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LOCAL RIGIDITY, BIFURCATION, AND STABILITY OF H_f-HYPERSURFACES IN WEIGHTED KILLING WARPED PRODUCTS

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Abstract: In a weighted Killing warped product $M^n_f\times_\rho\mathbb{R}$ with warping metric $\langle\,,\rangle_M+\rho^2\,dt$, where the warping function ρ is a real positive function defined on M^n and the weighted function f does not depend on the parameter $t\in\mathbb{R}$, we use equivariant bifurcation theory in order to establish sufficient conditions that allow us to guarantee the existence of bifurcation instants, or the local rigidity for a family of open sets $\{\Omega_\gamma\}_{\gamma\in I}$ whose boundaries $\partial\Omega_\gamma$ are hypersurfaces with constant weighted mean curvature. For this, we analyze the number of negative eigenvalues of a certain Schrödinger operator and study its evolution. Furthermore, we obtain a characterization of a stable closed hypersurface $x\colon\Sigma^n\hookrightarrow M^n_f\times_\rho\mathbb{R}$ with constant weighted mean curvature in terms of the first eigenvalue of the f-Laplacian of Σ^n.

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1. Introduction and statements of the results

According to Barbosa and do Carmo in [**5**], and Barbosa, do Carmo, and Eschenburg in [**6**], any closed hypersurface Σ^n with constant mean curvature (CMC) in a Riemannian manifold $\overline{M}^{n+1}$ ($n\geq 2$) is a critical point of the variational problem of minimizing the area functional for volume-preserving variations. Moreover, when $\overline{M}^{n+1}$ has constant sectional curvature c, they also established that geodesic spheres are the only stable critical points for this variational problem.

As observed in [**2, 9, 10**], the set of trial maps for the variational problem should be a collection of embeddings of CMC hypersurfaces Σ^n into $\overline{M}^{n+1}$. In order to detect solutions that are not isometrically congruent, one should take into consideration the action of the diffeomorphism group of Σ^n, acting by right composition in the space of embeddings, and the action of the isometry group of $\overline{M}^{n+1}$, acting by left composition on the space of embeddings. Note that the area and the volume functionals are invariant by the action of these two groups. The action of the diffeomorphism group of Σ^n on any set of embeddings of CMC hypersurfaces Σ^n into $\overline{M}^{n+1}$ is free, which suggests that one should consider a quotient of the space of embeddings by this action. This means that two embeddings of CMC hypersurfaces $x_1\colon \Sigma^n \hookrightarrow \overline{M}^{n+1}$ and $x_2\colon \Sigma^n \hookrightarrow \overline{M}^{n+1}$ will be considered *equivalent* if there exists a diffeomorphism $\phi\colon \Sigma^n \to \Sigma^n$ such that $x_2 = x_1 \circ \phi$. As to the left action of the isometry group of $\overline{M}^{n+1}$, this is not free; nevertheless, the group is compact, and one can study a *bifurcation* problem for its critical orbits. Thus, the variational problem described above provides us with a framework where we can study the *equivariant bifurcation* (cf. [**2, 10, 9, 28**]) in a set of equivalence classes of embeddings of CMC hypersurfaces Σ^n into $\overline{M}^{n+1}$.

In this context, Alías and Piccione in [**2**] studied the bifurcation of CMC Clifford torus of the form $x_r^{n,j}\colon \mathbb{S}^j(r) \times \mathbb{S}^{n-j}(\sqrt{1-r^2}) \hookrightarrow \mathbb{S}^{n+1}$ in unit Euclidean sphere $\mathbb{S}^{n+1}$, where $j \in \{1, \dots, n\}$ and $r \in (0,1)$. More precisely, they showed that the existence of two infinite sequences $x_{r_i}^{n,j}\colon \mathbb{S}^j(r_i) \times \mathbb{S}^{n-j}(\sqrt{1-r_i^2}) \hookrightarrow \mathbb{S}^{n+1}$ and $x_{s_l}^{n,j}\colon \mathbb{S}^j(s_l) \times \mathbb{S}^{n-j}(\sqrt{1-s_l^2}) \hookrightarrow \mathbb{S}^{n+1}$ that are not isometrically congruent to the CMC Clifford torus, and accumulating at some CMC Clifford torus, where $\{r_i\}_{i\geq 3}, \{s_l\}_{l\geq 3} \subset (0,1)$, are sequences of real numbers such that $\lim_{i\to\infty} r_i = 1$ and $\lim_{l\to\infty} s_l = 0$. Furthermore, they also showed that for all other values of $r \in (0,1)$ the family of CMC Clifford torus $x_r^{n,j}\colon \mathbb{S}^j(r)\times\mathbb{S}^{n-j}(\sqrt{1-r^2}) \hookrightarrow \mathbb{S}^{n+1}$ is *locally rigid*, in the sense that any CMC embedding of $\mathbb{S}^j(r) \times \mathbb{S}^{n-j}(\sqrt{1-r^2})$ into $\mathbb{S}^{n+1}$ which is sufficiently close to $x_r^{n,j}$ must be isometrically congruent to an embedding of the CMC Clifford family. Later, de Lima, de Lira, and Piccione ([**16**]) adapted the methods of [**2**] to obtain bifurcation and local rigidity results for a family of CMC Clifford torus in 3-dimensional Berger spheres $\mathbb{S}^3_\tau$, with $\tau > 0$.

More recently, Koiso, Palmer, and Piccione ([**23**]) proved bifurcation results for (compact portions of) nodoids in the 3-dimensional Euclidean space $\mathbb{R}^3$, whose boundary consists of two fixed coaxial circles of the same radius lying in parallel planes. Moreover, the same authors provide in [**24**] criteria for the existence of bifurcation branches of fixed boundary CMC surfaces in $\mathbb{R}^3$ and they discuss stability/instability issues for the surfaces in bifurcating branches.

Meanwhile, García-Martínez and Herrera in [**20**] deduced some bifurcation and local rigidity results for a certain family of CMC hypersurfaces in a class of *Riemannnian warped products* of the form $(I \times_\rho$

$M^n, dt^2+\rho^2\langle\,,\rangle_M)$, namely, in product manifolds $I\times M^n$ endowed with the warping metric $dt^2+\rho^2\langle\,,\rangle_M$, where $I\subset\mathbb{R}$ is an open interval, ρ is a real positive function defined on I, called *warping function*, and M^n is a closed Riemannian manifold with Riemannian metric $\langle\,,\rangle_M$, called *Riemannian fiber*. Such results are obtained considering some appropriate hypotheses that depend of the behavior of the eigenvalues of Laplacian operator on M^n.

On the other hand, on a complete Riemannian manifold $\overline{M}^{n+1}$, let us remember that the classical Laplace operator Δ on $\overline{M}^{n+1}$ can be defined as the differential operator associated to the standard Dirichlet form

$$\mathcal{Q}(u)=\int_{\overline{M}}|\nabla u|^2\,dV,\quad u\in C_c^\infty(\overline{M})\subset L^2(dV),$$

where $|\;|$ is the norm induced by the Riemannian metric $\langle\,,\rangle$ of $\overline{M}^{n+1}$, dV is the volume element on $\overline{M}^{n+1}$, $L^2(dV)$ denotes the set of measurable functions u on $\overline{M}^{n+1}$ such that the Lebesgue integral (with respect to dV) of $|u|^2$ exists and is finite, and $C_c^\infty(\overline{M})$ is the set of all smooth functions defined in $\overline{M}^{n+1}$ with compact support. Now, let $f\in C^\infty(\overline{M})$, that will be referred as a *weight function*. If we replace the measure dV with the *weighted measure* $d\sigma=e^{-f}\,dV$ in the definition of $\mathcal{Q}$, we obtain a new quadratic form $\mathcal{Q}_f$, and we will denote by Δ_f the elliptic operator on $C_c^\infty(\overline{M})\subset L^2(d\sigma)$ induced by $\mathcal{Q}_f$. In this sense, Δ_f arises as a natural generalization of the Laplacian. It is clearly symmetric, positive, and extends to a positive operator on $L^2(d\sigma)$. By Stokes' theorem,

$$\Delta_f(u)=\Delta u-\langle\nabla u,\nabla f\rangle,\quad u\in C_c^\infty(\overline{M}).$$

The triple $(\overline{M}^{n+1},\langle\,,\rangle,d\sigma)$ and the differential operator Δ_f defined above and acting on $C^\infty(\overline{M})$ will be called, respectively, the *weighted manifold* associated with $\overline{M}^{n+1}$ and f, which we simply denote by $\overline{M}^{n+1}_f$, and the *f-Laplacian*. In this setting, we recall that a notion of curvature for weighted manifolds goes back to Lichnerowicz [**25, 26**] and it was later developed by Bakry and Émery in their seminal work [**3**], where they introduced the following modified Ricci curvature $\overline{\mathrm{Ric}}_f=\overline{\mathrm{Ric}}+\overline{\mathrm{Hess}}\,f$, where $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{Hess}}$ are the standard Ricci tensor and the Hessian on $\overline{M}^{n+1}_f$, respectively. As is common in the current literature, we will refer to this tensor as being the *Bakry–Émery–Ricci tensor* of $\overline{M}^{n+1}_f$. We note that the interplay between the geometry of $\overline{M}^{n+1}$ and the behavior of the weighted function f is mostly taken into account by means of its Bakry–Émery–Ricci tensor Ric_f (cf. [**29**]).

