langle x, h_x(\gamma_l|p_l)\rangle | x \in X\}, \tag{1}$$

where $h_x(\gamma_l|p_l)$ represents the possible probabilistic membership degrees of $x \in X$ to the set H_p, and it is called P.H.F.E. $h_x(\gamma_l|p_l)$ consists of several possible membership degrees $\gamma_l(l = 1, 2, ..., |h_x|)$ with their probabilities $p_l(l = 1, 2, ..., |h_x|)$ such that $p_l \in [0, 1]$ and $\sum_{l=1}^{|h_x|} p_l = 1$. Here, $|h_x|$ denotes the number of the possible probabilistic membership degrees in $h_x(\gamma_l|p_l)$.

The expected value of a P.H.F.E. is defined as follows.

Definition 2 (Zhu & Xu, 2018). Assume that $h_x(\gamma_l|p_l) = \{(\gamma_l|p_l)|l = 1, 2, ..., |h_x|\}$ is a P.H.F.E. The expected value of $h_x(\gamma_l|p_l)$ is defined as

$$E[h_x(\gamma_l|p_l)] = \sum_{l=1}^{|h_x|} \gamma_l p_l \tag{2}$$

The expected value of $h_x(\gamma_l|p_l)$ is also considered as the score function of $h_x(\gamma_l|p_l)$ (Zhou & Xu, 2018). For any two P.H.F.E.s $h_1(\gamma_l|p_l)$ and $h_2(\gamma_l|p_l)$, if $E[h_1(\gamma_l|p_l)] > E[h_2(\gamma_l|p_l)]$, then $h_1(\gamma_l|p_l) > h_2(\gamma_l|p_l)$; if $E[h_1(\gamma_l|p_l)] = E[h_2(\gamma_l|p_l)]$, then $h_1(\gamma_l|p_l) = h_2(\gamma_l|p_l)$. Hence, the greater the expected value, the better the P.H.F.E. Also, the comparison rule can be improved (Zhang et al., 2017).

Obviously, as the probabilities of the elements in a .P.H.F.E. are equal, the P.H.F.E. can be degenerated into a hesitant fuzzy element (H.F.E.) (Xu & Zhou, 2017). It implies that the P.H.F.E. is an extension of the H.F.E.. In other words, the probability information will tend to be the main distinguishing feature. How to determine the probabilities of the elements in a P.H.F.E. will be a challenging problem, and it is also one of the emphases in this article.

2.2. Regret theory

Loomes and Sugden (1982) and Bell (1982), respectively proposed the regret theory, which is a behaviour and decision analysis theory. The regret and rejoice factors are imported into the calculation of utility value. According to regret theory, decision-makers rejoice that the selected alternative will bring better results than others; otherwise they will feel regret. As a human psychological and behaviour process, the regret aversion can be quantified (Bleichrodt, Cillo, & Diecidue, 2010).

Definition 3. Assume that x_1 and x_2 are the results acquired by choosing alternatives A_1 and A_2 respectively, then the perceived utility for alternative A_1 are defined as:

$$u(x_1, x_2) = v(x_1) + R\Big(v(x_1) - v(x_2)\Big). \tag{3}$$

where $v(\cdot)$ represents the utility function with $v'(\cdot) > 0$ and, $v''(\cdot) < 0$, and $R(\cdot)$ denotes the regret-rejoice function with $R(0) = 0, R'(\cdot) > 0$ and $R''(\cdot) < 0$. Furthermore, $\Delta v = v(x_1) - v(x_2)$ is adopted to measure the difference between two utility values of alternatives A_1 and A_2. If $R(\Delta v) > 0$, decision-makers are overjoyed at the chosen alternative; otherwise they will feel regretful.

The utility value is usually quantified using the power function $v(x) = x^\alpha$, where $\alpha \in (0, 1)$ is used to characterise the extent of risk aversion. Generally, the smaller the parameter α, the larger the risk aversion. Moreover, $R(\cdot)$ is strictly increasing and concave, which can be denoted as:

$$R(x) = 1 - \exp(-\delta \cdot x). \tag{4}$$

Here, $\delta \in [0, +\infty)$ represents the regret aversion coefficient. In general, the greater the parameter δ, the larger the regret aversion. However, we can frequently face the reality that the optimal alternative would be selected from multiple alternatives $A_i(i = 1, 2, ..., m)$. To cope with this situation, the regret theory was modified by Quiggin (1994). Assume that $x_i(i = 1, 2, ..., m)$ are the results of alternatives $A_i(i = 1, 2, ..., m)$ respectively. The perceived utility for alternative A_i is defined as:

$$u_i = v(x_i) + R\Big(v(x_i) - v(x^*)\Big). \tag{5}$$

where $x^* = \max_{1 \leq i \leq m}\{x_i\}$ and $R(v(x_i) - v(x^*)) \leq 0$. $R(v(x_i) - v(x^*))$ denotes the regret value. It indicates that the decision-maker will feel regretful after selecting alternative A_i instead of x^*. Actually, the regret theory implies that the classical utility function is modified through introducing a regret-rejoice term in the equation. Hence, the perceived utility value for an alternative is composed of two components, namely the utility value and regret-rejoice value.

3. Probabilistic hesitant fuzzy multiple attribute decision-making method

To handle the venture capital decision-making problem with unknown or partially known probability information, approaches to probabilistic hesitant fuzzy multiple attribute decision-making are put forward in this section. Considering that the venture capitalist often takes the anticipated regret into account, the regret theory is applied to the decision-making process of venture capital investment. Moreover, the occurrence probabilities of the elements in a P.H.E.F. are usually unknown or incompletely known, and two mathematical programming models are constructed for the probability calculation. Afterwards, the specific decision-making process for the selection of venture capital projects is offered.

3.1. Problem description

The decision-making problem on venture capital investment under probabilistic hesitant fuzzy environment is depicted in the following:

Let $\{Y_1, Y_2, ..., Y_m\}$ be the set of alternatives, and $\{C_1, C_2, ..., C_n\}$ be the set of attributes. Assume that $W = (w_1, w_2, ..., w_n)^T$ is the attribute weight vector, where w_j denotes the weight of attribute C_j such that $\sum_{j=1}^{n} w_j = 1$ and $0 \leq w_j \leq 1(j = 1, 2, ..., n)$. Suppose that some venture capitalists are required to assess alternatives $Y_i(i = 1, 2, ..., m)$ with respect to attributes $C_j(j = 1, 2, ..., n)$. And then the probabilistic hesitant fuzzy decision matrix $D = [h_{ij}(\gamma_{lij}|p_{lij})]_{m \times n}(i = 1, 2, ..., m, j = 1, 2, ..., n, l = 1, 2, ..., |h_{ij}|)$ can be obtained, where $h_{ij}(\gamma_{lij}|p_{lij})$ denotes a P.H.F.E. It represents the probabilistic membership degree of alternative Y_i with respect to attribute C_j, and p_{lij} is the occurrence probability of element γ_{lij} such that $0 \leq p_{lij} \leq 1$ and $\sum_{l=1}^{|h_{ij}|} p_{lij} \leq 1$.

