# An efficient importance sampling approach for reliability analysis of time-variant structures subject to time-dependent stochastic load

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**Abstract**

Structural performance is affected by deterioration processes and external loads. Both effects may change over time, posing a challenge for conducting reliability analysis. In such context, this contribution aims at assessing the reliability of structures where some of its parameters are modeled as random variables, possibly including deterioration processes, and which are subjected to stochastic load processes. The approach is developed within the framework of importance sampling and it is based on the concept of composite limit states, where the time-dependent reliability problem is transformed into a series system with multiple performance functions. Then, an efficient two-step importance sampling density function is proposed, which splits time-invariant parameters (random variables) from the time-variant ones (stochastic processes). This importance sampling scheme is geared towards a particular class of problems, where the performance of the structural system exhibits a linear dependency with respect to the stochastic load for fixed time. This allows calculating the reliability associated with the series system most efficiently. Practical examples illustrate the performance of the proposed approach.

*Keywords:* Time-variant structure, Stochastic load, Importance sampling, Composite limit state functions, Simulation-based method, Cumulative failure probability

## 1. Introduction

In the past decades, structural reliability theory for time-invariant problems has been widely investigated and developed. Following this framework, it is assumed that the system and its characteristics are static, and random variables are used to characterize their natural variability. Various methods have been developed to carry out static reliability analysis, which can be broadly classified as: asymptotic analytical methods, such as first/second order reliability method (FORM/SORM) [1]; and simulation-based methods, e.g., Monte Carlo simulation (MCS)[2][3], importance sampling (IS)[4, 5], line sampling [6] and subset simulation [7]. The efficiency of both classes of methods can be improved by applying surrogate methods such as response surface methods [8], Kriging [9, 10] support vector machines [11] and polynomial chaos expansion [12], among others.

Although advances in time-invariant reliability problems have been far reaching, in realistic engineering situations, the model parameters typically change as a function of time, which is termed as time-dependent (or time-variant). This is a result of the fact that engineering structures and systems are often exposed to severe operating or environmental conditions during their service life, which are responsible for the deterioration of structural strength and stiffness with time [13]. Furthermore, the intensity and frequency of loads acting on these systems may also vary with time. A reliability analysis can properly reflect and quantify the effect of time-variant factors by estimating the failure probability of a system/structure over a period of time. Because time is considered, more challenges are faced as compared with traditional, time-invariant reliability analysis. As such, the application of typically applied reliability engineering methods may not be direct in this context.

Reliability analysis considering time-variant properties and loadings has attracted much attention recently and a vast number of methods have been developed. These are roughly classified into three categories: (1) the out-crossing rate based methods; (2) the extreme value methods; and (3) the composite limit state methods. Methods based on out-crossing rate make use of the relationship between the failure probability and the expected mean number of out-crossings of the random process over a prescribed threshold. There are many different approximations to the out-crossing rate available in the literature, see e.g. [14, 15, 16]. However, the main drawback of methods based on out-crossing rate for reliability analysis is that they are based on the assumption of independence and Poisson distribution, which in certain cases may lead to a low accuracy. Methods based on extreme values consider the worst situation of system's performance over the time interval of interest, and whenever the extreme value of the limit state function exceeds a given threshold, failure occurs. The key challenge in extreme value methods lies in the construction of a proper surrogate model or a probability distribution for the output random process that characterizes the structural performance [17]. In this context, a Gaussian process (GP) model has been used in [18] to represent the extreme system response over time. Later, Hu and Mahadevan [19] proposed a single-loop GP approach where training points of random variables over time are generated at once (instead of tracking time and maximum responses separately). Qian et al. [20] also proposed a single-loop strategy for time-variant system reliability analysis by combining multiple response Gaussian process models. Many surrogate methods are only applicable to cases where no input random process are involved. In [21] and [22], surrogate model methods have been proposed to address this issue. However, discrete representation of stochastic processes increases the dimensionality of the problem, posing a challenge for surrogate modeling due to the so-called *curse of dimensionality*. An alternative strategy to surrogate modelling schemes is to fit a probability distribution for the extreme values by a suitable distribution estimation method. Hu and Du [17] proposed in this context a method for constructing an extreme value distribution, based on the expansion optimal linear estimation method (EOLE) and saddle-point approximation. However, the distribution of extreme values in some cases may be highly non-linear and/or follow a multimodal distribution, posing additional challenges. A third group of methods is based on the concept of a composite limit state function, which serves as an alternative to handle reliability problems with time-variant characteristics. The main idea behind composite limit state methods is that the original time-variant limit state function is discretized into a sequence of instantaneous ones, and then the concept of series system reliability is used to convert the time-variant reliability analysis into a time-invariant one. Jiang et al. [23] used time discretization to convert stochastic processes into random variables, and then the first-order reliability method (FORM) is adopted to compute the probability associated with the linearized limit-state function. Mourelatos et al. [24], based on the concept of composite limit state, used the total probability theorem and FORM to calculate reliability of time-dependent problems. Also, this composite limit state idea allows applying simulation-based methods for static system reliability analysis in time-variant problems. Recently, Li at al. [25] proposed a Generalized subset simulation (GSS) to handle high-dimensional time-variant reliability. Similarly, Du et al. [26] adopted parallel subset simulation to handle time-variant reliability with both deterioration in material properties and dynamic load.

Several of the methods for reliability analysis which have been developed so far consider load as stochastic excitation which is dependent on time, while parameters related with structural behavior are represented either as (static) random variables or even deterministic. For example, Au and Beck [27] proposed an efficient importance sampling method for linear systems; later, Misraji et al. [28] applied a directional importance sampling scheme for reliability analysis of structural systems subject to stochastic Gaussian loading. When uncertainties on different types of parameters are simultaneously considered, i.e., random structural variables, time-variant structural parameters (due to deterioration, etc.) and stochastic load processes, the reliability problem comprises a time-variant structure (system), which is subjected to time-variant loads. In this context, most of the current methods for reliability are based on approximate analytical methods, i.e., FORM [23][24], or resort to surrogate models, i.e., through building extreme value surrogate models, as in [29, 18, 19, 20, 21, 22], which have their own potential shortcomings. In the context of simulation-based methods, MCS and subset simulation can be used to solve reliability problems involving time-variant structures subject to stochastic load. However, their practical application may become unfeasible due to the prohibitively high computational associated with uncertainty propagation. Hence, there is still a large space for developing simulation-based methods for solving this kind of reliability problems in an efficient manner.

In view of the aforementioned difficulty in estimating the reliability of time-variant structures subject to time-dependent loads, this contribution focuses on a specific type of problems, namely *conditional linear time-variant systems*. By time-variant, it is meant that the time-variant structure (with time-variant structural properties) is subjected to time-variant loads (stochastic process); and by conditional linear, it is meant that at a particular instant of time and for a nominal structural parameter setting, the output response has a linear relationship with the load. The proposed approach is developed within the context of the composite limit state concept and transforms the time-variant problem into a series systems reliability problem. As linear time-variant systems are considered, an extremely efficient importance sampling density (ISD) function is proposed to compute the reliability of the transformed series system. The importance sampling scheme splits and explores the stochastic space spanned by the static random variables and the time-variant space spanned by the stochastic processes in two steps. First, samples are generated in the space associated with static parameters, after which conditional samples are generated according to a specially designed ISD in the time-variant space. This allows to compute the reliability associated with the transformed series system efficiently. The innovative aspects of this study with respect to the state-of-the-art are as follows:

- A new tool for structural time-variant reliability analysis is presented, in which input random variables, structural degradation processes as well as stochastic excitation processes are included.
- A simulation-based method which can produce satisfactory, accurate estimations of the failure probability is formulated.
- The most salient feature of the proposed approach is that it entails a single straightforward simulation scheme, where neither optimization is applied to find the extreme response values over time nor approximate linearization with respect to parameters is required.

This contribution is organized as follows. In Section 2, the definition and the transformation of time-dependent reliability problem is discussed. Then, the mathematical formulation of the proposed framework is developed in Section 3. Section 4 illustrates the performance of the proposed approach through a number of application examples. Section 5 closes the paper with discussions and an outlook for future work.

## 2. Reliability of structures with time-variant behavior

### *2.1. Reliability definition*

In general, reliability problems whose performance is time-variant can be classified into three groups according to the coupling and nature of the uncertain parameters [17]: (1) $G = g(\boldsymbol{x}, t)$ where $G$ is a response of interest, which is determined by the limit state function $g(\cdot)$; $t$ is the time instant; $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]$ is the vector of basic time-invariant random variables of the structure/system with probability density function $f(\boldsymbol{x})$; (2) $G = g(\boldsymbol{x}, \boldsymbol{Y}(t))$ where $\boldsymbol{Y}(t)$ are the time-dependent stochastic processes, which are implicit with respect to time $t$; (3) $G = g(\boldsymbol{x}, t, \boldsymbol{Y}(t))$ which is the most general type of problem with time-invariant random variables, explicit function terms with respect to time (such as structural degradation processes), and time-dependent stochastic processes (such as stochastic load processes).