On the other hand, it is well known that Killing vector fields are important objects which have been widely used in order to understand the geometry of submanifolds and, more particularly, of hypersurfaces immersed in Riemannian spaces. Into this branch, Alías, Dajczer, and Ripoll ([**1**]) extended the classical Bernstein's theorem [**8**] to the context of complete minimal surfaces in Riemannian spaces of nonnegative Ricci curvature carrying a Killing vector field. This was done under the assumption that the sign of the angle function between a global Gauss mapping and the Killing vector field remains unchanged along the surface. Afterwards, Dajczer, Hinojosa, and de Lira ([**15**]) defined a notion of Killing graph in a class of Riemannian manifolds endowed with a Killing vector field and solved the corresponding Dirichlet problem for prescribed mean curvature under hypothesis involving domain data and the Ricci curvature of the ambient space. Later on, Dajczer and de Lira ([**13**]) showed that an entire Killing graph of constant mean curvature contained in a slab must be a totally geodesic slice, under certain restrictions on the curvature of the ambient space. More recently, in [**14**] these same authors revisited this thematic treating the case when the entire Killing graph of constant mean curvature contained lies inside a possible unbounded region.

Also recently, the second author, jointly with Cunha, de Lima, Lima, Jr., and Medeiros ([**12**]), applied suitable maximum principles in order to obtain Bernstein type properties concerning CMC hypersurfaces Σ^n immersed in a *Killing warped product* $(M^n \times_\rho \mathbb{R}, \langle\,,\rangle_M + \rho^2\, dt)$, namely, in product manifolds $M^n \times \mathbb{R}$ endowed with the warping metric $\langle\,,\rangle_M + \rho^2\, dt$, where M^n is a Riemannian manifold with Riemannian tensor $\langle\,,\rangle_M$, called *Riemannian base*, and ρ is a real positive function defined on M^n, called *warping function.* To obtain these results, they assumed that M^n satisfies certain constraints and that ρ is concave on M^n. Afterwards, in [**17**] the second author, jointly with Lima, Jr., Medeiros, and Santos, obtained Liouville type results concerning hypersurfaces Σ^n immersed in a *weighted Killing warped product* $M^n_f \times_\rho \mathbb{R}$, where the weighted function f does not depend on the parameter $t \in \mathbb{R}$. For this, they assumed suitable boundedness on the Bakry–Émery–Ricci tensor of the base M^n. Furthermore, they also obtained rigidity results via constraints on the height function of the hypersurface.

Proceeding with the picture described above, our purpose in this paper is to study the notions of local rigidity, bifurcation instants, and stability for a family of open sets $\{\Omega_\gamma\}_\gamma$ of a weighted Killing warped product $M^n_f \times_\rho \mathbb{R}$ whose boundaries $\partial\Omega_\gamma$ are closed hypersurfaces with constant weighted mean curvature $H_f(\gamma)$ (in abbreviation, we say that

$\partial\Omega_\gamma$ is a closed $H_f(\gamma)$-*hypersurface*), where γ varies on a prescribed interval $I \subset \mathbb{R}$.

For this, in Section 2 we record some main facts about the hypersurfaces immersed in $M^n_f \times_\rho \mathbb{R}$. Next, in Subsection 3.1, for each Ω_γ, we establish the *variation* $X\colon (-\epsilon, \epsilon) \times \partial\Omega_\gamma \to M^n_f \times_\rho \mathbb{R}$ (see (3.1)) of $\partial\Omega_\gamma$ and we consider the variational problems:

(VP-1): *Minimizing the weighted area functional* $\mathcal{A}_f$ *(see* (3.5)*) for all variations of* $\partial\Omega_\gamma$ *that preserve the weighted volume of* Ω_γ*.*

(VP-2): *Minimizing the weighted area functional* $\mathcal{A}_f$ *(see* (3.5)*) for all variations of* $\partial\Omega_\gamma$*, not necessarily weighted volume-preserving variations of* Ω_γ*.*

By an analysis of the first variation of the associated *weighted Jacobi functional*

$$\mathcal{F}_f^{\lambda(\gamma)} = \mathcal{A}_f + \lambda(\gamma)\mathcal{V}_f, \quad \text{with } \lambda(\gamma) \in \mathbb{R}$$

(see (3.6)), where $\mathcal{V}_f$ is the weighted volume functional (see (3.4)), we obtain in Proposition 1 that the critical points of (VP-1) and (VP-2) are the open sets Ω_γ whose boundary $\partial\Omega_\gamma$ is a closed $H_f(\gamma)$-hypersurface with constant weighted mean curvature $H_f(\gamma) = \lambda(\gamma)/n$. For these critical points, in Proposition 2 we obtain the formula of the second variation of $\mathcal{F}_f^{\lambda(\gamma)}$.

Concerning the variational problem (VP-2), in Subsection 3.2 we use the equivariant bifurcation theory (cf. [**2, 10, 9, 28**]) to establish our notions of bifurcation instants and local rigidity in terms of the *Morse index* of the *weighted Jacobi operator* $\mathcal{J}_{f;\gamma}$ (see (3.21)). Then, in Section 4 we get some results of local rigidity and bifurcation instants in $M^n_f \times_\rho \mathbb{R}$ via the analysis the number of negative eigenvalues of $\mathcal{J}_{f;\gamma}$. Initially, we establish the following result of local rigidity.

Theorem 1. *Let* $\{\Omega_\gamma\}_{\gamma\in I}$ *be a family of open subsets of the weighted Killing warped product* $M^n_f \times_\rho \mathbb{R}$ *whose boundaries* $\partial\Omega_\gamma$ *are closed* $H_f(\gamma)$*-hypersurfaces. If, for all* $\gamma \in I$*, the function*

$$Q_f(\gamma) = \widetilde{\mathrm{Ric}}_f(N^*_\gamma, N^*_\gamma) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*_\gamma, N^*_\gamma) - \langle N_\gamma, Y\rangle^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A_\gamma|^2$$

is constant on $\partial\Omega_\gamma$ *and the first nonzero eigenvalue* $\mu^1_f(\gamma)$ *of the* f*-Laplacian* $\Delta_{f;\gamma}$ *on* $\partial\Omega_\gamma$ *satisfies*

$$\mu^1_f(\gamma) - Q_f(\gamma) > 0, \tag{1.1}$$

then $\{\Omega_\gamma\}_{\gamma\in I}$ *is locally rigid at each* γ*. In particular, such a family is locally rigid if one of the following conditions holds:*

(a) $\widetilde{\mathrm{Ric}}_f(N_\gamma^*,N_\gamma^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N_\gamma^*,N_\gamma^*) - \langle N_\gamma, Y\rangle^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} \le -|A_\gamma|^2$;

(b) *either*

$$\widetilde{\mathrm{Ric}}_f(N_\gamma^*,N_\gamma^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N_\gamma^*,N_\gamma^*) - \langle N_\gamma, Y\rangle^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} < 0 \ \textit{and}\ \mu_f^1(\gamma) \ge |A_\gamma|^2,$$

or

$$\widetilde{\mathrm{Ric}}_f(N_\gamma^*,N_\gamma^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N_\gamma^*,N_\gamma^*) - \langle N_\gamma, Y\rangle^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} \le 0 \ \textit{and}\ \mu_f^1(\gamma) > |A_\gamma|^2.$$

In Theorem 1, Y is the Killing vector field defined on the weighted Killing warped product $M_f^n \times_\rho \mathbb{R}$, $\rho = |Y| > 0$ is the warping function, N_γ is the unit normal vector field on $\partial\Omega_\gamma$, $\widetilde{\Delta}_f$ represents the f-Laplacian on M_f^n, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry–Émery–Ricci tensor and the Hessian operator on M_f^n, $|A|^2$ stands for the square of the norm of the shape operator A of $\partial\Omega_\gamma$ with respect to the orientation given by N_γ, and N_γ^* is the orthogonal projection of N onto the tangent bundle of M^n. These notations will also be used in the statements of the next theorems.