Generally, in the decision-making process under probabilistic hesitant fuzzy environment, the information on the occurrence probabilities of elements in a P.H.F.E. is often unknown, and the probability vector of all the elements in a P.H.F.E. is defined by $P = (p_1, p_2, \ldots, p_n)^T$ such that $0 \leq p_l \leq 1$ and $\sum_{l=1}^{n} p_l = 1$. Here, p_l is the occurrence probability of element γ_l in P.H.F.E. However, we often encounter the situations that partial ignorance on probability information exists and the information on the occurrence probabilities of elements in a P.H.F.E. is incomplete due to the decision-maker's limited expertise, lack of data and so on (Zhang et al., 2017). At this point, $\sum_{l=1}^{n} p_l < 1$. Assume that Δ is the set of known probability information. Motivated by the ideas of Park and Kim (1997) and Kim, Choi, and Kim (1999), the incomplete probability information can be taken by the form of linear inequalities or rankings, which can be constructed by the following forms:

1. A weak ranking: $p_i \geq p_j$;
2. A strict ranking: $p_i - p_j \geq \alpha_i, \alpha_i > 0$;
3. A ranking of differences: $p_i - p_j \geq p_m - p_n$, for $j \neq m \neq n$;
4. A ranking with multiples: $p_i \geq \beta_i p_j, \beta_i \in [0, 1]$;
5. An interval form: $\alpha_i \leq p_i \leq \alpha_i + \varepsilon_i, 0 \leq \alpha_i < \alpha_i + \varepsilon_i \leq 1$.
6. A partially ignorant form: $\sum_{l=1}^{n} p_l < 1$.

Such linear partial information above is defined as incomplete information and can be provided by decision-maker. It is worth mentioning that we take the partial ignorance on probability information into account. In fact, in the decision-making process, some decision-makers may not provide their perceptive information because of limited capacity (Zhang et al., 2017), and then partial ignorance exits, which implies that $\sum_{l=1}^{n} p_l < 1$.

3.2. Probability calculation for elements in P.H.F.E. based on water-filling theory and the maximum entropy principle

The occurrence probabilities of elements in P.H.F.E. are the main factors affecting the decision results. They are often difficult to acquire through subjective evaluation of decision-makers, especially when the information on the occurrence probabilities of elements in P.H.F.E. is incomplete. In this section, we will extend water-filling theory in the wireless communication area and the maximum entropy principle to accommodate probabilistic hesitant fuzzy environment and develop an approach for determining the occurrence probabilities of elements in P.H.F.E.

Water-filling theory was initially utilised to resolve the problem of power optimisation allocation in the field of wireless communication (Palomar & Fonollosa, 2005). Taking aim at maximising the channel capacity, the water-filling theory adopts the signal to noise ratio (S.N.R.) of each sub-channel to conduct adaptive allocation of transmission power. If the S.N.R. is low, the sub-channel is assigned a small power, and vice versa. The basic idea of the theory can be described using the formula as follows:

$$T = \sum_{j=1}^{n} \log_2 \left(1 + \frac{\alpha_j^2 P_j}{\sigma_j^2} \right), \tag{6}$$

where T represents the channel capacity. P_j, σ_j and α_j are used to denote the assigned power, noise variance and gain of the j th sub-channel respectively. In this theory, the S.N.R. is the quality index used for evaluating the sub-channel, and then the gain and noise variance of the sub-channel are considered as two main scale targets. Likewise, in a P.H.F.E., if we compare each element in the P.H.F.E. to each sub-channel, the occurrence probability of each element in a P.H.F.E. can be understood as the assigned power of the sub-channel. Then the element and its deviation, similar to the gain and noise variance mentioned above, can be regarded as indexes to evaluate the performance of the channel capacity, and thus the occurrence probabilities of all the elements in a P.H.F.E. are acquired accordingly. Analogous to the channel capacity, the total capacity of a P.H.F.E. can be defined as below.

Definition 4. Suppose that p_l is the occurrence probability of the element γ_l in PHFE $h_x(\gamma_l|p_l)$, such that $\sum_{l=1}^{n} p_l = 1$ and $p_l \in [0, 1]$, then:

$$T_{h_x} = \sum\nolimits_{l=1}^{n} \log_2 \left[1 + \left(\frac{\gamma_l}{\sigma_l} \right)^2 p_l \right], \tag{7}$$

is called the total capacity of PHFE $h_x(\gamma_l|p_l)$. Here, γ_l and σ_l denote the element in PHFE $h_x(\gamma_l|p_l)$ and its deviation respectively, where:

$$\sigma_l = \sqrt{\frac{1}{n-1} \sum_{k=1, k \neq l}^{n} (\gamma_l - \gamma_k)^2}. \tag{8}$$

T_{h_x} denotes the total capacity of PHFE $h_x(\gamma_l|p_l)$, which can be used for measuring the amount of information contained in PHFE $h_x(\gamma_l|p_l)$. In fact, a higher value of T_{h_x} means that a larger amount of information can be provided (Liu, Wang, & Zhang, 2018b; Zhao, Yan, & Wang, 2014), and thus the element γ_l in PHFE $h_x(\gamma_l|p_l)$ should be assigned a higher probability. Based on these analyses, a mathematical programming model is constructed to determine the occurrence probability of each element in PHFE $h_x(\gamma_l|p_l)$ as follows:

$$\begin{cases} \max T_{h_x} = \sum\nolimits_{l=1}^{n} \log_2 \left[1 + \left(\frac{\gamma_l}{\sigma_l} \right)^2 p_l \right] \\ s.t. \sum\nolimits_{l=1}^{n} p_l = 1, 0 \leq p_l \leq 1 \end{cases}, \tag{M-1}$$

The mathematical model (M-1) takes the uncertain data into account. In addition, we should identify the unknown distribution of probabilities based on the limited information for the calculation of probabilities of elements in the P.H.F.E. Then, the information entropy of the distribution of probabilities $p_l(l = 1, 2, \ldots n)$ is introduced and defined as (Shannon, 1948)

$$H(p) = -\sum_{l=1}^{n} p_l \log p_l, \tag{9}$$

Here, $0 \log 0 = 0$. The uncertainty of a distribution can be measured by the information entropy.[1] Jaynes (1957) adopted the information entropy concept to ascertain the unknown distribution of probabilities, which is known as the maximum entropy principle. Under this principle, people can select the distribution for which the data is just enough to fix the probability assignment. In other words, people can select the distribution, among those that consistent with known information, which maximises the entropy (Wu, 2009). Moreover, the maximum entropy principle shows that the minimal amount of information is added (Zhang & Singh, 2012). Therefore, according to the maximum entropy principle, another mathematical model to determine the occurrence probability of each element in PHFE $h_x(\gamma_l|p_l)$ is constructed as follows:

$$\begin{cases} \max f(p) = -\sum_{l=1}^{n} p_l \log p_l \\ s.t. \sum_{l=1}^{n} p_l = 1, 0 \le p_l \le 1 \end{cases} \tag{M-2}$$