In this contribution, the last, most general type of problems is considered. The corresponding cumulative failure probability over the time period is given by:

$$P_f(0, T) = P\{G = g(\boldsymbol{x}, t, \boldsymbol{Y}(t)) \le 0, \exists t \in [0, T]\} \tag{1}$$

where $[0, T]$ is the time interval of interest; $\exists$ stands for 'there exists at least one'; $\boldsymbol{Y}(t) = [Y_1(t), \ldots, Y_{n_Y}(t)]$ is the vector of time-dependent stochastic processes with respect to time $t$, which refers to load; $g(\cdot)$ is the corresponding limit state function. In this contribution, the focus is on conditional linear time-variant structural systems. That is, the response of interest is linear with respect to the input load conditional on a given time instant and structural parameters fixed at certain values; from a mathematical viewpoint, the load terms appears in the form of a linear term in the limit state function, which can be given by (in case it is explicitly available):

$$g(\boldsymbol{x}, t, \boldsymbol{Y}(t)) = a(\boldsymbol{x}, t) + \sum_{i=1}^{n_Y} b_i(\boldsymbol{x}, t) Y_i(t) \tag{2}$$

where $a(\boldsymbol{x}, t)$ and $b_i(\boldsymbol{x}, t)$ can be any type of function(implicit or explicit) of structural random variables and time, and $Y_i(t)$ appears as a linear term. Note that even if the limit state function cannot be stated explicitly, in case that a linear relationship between the response and load at fixed time is verified, the proposed approach can also be applied. Such is the case, for example, in linear structural analysis, where the response (deformation, stress and strain, etc.) is linear with respect to the load.

### *2.2. Composite limit states transformation of reliability problem with time-variant behavior*

The composite limit state is a useful approach to handle reliability problems with time-dependent behavior. Its basic idea is to use the concept of series system reliability of the instantaneous limit state functions to covert the time-dependent reliability problem into a time-invariant one. First, the time interval $[0, T]$ is discretized using a time step size $\triangle t$. Then, a time sequence $[t_0, \ldots, t_l \ldots, t_m] = [0, \ldots, l\triangle t, \ldots, m\triangle t]$ is generated, where $l = 0, \ldots, m$ is the time index, $t_0 = 0$ and $t_m = m\triangle t = T$. Based on this time discretization, the instantaneous failure probability is:

$$P_f(t_l) = P\{G_l = g(\boldsymbol{x}, t_l, \boldsymbol{Y}(t_l)) \leq 0)\} \tag{3}$$

where $G_l = g(x, t_l, \boldsymbol{Y}(t_l))$ is the instantaneous limit state function at time instant $t_l$.

Based on the series system reliability formulation, the cumulative failure probability at time instant $T$ is given by:

$$P_f(0, T) = P\left\{\bigcup_{l=0}^{m} F_l\right\} = P\left\{\min_{l=0,\ldots,m} g(\boldsymbol{x}, t_l, \boldsymbol{Y}(t_l)) \leq 0\right\} \tag{4}$$

where $F_l = \{G_l \leq 0\}$ is the failure region associated with the limit state function at the $l$-th time instant, that is $G_l$. The failure probability can be expressed as an integral given by:

$$P_f(0,T) = \iint I_F[\boldsymbol{x}, t_0, \ldots, t_m, \boldsymbol{Y}(t_1), \ldots, \boldsymbol{Y}(t_m)] f(\boldsymbol{x}) \\ f(\boldsymbol{Y}(t_0)) \ldots f(\boldsymbol{Y}(t_m)) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{Y}(t_0) \ldots \mathrm{d}\boldsymbol{Y}(t_m) \tag{5}$$

where $F = \bigcup_{l=0}^{m} \{F_l : G_l \leq 0\}$ is the failure of the series system; $f(\boldsymbol{x})$ is the probability distribution function of the (static) random variables; and $I_F[\cdot]$ is the indicator function, which is equal to 1 in case failure occurs at any time instant within $[0, T]$. Note that the computation of this failure probability is far from trivial, as it is a series system problem [25, 30]. Of all simulation-based reliability analysis methods, Monte Carlo simulation is a widely used technique which can be applied to estimate this failure probability. However, its drawback is its low efficiency for estimating small failure probabilities.

## 3. Proposed approach

This section presents an efficient importance sampling reliability analysis approach for systems with time-dependent properties. Section 3.1 first describes the spectral decomposition method for modeling the input stochastic process associated with the load. The strategy for coping with time-dependent limit state functions by transforming the problem to a series system is discussed in Section 3.2. Section 3.3 presents the proposed importance sampling density function and Section 3.4 summarizes the procedure of proposed approach.

### *3.1. Spectral decomposition of input random process*

In case an analyst is faced with aleatory uncertainties, the uncertain quantities are usually modeled as random variables. However, when these quantities vary over time (or space), they should be treated as random processes. Note that there are different kinds of random processes [31]. In the context of reliability analysis, a common approach is to transform them into traditional random variables via an appropriate spectral decomposition such as the Karhunen-Loève (K-L) expansion [32, 33] or the Expansion Optimal Linear Estimator (EOLE) [34].

Consider an Gaussian, nonstationary stochastic process $Y(t)$ defined by the mean function $\mu_Y(t)$, the standard deviation function $\sigma_Y(t)$, and the auto-correlation function $\rho_Y(t_l, t_j)$. The covariance function of this process between time $t_l$ and $t_j$ is calculated by:

$$Cov(t_l, t_j) = \rho_Y(t_l, t_j)\sigma_Y(t_l)\sigma_Y(t_j) \tag{6}$$

When the time interval is discretized as $[t_0, \dots, t_l \dots, t_m]$, the corresponding covariance matrix $\boldsymbol{\Sigma}_Y$ is formed as

$$\boldsymbol{\Sigma}_Y = \begin{bmatrix} Cov\,(t_0, t_0) & Cov\,(t_0, t_1) & \cdots & Cov\,(t_0, t_m) \\ Cov\,(t_1, t_0) & Cov\,(t_1, t_1) & \cdots & Cov\,(t_1, t_m) \\ \vdots & \vdots & \ddots & \vdots \\ Cov\,(t_m, t_0) & Cov\,(t_m, t_1) & \cdots & Cov\,(t_m, t_m) \end{bmatrix} \tag{7}$$

In case $\boldsymbol{\Sigma}_Y$ is symmetric, bounded and positive-definite, $Y(t)$ can be represented following the K-L expansion as [32]:

$$Y(t) = \mu_Y(t) + \sum_{p=1}^{k} \sqrt{\lambda_p}\Psi_p(t) Z_p \tag{8}$$

where $\Psi_p(t)$ corresponds to the element of matrix $\boldsymbol{\Psi}$ located in the row associated with time $t$ and the $p$-th column; $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \dots, \boldsymbol{\Psi}_k]$ is the matrix of orthogonal eigenvectors and $\boldsymbol{\Lambda} = \mathrm{diag}\,[\lambda_1, \lambda_2, \dots, \lambda_m]$ is a diagonal matrix that contains the corresponding non-negative eigenvalues obtained by performing eigendecomposition $\boldsymbol{\Sigma}_Y = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Psi}^T$; $k \leq m$ is the number of identified dominant eigenfunctions, where $k$ must be selected such that a well-selected error on the reconstructed variance of the process is minimized [35]. Since $Y(t)$ is a Gaussian process, $Z_p,\ p = 1, \dots, k$ are i.i.d. standard Normal random variables. This means that the random process $Y(t)$ can be represented by a number of standard Gaussian random variables, and its standard deviation $\sigma_Y(t)$ can be obtained as:

$$\sigma_Y(t) = \sqrt{\sum_{p=1}^{k} \lambda_p \Psi_p^2(t)} \tag{9}$$

Note that the approach that is introduced in the following section can be applied to various kinds of stochastic process once they are represented by independent normal variables, no matter what series expansion methods are used. As an additional remark, it should be noted that the proposed framework is applicable for problems involving Gaussian stochastic processes only. In fact, non Gaussian processes cannot be considered directly within the proposed scheme.

### *3.2. Reliability expressed in terms of a series system*

According to Eq. (8), the stochastic process $Y_i(t)$ at time instant $t$ is given by:

$$Y_i(t) = \mu_{Y_i}(t) + \sum_{p=1}^{k_i} \sqrt{\lambda_{ip}}\Psi_{ip}(t) Z_{ip} \tag{10}$$

where $k_i$ is the number of retained eigenfunctions in the K-L expansion, and the corresponding standard deviation can be obtained as:

$$\sigma_{Y_i}(t) = \sqrt{\sum_{p=1}^{k_i} \lambda_{ip} \Psi_{ip}^2(t)} \tag{11}$$

Then the instantaneous limit state function in Eq. (2) can be rewritten as:

$$g_Z(\boldsymbol{x}, t_l, \boldsymbol{Z}) = a(\boldsymbol{x}, t_l) + \sum_{i=1}^{n_Y} b_i(\boldsymbol{x}, t_l)\mu_{Y_i}(t_l) + \sum_{i=1}^{n_Y}\sum_{p=1}^{k_i} b_i(\boldsymbol{x}, t_l)\sqrt{\lambda_{ip}}\Psi_{ip}(t) Z_{ip} \tag{12}$$

where $\mathbf{Z} = [Z_{11}, \cdots, Z_{1k_1}, Z_{21}, \cdots, Z_{2k_2}, \cdots, Z_{n_Y 1}, \cdots, Z_{n_Y k_{n_Y}}]$ is the vector collecting all Gaussian random variables involved.