In turn, the bifurcation instants of the family $\{\Omega_\gamma\}_{\gamma\in I}$ are established in the following result.

Theorem 2. *Let $\{\Omega_\gamma\}_\gamma$ be a family of open subsets of the weighted Killing warped product $M_f^n \times_\rho \mathbb{R}$ whose boundaries $\partial\Omega_\gamma$ are closed $H_f(\gamma)$-hypersurfaces. Suppose that, for all $\gamma \in I$, the function*

$$Q_f(\gamma) = \widetilde{\mathrm{Ric}}_f(N_\gamma^*,N_\gamma^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N_\gamma^*,N_\gamma^*) - \langle N_\gamma, Y\rangle^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A_\gamma|^2$$

is constant on $\partial\Omega_\gamma$. If there are two values γ_1 and γ_2, with $\gamma_1 < \gamma_2$, such that the eigenvalues $\widehat{\mu}_f^j(\gamma_1)$ and $\widehat{\mu}_f^j(\gamma_2)$ of the weighted Jacobi operators $\mathcal{J}_{f;\gamma_1}$ and $\mathcal{J}_{f;\gamma_2}$ (respectively) satisfy

(a) *$\widehat{\mu}_f^j(\gamma_1) \ne 0$ and $\widehat{\mu}_f^j(\gamma_2) \ne 0$ for all $j \in \{0,1,2,\dots\}$,*

(b) *there exists $j_0 \in \{0,1,2,\dots\}$ such that $(\widehat{\mu}_f^{j_0}(\gamma_1))(\widehat{\mu}_f^{j_0}(\gamma_2)) < 0$,*

then there exists a bifurcation instant $\gamma_* \in (\gamma_1,\gamma_2)$.

Furthermore, in Section 4, when M^n is closed Riemannian manifold, we give sufficient conditions for both the existence and nonexistence of bifurcation instants of a certain family $\{\Omega_\gamma\}_\gamma$ of open subsets of the weighted Killing warped product $M_f^n \times_\rho \mathbb{R}$ (see (4.2)) whose boundaries $\partial\Omega_\gamma$ are f-minimal hypersurfaces; namely, each $\partial\Omega_\gamma$ is a hypersurface with f-mean curvature equal to zero (cf. Corollaries 1 and 2).

Finally, in Section 5 we study the notion of stability for a critical point of the variational problem (VP-1). More precisely, we established a notion of f-stability for a closed H_f-hypersurface Σ^n immersed in $M^n_f \times_\rho \mathbb{R}$ and, with the help of the f-Laplacian Δ_f of Σ^n of a certain angle function Θ given in Proposition 3, we obtain the following characterization for the f-stability:

Theorem 3. *Let $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ be a closed H_f-hypersurface immersed into weighted Killing warped product $M^n_f \times_\rho \mathbb{R}$. If*

$$\mu = \widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*, N^*) - \Theta^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2$$

is constant, then $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ is f-stable if and only if μ is the first eigenvalue of drift Laplacian Δ_f on Σ^n.

2. Hypersurfaces in weighted Killing warped products

Unless stated otherwise, all manifold considered in this work will be connected, while *closed* means compact without boundary. Throughout this paper, we will consider an $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$ ($n \geq 2$) endowed with a Killing vector field Y. Suppose that the distribution of all vector fields of $\overline{M}^{n+1}$ that are orthogonal to Y is of constant rank and integrable. Given an integral leaf M^n of that distribution, let $\Psi\colon \mathbb{I} \times M^n \to \overline{M}^{n+1}$ be the flow generated by Y with initial values in M^n, where $\mathbb{I}$ is a maximal interval of definition. Without loss of generality, in what follows we will consider $\mathbb{I} = \mathbb{R}$.

In this setting, our space $\overline{M}^{n+1}$ can be regarded as the *Killing warped product* $M^n \times_\rho \mathbb{R}$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the warping metric

$$\langle\,,\rangle = \pi_M^*(\langle\,,\rangle_M) + (\rho \circ \pi_P)^2 \pi_{\mathbb{R}}^*(dt^2), \tag{2.1}$$

where π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M^n \times \mathbb{R}$ onto each factor, $\langle\,,\rangle_M$ is the induced Riemannian metric on the base M^n, dt^2 denotes the usual Riemannian metric in $\mathbb{R}$, and $\rho = |Y| > 0$ is the warping function. By $C^\infty(M^n \times_\rho \mathbb{R})$ we mean the ring of real functions of class C^∞ on $M^n \times_\rho \mathbb{R}$, and by $\mathfrak{X}(M^n \times_\rho \mathbb{R})$ the $C^\infty(M^n \times_\rho \mathbb{R})$-module of vector fields of class C^∞ on $M^n \times_\rho \mathbb{R}$. Let $\overline{\nabla}$ and $\widetilde{\nabla}$ be the Levi–Civita connections of $M^n \times_\rho \mathbb{R}$ and M^n, respectively.

Now, let $(M^n \times_\rho \mathbb{R})_f$ be a *weighted Killing warped product*, namely, a Killing warped product $M^n \times_\rho \mathbb{R}$ endowed with a weighted volume form $d\overline{\sigma} = e^{-f} d\overline{v}$, where $f \in C^\infty(M^n \times_\rho \mathbb{R})$ is a real-valued function, called *weighted function* (or *density function*), and $d\overline{v}$ is the volume element induced by the warping metric $\langle\,,\rangle$ defined in (2.1). For $(M^n \times_\rho \mathbb{R})_f$, the Bakry–Émery–Ricci tensor $\overline{\mathrm{Ric}}_f$ is defined by

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}}\, f, \tag{2.2}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{Hess}}$ are the Ricci tensor and the Hessian operator in $M^n \times_\rho \mathbb{R}$, respectively.

Throughout this work, we will deal with hypersurfaces $x\colon \Sigma^n \hookrightarrow (M^n \times_\rho \mathbb{R})_f$ immersed in a weighted Killing warped product $(M^n \times_\rho \mathbb{R})_f$ and which are *two-sided*. This condition means that there is a globally defined unit normal vector field N. We let ∇ denote the Levi–Civita connection of Σ^n.

In this setting, let A denote the shape operator of Σ^n with respect to N, so that at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map

$$\begin{aligned} A_p\colon T_p\Sigma &\to T_p\Sigma \\ v &\mapsto A_p v = -\overline{\nabla}_v N. \end{aligned}$$

According to Gromov [**21**], the *weighted mean curvature* H_f, or simply the *f-mean curvature*, of $x\colon \Sigma^n \hookrightarrow (M^n \times_\rho \mathbb{R})_f$ is given by

$$nH_f = nH + \langle \overline{\nabla} f, N\rangle, \tag{2.3}$$

where H denotes the standard mean curvature of $x\colon \Sigma^n \hookrightarrow (M^n \times_\rho \mathbb{R})_f$ with respect to its orientation N. When required, if a hypersurface $x\colon \Sigma^n \hookrightarrow (M^n \times_\rho \mathbb{R})_f$ has constant f-mean curvature H_f, then for short we will say that $x\colon \Sigma^n \hookrightarrow (M^n \times_\rho \mathbb{R})_f$ is an *H_f-hypersurface*. Moreover, we recall that $x\colon \Sigma^n \hookrightarrow (M^n \times_\rho \mathbb{R})_f$ is called *f-minimal* when its f-mean curvature vanishes identically.

The *f-divergence* on Σ^n is defined by

$$\begin{aligned} \mathrm{div}_f\colon \mathfrak{X}(\Sigma^n) &\to C^\infty(\Sigma^n) \\ X &\mapsto \mathrm{div}_f X = \mathrm{div}\, X - \langle \nabla f, X\rangle, \end{aligned}$$

where $\mathrm{div}(\cdot)$ denotes the standard divergence on Σ^n. We define the *drift Laplacian* of Σ^n by

$$\begin{aligned} \Delta_f\colon C^\infty(\Sigma^n) &\to C^\infty(\Sigma^n) \\ u &\mapsto \Delta_f(u) = \mathrm{div}_f \nabla u = \Delta u - \langle \nabla f, \nabla u\rangle, \end{aligned} \tag{2.4}$$

where Δ is the standard Laplacian on Σ^n. We will also refer to such an operator as the *f-Laplacian* of Σ^n.