To ultimately determine the occurrence probability of each element in PHFE $h_x(\gamma_l|p_l)$, the multi-objective model is converted into a single-objective one as below:

$$\begin{cases} \max \tilde{f}(p) = \sum_{l=1}^{n} \left\{ \log_2 \left[1 + \left(\frac{\gamma_l}{\sigma_l}\right)^2 p_l \right] - p_l \log p_l \right\}, \\ s.t. \sum_{l=1}^{n} p_l = 1, 0 \le p_l \le 1 \end{cases} \tag{M-3}$$

In fact, *the maximum entropy estimate is the least biased estimate possible on the given information; i.e. it is maximally noncommittal with regard to missing information* (Greiff, 1999; Jaynes, 1957). Therefore, the model (M-3) not only considers the known decision information, but also takes into account the uncertainty of a distribution that measured by the information entropy. By solving the mathematical model (M-3), we can obtain the probabilities of all the possible elements in the P.H.F.E.

Proposition 1. The optimal solution for model (M-3) exists.

Proof. Assume that $\Omega = \{p_l | 0 \le p_l \le 1, l = 1, 2, \ldots, n, \sum_{l=1}^{n} p_l = 1\}$ is the feasible region of model (M-1). Since there exists $p_l \in [0, 1]$ such that $\sum_{l=1}^{n} p_l = 1$, the feasible region Ω is nonempty. It is obvious that Ω occupies a bounded closed region, and $\tilde{f}(p)$ is a continuous function in bounded closed domain Ω. Therefore, according to the maximum value theory of multivariate functions, there are maximum and minimum values in a bounded closed domain Ω for the objective function $\tilde{f}(p)$(Larson, 2009). On the other hand, the model (M-3) can be transformed into the following form:

$$\begin{cases} \min \widehat{f}(p) = \sum_{l=1}^{n} \left\{ -\log_2 \left[1 + \left(\frac{\gamma_l}{\sigma_l}\right)^2 p_l \right] + p_l \log p_l \right\} \\ s.t. \sum_{l=1}^{n} p_l - 1 \ge 0 \\ 1 - \sum_{l=1}^{n} p_l \ge 0 \\ p_l \ge 0, 1 - p_l \ge 0, l = 1, 2, \ldots n \end{cases} \tag{M-4}$$

which implies that the objective function is convex and the feasible region is also a convex set. Then model (M-4) is a convex programming model, and the local optimal solution is also the global optimal solution (Stephen, 2004). Besides, the objective function $\widehat{f}(p)$ is strictly convex. Model (M-4) thus has the unique optimal solution, which completes the proof.

Model (M-3) is a nonlinear programming model, which can be executed by utilising the L.I.N.G.O. mathematics software. After finding the optimal solution of model (M-3), we can determine the occurrence probabilities of all elements in P.H.F.E. However, in the actual decision-making process, the information on the occurrence probabilities of elements in a P.H.F.E. may be not completely unknown but partially known. Then based on the set of known probability information, Δ, an optimisation model is constructed as below:

$$\begin{cases} \max \tilde{f}(p) = \sum_{l=1}^{n} \left\{ \log_2 \left[1 + \left(\frac{\gamma_l}{\sigma_l} \right)^2 p_l \right] - p_l \log p_l \right\}, \\ s.t.\ \ p_l \in \Delta, 0 \le p_l \le 1, l = 1, 2, \ldots, n \end{cases} \tag{M-5}$$

Here, Δ denotes the set of known probability information, which is defined in Section 3.1. In this case, the information on the occurrence probabilities of elements in a P.H.F.E. is partially known, and we can obtain the probabilities of elements in the P.H.F.E. by solving the mathematical model (M-5).

Proposition 2. The optimal solution for model (M-5) exists.

Proof. The process of proof is similar to that of Proposition 1. Therefore, we will not go into much detail here.

If the probabilistic information for a P.H.F.E. is incomplete, the normalisation should be carried out first, especially when a partially ignorant form exists.

Definition 5 (Zhang et al., 2017). Suppose that $h_x(\gamma_l|p_l) = \{(\gamma_l|p_l)|l = 1, 2, \ldots, |h_x|\}$ is a P.H.F.E. with $\sum_{l=1}^{|h_x|} p_l < 1$. Then the normalised form of $h_x(\gamma_l|p_l)$ is defined as $h(\gamma_l|\tilde{p}_l) = \{(\gamma_l|\tilde{p}_l)|l = 1, 2, \ldots |h|\}$, where, $\tilde{p}_l = p_l / \sum_{l=1}^{|h_x|} p_l, l = 1, 2, \ldots, |h_x|$.

Example 1. Suppose that $h(\gamma_l|p_l) = \{0.2|p_1, 0.3|p_2, 0.5|p_3\}$ is a P.H.F.E. Then the occurrence probabilities of elements in PHFE $h(\gamma_l|p_l)$ can be determined by utilising model (M-3).

For PHFE $h(\gamma_l|p_l) = \{0.2|p_1, 0.3|p_2, 0.5|p_3\}$, we construct a mathematical programming model to determine the occurrence probabilities of elements in PHFE $h(\gamma_l|p_l)$ as follows:

$$\begin{cases} \max \tilde{f}(p) = \log_2 \left(1 + \frac{0.04}{0.05} p_1 \right) + \log_2 \left(1 + \frac{0.09}{0.025} p_2 \right) + \log_2 \left(1 + \frac{0.25}{0.065} p_3 \right) - \sum_{l=1}^{n} p_l \log p_l \\ s.t.\ \ p_1 + p_2 + p_3 = 1, 0 \le p_1, p_2, p_3 \le 1 \end{cases}$$

By solving the model above, we obtain $p_1 = 0.186$, $p_2 = 0.403$ and $p_3 = 0.411$. Therefore, $h(\gamma_l|p_l) = \{0.2|0.1860, 0.3|0.4028, 0.5|0.4112\}$.