Considering the limit state function (LSF) given in Eq. (12), the corresponding failure probability, as defined in Eq. (4), is expressed as:

$$P_f(0, T) = P_f(0, t_m) = \iint I_{F_m^U}[\boldsymbol{x}, t_0, \cdots, t_m, \boldsymbol{Z}]\phi(\boldsymbol{Z}) f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{Z} \tag{13}$$

where $F_m^U = \bigcup_{l=0}^{m} \{F_l : g_Z(\boldsymbol{x}, t_l, \boldsymbol{Z}) \leq 0\}$ is the union of failure events of the series system; $I_{F_m^U}(\cdot)$ is the indicator function of $F_m^U$;and $\phi(\cdot)$ is the joint PDF of i.i.d. standard Gaussian variables.

Note that direct solution of this equation using standard quadrature schemes is not possible due to the typically high number of random variables involved. Therefore, simulation methods are preferred to approximate this integral, which may require a high number of LSF evaluations to obtain an acceptable coefficient of variation [36]. This motivates the use of importance sampling methods [37] to reduce the required number of LSF evaluations, as recently also successfully applied in the context of calculating first passage probabilities in linear dynamical systems [28, 38]. The next section deals with the development of an importance sampling density for this specific case.

### *3.3. Importance sampling density function*

In this section, an efficient importance sampling density (ISD) function is formulated, which has the following form:

$$H(\boldsymbol{x}, \boldsymbol{Z}) = H(\boldsymbol{Z}|\boldsymbol{x})H(\boldsymbol{x}) \tag{14}$$

Inspection of Eq.(14) reveals that this importance sampling density consists of two parts: one part related with the (static) random variables $\boldsymbol{x}$ and another part to handle the stochastic process $Y(t)$ via its associated random variables $\boldsymbol{Z}$. Recall that $\boldsymbol{x}$ is regarded as static because it does not vary with time. In other words, when a sample of $\boldsymbol{x}$ is generated, it becomes a constant during the time interval $[0, T]$.

#### 3.3.1. *Importance sampling density function for static random variables*

The fist step of the proposed framework is to determine $H(\boldsymbol{x})$. The most straightforward way is to just choose the original distribution as the ISD, which is :

$$H(\boldsymbol{x}) = f(\boldsymbol{x}) \tag{15}$$

Undoubtedly, $H(\boldsymbol{x})$ may also be constructed based on information from the failure region, considering design points or an adaptive importance sampling density function. Nonetheless, it may be challenging to determine a proper importance sampling function associated with the series system in Eq. (2). Besides, extra computational burden is involved [30]. It is also worthy to note that sampling of $H(\boldsymbol{x})$ can be done applying low-discrepancy sequences [5], which are especially suitable for cases where a small number of samples is used. It will be shown in examples section that the proposed approach just needs hundreds of samples, and it expected to achieve an improved convergence rate in case the low-discrepancy sequence is adopted.

#### 3.3.2. *Importance sampling density function for* $\boldsymbol{Z}$

The next step is to formulate the probability function associated with $\boldsymbol{Z}$ conditioned on $\boldsymbol{x}$. Once the ISD of $H(\boldsymbol{x})$ is determined, a number of samples $\{\boldsymbol{x}^{(j)} : (j = 1, \ldots, N)\}$ can be generated according to $H(\boldsymbol{x})$. Hence, in the following, it is considered that $\boldsymbol{x}$ has assumed a fixed value. Then, $H(\boldsymbol{Z}|\boldsymbol{x})$ can be determined based on this fixed value.

Recall the transformed instantaneous LSF $g_Z(\boldsymbol{x}, t_l, \boldsymbol{Z})$ of Eq. (12). It is readily noticed that for a fixed value $\boldsymbol{x}$, the LSF is only an expression depending on $\boldsymbol{Z}$. As $\boldsymbol{Z}$ is a vector of i.i.d. normal variables, the corresponding mean and standard deviation of LSF (conditioned on $\boldsymbol{x}$) can be easily obtained:

$$\mu^l_{g_Z}(\boldsymbol{x}) = \mu_{g_Z}(\boldsymbol{x}, t_l) = a(\boldsymbol{x}, t_l) + \sum_{i=1}^{n_Y} b_i(\boldsymbol{x}, t_l)\mu_{Y_i}(t_l) \tag{16}$$

$$\sigma^l_{g_Z}(\boldsymbol{x}) = \sigma_{g_Z}(\boldsymbol{x}, t_l) = \sqrt{\sum_{i=1}^{n_Y}\sum_{p=1}^{k_i} b_i^2(\boldsymbol{x}, t_l)\lambda_{ip}\Psi^2_{ip}(t)} = \sqrt{\sum_{i=1}^{n_Y} b_i^2(\boldsymbol{x}, t_l)\sigma^2_{Y_i}(t_l)} \tag{17}$$

The reliability index for the instantaneous limit state function at time instant $t_l$ in Eq. (12) can be obtained as:

$$\beta(\boldsymbol{x}, t_l) = \frac{\mu^l_{g_Z}(\boldsymbol{x})}{\sigma^l_{g_Z}(\boldsymbol{x})} \tag{18}$$

and the corresponding design point $\boldsymbol{Z}_l^*(\boldsymbol{x}) = [Z_{l,1}^*(\boldsymbol{x}), \cdots, Z_{l,p}^*(\boldsymbol{x}), \cdots, Z_{l,n_Z}^*(\boldsymbol{x})]$ can be obtained as:

$$Z_{l,p}^*(\boldsymbol{x}) = -\frac{b_i(\boldsymbol{x}, t_l)\sqrt{\lambda_{ip}}\Psi_{ip}(t)\mu_{gZ}^l(\boldsymbol{x})}{\sum_{i=1}^{n_Y} b_i^2(\boldsymbol{x}, t_l)\sigma_{Y_i}^2(t_l)} \tag{19}$$

The distribution of the random variables ($\boldsymbol{Z}$) conditional on the failure region $F_l(\boldsymbol{x})$ (which is defined as $F_l(\boldsymbol{x}) = \{g_Z(\boldsymbol{x}, t_l, \boldsymbol{Z}) \leq 0\}$) is given by [27]:

$$f(\boldsymbol{Z}|F_l(\boldsymbol{x})) = \frac{\phi(\boldsymbol{Z})I_{F_l}(\boldsymbol{x}, \boldsymbol{Z})}{\Phi\left[-\beta(\boldsymbol{x}, t_l)\right]} \tag{20}$$

where $I_{F_l}(\cdot)$ is the indicator function of $F_l(\boldsymbol{x})$; $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard Gaussian variable. An expression for generating samples of $\boldsymbol{Z}$ conditional on $F_l(\boldsymbol{x})$ is given by [27] :

$$\boldsymbol{Z} = \alpha \boldsymbol{e}_l^*(\boldsymbol{x}) + \boldsymbol{U}^{\perp} \tag{21}$$

where $\alpha$ is a standard Gaussian random variable conditional on $\alpha \geq \beta(\boldsymbol{x}, t_l)$; $\boldsymbol{e}_l^*(\boldsymbol{x})$ is a unit vector in the direction of the design point which is given by:

$$\boldsymbol{e}_l^*(\boldsymbol{x}) = \frac{\boldsymbol{Z}_l^*(\boldsymbol{x})}{||\boldsymbol{Z}_l^*(\boldsymbol{x})||} = \frac{\boldsymbol{Z}_l^*(\boldsymbol{x})}{\beta(\boldsymbol{x}, t_l)} \tag{22}$$

and $\boldsymbol{U}^{\perp}$ is a standard Gaussian random vector orthogonal to $\boldsymbol{e}_l^*(\boldsymbol{x})$, which is described by [27]:

$$\boldsymbol{U}^{\perp} = \boldsymbol{U} - \langle \boldsymbol{U}, \boldsymbol{e}_l^*(\boldsymbol{x})\rangle \boldsymbol{e}_l^*(\boldsymbol{x}) \tag{23}$$

with $\langle \cdot, \cdot \rangle$ denoting the inner product operator; and $\boldsymbol{U}$ is a $n$-dimensional standard Gaussian vector. Note that $n$ represents the total number of random variables associated with the representation of the stochastic processes, that is $n = \sum_{i=1}^{n_Y} k_i$. Thus, the final expression for generating samples of $\boldsymbol{Z}$ which are distributed as $f(\boldsymbol{Z}|F_l(\boldsymbol{x}))$ is:

$$\boldsymbol{Z} = \boldsymbol{U} + (\alpha - \langle \boldsymbol{U}, \boldsymbol{e}_l^*(\boldsymbol{x})\rangle)\boldsymbol{e}_l^*(\boldsymbol{x}) \tag{24}$$

This means that according to Eq. (24), a sample $\boldsymbol{Z}$ will fall in the failure region $F_l(\boldsymbol{x})$ and lead to $I_{F_l}(\boldsymbol{x}, \boldsymbol{Z}) = 1$. A schematic representation illustrating the capabilitiy of Eq. (24) for generating failure samples is shown in Fig. 1.