Remark 1. We observe that the Killing vector field Y determines in $M^n \times_\rho \mathbb{R}$ a codimension one foliation by totally geodesic slices $M^n \times \{t\}$, $t \in \mathbb{R}$, with respect to orientation determined by Y. Moreover, assuming that the weighted function $f \in C^\infty(M^n \times_\rho \mathbb{R})$ is invariant along the flow determined by Y, that is, $\langle\overline{\nabla} f, Y\rangle = 0$, from (2.3) we get that each slice $M^n \times \{t\}$ is f-minimal.

Remark 2. We observe that the following result is a consequence of a Cheeger–Gromoll type splitting theorem due to G. Wei and W. Wylie (cf. Theorem 6.1 of [**29**]; see also Theorem 1.1 of [**19**]): *Let $\overline{M}_f^{n+1}$ be a weighted Riemannian manifold that contains a line. If the Bakry–Émery–Ricci tensor of $\overline{M}_f^{n+1}$ is nonnegative and the weighted function f is bounded, then f must be constant along the line.* Consequently, in any weighted Killing warped product $(M^n \times_\rho \mathbb{R})_f$ having nonnegative Bakry–Émery–Ricci tensor and with bounded weighted function f, we have that f does not depend on the parameter of the flow associated to the Killing vector field Y.

Motivated by Remarks 1 and 2, in this work we will consider Killing warped products $M^n \times_\rho \mathbb{R}$ endowed with a weighted function f does not depend on the parameter $t \in \mathbb{R}$, that is, $\langle\overline{\nabla} f, Y\rangle = 0$. For the sake of simplicity, we will denote such an ambient space by

$$M_f^n \times_\rho \mathbb{R}.$$

3. The variational problem and the notion of bifurcation instants

Let $\mathcal{M}$ be the space of open subsets Ω of $M_f^n \times_\rho \mathbb{R}$ with compact closure $\overline{\Omega}$ and whose smooth compact boundary $\partial\Omega$ is a closed, connected, and orientable hypersurface. For any $\Omega \in \mathcal{M}$,

$$\mathrm{Vol}_f(\Omega) \quad \text{and} \quad \mathrm{Area}_f(\partial\Omega)$$

will denote the f-*volume* and f-*area* of Ω and $\partial\Omega$, respectively.

3.1. Description of the variational problem. If $\Omega \in \mathcal{M}$, the globally unit normal vector field defined on $\partial\Omega$ will be denoted by N. For $\Omega \in \mathcal{M}$, we define a *variation* of $\partial\Omega$ as being the smooth mapping

$$\begin{aligned} X\colon (-\epsilon,\epsilon)\times\partial\Omega &\to M_f^n \times_\rho \mathbb{R} \\ (s,p) &\mapsto X(s,p), \end{aligned} \tag{3.1}$$

satisfying the following two conditions:

(1) for all $s \in (-\epsilon,\epsilon)$, the map

$$\begin{aligned} X_s\colon \partial\Omega &\hookrightarrow M_f^n \times_\rho \mathbb{R} \\ p &\mapsto X_s(p) = X(s,p) \end{aligned} \tag{3.2}$$

is an immersion;

(2) $X(0,p) = \iota(p)$ for all $p \in \partial\Omega$, where $\iota\colon \partial\Omega \hookrightarrow \overline{\Omega}$ is the inclusion map.

In this context, given $\Omega \in \mathcal{M}$ and a variation $X\colon (-\epsilon,\epsilon) \times \partial\Omega \to M^n_f \times_\rho \mathbb{R}$ of $\partial\Omega$ we adopt the notation $\partial\Omega_s = X_s(\partial\Omega)$. For values of s small enough, $\partial\Omega_s$ is also a connected and oriented n-dimensional smooth submanifold. Moreover, it bounds an open subset Ω_s whose closure is also compact. Thus, the variation $X\colon (-\epsilon,\epsilon) \times \partial\Omega \to M^n_f \times_\rho \mathbb{R}$ described above induces a variation of the open subset Ω denoted by Ω_s, which is also an element of $\mathcal{M}$.

In all that follows, we let $d(\partial\Omega_s)$ denote the volume element of the metric induced on $\partial\Omega_s$ by X_s and N_s, the unit normal vector field along X_s. Moreover, we also consider in $\partial\Omega_s$ the weighted volume form given by $d\sigma_s = e^{-f} d(\partial\Omega_s)$. When $s = 0$ all these objects coincide with ones defined in $\partial\Omega$, respectively.

The *variational field* associated to the variation $X\colon (-\epsilon,\epsilon) \times \partial\Omega \to M^n_f \times_\rho \mathbb{R}$ is the vector field $\frac{\partial X}{\partial s}\big|_{s=0}$. Letting

$$u_s = \left\langle \frac{\partial X}{\partial s}, N_s \right\rangle, \tag{3.3}$$

we get

$$\frac{\partial X}{\partial s}\bigg|_{s=0} = u_0 N + \left(\frac{\partial X}{\partial s}\bigg|_{s=0} \right)^{\top},$$

where $(\cdot)^\top$ stands for tangential components.

The *weighted volume functional* associated to the variation $X\colon (-\epsilon,\epsilon)\times \partial\Omega \to M^n_f \times_\rho \mathbb{R}$ is

$$\begin{aligned} \mathcal{V}_f\colon (-\epsilon,\epsilon) &\to \mathbb{R} \\ s \quad &\mapsto \mathcal{V}_f(s) = \mathrm{Vol}_f(\Omega_s) = \int_{\Omega_s} d\overline{\sigma}, \end{aligned} \tag{3.4}$$

and we say that $X\colon (-\epsilon,\epsilon) \times \partial\Omega \to M^n_f \times_\rho \mathbb{R}$ is *weighted volume-preserving* of Ω if $\mathcal{V}_f(s) = \mathcal{V}_f(0)$ for all $s \in (-\epsilon,\epsilon)$.

The following result is well known and, in the context of weighted manifolds, can be found in [**11**].

Lemma 1. *If $\Omega \in \mathcal{M}$ and $X\colon (-\epsilon,\epsilon) \times \partial\Omega \to M^n_f \times_\rho \mathbb{R}$ is a variation of $\partial\Omega$, then*

$$\frac{d}{dt}\mathcal{V}_f(s) = \int_{\partial\Omega_s} u_s \, d\sigma_s \quad \textit{for all } s \in (-\epsilon,\epsilon),$$

where u_s is the function defined in (3.3). *In particular, $X\colon (-\epsilon,\epsilon)\times\partial\Omega \to M^n_f \times_\rho \mathbb{R}$ is weighted volume-preserving of Ω if and only if $\int_{\partial\Omega_s} u_s \, d\sigma_s = 0$ for all $s \in (-\epsilon,\epsilon)$.*

Remark 3. We observe that is not difficult to verify that Lemma 2.2 of [**6**] still remains valid for the context of weighted Riemannian manifolds, that is, if $u \in C^\infty(\partial\Omega)$ is such that $\int_{\partial\Omega} u\, d\sigma = 0$, then there exists a weighted volume-preserving variation $X\colon (-\epsilon,\epsilon)\times\partial\Omega \to M^n_f \times_\rho \mathbb{R}$ of $\partial\Omega$ whose variational field is $\frac{\partial X}{\partial s}\big|_{s=0} = uN$.

The *weighted area functional* associated to the variation X is given by

$$\begin{aligned} \mathcal{A}_f\colon (-\epsilon,\epsilon) &\to \mathbb{R} \\ s \quad &\mapsto \mathcal{A}_f = \mathrm{Area}_f(\partial\Omega_s) = \int_{\partial\Omega_s} d\sigma_s. \end{aligned} \tag{3.5}$$

Following the same steps of the proof of Lemma 3.2 of [**11**], it is not difficult to see that we get the following

Lemma 2. *If $\Omega \in \mathcal{M}$ and $X\colon (-\epsilon,\epsilon)\times\partial\Omega \to M^n_f \times_\rho \mathbb{R}$ is a variation of $\partial\Omega$, then*

$$\frac{d}{ds}\mathcal{A}_f(s) = -n\int_{\partial\Omega_s} (H_f)_s u_s\, d\sigma_s \quad \textit{for all } s \in (-\epsilon,\epsilon),$$

where u_s is the function given in (3.3) *and $(H_f)_s = H_f(s,\cdot)$ denotes the f-mean curvature of $\partial\Omega_s$ with respect to the metric induced by the immersion X_s defined in* (3.2).