If the information on the occurrence probabilities of elements in $h(\gamma_l|p_l) = \{0.2|p_1, 0.3|p_2, 0.5|p_3\}$ is partially known, such as $0 \le p_1 + p_2 + p_3 \le 0.8$ and $0.1 \le p_1 - p_2$, we construct a mathematical model as below:

$$\begin{cases} \max \tilde{f}(p) = \log_2\left(1 + \dfrac{0.04}{0.05}p_1\right) + \log_2\left(1 + \dfrac{0.09}{0.025}p_2\right) + \log_2\left(1 + \dfrac{0.25}{0.065}p_3\right) - \sum_{l=1}^{n} p_l \log p_l \\ s.t.\ \ 0 \le p_1 + p_2 + p_3 \le 0.8, \\ \quad\ \ 0.1 \le p_1 - p_2 \\ \quad\ \ 0 \le p_1, p_2, p_3 \le 1 \end{cases}.$$

Solving the equations above, we obtain $p_1 = 0.2889$, $p_2 = 0.1889$ and $p_3 = 0.3222$. Therefore, $h(\gamma_l|p_l) = \{0.2|0.2889, 0.3|0.1889, 0.5|0.3222\}$, and a normalised P.H.F.E. can be obtained as below:

$$h(\gamma_l|\tilde{p}_l) = \{0.2|0.3611, 0.3|0.2361, 0.5|0.4028\}.$$

3.3. Probabilistic hesitant fuzzy multiple attribute decision-making based on regret theory

As an important behavioural decision theory, regret theory considers the anticipated regret and rejoicing of the decision-maker. Because of its strong ability to describe the behaviour characteristics of investors as well as the simple calculation procedure, the regret theory is used to evaluate venture capital projects in this article. We will expand the application scope of regret theory, and present a novel approach to probabilistic hesitant fuzzy multiple attribute decision-making in this section.

After determining the occurrence probabilities of all elements in a P.H.F.E., we can obtain the expected value of P.H.F.E. $h(\gamma_l|\tilde{p}_l)$ by eq. (2), which is denoted as $E[h(\gamma_l|\tilde{p}_l)]$. Then the regret-rejoice value for alternative A_i with respect to attribute C_j is defined as:

$$R_{ij} = R\left(v\left(E[h_{ij}(\gamma_l|\tilde{p}_l)]\right) - v\left(E[h_j^*(\gamma_l|\tilde{p}_l)]\right)\right), i = 1, 2, \dots, m, j = 1, 2, \dots, n, \tag{10}$$

where $R(\cdot)$ denotes the regret-rejoice function described in eq. (4), and $v(\cdot)$ denotes the power function. $E(\cdot)$ represents the expected value of P.H.F.E. and $E[h_j^*(\gamma_l|\tilde{p}_l)] = \max_{1 \le i \le m}\{E[h_{ij}(\gamma_l|\tilde{p}_l)]\}$. Hence, $E[h_{ij}(\gamma_l|\tilde{p}_l)] \le E[h_j^*(\gamma_l|\tilde{p}_l)]$, which indicates $R_{ij} < 0$. At this point, R_{ij} denotes the regret value. Then the perceived utility matrix $U = (u_{ij})_{m \times n}$ can be obtained, where:

$$u_{ij} = v\left(E[h_{ij}(\gamma_l|\tilde{p}_l)]\right) + R\left(v\left(E[h_{ij}(\gamma_l|\tilde{p}_l)]\right) - v\left(E[h_j^*(\gamma_l|\tilde{p}_l)]\right)\right), i = 1, 2, \dots, m, \\ j = 1, 2, \dots, n. \tag{11}$$

Accordingly, we can acquire the total perceived utility value for alternatives $Y_i (i = 1, 2, \dots, m)$ in the following:

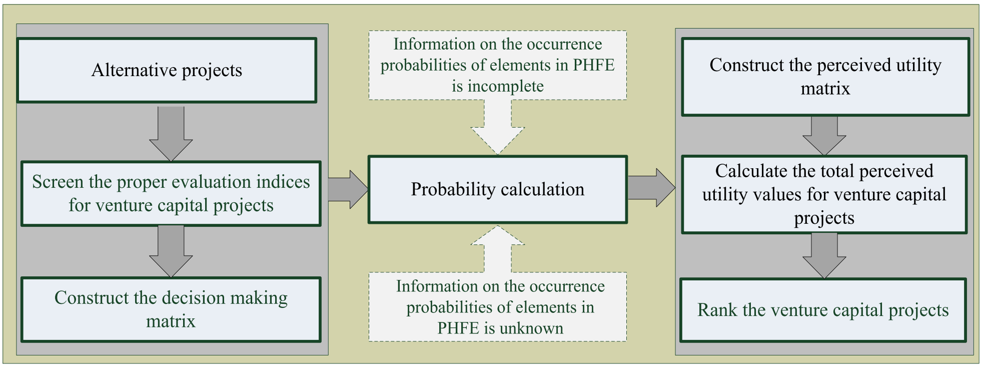

Figure 1. The decision-making process of the proposed method.

$$u_i = \sum_{j=1}^{n} u_{ij} w_j, i = 1, 2, ..., m. \tag{12}$$

Here, $w_j (j = 1, 2, ..., n)$ represents the weight of the j th attribute C_j. Generally, the larger the u_i, the better the alternative A_i. Therefore, the alternatives can be ranked by the u_i.

3.4. Decision-making procedure under probabilistic hesitant fuzzy environment

In this section, an approach based on regret theory for probabilistic hesitant fuzzy multiple attribute decision-making is developed. Based on analysis of the previous sections, a detailed procedure for evaluating the venture capital projects is summarised as below and the specific process of the proposed method is shown in Figure 1.

Step 1. For a probabilistic hesitant fuzzy multiple attribute decision-making problem, some venture capitalists are required to assess alternatives $Y_i (i = 1, 2, ..., m)$ with respect to attributes $C_j (j = 1, 2, ..., n)$, where the assessment values are in the form of P.H.F.Es. Then the probabilistic hesitant fuzzy decision matrix $D = [h_{ij}(\gamma_{lij} | p_{lij})]_{m \times n} (i = 1, 2, ..., m, j = 1, 2, ..., n, l = 1, 2, ..., |h_{ij}|)$ can be obtained.
Step 2. Determine the occurrence probabilities of all the elements in a P.H.F.E. If the information on the occurrence probabilities of elements in a P.H.F.E. is unknown, we use model (3) to determine the occurrence probabilities; if the information on the occurrence probabilities of elements in a P.H.F.E. is incompletely known, model (5) is adopted.
Step 3. Construct the perceived utility matrix according to eq. (11).
Step 4. The total perceived utility value u_i for alternatives $Y_i (i = 1, 2, ..., m)$ can be calculated according to eq. (12).
Step 5. Rank the alternatives $Y_i (i = 1, 2, ..., m)$ according to u_i, and the optimal alternative can be determined.
Step 6. End.