In time-dependent problems, there are several limit state functions associated with a certain realization $\boldsymbol{x}$ at different time instants, i.e. $g_Z(\boldsymbol{x}, \boldsymbol{Z}, t_l)$, $l = 0, \ldots, m$ (see Eq. (12)). Taking this fact into account, the proposed importance sampling density function $H(\boldsymbol{Z}|\boldsymbol{x})$ is constructed

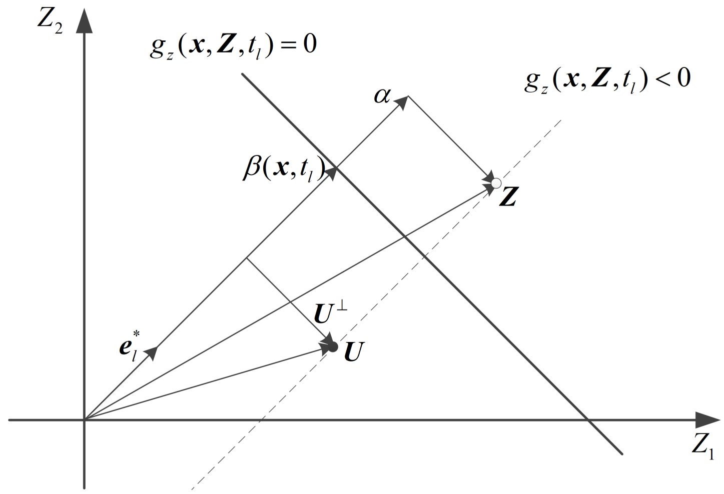


Figure 1: Schematic diagram of sampling of conditional $Z$.

based on a combination of individual optimal sampling functions $f(\boldsymbol{Z}|F_l(\boldsymbol{x}))$ which is given by [27]:

$$H(\boldsymbol{Z}|\boldsymbol{x}) = \sum_{l=0}^{m} w_l(\boldsymbol{x}) f(\boldsymbol{Z}|F_l(\boldsymbol{x})) = \sum_{l=0}^{m} w_l(\boldsymbol{x}) \frac{\phi(\boldsymbol{Z}) I_{F_l}(\boldsymbol{x},\boldsymbol{Z})}{\Phi\left[-\beta(\boldsymbol{x},t_l)\right]} \tag{25}$$

where $w_l(\boldsymbol{x}) \geq 0$ are the weights of individual sampling functions, which fulfill the conditions that $w_l(\boldsymbol{x}) \geq 0$ and $\sum_{l=1}^{m} w_l(\boldsymbol{x}) = 0$. In this contribution, the weights are chosen to be proportional to the probability content of $F_l(\boldsymbol{x}) = \{g_Z(\boldsymbol{x},\boldsymbol{Z},t_l) \leq 0\}$ [39]:

$$w_l(\boldsymbol{x}) = \frac{P(F_l(\boldsymbol{x}))}{\sum_{s=0}^{m} P(F_s(\boldsymbol{x}))} = \frac{\Phi\left[-\beta(\boldsymbol{x},t_l)\right]}{\sum_{s=0}^{m} \Phi\left[-\beta(\boldsymbol{x},t_s)\right]} \tag{26}$$

Substituting Eq. (26) into (25), the proposed importance sampling density function $H(\boldsymbol{Z}|\boldsymbol{x})$ is finally expressed by:

$$H(\boldsymbol{Z}|\boldsymbol{x}) = \phi(\boldsymbol{Z}) \frac{\sum_{l=0}^{m} I_{F_l}(\boldsymbol{x},\boldsymbol{Z})}{\sum_{l=0}^{m} \Phi\left[-\beta(\boldsymbol{x},t_l)\right]} \tag{27}$$

*3.3.3. Estimation of the failure probability*

Using the newly introduced ISD $H(\boldsymbol{Z}|\boldsymbol{x})$ in Eq. (27) as the importance sampling density, the time-dependent failure probability in Eq. (13) is expressed as:

$$\begin{aligned} P_f(0,T) &= \iint I_{F_m^U}[\boldsymbol{x},t_0,\cdots,t_m,\boldsymbol{Z}] \frac{\phi(\boldsymbol{Z})}{H(\boldsymbol{Z}|\boldsymbol{x})} H(\boldsymbol{Z}|\boldsymbol{x}) f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{Z} \\ &= E\left[I_{F_m^U}(\boldsymbol{x},t_0,\cdots,t_m,\boldsymbol{Z}) \frac{\phi(\boldsymbol{Z})}{H(\boldsymbol{Z}|\boldsymbol{x})}\right] \\ &= E\left[\frac{\sum_{l=0}^{m} \Phi\left[-\beta(\boldsymbol{x},t_l)\right]}{\sum_{l=0}^{m} I_{F_l}(\boldsymbol{x},\boldsymbol{Z})}\right] \end{aligned} \tag{28}$$

Note that $I_{F_m^U}[\boldsymbol{x}, t_0, \cdots, t_m, \boldsymbol{Z}] = 1$ has been considered within Eq. (28). The reason is that every sample $\boldsymbol{Z}$ generated from $H(\boldsymbol{Z}|\boldsymbol{x})$ is – by construction – located in at least one of the failure domains $F_l(\boldsymbol{x})$ (see Eq. 24), and as $F_l(\boldsymbol{x}) \subset F_m^U$, thus $\boldsymbol{Z} \in F_m^U$ holds, leading to $I_{F_m^U}[\boldsymbol{x}, t_0, \cdots, t_m, \boldsymbol{Z}] = 1$.

The expression for the failure probability as cast in Eq. (28) can be evaluated by means of simulation. For that purpose, samples of the static random variables $\boldsymbol{x}^{(j)} : j = 1, \ldots, N$ distributed according to $f(\boldsymbol{x})$ are generated in the first place. Then, samples of $\boldsymbol{Z}$ are generated such that they follow $H(\boldsymbol{Z}|\boldsymbol{x})$, yielding a set of samples, $(\boldsymbol{x}^{(j)},\ \boldsymbol{Z}^{(j)}), j = 1, \ldots, N$ which is distributed as $H(\boldsymbol{Z}|\boldsymbol{x})f(\boldsymbol{x})$. Thus, $P_f(0, T)$ is estimated as:

$$\hat{P}_f(0, T) = \frac{1}{N} \sum_{j=1}^{N} \frac{\sum_{l=0}^{m} \Phi\left[-\beta(\boldsymbol{x}^{(j)}, t_l)\right]}{\sum_{l=0}^{m} I_{F_l}(\boldsymbol{x}^{(j)}, \boldsymbol{Z}^{(j)})} \tag{29}$$

Obviously, the estimate $\hat{P}_f(0, T)$ is unbiased. The variance as well as the coefficient of variation (c.o.v., denoted as $\delta$) of the estimate $\hat{P}_f(0, T)$ can be obtained straightforwardly:

$$Var\left[\hat{P}_f(0, T)\right] \approx \frac{1}{N-1} \left\{ \frac{1}{N} \sum_{j=1}^{N} \left[ \frac{\sum_{l=0}^{m} \Phi\left[-\beta(\boldsymbol{x}^{(j)}, t_l)\right]}{\sum_{l=0}^{m} I_{F_l}(\boldsymbol{x}^{(j)}, \boldsymbol{Z}^{(j)})} \right]^2 - \hat{P}_f^2(0, T) \right\} \tag{30}$$

$$\delta[\hat{P}_f(0, T)] \approx \frac{\sqrt{Var[\hat{P}_f(0, T)]}}{\hat{P}_f(0, T)} \tag{31}$$

In conclusion, the proposed approach utilizes the information on the 'conditional linear' relationship between the response and the applied load. Based on this fact, an efficient importance sampling density function is constructed based on analytical investigation of the failure regions. It can be seen from Eq. (29) that as the probability estimator is simulation-based, there is no need for fitting a distribution, or conducting approximate analytical calculations. Hence, the accuracy can be guaranteed as the simulation proceeds, i.e. as the number of samples increases. In the following numerical applications, it is shown that the proposed approach exhibits excellent efficiency, and that the proposed two-step designed importance sampling density function is highly rewarding.

### *3.4. Summary of the proposed approach*

The approach for estimating the failure probability of a time-variant structure subject to time-variant load proposed in this section is summarized as follows.

1. Stochastic processes are represented by spectral decomposition as in Eq. (8), and the equivalent composite limit functions are obtained according to Eq. (12).
2. Generate samples $\boldsymbol{x}^{(j)}, j = 1, \ldots, N$ following $f(\boldsymbol{x})$.
3. Draw a set of samples $\boldsymbol{Z}^{(j)}, j = 1, \ldots, N$ according to the proposed ISD given by Eq. (27). Specifically:

   (a) Draw an index $l$ within the discrete set $\{0, \ldots, m\}$ with probability proportional to $w_l(\boldsymbol{x}^{(j)})$.