In order to characterize open subsets Ω of $M^n_f \times_\rho \mathbb{R}$ whose boundaries are closed hypersurfaces with constant f-mean curvature (possibly equal to zero), we consider the variational problem (VP-1) described in Section 1. The Lagrange multiplier method leads us then to the associated *weighted Jacobi functional*

$$\begin{aligned} \mathcal{F}^\lambda_f\colon (-\epsilon,\epsilon) &\to \mathbb{R} \\ s \quad &\mapsto \mathcal{F}^\lambda_f(s) = \mathrm{Area}_f(\partial\Omega_s) + \lambda\,\mathrm{Vol}_f(\Omega_s), \end{aligned} \tag{3.6}$$

where λ is a constant to be determined (eventually λ can be zero, and in this case, for $\Omega \in \mathcal{M}$, our variational problem reduces to minimizing the functional $\mathcal{A}_f$ for all variations of $\partial\Omega$).

As an immediate consequence of Lemmas 1 and 2 we get that the first variation of $\mathcal{F}^\lambda_f$ takes the following form

$$\frac{d}{ds}\mathcal{F}^\lambda_f(s) = \frac{d}{ds}\mathcal{A}_f(s) + \lambda\frac{d}{ds}\mathcal{V}_f(s) = \int_{\partial\Omega_s}\{-n(H_f)_s + \lambda\}u_s\, d\sigma_s. \tag{3.7}$$

Thinking about making the best possible choice of λ, let

$$\overline{\mathcal{H}} = \frac{1}{\mathrm{Area}_f(\partial\Omega)} \int_{\partial\Omega} H_f \, d\sigma \tag{3.8}$$

be an integral mean of the f-mean curvature H_f on $\partial\Omega$. We call the attention to the fact that, in case H_f is constant, we have

$$\overline{\mathcal{H}} = H_f, \tag{3.9}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\lambda = n\overline{\mathcal{H}}$, from (3.7) we arrive at

$$\frac{d}{ds}\mathcal{F}_f^\lambda(s) = -n \int_{\partial\Omega_s} \{(H_f)_s - \overline{\mathcal{H}}\} u_s \, d\sigma_s. \tag{3.10}$$

In particular,

$$\frac{d}{ds}\mathcal{F}_f^\lambda(0) = -n \int_{\partial\Omega} \{H_f - \overline{\mathcal{H}}\} u_0 \, d\sigma. \tag{3.11}$$

Now, from (3.11) and following the same ideas of Proposition 2.7 of [**5**] we can establish the following result.

Proposition 1. *Let $\Omega \in \mathcal{M}$. The following statements are equivalent:*

(a) *$\partial\Omega$ is a closed H_f-hypersurface with constant f-mean curvature H_f equal to $H_f = \lambda/n$;*

(b) *for all weighted volume-preserving variations $X\colon (-\epsilon, \epsilon) \times \partial\Omega \to M_f^n \times_\rho \mathbb{R}$ of $\partial\Omega$, we have $\frac{d}{ds}\mathcal{A}_f(0) = 0$;*

(c) *for all variations $X\colon (-\epsilon, \epsilon) \times \partial\Omega \to M_f^n \times_\rho \mathbb{R}$ of $\partial\Omega$, we have $\frac{d}{ds}\mathcal{F}_f^\lambda(0) = 0$.*

Hence, from Proposition 1 we have that the critical points of (VP-1) are open subsets Ω of $M_f^n \times_\rho \mathbb{R}$ whose boundary $\partial\Omega$ is a closed H_f-hypersurface with constant second mean curvature H_f equal to

$$H_f = \frac{\lambda}{n}, \tag{3.12}$$

with $\lambda \in \mathbb{R}$. On the other hand, if we change (VP-1) to (VP-2) (see Section 1), from Proposition 1 we obtain that the respective critical points of (VP-2) coincide with the same critical points of the initial variational problem (VP-1).

Remark 4. If $\lambda = 0$, we observe that the two variational problems (VP-2) and (VP-1) coincide, in which case the respective critical points are open subsets Ω of $M_f^n \times_\rho \mathbb{R}$ whose boundary $\partial\Omega$ are closed f-minimal hypersurfaces. Furthermore, from (3.6) we can observe that $\mathcal{F}_f^0$ coincides with the weighted area functional $\mathcal{A}_f$ and, for each $\Omega \in \mathcal{M}$, this whole situation comes down to the variational problem of minimizing $\mathcal{A}_f$ for all variations of $\partial\Omega$ (not necessarily for those that preserve the weighted volume of Ω).

Remark 5. As observed in [**20**], our approach is valid for the following more general configuration. Assume that $\mathcal{M}$ is the space of open subsets $\Omega \subset M^n_f \times_\rho \mathbb{R}$ whose boundary $\partial\Omega$ is the union of two disjoint sets $\partial\Omega = \Sigma^n_1 \cup \Sigma^n_2$. We will assume that one of them, say Σ^n_1, is a fixed set so that the variations considered of $\partial\Omega$ only affect Σ^n_2. Under this assumption, the critical points of (VP-1) or (VP-2) will be open subsets Ω such that their boundaries are the union of a (fixed) set Σ^n_1 and a closed H_f-hypersurface Σ^n_2 with constant f-mean curvature H_f given by (3.12).

For such a critical point (for either of the two variational problems described above), the formula for the second variation of $\mathcal{F}^\lambda_f$ is given in the following result.

Proposition 2. *Let $\Omega \in \mathcal{M}$ be an open subset of $M^n_f \times_\rho \mathbb{R}$ whose boundary $\partial\Omega$ is a compact H_f-hypersurface with constant f-mean curvature H_f given by* (3.12). *Then the second variation $\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)$ of the weighted Jacobi functional $\mathcal{F}^\lambda_f$ is given by*

$$\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)(u) = -\int_{\partial\Omega} u\,\mathcal{J}_f(u)\,d\sigma, \tag{3.13}$$

for any $u \in C^\infty(\partial\Omega)$, where $\mathcal{J}_f\colon C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ is the weighted Jacobi operator given by

$$\mathcal{J}_f = \Delta_f + \widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*, N^*) - \langle N, Y\rangle^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2. \tag{3.14}$$

Here, Y is the Killing vector field on $M^n_f \times_\rho \mathbb{R}$, $\rho = |Y| > 0$, N is the unit normal vector field on $\partial\Omega$, Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on $\partial\Omega$ and M^n_f, respectively, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry–Émery–Ricci tensor and the Hessian operator on M^n_f, $|A|^2$ represents the square of the norm of the shape operator A of $\partial\Omega$ with respect to the orientation given by N, and N^* is the orthogonal projection of N on the tangent bundle of M^n. With respect to the functions on $\partial\Omega$ to be evaluated in $\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)$ for a critical point of (VP-1), *they have to be considered according to Remark 3, that is, smooth functions on $\partial\Omega$ whose integral mean is zero; and, on the other hand, any smooth function on $\partial\Omega$ can be evaluated in $\frac{d^2}{ds^2}\mathcal{F}^\lambda(0)$ for a critical point of* (VP-2).

Proof: Initially, for any variation $X\colon (-\epsilon,\epsilon)\times\partial\Omega\to M^n_f\times_\rho\mathbb{R}$ of $\partial\Omega$ we consider the function $u_0\in C^\infty(\partial\Omega)$ defined in (3.3). Since H_f is constant, from (3.10) and (3.9) we have that

$$\begin{aligned}\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)(u_0) = &-n\int_{\partial\Omega}\left(\frac{\partial(H_f)_s}{\partial s}\bigg|_{s=0}\right)u_0\,d\sigma\\ &-n\int_{\partial\Omega}\Big(\underbrace{H_f-\overline{\mathcal{H}}}_{0}\Big)\frac{\partial}{\partial s}(u_s\,d\sigma_s)\bigg|_{s=0}.\end{aligned}$$

Reasoning as in the proof of equation (3.5) of [**11**], we obtain

$$n\frac{\partial(H_f)_s}{\partial s}\bigg|_{s=0} = \Delta_f(u_0)+\{\overline{\mathrm{Ric}}_f(N,N)+|A^2|\}u_0.$$

Hence,

$$\frac{d^2}{ds^2}\mathcal{F}^\lambda_f(0)(u_0) = -\int_{\partial\Omega}\{\Delta_f(u_0)+\{\overline{\mathrm{Ric}}_f(N,N)+|A|^2\}u_0\}u_0\,d\sigma. \tag{3.15}$$

On the other hand, denoting by N^* and $N^\perp$ the orthogonal projections of N over the tangent and normal bundles of M^n, respectively, and taking into account that f is invariant along the flow determined by Y, from [**27**, Proposition 7.35] we obtain

$$\begin{aligned}\overline{\mathrm{Hess}}\,f(N,N) &= \langle\overline{\nabla}_N\overline{\nabla}f,N\rangle\\ &= \langle\overline{\nabla}_N\widetilde{\nabla}f,N^*+N^\perp\rangle\\ &= \widetilde{\mathrm{Hess}}\,f(N^*,N^*)+\frac{1}{\rho}\langle\widetilde{\nabla}f,\widetilde{\nabla}\rho\rangle|N^\perp|^2\\ &= \widetilde{\mathrm{Hess}}\,f(N^*,N^*)+\frac{1}{\rho^3}\langle\widetilde{\nabla}f,\widetilde{\nabla}\rho\rangle\langle N,Y\rangle^2.\end{aligned}\tag{3.16}$$