4. The application process of the proposed method

In this section, we present an example on the evaluation of venture capital projects (adapted from Zhang, Du, et al., 2018) to illustrate the application process of the proposed method. In addition, some comparisons are also conducted to verify the effectiveness of the proposed method.

4.1. Case study

With the development of science and technology, the process of human civilisation is promoted tremendously. As mentioned by Chinese President Xi, the innovation of engineering science and technology will offer an inexhaustible driving force for human civilisation. Especially in China, the development of science and technology has greatly driven China's economic growth and improved the investment environment in China. In addition, many favorable policies for attracting investment have been introduced. It is evident that the investment market in China has broad prospects. This section is focused on the optional investment projects for venture capitalists. After preliminary screening, there are four investment projects left to be further investigated: Y_1 : an internet company; Y_2 : a new energy automobile company; Y_3 : a solar photovoltaic company; and Y_4 : a real estate company;

To assess the investment projects above, a group of venture capitalists are invited, and four decision attributes are taken into account in the following.

C_1: Management team. In project management, an efficient project team is vital to the success of the project. Through the joint efforts of all team members, goals can be achieved easily. A good management team full of passion and energy is a guarantee of success for the start-ups. In addition, the levels of education and work experience of team members have a positive effect on corporate performance;

C_2: Financial situation, which refers to financial support. Undoubtedly, shortage of funds leads to the failure of promising start-ups. The ultimate goal of venture capitalists is to get more benefits, but the current financial situation of the venture capital project is a problem that does not allow to be neglected;

C_3: Market condition. As one of the most important factors for the success of start-ups, market demand is the driving force for offering products and services. Most of the failures of start-ups are often caused by lack of market demand. Market condition covers a wide range of areas, including market growth rate, market prospect, market competition level and so on.

C_4: Product and service, which refers to the products that consumer purchases and after-sales service. When consumers purchase products, they also hope to enjoy reliable and considerate service. Therefore, it is a primary concern for venture capitalists that whether the product and service offered by the optional start-up project are competitive in the fierce market.

4.2. The decision steps

To determine the optimal investment project, the proposed method is used, which includes the following two cases:

Table 1. The evaluation information of projects provided by venture capitalists.

	C_1	C_2	C_3	C_4
Y_1	$\{0.55\|p_{111}, 0.68\|p_{211}, 0.73\|p_{311}\}$	$\{0.6\|p_{112}, 0.66\|p_{212}\}$	$\{0.62\|p_{113}, 0.68\|p_{213}\}$	$\{0.64\|p_{114}, 0.72\|p_{214}\}$
Y_2	$\{0.62\|p_{121}, 0.77\|p_{221}\}$	$\{0.68\|p_{122}, 0.77\|p_{222}\}$	$\{0.6\|p_{123}, 0.73\|p_{223}, 0.85\|p_{323}\}$	$\{0.77\|p_{124}, 0.88\|p_{224}\}$
Y_3	$\{0.63\|p_{131}, 0.71\|p_{231}, 0.77\|p_{331}\}$	$\{0.66\|p_{132}, 0.71\|p_{232}\}$	$\{0.68\|p_{133}, 0.74\|p_{233}\}$	$\{0.71\|p_{134}, 0.78\|p_{234}, 0.81\|p_{334}\}$
Y_4	$\{0.67\|p_{141}, 0.72\|p_{241}\}$	$\{0.62\|p_{142}, 0.69\|p_{242}\}$	$\{0.67\|p_{143}, 0.71\|p_{243}\}$	$\{0.68\|p_{144}, 0.73\|p_{244}, 0.79\|p_{344}\}$

Case 1. Suppose that the information on the occurrence probabilities of elements in a P.H.F.E. is unknown and the decision-making process can be described as below:
Step 1. Several venture capitalists are invited to evaluate the alternatives $Y_i(i = 1, 2, 3, 4)$ with respect to the attributes $C_j(j = 1, 2, 3, 4)$. The evaluation information is shown in Table 1 and the attribute weight vector is assumed to be $W = (0.395, 0.112, 0.224, 0.269)^T$(Zhang, Du, et al., 2018).
Step 2. Determine the occurrence probabilities of all the elements in a P.H.F.E. If the information on the occurrence probabilities of elements in a P.H.F.E. is unknown, model (M-3) is used to determine the occurrence probabilities, and we get the probabilistic hesitant fuzzy decision matrix $\tilde{D}$ shown in Table 2.
Step 3. Construct the perceived utility matrix according to eq. (11). This requires calculating the expected value of each P.H.F.E. in Table 3. According to eq. (2), the expected value matrix is acquired as below:

Then the perceived utility matrix can be obtained in the following (See Table 4). Here, we set $\alpha = 0.88$ and $\delta = 0.3$ as in Tversky and Kahneman (1992) and Zhang et al. (2016) (which used experimental verification).

Step 4. According to eq. (12), the total perceived utility value u_i for each alternative $Y_i(i = 1, 2, 3, 4)$ can be calculated in the following: $u_1 = 0.6683$, $u_2 = 0.7679, u_3 = 0.7422, u_4 = 0.7178$.
Step 5. Rank the alternatives $Y_i(i = 1, 2, 3, 4)$ according to u_i. Therefore, $Y_2 \succ Y_3 \succ Y_4 \succ Y_1$.
Case 2. Suppose that the information on the occurrence probabilities of elements in a P.H.F.E. is partially known, and the known probability information is given as below:

$$\Delta = \left\{ \begin{array}{ll} \left\{ \begin{array}{l} 0.1 \le p_{1ij}, 0 \le p_{3ij} \le p_{2ij}, p_{2ij} - p_{1ij} \ge 0.3, \\ p_{1ij} + p_{2ij} + p_{3ij} \le 0.9, i, j = 1, 2, 3, 4 \end{array} \right. & \textit{for PHFE with three elements} \\ \left\{ \begin{array}{l} 0 \le p_{1ij}, 0 \le p_{2ij}, \\ p_{1ij} + p_{2ij} \le 0.7, i, j = 1, 2, 3, 4 \end{array} \right. & \textit{for PHFE with two elements} \end{array} \right\}$$

Step 1'. See Step 1.
Step 2'. Determine the occurrence probabilities of all the elements in a P.H.F.E. Model (M-5) is used to determine the occurrence probabilities, and we get the probabilistic hesitant fuzzy decision matrix $\widehat{D}$ shown in Table 5.

According to Definition 5, the normalizsed P.H.F.E.s can be given as in Table 6.

Table 2. The probabilistic hesitant fuzzy decision matrix $\tilde{D}$.