   (b) Simulate $\boldsymbol{U}$ as a $n$-dimensional standard Gaussian vector with independent components, and $u$ as uniform variable on [0,1]. Compute

$$\alpha = \Phi^{-1}\left[u + (1-u)\,\Phi\left(\beta(\boldsymbol{x}^{(j)}, t_l)\right)\right] \tag{32}$$

   or

$$\alpha = -\Phi^{-1}\left[(1-u)\,\Phi\left(-\beta(\boldsymbol{x}^{(j)}, t_l)\right)\right] \tag{33}$$

   and set

$$\boldsymbol{Z}^{(j)} = \boldsymbol{U} + \left(\alpha - \left\langle \boldsymbol{U}, \boldsymbol{e}_l^*(\boldsymbol{x}^{(j)})\right\rangle\right) \boldsymbol{e}_l^*(\boldsymbol{x}^{(j)}) \tag{34}$$

4. Based on the generated samples set $(\boldsymbol{x}^{(j)}, \boldsymbol{Z}^{(j)}),\ j = 1, \ldots, N$, compute the failure probability estimator according to Eq. (29) as well as the coefficient of variation of the estimator according to Eq. (31).

## 4. Examples

In this section, examples are given to illustrate the performance of the proposed method in terms of accuracy and efficiency. Direct Monte Carlo simulation (MCS) and Subset simulation (SS) are used for comparison. Note that the unit coefficient of variation (c.o.v.) $\Delta$ is calculated in all examples considered. Since any simulation algorithm for estimating failure probabilities has a c.o.v. of the form $\delta = \Delta/\sqrt{N}$ [27], the 'unit c.o.v.' $\Delta$ is adopted as a measure of efficiency which is inherent to the algorithm. That is, it is in theory invariant to the accuracy achieved and the computational effort spent, where smaller values of $\Delta$ correspond to a higher computational efficiency. In addition, the computational cost is measured in terms of the number of samples $N$ considered for the evaluation of the failure probability.

For Examples 1 to 3, a time period of [0,10] years is considered, and a constant time interval $\Delta t = 0.1$ year is adopted to discretize the time interval in the calculation. As such, a number of

$m = 101$ discrete time instants are obtained, i.e., $t_l = l\Delta t,\ l = 0, \ldots, m$. Also, the number of identified dominating eigenfunctions in K-L expansion is fixed at $k = 30$ which has been found to be reasonable by carrying out some numerical validations. For Example 4, the time interval and the number of expansion terms for the stochastic process are considered as $\Delta t = 0.25$ year and $k = 30$, respectively, as it involves finite element analysis. Such setting is used in order to alleviate the computation burden associated with MCS.

### *4.1. Example 1: A two-bar frame*

A two-bar frame shown in Fig. 2 is considered for the first example. It has been investigated in [40] and [41] (among others) and is modified to suit the purposes of this work. This frame is subjected to a dynamic force $F(t)$ which is described by a stochastic process. The yield strength of the bar A degrades with time, i.e. $S(\mathrm{t}) = S_0 e^{(-0.05t)}$, in which $S_0$ is the initial yield strength of the bar. Structural failure is defined as the maximum stress of the bar exceeding the corresponding yield strength. Thus, the limit state function of this two-bar frame is expressed as follows.

$$g(\boldsymbol{x}, t, \boldsymbol{Y}(t)) = \pi l_2 D^2 S(t) - 4F(t)\sqrt{l_1^2 + l_2^2} \tag{35}$$

where $\boldsymbol{x} = [D, l_1, l_2, S_0]$ is the vector of structural random variables; $\boldsymbol{Y}(t) = F(t)$ is the vector of stochastic load; $t \in [0, 10]$ year. Distribution information of the input parameters is listed in Table 1. As can be noted from this table, the load is described by a Gaussian stochastic process with mean $\mu = 2.2 \times 10^6$ (N) and standard deviation $\sigma = 2.2 \times 10^5$ (N). The autocorrelation of the process is given by a squared exponential correlation function with correlation length equal to 1 year.

Table 1: Information of variables and parameters for two-bar frame (Example 1)

| Parameter | Distribution | Mean | Standard deviation | Auto-correlation function |
|---|---|---|---|---|
| $D$(m) | Normal | 0.2 | $2 \times 10^{-3}$ | – |
| $l_1$(m) | Normal | 0.4 | $4 \times 10^{-3}$ | – |
| $l_2$(m) | Normal | 0.3 | $3 \times 10^{-3}$ | – |
| $S_0$(Pa) | Lognormal | $2.5 \times 10^8$ | $2.5 \times 10^6$ | – |
| $F(t)$(N) | Gaussian process | $2.2 \times 10^6$ | $2.2 \times 10^5$ | $e^{-\tau^2}$ |

The proposed approach is applied to solve this problem. In addition, Subset simulation and Direct Monte Carlo simulation are also applied for comparison. The details of the results obtained

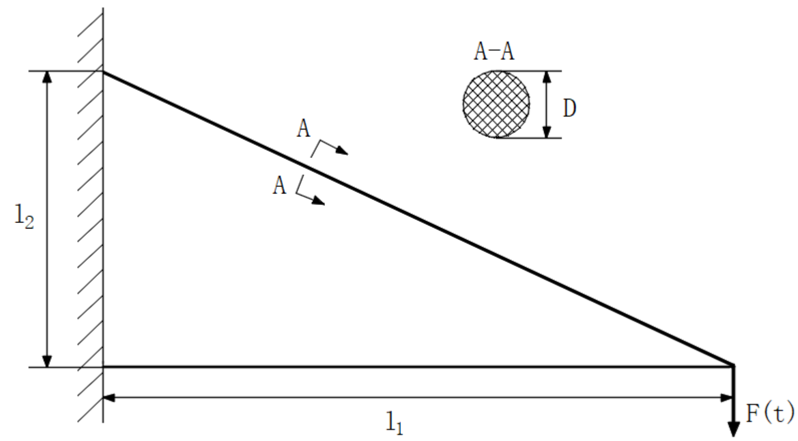


Figure 2: A two-bar frame

Table 2: Results by different methods for two-bar frame beam (Example 1)

| Methods | $P_f(0,T)$ | c.o.v. | $N$ | Unit c.o.v. |
|---|---|---|---|---|
| Proposed approach | $3.01\times10^{-3}$ | 0.12 | 100 | 1.2 |
| SS | $3.79\times10^{-3}$ | 0.20 | 3000 | 10.9 |
| MCS | $3.05\times10^{-3}$ | 0.02 | $10^6$ | 18.1 |

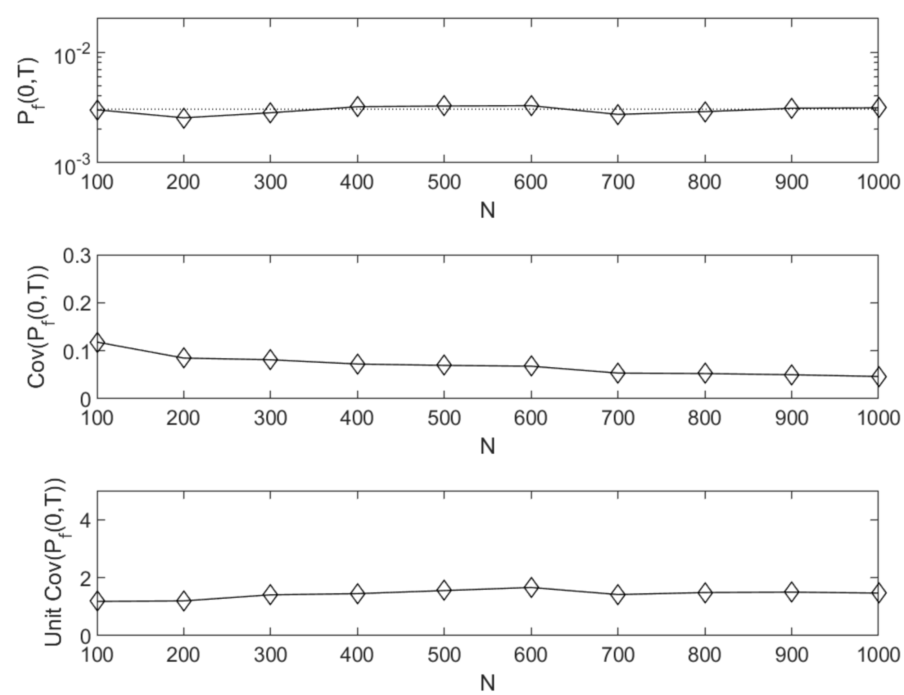


Figure 3: Evolution of different estimators with respect to the number of samples employing the proposed approach (Example 1).