Moreover, from [**27**, Corollary 7.43] we get

$$\overline{\mathrm{Ric}}(N,N) = \widetilde{\mathrm{Ric}}(N^*,N^*)-\frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*,N^*)-\langle N,Y\rangle^2\frac{\widetilde{\Delta}(\rho)}{\rho^3}. \tag{3.17}$$

Now, from equations (3.16) and (3.17), we have

$$\overline{\mathrm{Ric}}_f(N,N) = \widetilde{\mathrm{Ric}}_f(N^*,N^*)-\frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*,N^*)-\langle N,Y\rangle^2\frac{\widetilde{\Delta}_f(\rho)}{\rho^3}. \tag{3.18}$$

Therefore, from equations (3.18) and (3.15) we obtain

$$\frac{d^2}{ds^2}\mathcal{F}_f^\lambda(0)(u_0) = -\int_{\partial\Omega} u_0\,\mathcal{J}_f(u_0)\,d\sigma, \tag{3.19}$$

where $\mathcal{J}_f$ is given in (3.14).

Now, for any $u \in C^\infty(\partial\Omega)$, considering variations $X\colon(-\epsilon,\epsilon)\times\partial\Omega\to M_f^n\times_\rho\mathbb{R}$ of $\partial\Omega$ whose variational field is $\frac{\partial X}{\partial t}\big|_{t=0} = uN$, we obtain that the last expression (3.19) is also valid for every $u \in C^\infty(\partial\Omega)$. Taking into account the set of functions on $\partial\Omega$ that are admissible for a critical point of (VP-2), we conclude that all the arguments stated above are valid to provide the formula of the second variation of $\mathcal{F}_f^\lambda$ for critical points of (VP-2).

For those critical points of (VP-1), if $X\colon(-\epsilon,\epsilon)\times\partial\Omega\to M_f^n\times_\rho\mathbb{R}$ is a variation of $\partial\Omega$ which preserve the weighted volume of Ω, then for $u_0 \in C^\infty(\partial\Omega)$ defined in (3.3), we have from Lemma 1 that $\int_{\partial\Omega} u_0\,dV = 0$ and, in addition, the expression (3.19) is valid for such u_0. Finally, for any function $u \in C^\infty(\partial\Omega)$ such that $\int_{\partial\Omega} u\,dV = 0$, from Remark 3 we get a variation $X\colon(-\epsilon,\epsilon)\times\partial\Omega\to M_f^n\times_\rho\mathbb{R}$ of $\partial\Omega$ which preserves the weighted volume of Ω such that the variational field is $\frac{\partial X}{\partial t}\big|_{t=0} = uN$, and it follows immediately that (3.19) is retrieved for such a u. □

We conclude this subsection by noting that the weighted Jacobi operator $\mathcal{J}_f$ given in (3.14) belongs to a class of differential operators which are usually referred to as Schrödinger operators, that is, operators of the form $\Delta + q$, where Δ is the standard Laplacian on $\partial\Omega$ and q is any continuous function on $\partial\Omega$ (see, for instance, [**18**]). In particular, we can highlight that the behavior of the eigenvalues of $\mathcal{J}_f$ is well known, and this behavior will play an important role in obtaining the main results of this work.

3.2. The notion of bifurcation instants for H_f-hypersurfaces in $M_f^n\times_\rho\mathbb{R}$. In what follows, we consider the one-parameter family $\{\Omega_\gamma\}_\gamma$ of open subsets in weighted Killing warped product $M_f^n\times_\rho\mathbb{R}$ such that the boundary of each Ω_γ, denoted by $\partial\Omega_\gamma$, is a closed $H_f(\gamma)$-hypersurface with constant f-mean curvature $H_f(\gamma)$, where γ varies on a prescribed interval $I\subset\mathbb{R}$. In this context, as a consequence of our study of Subsection 3.1, we have that each Ω_γ is a critical point of a certain variational problem of type (VP-2). More specifically, each Ω_τ is a critical point for the one-parameter family of weighted Jacobi functionals

$$I \ni \gamma \mapsto \mathcal{F}_f^{\lambda(\gamma)} = \mathcal{A}_f + \lambda(\gamma)\mathcal{V}_f$$

defined in (3.6), where

$$\lambda(\gamma) = nH_f(\gamma).$$

Moreover, from Proposition 2, associated with each closed $H_2(\gamma)$-hypersurface $\partial\Omega_\gamma$ we have that the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma)}(0)$ of $\mathcal{F}_f^{\lambda(\gamma)}$ is given by

$$\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\tau)}(0)(u) = -\int_{\partial\Omega} u\,\mathcal{J}_{f;\gamma}(u)\,d\sigma, \tag{3.20}$$

for any $u \in C^\infty(\partial\Omega_\gamma)$, where

$$\begin{aligned}\mathcal{J}_{f;\gamma} &= \Delta_{f;\gamma} + \widetilde{\mathrm{Ric}}_f(N_\gamma^*, N_\gamma^*)\\ &\quad - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N_\gamma^*, N_\gamma^*) - \langle N_\gamma, Y\rangle^2\frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A_\gamma|^2\end{aligned} \tag{3.21}$$

is the weighted Jacobi operator on $\partial\Omega_\gamma$. Here, $\Delta_{f;\gamma}$ and $\widetilde{\Delta}_f$ are the f-Laplacians on $\partial\Omega_\gamma$ and M_f^n, respectively, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry–Émery–Ricci tensor and the Hessian operator in M_f^n, A_γ is the shape operator of $\partial\Omega_\gamma$ with respect to normal vector field N_γ, and N_γ^* is the orthogonal projection of N_γ on the tangent bundle of M^n.

With respect to our family $\{\Omega_\gamma\}_{\gamma\in I}$ of critical points of (VP-2), we need to adopt some notions and results that correspond to equivariant bifurcation theory for geometric variational problems. For more details on this subject, we recommend the references [**2**], [**10**], [**9**], and [**28**].

Let us first recall that two elements Ω_{γ_1} and Ω_{γ_2} of $\{\Omega_\gamma\}_{\gamma\in I}$ are said to be *isometrically congruent* when there is an isometry ψ of $M_f^n \times_\rho \mathbb{R}$ that carries the image of $x_1\colon \partial\Omega_{\gamma_1} \hookrightarrow M_f^n \times_\rho \mathbb{R}$ onto the image of $x_2\colon \partial\Omega_{\gamma_2} \hookrightarrow M_f^n \times_\rho \mathbb{R}$ (cf. Subsection 1.2 of [**2**]), where x_1 and x_2 are the immersions of $\partial\Omega_{\gamma_1}$ and $\partial\Omega_{\gamma_2}$ into $M_f^n \times_\rho \mathbb{R}$, respectively, i.e., if there exists a diffeomorphism $\phi\colon \partial\Omega_{\gamma_1} \to \partial\Omega_{\gamma_2}$ and an isometry ψ of $M_f^n \times_\rho \mathbb{R}$ such that the following diagram commutes:

$$\begin{array}{ccc}\partial\Omega_{\gamma_1} & \xrightarrow{x_1} & M_f^n \times_\rho \mathbb{R}\\ \phi\downarrow & & \downarrow\psi\\ \partial\Omega_{\gamma_2} & \xrightarrow[x_2]{} & M_f^n \times_\rho \mathbb{R}\,.\end{array}$$

Taking into account the studies reported in [**9**], $\widetilde{\gamma} \in I$ is said to be a *bifurcation instant* for the family $\{\Omega_\gamma\}_{\gamma\in I}$ if there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset I$ and a sequence $\{\Omega_{\gamma_n}\}_{n\in\mathbb{N}} \subset \{\Omega_\gamma\}_{\gamma\in I}$ such that

(a) $\lim_{n\to\infty} \gamma_n = \widetilde{\gamma}$,
(b) $\lim_{n\to\infty} x_n = \widetilde{x}$, where $x_n\colon \Omega_{\gamma_n} \hookrightarrow M_f^n \times_\rho \mathbb{R}$ and $\widetilde{x}\colon \Omega_{\widetilde{\gamma}} \hookrightarrow M_f^n \times_\rho \mathbb{R}$ are the immersions of Ω_{γ_n} and $\Omega_{\widetilde{\gamma}}$ into $M_f^n \times_\rho \mathbb{R}$, respectively, and
(c) for all $n \in \mathbb{N}$, x_n is not isometrically congruent to $\widetilde{x}$.