	C_1	C_2	C_3	C_4
Y_1	$\{0.55\vert 0.3081, 0.68\vert 0.3501, 0.73\vert 0.3418\}$	$\{0.6\vert 0.4994, 0.66\vert 0.5006\}$	$\{0.62\vert 0.4995, 0.68\vert 0.5005\}$	$\{0.64\vert 0.4989, 0.72\vert 0.5011\}$
Y_2	$\{0.62\vert 0.4936, 0.77\vert 0.5064\}$	$\{0.68\vert 0.4987, 0.77\vert 0.5013\}$	$\{0.6\vert 0.3026, 0.73\vert 0.3570, 0.85\vert 0.3404\}$	$\{0.77\vert 0.4984, 0.88\vert 0.5016\}$
Y_3	$\{0.63\vert 0.3246, 0.71\vert 0.3412, 0.77\vert 0.3342\}$	$\{0.66\vert 0.4997, 0.71\vert 0.5003\}$	$\{0.68\vert 0.4996, 0.74\vert 0.5004\}$	$\{0.71\vert 0.3293, 0.78\vert 0.3367, 0.81\vert 0.3340\}$
Y_4	$\{0.67\vert 0.4998, 0.72\vert 0.5002\}$	$\{0.62\vert 0.4992, 0.69\vert 0.5008\}$	$\{0.67\vert 0.4999, 0.71\vert 0.5001\}$	$\{0.68\vert 0.3301, 0.73\vert 0.3375, 0.79\vert 0.3324\}$

Table 3. The expected value matrix $E(\tilde{D})$.

	C_1	C_2	C_3	C_4
Y_1	0.6570	0.6300	0.6500	0.6801
Y_2	0.6960	0.7251	0.7315	0.8252
Y_3	0.7041	0.6850	0.7100	0.7670
Y_4	0.6950	0.6551	0.6900	0.7334

Table 4. The perceived utility matrix $U(\tilde{D})$.

	C_1	C_2	C_3	C_4
Y_1	0.6779	0.6393	0.6617	0.6719
Y_2	0.7247	0.7536	0.7595	0.8444
Y_3	0.7344	0.7057	0.7339	0.7759
Y_4	0.7235	0.6697	0.7099	0.7359

Step 3'. Construct the perceived utility matrix according to eq. (11). This requires calculating the expected value of each P.H.F.E. in Table 7. According to eq. (2), the expected value matrix is acquired as below:

Then the perceived utility matrix can be obtained in the following (See Table 8). Here, $\alpha = 0.88$ and $\delta = 0.3$ (Tversky & Kahneman, 1992; Zhang et al., 2016).

Step 4'. According to eq. (12), the total perceived utility value u_i for each alternative $Y_i(i = 1, 2, 3, 4)$ can be calculated in the following: $u_1 = 0.6740$, $u_2 = 0.7701$, $u_3 = 0.7473, u_4 = 0.7174$.

Step 5'. Rank the alternatives $Y_i(i = 1, 2, 3, 4)$ according to u_i. Therefore, $Y_2 \succ Y_3 \succ Y_4 \succ Y_1$.

4.3. Comparative analysis and discussions

4.3.1. Comparison of the proposed method with probabilistic hesitant fuzzy T.O.D.I.M. method

The above problem was also studied by Zhang, Du, et al. (2018), and the T.O.D.I.M. method has been extended to accommodate probabilistic hesitant fuzzy environment for the evaluation of venture capital projects. With the probabilistic hesitant fuzzy T.O.D.I.M. method proposed in Zhang, Du, et al. (2018), we first need to add values into the shorter P.H.F.E. until the compared P.H.F.E.s have the same length. The risk preference of the venture capitalist is assumed to be risk-seeking. The largest value would be added into the shorter P.H.F.E., and the probabilities of the added values are zero. Moreover, the probability information is provided by investors in advance (See Zhang et al. [2018] for details) and is normalised according to Definition 5.

Step 1. The probabilistic hesitant fuzzy decision information can be obtained and is shown in Table 9.

Step 2. The attribute weight vector is assumed to be $W = (0.395, 0.112, 0.224, 0.269)^T$(Zhang, Du, et al., 2018), and the relative attribute weight according to

Table 5. The probabilistic hesitant fuzzy decision matrix $\widehat{D}$.

	C_1	C_2	C_3	C_4
Y_1	$\{0.55 \mid 0.1587, 0.68 \mid 0.4587, 0.73 \mid 0.2826\}$	$\{0.6 \mid 0.3493, 0.66 \mid 0.3507\}$	$\{0.62 \mid 0.3494, 0.68 \mid 0.3506\}$	$\{0.64 \mid 0.3488, 0.72 \mid 0.3512\}$
Y_2	$\{0.62 \mid 0.3429, 0.77 \mid 0.3571\}$	$\{0.68 \mid 0.3486, 0.77 \mid 0.3514\}$	$\{0.6 \mid 0.1574, 0.73 \mid 0.4574, 0.85 \mid 0.2852\}$	$\{0.77 \mid 0.3483, 0.88 \mid 0.3517\}$
Y_3	$\{0.63 \mid 0.1670, 0.71 \mid 0.4670, 0.77 \mid 0.2660\}$	$\{0.66 \mid 0.3497, 0.71 \mid 0.3503\}$	$\{0.68 \mid 0.3495, 0.74 \mid 0.3505\}$	$\{0.71 \mid 0.1691, 0.78 \mid 0.4691, 0.81 \mid 0.2618\}$
Y_4	$\{0.67 \mid 0.3497, 0.72 \mid 0.3503\}$	$\{0.62 \mid 0.3491, 0.69 \mid 0.3509\}$	$\{0.67 \mid 0.3499, 0.71 \mid 0.3501\}$	$\{0.68 \mid 0.1698, 0.73 \mid 0.4698, 0.79 \mid 0.2604\}$

Table 6. The probabilistic hesitant fuzzy decision matrix $\breve{D}$.

	C_1	C_2	C_3	C_4
Y_1	$\{0.55 \mid 0.1763, 0.68 \mid 0.0.5097, 0.73 \mid 0.3140\}$	$\{0.6 \mid 0.4990, 0.66 \mid 0.5010\}$	$\{0.62 \mid 0.4991, 0.68 \mid 0.5009\}$	$\{0.64 \mid 0.4983, 0.72 \mid 0.5017\}$
Y_2	$\{0.62 \mid 0.4899, 0.77 \mid 0.5101\}$	$\{0.68 \mid 0.4980, 0.77 \mid 0.5020\}$	$\{0.6 \mid 0.1749, 0.73 \mid 0.5082, 0.85 \mid 0.3169\}$	$\{0.77 \mid 0.4976, 0.88 \mid 0.5024\}$
Y_3	$\{0.63 \mid 0.1856, 0.71 \mid 0.5189, 0.77 \mid 0.2955\}$	$\{0.66 \mid 0.4996, 0.71 \mid 0.5004\}$	$\{0.68 \mid 0.4993, 0.74 \mid 0.5007\}$	$\{0.71 \mid 0.1879, 0.78 \mid 0.5212, 0.81 \mid 0.2909\}$
Y_4	$\{0.67 \mid 0.4996, 0.72 \mid 0.5004\}$	$\{0.62 \mid 0.4987, 0.69 \mid 0.5013\}$	$\{0.67 \mid 0.4998, 0.71 \mid 0.5002\}$	$\{0.68 \mid 0.1887, 0.73 \mid 0.5220, 0.79 \mid 0.2893\}$

Table 7. The expected value matrix $E(\breve{D})$.