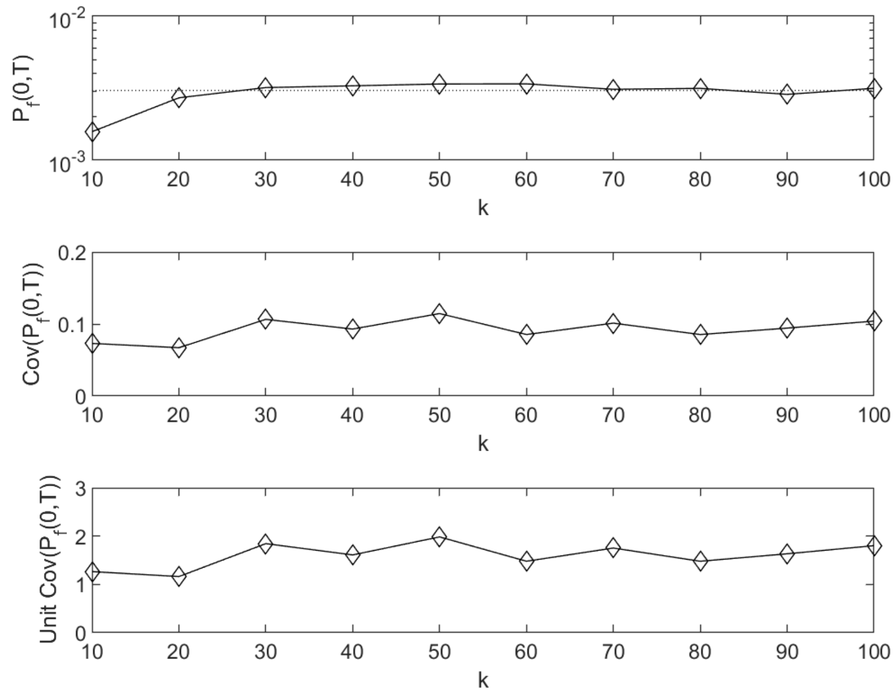


Figure 4: Evolution of different estimators with respect to the number of K-L expansion terms employing the proposed approach (Example 1).

with each approach are listed in Table 2, which includes the estimate of failure probability, the c.o.v. of estimate, the number of samples used $N$, and the unit c.o.v. of the method. It can be seen from the table that the results that are obtained by these methods agree very well with each other. Note that the proposed approach has been applied with only 100 samples, leading to a c.o.v. of around 0.10. On the other hand, Subset simulation demanded 3000 samples, whereas the c.o.v. is 0.20. As such, it is clear that the proposed approach is highly efficient. Such conclusion is reinforced when examining the value of the unit c.o.v., which is an index of efficiency for a simulation method. It can be seen that the proposed method owns the smallest unit c.o.v., which is nearly 1/20 of that of MCS and 1/10 of that of Subset simulation for this example. Due to the small unit c.o.v., the proposed approach can significantly reduce the computation cost.

In order to investigate the performance of the proposed method more clearly, several runs with different number of samples are carried out. Fig. 3 shows the corresponding results obtained by the proposed method as a function of the number of samples used. It can be seen from the figure that, even when a small number of samples (100) is used, the proposed approach can produce quite accurate estimates. Furthermore, when the number of samples increases from 100 to 1000, the c.o.v. of the failure probability estimate gradually decreases from about 0.1 to about 0.04. At the same time, the unit c.o.v. fluctuates between approximate 1 and 2.

Also, the sensitivity of the probability with respect to the number of terms ($k$) in the K-L expansion is shown in Fig. (4). It can be seen from this figure that 1) the failure probability is first underestimated and becomes stable when $k$ is larger than 30; 2) the c.o.v. and unit c.o.v. of the estimate fluctuate when $k > 30$, implying that the efficiency of the proposed approach seems insensitive to the number of terms $k$, under the condition that $k$ is large enough such that the error is controlled appropriately. This shows that the method is insensitive to the number of transformed variables in this example ( according to Eq. (12)).

In conclusion, from the figure and table, the obtained results show that, for this kind of problem, the proposed method exhibits high efficiency when compared to other existing methods. The reason for this behavior is that the proposed method exploits the linearity with respect to the stochastic process.

### *4.2. Example 2: Composite beam*

This second example involves the composite beam shown in Fig.5, which has been borrowed from [42] and [43] with appropriate modifications. The beam possesses a cross section of width $A$(mm) and height $B$(mm) while its total length is $L$(mm). Its Young's modulus is denoted as $E_w(t)$(GPa). The beam has attached on its bottom face an aluminum plate with Young's modulus $E_a(t)$(GPa), whose cross section is $C$(mm) wide by $D$(mm) high. Along the beam, six time-dependent stochastic loads, $P_1(t)$, $P_2(t)$, $P_3(t)$, $P_4(t)$, $P_5(t)$ and $P_6(t)$(kN) are applied at six different locations, $L_1$, $L_2$, $L_3$, $L_4$, $L_5$ and $L_6$(mm). Failure is defined whenever the maximum bending normal stress of the beam exceeds the allowable tensile stress (yield strength) $S(t)$. The limit state function is then given by

$$g(\boldsymbol{x}, t, \boldsymbol{Y}(t)) = S(t) - \frac{\left[\frac{L_3}{L}\sum_{i=1}^{6} P_i(t)\,(L - L_i) - \sum_{i=1}^{2} P_i(t)\,(L_k - L_i)\right] d(\boldsymbol{x})}{W(\boldsymbol{x})} \tag{36}$$

where $\boldsymbol{Y}(t) = [P_1(t), P_2(t), P_3(t), P_4(t), P_5(t), P_6(t)]$ is the vector of stochastic processes;

$$d(\boldsymbol{x}) = \frac{0.5AB^2 + DC(B+D)E_a(t)/E_w(t)}{AB + DCE_a/E_w} \tag{37}$$

$$W(\boldsymbol{x}) = \frac{AB^3}{12} + AB\left[d(\boldsymbol{x}) - \frac{B}{2}\right]^2 + \frac{CD^3E_a(t)}{12E_w(t)} + \frac{CDE_a(t)}{E_w(t)}\left[\frac{D}{2} + B - d(\boldsymbol{x})\right]^2 \tag{38}$$

and

$$S(t) = S_0 e^{-0.01t} \tag{39}$$

$$E_a(t) = E_{a0}\left[1 - 0.1\log(t/2+1)\right] \tag{40}$$

$$E_w(t) = E_{w0}\left[1 - 0.3\log(t+1)\right] \tag{41}$$

are three random processes related with structural parameters' deterioration. In this context, $S_0$ is the initial yield strength, and $E_{a0}$, $E_{w0}$ are the initial Young's moduli of the beam and the aluminum plate, respectively. There are 14 random variables and 6 stochastic load process considered in this example. The distribution information of these parameters is given in Table 3.

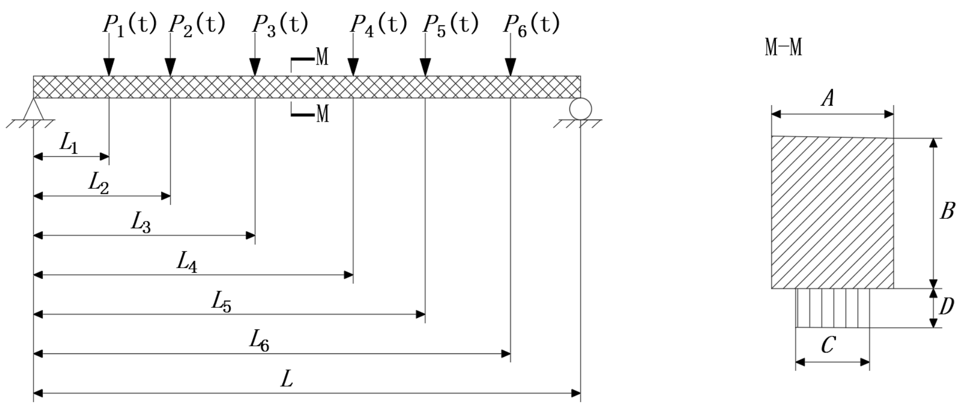


Figure 5: A composite beam.

In this example, six stochastic processes are considered simultaneously. The proposed approach, as well as Subset simulation and Direct Monte Carlo simulation are applied to solve this problem. Table 4 exhibits the obtained results by these methods, showing consistency between methods with respect to the failure probability estimates. In this case, MCS shows a relatively low efficiency, where $10^6$ samples are used and the unit c.o.v is 103.7. Subset simulation produces a probability estimate with a relatively large c.o.v. (0.29). The proposed approach has been carried out with 500 samples and produces an estimate with c.o.v. around 0.10. In addition, the proposed method owns the smallest unit c.o.v. which is nearly 1/8 of that of MCS and 1/40 of that of Subset simulation in this example. The advantage on efficiency of the proposed approach is obvious.