Furthermore, according to the ideas set out in [**10**], if $\widetilde{\gamma} \in I$ is not a bifurcation instant, the family $\{\Omega_\gamma\}_{\gamma\in I}$ is said to be *locally rigid* at $\widetilde{\gamma}$.

One of the classical criteria to determine when a instant $\widetilde{\gamma} \in I$ is of bifurcation is related with the so-called *Morse index* associated with the variational problem in question (see, for instance, [**2**] and [**9**]). Following this philosophy, we define the *Morse index* of Ω_γ, which will be denoted by $\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma)$, as the dimension of the maximal subspace where the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma)}(0)$ of the weighted Jacobi functional $\mathcal{F}_f^{\lambda(\gamma)}$ is negative definite. Equivalently, $\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma)$ is the number of negative eigenvalues (counted with multiplicity) of the weighted Jacobi operator $\mathcal{J}_{f;\gamma}$ given in (3.21). With our notations, a real number $\widehat{\mu}(\gamma)$ is an *eigenvalue* of $\mathcal{J}_{f;\gamma}$ if and only if $\mathcal{J}_{f;\gamma}(u) + \widehat{\mu}(\gamma)u = 0$ for some function $u \in C^\infty(\partial\Omega_\gamma)$. Moreover, using the same arguments of Proposition 2.7 of [**2**] we obtain that $\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma)$ is finite on $I \subset \mathbb{R}$. Intuitively, $\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma)$ measures the number of independent directions in which the $H_f(\gamma)$-hypersurface $\partial\Omega_\gamma$ fails to minimize the weighted area functional $\mathcal{A}_f$ defined in (3.5).

Essentially, a variation of $\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma)$ along the interval $I \subset \mathbb{R}$ will indicate the existence of a bifurcation instant. More precisely, under suitable Fredholmness assumptions (cf. [**2**] and [**9**]), we have that if there are $\gamma_1, \gamma_2 \in I$ with $\gamma_1 < \gamma_2$ such that the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma_j)}(0)$ of the weighted Jacobi functional $\mathcal{F}_f^{\lambda(\gamma_j)}$ is nonsingular (namely, the eigenvalues of the weighted Jacobi operator $\mathcal{J}_{f;\gamma_j}$ are nonzero) for $j \in \{1, 2\}$, and

$$\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma_1)}, \Omega_{\gamma_1}) \neq \mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma_2)}, \Omega_{\gamma_2}), \tag{3.22}$$

then $\{\Omega_\gamma\}_{\gamma\in I}$ admits a bifurcation instant at some $\gamma_* \in (\gamma_1, \gamma_2)$. On the other hand, according to [**10**], using the Implicit Function Theorem we obtain that if $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\widetilde{\gamma})}(0)$ is nonsingular for some $\widetilde{\gamma} \in I$, then the family $\{\Omega_\gamma\}_{\gamma\in I}$ is locally rigid at $\widetilde{\gamma}$. In particular, when $\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma)}, \Omega_\gamma) = 0$ for all $\gamma \in I$, $\{\Omega_\gamma\}_{\gamma\in I}$ does not have bifurcation instants.

Remark 6. We observe that the change in the Morse index of a family of hypersurfaces given by condition (3.22) is not sufficient to guarantee the bifurcation of the family $\{\Omega_\gamma\}_{\gamma\in I}$. Indeed, considering the standard context, the family of CMC spherical caps, starting with a pole and terminating with the entire sphere has a change in the Morse index from 0 to 1 at the hemisphere, but there is no bifurcation (for more details, see [**4**]). Hence, our assumption that $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma_j)}(0)$ is nonsingular for $j \in \{1, 2\}$ is a necessary condition to reach at the bifurcation.

In this paper we will study the local rigidity and the bifurcation instants of $\{\Omega_\gamma\}_{\gamma\in I}$ by analyzing the spectrum of $\mathcal{J}_{f;\gamma}$ for all $\gamma \in I$. Essentially, we will determine the number of negative eigenvalues for each γ (counting its multiplicity) and we will study the evolution of such a number.

4. Local rigidity and bifurcation instants in $M_f^n \times_\rho \mathbb{R}$

Our first result given in Theorem 1 provides some simple sufficient conditions to get the local rigidity of the family $\{\Omega_\gamma\}_{\gamma\in I}$ of critical points of the variational problem (VP-2) described in Subsection 3.2.

Proof of Theorem 1: Since $Q_f(\gamma)$ is constant, from (3.21) we have that the eigenfunctions of the weighted Jacobi operator $\mathcal{J}_{f;\gamma}$ will coincide with the eigenfunctions of f-Laplacian $\Delta_{f;\gamma}$. More specifically, if u is an eigenfunction of $\Delta_{f;\gamma}$ associated with an eigenvalue $\mu_f(\gamma)$, then u is eigenfunction of $\mathcal{J}_{f;\gamma}$ with eigenvalue

$$\widehat{\mu}_f(\gamma) = \mu_f(\gamma) - Q_f(\gamma).$$

Moreover, by the spectral theorem we know that all the eigenvalues of $\Delta_{f;\gamma}$ are given by a sequence $\{\mu_f^j(\gamma)\}_{j=0}^{+\infty}$ satisfying

$$0 = \mu_f^0(\gamma) < \mu_f^1(\gamma) \le \cdots \le \mu_f^j(\gamma) \le \mu_f^{j+1}(\gamma) \le \cdots,$$

repeated according to their multiplicity, and

$$\lim_{j\to+\infty} \mu_f^j(\gamma) = +\infty$$

(see, for instance, Section 1 of [**30**]). So, all the eigenvalues $\widehat{\mu}_f^j(\gamma)$ of $\mathcal{J}_{f;\gamma}$ have the following form

$$\widehat{\mu}_f^j(\gamma) = \mu_f^j(\gamma) - Q_f(\gamma) \quad \text{for every } j \in \{0, 1, 2, \dots\}. \tag{4.1}$$

So, from (1.1) and (4.1) we obtain

$$\widehat{\mu}_f^j(\gamma) = \mu_f^j(\gamma) - Q_f(\gamma) \ge \mu_f^1(\gamma) - Q_f(\gamma) > 0 \quad \text{for every } j \in \{0, 1, 2, \dots\}.$$

Hence, the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma)}(0)$ given in (3.20) is nonsingular for all $\gamma \in I$ and, therefore, the family $\{\Omega_\gamma\}_{\gamma\in I}$ is locally rigid at each $\gamma \in I$. □

In Theorem 2 we obtain a criterion that guarantees the existence of bifurcation instants of the family $\{\Omega_\gamma\}_{\gamma\in I}$.

Proof of Theorem 2: Initially, from (3.21) and (3.20) we note that the condition about $Q_f(\gamma)$ and hypothesis (a) assure us that the second variation $\frac{d^2}{ds^2}\mathcal{F}_f^{\lambda(\gamma_j)}(0)$ of the weighted Jacobi functional $\mathcal{F}_f^{\lambda(\gamma_j)}$ is nonsingular for $j \in \{1, 2\}$. On the other hand, we observe that hypothesis (b) assures us that the eigenvalue of the weighted Jacobi operator which corresponds to $j = j_0$ admits a change of the sign between γ_1 and γ_2. Moreover, as the eigenvalues of the one-parameter family of weighted Jacobi functionals are ordered, we can ensure that the number of negative eigenvalues between γ_1 and γ_2 has changed. Therefore,

$$\mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma_1)}, \Omega_{\gamma_1}) \neq \mathrm{Ind}_f(\mathcal{F}_f^{\lambda(\gamma_2)}, \Omega_{\gamma_2})$$

and the result follows. □

When M^n is closed, the weighted Killing warped product $M^n_f \times_\rho \mathbb{R}$ naturally admits a family of open subsets that can be realized as critical points of the weighted area functional $\mathcal{A}_f$ defined in (3.5). To visualize this, for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, we consider the family of open subsets $\{\Omega_\gamma\}_{\gamma\in(t_1,t_2]}$ given by

$$\Omega_\gamma = M^n \times (t_1, \gamma), \quad \gamma \in (t_1, t_2], \tag{4.2}$$

whose boundary $\partial\Omega_\gamma$ of each Ω_γ is formed by the disjoint union