	C_1	C_2	C_3	C_4
Y_1	0.6728	0.6301	0.6501	0.6801
Y_2	0.6965	0.7252	0.7453	0.8253
Y_3	0.7129	0.6850	0.7101	0.7756
Y_4	0.6950	0.6551	0.6901	0.7379

Table 8. The perceived utility matrix $U(\breve{D})$.

	C_1	C_2	C_3	C_4
Y_1	0.6944	0.6394	0.6580	0.6718
Y_2	0.7229	0.7537	0.7721	0.8445
Y_3	0.7424	0.7057	0.7302	0.7860
Y_4	0.7211	0.6696	0.7062	0.7413

reference attribute can be calculated: $w'_j = w_j/w_o (j = 1,2,3,4)$, where w_j denotes the weight of attribute C_j and $w_o = \max_j\{w_j\}$.

Step 3. The relative dominance of gains or losses for alternative Y_i to Y_k, which is represented as $\vartheta_j(Y_i, Y_k)$, can be obtained:

$$\vartheta_j(Y_i, Y_k) = \begin{cases} \left.\begin{cases} \sqrt{\dfrac{w'_j}{\sum_{j=1}^4 w'_j} d\Big(h_{ij}(\gamma_{lij}|p_{lij}), h_{kj}(\gamma_{lkj}|p_{lkj})\Big)} & h_{ij}(\gamma_{lij}|p_{lij}) > h_{kj}(\gamma_{lkj}|p_{lkj}) \\ 0 & h_{ij}(\gamma_{lij}|p_{lij}) = h_{kj}(\gamma_{lkj}|p_{lkj}) \\ -\dfrac{1}{\theta}\sqrt{\dfrac{\sum_{j=1}^4 w'_j}{w'_j} d\Big(h_{ij}(\gamma_{lij}|p_{lij}), h_{kj}(\gamma_{lkj}|p_{lkj})\Big)} & h_{ij}(\gamma_{lij}|p_{lij}) < h_{kj}(\gamma_{lkj}|p_{lkj}) \end{cases}\right\} \text{for benefit attribute} \\ \left.\begin{cases} -\dfrac{1}{\theta}\sqrt{\dfrac{\sum_{j=1}^4 w'_j}{w'_j} d\Big(h_{ij}(\gamma_{lij}|p_{lij}), h_{kj}(\gamma_{lkj}|p_{lkj})\Big)} & h_{ij}(\gamma_{lij}|p_{lij}) > h_{kj}(\gamma_{lkj}|p_{lkj}) \\ 0 & h_{ij}(\gamma_{lij}|p_{lij}) = h_{kj}(\gamma_{lkj}|p_{lkj}) \\ \sqrt{\dfrac{w'_j}{\sum_{j=1}^4 w'_j} d\Big(h_{ij}(\gamma_{lij}|p_{lij}), h_{kj}(\gamma_{lkj}|p_{lkj})\Big)} & h_{ij}(\gamma_{lij}|p_{lij}) < h_{kj}(\gamma_{lkj}|p_{lkj}) \end{cases}\right\} \text{for cost attribute} \end{cases} \tag{13}$$

Here, θ denotes the attenuation factor of the losses and $d(h_{ij}(\gamma_{lij}|p_{lij}), h_{kj}(\gamma_{lkj}|p_{lkj}))$ denotes the Hamming distance measure between P.H.F.E.s $h_{ij}(\gamma_{lij}|p_{lij})$ and $h_{kj}(\gamma_{lkj}|p_{lkj})$(Zhang, Du, et al., 2018). Then the relative dominance between alternatives with respect to each attribute can be determined. To save space, we would not list the relative dominance between alternatives (See Zhang et al., 2018) for details).

Step 4. By aggregating the dominance of alternative Y_i to Y_k, we obtain the overall dominance of alternative Y_i :

$$O(Y_i) = \frac{\sum_{k=1}^4 \theta(Y_i, Y_k) - \min_i\{\sum_{k=1}^4 \theta(Y_i, Y_k)\}}{\max_i\{\sum_{k=1}^4 \theta(Y_i, Y_k)\} - \min_i\{\sum_{k=1}^4 \theta(Y_i, Y_k)\}}, i = 1,2,3,4 \tag{14}$$

Here, $\theta(Y_i, Y_k) = \sum_{j=1}^4 \vartheta_j(Y_i, Y_k)$. Therefore, $O(Y_1) = 0, O(Y_2) = 1, O(Y_3) = 0.43$ and $O(Y_4) = 0.34$.

Table 9. The normalised evaluation information of projects.

	C_1	C_2	C_3	C_4
Y_1	$\{0.55\mid 0.22, 0.68\mid 0.51, 0.73\mid 0.27\}$	$\{0.6\mid 0.61, 0.66\mid 0.39\}$	$\{0.62\mid 0.77, 0.68\mid 0.23, 0.68\mid 0\}$	$\{0.64\mid 0.67, 0.72\mid 0.33, 0.72\mid 0\}$
Y_2	$\{0.62\mid 0.31, 0.77\mid 0.69, 0.77\mid 0\}$	$\{0.68\mid 0.29, 0.77\mid 0.71\}$	$\{0.6\mid 0.18, 0.73\mid 0.21, 0.85\mid 0.61\}$	$\{0.77\mid 0.625, 0.88\mid 0.375, 0.88\mid 0\}$
Y_3	$\{0.63\mid 0.35, 0.71\mid 0.52, 0.77\mid 0.13\}$	$\{0.66\mid 0.48, 0.71\mid 0.52\}$	$\{0.68\mid 0.65, 0.74\mid 0.35, 0.74\mid 0\}$	$\{0.71\mid 0.53, 0.78\mid 0.22, 0.81\mid 0.25\}$
Y_4	$\{0.67\mid 0.53, 0.72\mid 0.47, 0.72\mid 0\}$	$\{0.62\mid 0.55, 0.69\mid 0.45\}$	$\{0.67\mid 0.70, 0.71\mid 0.30, 0.71\mid 0\}$	$\{0.68\mid 0.39, 0.73\mid 0.45, 0.79\mid 0.16\}$

Table 10. The hesitant fuzzy decision-making information.