### *4.3. Example 3: Ten-bar truss structure*

This example involves a ten-bar aluminum truss, which corresponds to a modified version of the problem considered in [44] and [45] in order to suit the purposes of this contribution. The truss

Table 3: Information parameters for composite beam (Example 2)

| Parameter | Distribution | Mean | Standard deviation | Autocorrelation function |
|---|---|---|---|---|
| $A$(mm) | Normal | 100 | 1 | – |
| $B$(mm) | Normal | 200 | 2 | – |
| $C$(mm) | Normal | 80 | 0.8 | – |
| $D$(mm) | Normal | 20 | 0.2 | – |
| $L_1$(mm) | Normal | 200 | 2 | – |
| $L_2$(mm) | Normal | 400 | 4 | – |
| $L_3$(mm) | Normal | 600 | 6 | – |
| $L_4$(mm) | Normal | 800 | 8 | – |
| $L_5$(mm) | Normal | 1000 | 10 | – |
| $L_6$(mm) | Normal | 1200 | 12 | – |
| $L$(mm) | Normal | 1400 | 14 | – |
| $E_{a0}$(GPa) | Extreme value | 70 | 0.7 | – |
| $E_{w0}$(GPa) | Extreme value | 8.75 | 0.0875 | – |
| $S_0$(GPa) | Lognormal | $2.7 \times 10^{-2}$ | $2.7 \times 10^{-4}$ | – |
| $P_1(t)$(kN) | Gaussian process | 15 | 1.5 | $\cos(\pi\tau)$ |
| $P_2(t)$(kN) | Gaussian process | 15 | 1.5 | $\cos(\pi\tau)$ |
| $P_3(t)$(kN) | Gaussian process | 15 | 1.5 | $\cos(\pi\tau)$ |
| $P_4(t)$(kN) | Gaussian process | 15 | 1.5 | $\cos(\pi\tau)$ |
| $P_5(t)$(kN) | Gaussian process | 15 | 1.5 | $\cos(\pi\tau)$ |
| $P_6(t)$(kN) | Gaussian process | 15 | 1.5 | $\cos(\pi\tau)$ |

Table 4: Results by different methods for composite beam (Example 2)

| Methods | $P_f(0,T)$ | c.o.v. | $N$ | Unit c.o.v. |
|---|---|---|---|---|
| Proposed approach | $9.1 \times 10^{-5}$ | 0.11 | 500 | 2.59 |
| SS | $8.6 \times 10^{-5}$ | 0.29 | 5000 | 20.6 |
| MCS | $9.3 \times 10^{-5}$ | 0.10 | $10^6$ | 103.7 |

is shown in Fig. 6. In this example, the length of the vertical and horizontal bars is $L$, the modulus of elasticity is $E(t)$, and the cross-sectional area of its members is denoted as $A_j$ $j = 1, 2, \ldots, 10$. These quantities are all modelled as basic random variables. This truss is subjected to two vertical stochastic process loads $F_1(t)$ and $F_2(t)$, and a horizontal stochastic process load $F_3(t)$. The limit state function is cast as the difference between the allowable displacement $d(t)$ and the vertical displacement $\delta_2$ of joint 2, that is:

$$g(\boldsymbol{x}, t, \boldsymbol{Y}(t)) = d(t) - \left( \sum_{i=1}^{6} \frac{N_i N_i^0}{A_i} + \sqrt{2} \sum_{i=7}^{10} \frac{N_i N_i^0}{A_i} \right) \frac{L}{E(t)} \tag{42}$$

where

$$d(t) = d_0 e^{(-0.05t)} \tag{43}$$

$$E(t) = E_0[1 - 0.1 \log(t+1)] \tag{44}$$

are two stochastic processes related with structural deterioration, respectively; $\boldsymbol{x} = [A_1, \cdots, A_{10}, L, E_0, d_0]$ is the vector of basic random variables; $N_j (j = 1, 2, \ldots, 10)$ are the axial forces which can be obtained from the equilibrium and compatibility equations:

$$\begin{cases} N_1 = F_2 - \sqrt{2} N_8/2 \\ N_2 = -\sqrt{2} N_{10}/2 \\ N_3 = -F_1(t) - 2F_2(t) + F_3(t) - \sqrt{2} N_8/2 \\ N_4 = -F_2(t) + F_3(t) - \sqrt{2} N_{10}/2 \\ N_5 = -F_2(t) - \sqrt{2} N_8/2 - \sqrt{2} N_{10}/2 \\ N_6 = -\sqrt{2} N_{10}/2 \\ N_7 = \sqrt{2}\left(F_1(t) + F_2(t)\right) + N_8 \\ N_8 = \left(a_{22} b_1 - a_{12} b_2\right) / \left(a_{11} a_{22} - a_{12} a_{21}\right) \\ N_9 = \sqrt{2} F_2(t) + N_{10} \\ N_{10} = \left(a_{11} b_2 - a_{21} b_1\right) / \left(a_{11} a_{22} - a_{12} a_{21}\right) \end{cases} \tag{45}$$

where

$$\begin{cases} a_{11} = \left(1/A_1 + 1/A_3 + 1/A_5 + 2\sqrt{2}/A_7 + 2\sqrt{2}/A_8\right) L/(2E) \\ a_{12} = a_{21} = L/(2A_5 E) \\ a_{22} = \left(1/A_2 + 1/A_4 + 1/A_6 + 2\sqrt{2}/A_9 + 2\sqrt{2}/A_{10}\right) L/(2E) \\ b_1 = \left(F_2(t)/A_1 - \left(F_1(t) + 2F_2(t) - F_3(t)\right)/A_3 - F_2(t)/A_5 - 2\sqrt{2}\left(F_1(t) + F_2(t)\right)/A_7\right) \sqrt{2} L/(2E) \\ b_2 = \left(\sqrt{2}\left(F_3(t) - F_2(t)\right)/A_4 - \sqrt{2} F_2(t)/A_5 - 4F_2(t)/A_9\right) L/(2E) \end{cases} \tag{46}$$

where $\boldsymbol{Y}(t) = [F_1(t), F_2(t), F_3(t)]$ is the vectors of load stochastic processes; $N_i^0$ is obtained by assuming $F_1 = F_3 = 0$ and $F_2 = 1$ in Eq. 45.

The information of basic random variables and the stochastic processes is given in Table 5.

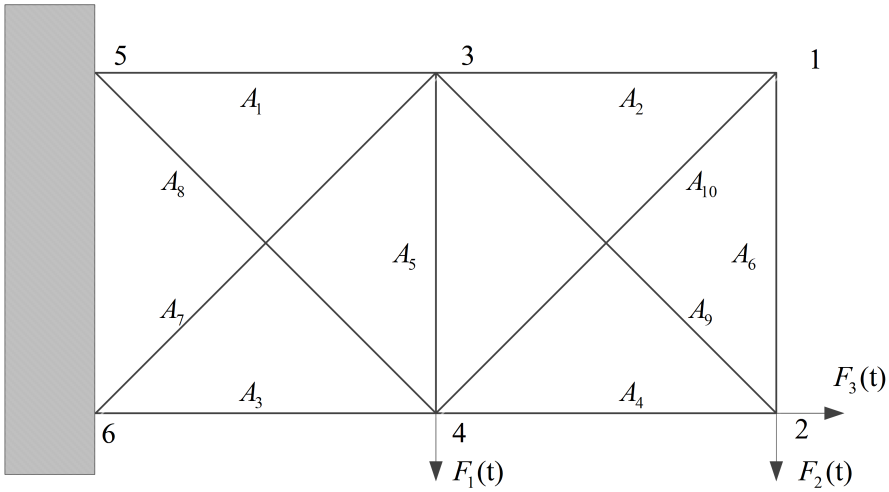


Figure 6: Ten-bar truss structure.

Table 5: Information of variables and parameters for ten-bar truss (Example 3)

| Parameter | Distribution | Mean | Standard deviation | Autocorrelation function |
|---|---|---|---|---|
| $A_1, \cdots, A_{10}$(in) | Normal | 10 | 0.1 | – |
| $L$(in) | Normal | 360 | 3 | – |
| $E_0$(ksi) | Normal | $1.5 \times 10^4$ | $1.5 \times 10^2$ | – |
| $d_0$(in) | Lognormal | 5 | 0.05 | – |
| $F_1(t)$(kip) | Gaussian process | 100 | 10 | $e^{-\tau^2}$ |
| $F_2(t)$(kip) | Gaussian process | 120 | 12 | $e^{-\tau^2}$ |
| $F_3(t)$(kip) | Gaussian process | 400 | 40 | $e^{-\tau^2}$ |

In this example, three stochastic processes are considered. The proposed approach, Subset simulation and Monte Carlo simulation are applied to solve this problem. Table 6 shows the obtained results by these methods. It can be seen from the table that the results by these methods are consistent with each other. Note that the failure probability is quite small ($\approx 10^{-4}$). MCS shows a low efficiency with unit c.o.v. bigger than 100; SS also obtained an estimate with relatively large c.o.v. of 0.25 (1000 samples were considered for each level). The proposed approach has been carried out with only 100 samples and obtains a estimate with c.o.v. below 0.10. Clearly,

Table 6: Results by different methods for ten-bar truss (Example 3)

| Methods | $P_f(0,T)$ | c.o.v. | Total number of samples | Unit c.o.v. |
|---|---|---|---|---|
| Proposed approach | $8.2 \times 10^{-5}$ | 0.08 | 100 | 0.8 |
| SS | $6.5 \times 10^{-5}$ | 0.25 | 5000 | 22.5 |
| MCS | $8.9 \times 10^{-5}$ | 0.11 | $10^6$ | 105.9 |

the proposed method owns the smallest unit c.o.v. which is nearly 1/100 of that of MCS and 1/25 of that of Subset simulation in this example.