	C_1	C_2	C_3	C_4
Y_1	$\{0.55, 0.68, 0.73\}$	$\{0.6, 0.66\}$	$\{0.62, 0.68\}$	$\{0.64, 0.72\}$
Y_2	$\{0.62, 0.77\}$	$\{0.68, 0.77\}$	$\{0.6, 0.73, 0.85\}$	$\{0.77, 0.88\}$
Y_3	$\{0.63, 0.71, 0.77\}$	$\{0.66, 0.71\}$	$\{0.68, 0.74\}$	$\{0.71, 0.78, 0.81\}$
Y_4	$\{0.67, 0.72\}$	$\{0.62, 0.69\}$	$\{0.67, 0.71\}$	$\{0.68, 0.73, 0.79\}$

Step 5. Rank the alternatives $Y_i(i = 1, 2, 3, 4)$ according to $O(Y_i)$. Therefore, $Y_2 \succ Y_3 \succ Y_4 \succ Y_1$.

Obviously, the ranking results obtained by the probabilistic hesitant fuzzy T.O.D.I.M. method are the same as that obtained by the proposed method in this article, which also demonstrates the effectiveness of the proposed method. Even so, there is reason to believe that the proposed method has some desirable advantages over the Zhang et al.'s method as below:

1. Zhang, Du, et al. (2018) extended the T.O.D.I.M. method, which is based on prospect theory, to deal with the probabilistic hesitant fuzzy multiple attribute decision-making problems. In this article, we present an approach based on regret theory for the evaluation of venture capital projects. The former method considers prospect preference in the decision-making process, while the latter method takes regret aversion into consideration. These two methods can effectively capture the decision-maker's psychological behaviour. However, in Zhang et al.'s method, the shorter P.H.F.E. should be extended by adding some values into it until the compared P.H.F.E.s have the same length, which could affect the decision outcomes.
2. To find a promising venture capital project, Zhang, Du, et al. (2018) adopted P.H.F.E.s to model the uncertain circumstance of venture capital, and the occurrence probabilities of elements in a P.H.F.E. are given by the investors in advance. It is noted that the P.H.F.E. is a generalised H.F.E. In the actual decision-making process, the H.F.E.s could be provided easily by decision-makers based on their knowledge and experience (Zhou & Xu, 2018). However, it is hard to determine the occurrence probabilities of these elements in P.H.F.E.s by subjective evaluation. To address this issue, an approach based on water-filling theory and the maximum entropy principle in this article is presented to determine the occurrence probabilities of elements in a P.H.F.E. By utilising the proposed method, we can objectively determine the occurrence probabilities of all elements in the P.H.F.E.s, whatever the information on the occurrence probabilities of elements in a P.H.F.E. is completely unknown or not.

4.3.2. Comparison of the proposed method with the method based on regret theory and H.F.S.s

As we know, the P.H.F.S. is an enhanced version of H.F.S. and can reserve more original information than H.F.S. Therefore, a comparative analysis with the hesitant

Table 11. The perceived utility matrix $U(\bar{D})$.

	C_1	C_2	C_3	C_4
Y_1	0.6736	0.6393	0.6631	0.6718
Y_2	0.7237	0.7535	0.7551	0.8443
Y_3	0.7336	0.7057	0.7352	0.7756
Y_4	0.7237	0.6696	0.7113	0.7358

fuzzy decision-making method based on regret theory will be conducted in this section. The specific decision process is as follows:

Step 1. The P.H.F.E.s in Table 1 are reduced to H.F.E.s as listed in Table 10.
Step 2. Construct the perceived utility matrix according to eq. (11), and the expected value of PHFE $h_{ij}(\gamma_l|p_l)$ is reduced to the score value of HFE $h_{ij}(\gamma_l)$(Xia & Xu, 2011):

$$s(h_{ij}(\gamma_l)) = \frac{1}{|h_{ij}|}\sum\nolimits_{l=1}^{|h_{ij}|}\gamma_l \tag{15}$$

Then the perceived utility matrix can be obtained in the following (See Table 11). Here, $\alpha = 0.88$ and $\delta = 0.3$ (Tversky & Kahneman, 1992; Zhang et al., 2016).

Step 3. The attribute weight vector is assumed to be $W = (0.395, 0.112, 0.224, 0.269)^T$(Zhang, Du, et al., 2018). According to eq. (12), the total perceived utility value u_i for each alternative $Y_i(i = 1, 2, 3, 4)$ can be calculated in the following: $u_1 = 0.6669, u_2 = 0.7665, u_3 = 0.7421, u_4 = 0.7181$.
Step 4. Rank the alternatives $Y_i(i = 1, 2, 3, 4)$ according to u_i. Therefore, $Y_2 \succ Y_3 \succ Y_4 \succ Y_1$.

It is noted that the ranking results have remained unchanged, and Y_2 is still considered to be the most suitable investment project for investors. Even so, the proposed method with probabilistic hesitant fuzzy information is superior to the method with hesitant fuzzy information. As mentioned above, the P.H.F.S. is an enhanced version of H.F.S., and thus can depict different probability of each evaluation value. When the investment projects are assessed in terms of H.F.S.s, the probability of each evaluation value is considered to be the same, which could not be in accordance with the fact and lead to improper decision results. Therefore, the proposed method, which can reserve more original information, is more effective for aiding decision-making.

5. Conclusion

In this article, an approach to probabilistic hesitant fuzzy multiple attribute decision-making is presented. In many cases, the occurrence probabilities of the elements in a P.H.F.E. are assumed to be known. However, it is usually difficult to determine these probabilities through subjective evaluation of decision-makers. Therefore, this article is concentrated on how to objectively identify the occurrence probabilities of all the possible elements in a P.H.F.E.

To this end, this article has proposed two nonlinear mathematical programming models to determine the probabilities of the elements in a P.H.F.E. The proposed methods in this article possess the following advantages: Firstly, based on the water-filling theory in the field of wireless communication and information entropy theory, a hybrid model for calculating the probabilities of elements in a P.H.F.E., where the information on the occurrence probabilities of elements in a P.H.F.E. is completely unknown, is put forward. Secondly, the information on the occurrence probabilities of elements in a P.H.F.E. is sometimes partially known. To handle this situation, a different mathematical programming model is used. Last but not least, based on regret theory, this article has presented a method with probabilistic hesitant fuzzy information for the evaluation of venture capital projects, where the psychological behaviours of venture capitalists are integrated into decision analysis.

In conclusion, the research on P.H.F.S.s is in its infancy, and there are many issues to be studied further. In future research, the interactive characteristics for decision attributes and the extensions of the proposed method for different types of information will be analysed.

Note

1. Consistent with the rest of this article, all uses of the log symbol will refer to base 2.

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