### *4.4. Example 4: Bracket structure*

This example involves a bracket structure. Its 3-D finite element model is shown schematically in Fig. 7. The rear face of the bracket is fixed, and a distributed stochastic process load $F(t)$ is applied in the negative vertical direction over the front face. In this example, the maximum allowable deflection of the tip of the bracket in the vertical direction is $d_0$, which is characterized as a random variable. The limit state function is defined as:

$$g(\boldsymbol{x}, t, \boldsymbol{Y}(t)) = d_0 - \delta_{max}\left(E(t), \gamma, P(t)\right) \tag{47}$$

where $\gamma$ is Poisson's ratio; $\delta_{max}(\cdot)$ is the maximum displacement of the bracket, which is determined through a finite element analysis in Matlab utilizing the Partial Differential Equation (PDE) toolbox; $E(t)$ is the Young's modulus, which is a random time-variant process given by

$$E(t) = E_0(1 - 0.2\log(t+1)) \tag{48}$$

The distribution information of the parameters is given in Table 7.

In this linear elastic problem, the response (maximum vertical displacement) has a linear relationship with load (provided that strains and stresses are below the elastic limit). Therefore, the proposed approach can be applied, even though it actually involves an implicit limit state function.

Table 8 shows the results obtained by different reliability methods. It can be seen from the table that the results by these methods are consistent with each other. The proposed approach has been carried out with only 30 samples and obtains an estimate of the failure probability with a c.o.v. around 0.10. Figure 8 illustrates the evolution of the estimates generated with the proposed approach, confirming that 30 samples suffice for an accurate analysis. As in the previous examples, the proposed method owns the smallest unit c.o.v. which is nearly 1/24 and 1/15 of those associated with MCS and Subset simulation, respectively. The advantage on efficiency of the proposed approach is remarkable, as it comprises an implicit LSF whose evaluation is time-consuming because of the finite element model.

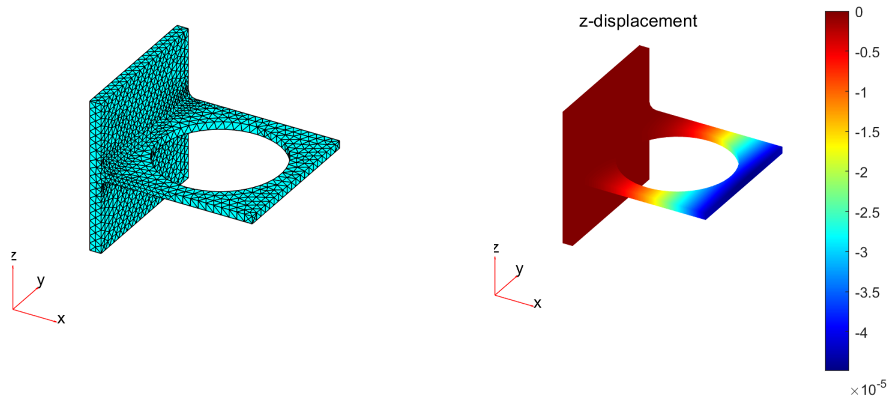


Figure 7: The finite model and response of bracket structure.

Table 7: Information of variables and parameters for bracket structure (Example 4)

| Parameter | Distribution | Mean | Standard deviation | Autocorrelation function |
|---|---|---|---|---|
| $E_0$(GPa) | Lognormal | 200 | 2 | – |
| $\gamma$ | Lognormal | 0.3 | 0.03 | – |
| $d_0$(m) | Extreme value | $2.0 \times 10^{-4}$ | $2.0 \times 10^{-6}$ | – |
| $P(t)$(N/m$^2$) | Gaussian process | $1.8 \times 10^4$ | $1.8 \times 10^3$ | $e^{-\tau^2}$ |

Table 8: Results by different methods for bracket example (Example 4)

| Methods | $P_f(0,T)$ | c.o.v. | Total number of samples | Unit c.o.v. |
|---|---|---|---|---|
| Proposed approach | $3.4 \times 10^{-3}$ | 0.12 | 30 | 0.68 |
| SS | $5.1 \times 10^{-3}$ | 0.19 | 3000 | 10.3 |
| MCS | $3.7 \times 10^{-3}$ | 0.12 | $2 \times 10^4$ | 16.4 |

## 5. Conclusions

A new efficient importance sampling method has been proposed to estimate the reliability of a time-variant structure subject to time-variant load, where the limit state function includes random variables, structural degradation parameter processes and Gaussian stochastic load processes.

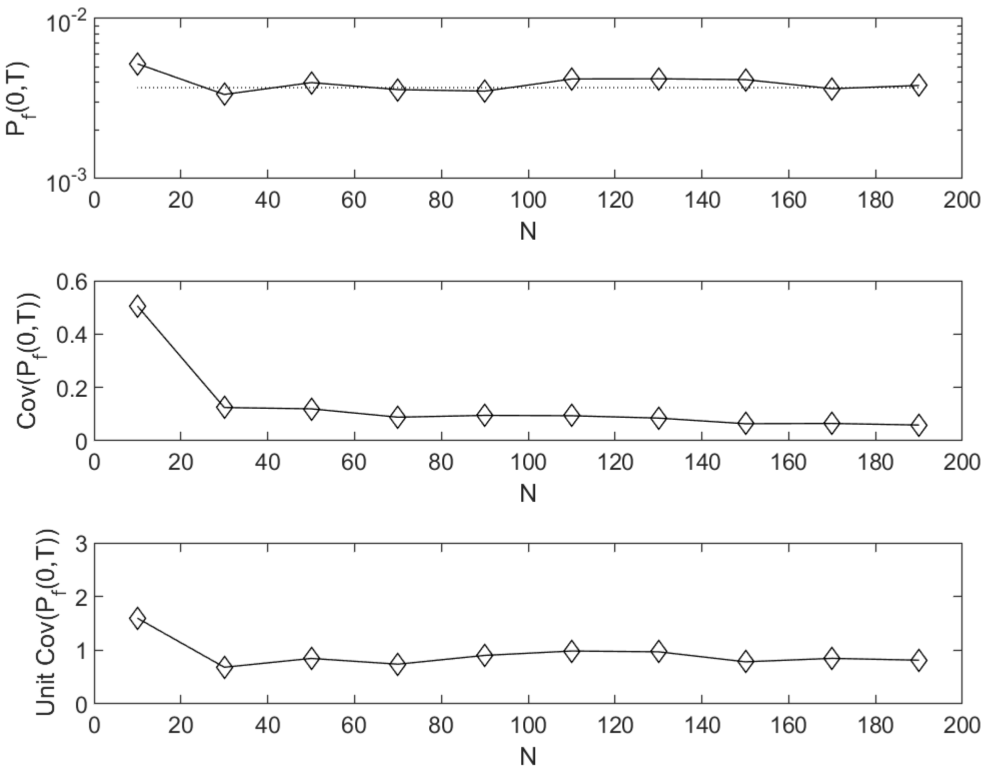


Figure 8: Evolution of estimators with respect to the number of samples by the proposed approach (Example 4).

This approach first utilizes the series expansion methods to discretize the stochastic process and transforms the time-variant problem into a series system. Then, a two-step importance sampling density function is constructed to carry out the reliability analysis by means of simulation. Examples have been presented to investigate the performance of the proposed approach. It is shown that the proposed approach has advantages in the following aspects: (1) It owns high efficiency, as it can obtain a reliable estimate of the probability with only a hundred of samples in the given examples. (2) The accuracy of the failure probability evaluated by proposed approach is relatively insensitive to the magnitude of the probability itself. (3) No approximate analytical calculation is involved.

Note that the application of the proposed approach is limited to problems involving first order stochastic processes. That is, the performance of the structural system must exhibit a linear dependency with respect to the stochastic load for a fixed time. It is expected that it can also be applied to weakly non-linear stochastic processes. It should also be recalled that the proposed approach is applicable for those cases where the limit state function is linear with respect to the loading terms. Another aspect that should be also pointed out is that the proposed approach is based on composite LSF idea which is sensitive to the time discretization. Then, a smaller interval will result in an increased number of composite LSFs, which involves bigger computational burden. It is also found that the efficiency (indicated by the unit c.o.v.) may vary for different problems. Generally speaking, the proposed approach may be highly efficient in case the transformed series system is dominated by one or few of the composite LSF.

Future research efforts will aim at analyzing the application of the proposed approach considering linearization of nonlinear stochastic processes. Another aspect to be examined is the formulation of an importance sampling density function associated with the so-called static random variables.

## CRediT authorship contribution statement

Xiukai Yuan: Conceptualization, Methodology, Software, Validation, Writing - original draft, Supervision, Funding acquisition. Shaolong Liu: Methodology, Software, Writing - original draft. Matthias Faes: Supervision, Writing - review & editing, Funding acquisition. Marcos A. Valdebenito: Supervision, Writing - review & editing, Funding acquisition. Micheal Beer: Supervision, Writing - review & editing.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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