$$\partial\Omega_\gamma = \Sigma_1^n \cup \Sigma_2^n(\gamma)$$

of a fixed set $\Sigma_1^n = M^n \times \{t_1\}$ and other set $\Sigma_2^n(\gamma) = M^n \times \{\gamma\}$. From Remark 1 we have that each $\Sigma_2^n(\gamma)$, $\gamma \in (t_1, t_2]$, is an f-minimal totally geodesic closed hypersurface. So, since the variations of $\partial\Omega_\tau$ only affect $\Sigma_2^n(\gamma)$, from Remarks 4 and 5 we conclude that each element of the family $\Omega_{\gamma\in(t_1,t_2]}$ is a critical point of $\mathcal{A}_f$. For these critical points, noting that ∂_t is the vector field on $M^n_f \times_\rho \mathbb{R}$ that determines the orientation of each $\Sigma_2^n(\gamma)$, $\gamma \in (t_1, t_2]$, we have that the second variation of the weighted Jacobi functional $\mathcal{F}_f^0 = \mathcal{A}_f$ and the weighted Jacobi operator on each $\partial\Omega_\gamma$, given by the expressions (3.20) and (3.21), are reduced to

$$\frac{d^2}{ds^2}\mathcal{A}_f(0)(u) = -\int_{\Sigma_2(\gamma)} u\,\mathfrak{J}_{f;\gamma}(u)\,d\sigma$$

and

$$\mathfrak{J}_{f;\gamma}(u) = \Delta_{f;\gamma}(u) - \frac{1}{\rho}\widetilde{\Delta}_f(\rho)\,u$$

for any $u \in C^\infty(\Sigma_2^n(\gamma))$, respectively, where $\Delta_{f;\gamma}$ represents the f-Laplacian on $\Sigma_2^n(\gamma)$, $\widetilde{\Delta}_f$ is the f-Laplacian on M^n_f, $\rho = |Y| > 0$, and Y is the Killing vector field that determines the foliation on $M^n_f \times_\rho \mathbb{R}$ by totally geodesic closed slices $M^n \times \{t\}$, $t \in \mathbb{R}$. In addition, if ρ is an eigenfunction of $\widetilde{\Delta}_f$, with associated eigenvalue c, we have that $\mathfrak{J}_{f;\gamma}$ can be written simply as

$$\mathfrak{J}_{f;\gamma} = \Delta_{f;\gamma} + c.$$

In this scenario, we observe that the arguments of the proofs of Theorems 1 and 2 are valid, and even more, the statements can be refined in the sense that we now ask as hypotheses a certain behavior of the spectrum of the drift Laplacian $\widetilde{\Delta}_f$ of the closed manifold M^n_f.

Corollary 1. *Let M^n be an n-dimensional closed Riemannian manifold and, for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, let $\Omega_{\gamma\in(t_1,t_2]}$ be the family of open subsets of the weighted Killing warped product $M^n_f \times_\rho \mathbb{R}$ given by (4.2). Let $\widetilde{\Delta}_f$ be the f-Laplacian on M^n_f. If ρ is an eigenfunction of $\widetilde{\Delta}_f$ (with associated eigenvalue c) and the first nonzero eigenvalue $\mu^1_f(\gamma)$ of the f-Laplacian $\Delta_{f;\gamma}$ on $\Sigma_2(\gamma) = M^n \times \{\gamma\}$, $\gamma \in (t_1, t_2]$, satisfies*

$$\mu^1_f(\gamma) > c,$$

then $\{\Omega_\gamma\}_{\gamma\in(t_1,t_2]}$ is locally rigid at each $\gamma \in (t_1, t_2]$.

Proof: Initially, it is immediate to note that the function $Q_f(\gamma)$ of Theorem 1 reduces to the nonnegative constant c. Then, as in the steps of the proof of Theorem 1, we make an analysis of the eigenvalues of $\mathfrak{J}_{f;\gamma}$ that contribute to $\mathrm{Ind}_f(\mathcal{A}_f, \Omega_\gamma)$ and the result follows. □

Remark 7. Considering once more the behavior of the eigenvalues of the f-Laplacian $\Delta_{f;\gamma}$ on an arbitrary closed weighted manifold M^n_f, from Corollary 1 we obtain the following consequence: The family of open subsets of the weighted product $M^n_f \times \mathbb{R}$ given by (4.2) is always locally rigid at each $\gamma \in (t_1, t_2]$.

Thinking similarly, from Theorem 2 we obtain the following result.

Corollary 2. *Let M^n be an n-dimensional closed Riemannian manifold and, for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, let $\Omega_{\gamma\in(t_1,t_2]}$ be the family of open subsets of the weighted Killing warped product $M^n_f \times_\rho \mathbb{R}$ given by (4.2). Let $\widetilde{\Delta}_f$ be the f-Laplacian on M^n_f. If ρ is an eigenfunction of $\widetilde{\Delta}_f$ (with associated eigenvalue c) and if there are two values $\gamma_1, \gamma_2 \in (t_1, t_2]$, with $\gamma_1 < \gamma_2$, such that the eigenvalues $\widehat{\mu}^j_f(\gamma_1)$ and $\widehat{\mu}^j_f(\gamma_2)$ of the Jacobi operators $\mathfrak{J}_{f;\gamma_1}$ and $\mathfrak{J}_{f;\gamma_2}$ (respectively) satisfy*

(a) *$\widehat{\mu}^j_f(\gamma_1) \neq 0$ and $\widehat{\mu}^j_f(\gamma_2) \neq 0$ for all $j \in \{0, 1, 2, \dots\}$, and*

(b) *there exists $j_0 \in \{0, 1, 2, \dots\}$ such that $(\widehat{\mu}^{j_0}_f(\gamma_1))(\widehat{\mu}^{j_0}_f(\gamma_2)) < 0$,*

then there exists a bifurcation instant $\gamma_* \in (\gamma_1, \gamma_2)$.

5. Stability of H_f-hypersurfaces in $M^n_f \times_\rho \mathbb{R}$

It is important to remark that, for all calculations in Section 3, there is no real dependence on the open set $\Omega \in \mathcal{M}$ but on the hypersurface $\partial\Omega$. In fact, in the literature it is more common to work in terms of hypersurfaces (for instance, see [**5, 6**] for the classical context, and [**11, 22**] for the weighted context). In this scenario, $\mathcal{M}$ becomes the space of all closed orientable hypersurfaces of $M^n_f \times_\rho \mathbb{R}$.

In this last section we study the notion of stability associated with problem (VP-1) described in Section 3 for this new set $\mathcal{M}$. We begin this study by recalling that if $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ is such a hypersurface, then the weighted volume and weighted area associated with a variation $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to M^n_f \times_\rho \mathbb{R}$ are given by

$$\begin{aligned} \mathcal{V}_f\colon (-\epsilon,\epsilon) &\to \mathbb{R} \\ s &\mapsto \mathcal{V}_f(s) = \mathrm{Vol}_f(\Sigma^n \times [0,s]) = \int_{\Sigma^n\times[0,s]} X^*(d\overline{\sigma}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_f\colon (-\epsilon,\epsilon) &\to \mathbb{R} \\ s &\mapsto \mathcal{A}_f(s) = \mathrm{Area}_f(X_s(\Sigma^n)) = \int_{\Sigma^n} d\sigma_s, \end{aligned}$$

respectively. Furthermore, the variational problem of minimizing the functional $\mathcal{A}_f$ for all variations of $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ that preserve the weighted volume $\mathcal{V}_f$ is addressed by the study of the weighted Jacobi functional

$$\begin{aligned} \mathcal{F}_f\colon (-\epsilon,\epsilon) &\to \mathbb{R} \\ s &\mapsto \mathcal{F}_f(s) = \mathcal{A}_f(s) + n\overline{\mathcal{H}}\,\mathcal{V}_f(s), \end{aligned}$$

where $\overline{\mathcal{H}}$ is the constant defined in (3.8), and their respective critical points are the closed H_f-hypersurfaces of $M^n_f \times_\rho \mathbb{R}$. For these critical points, the stability of the corresponding variational problem is given by the second variation

$$\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) = -\int_{\Sigma^n} u\,\mathcal{J}_f(u)\,d\sigma,$$

where $\mathcal{J}_f\colon C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ is the weighted Jacobi operator given in (3.14). The above discussion motivates the following notion of stability.

We say that a closed H_f-hypersurface $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ is *f-stable* if

$$\frac{d^2}{ds^2}\mathcal{A}_f(0) \geq 0$$

for all weighted volume-preserving variations $X\colon \Sigma^n \times (-\epsilon,\epsilon) \to M^n_f \times_\rho \mathbb{R}$ of $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$.

Remark 8. Let $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ be a closed H_f-hypersurface as described in the last definition above. We consider the set

$$\mathcal{G} = \left\{ u \in C^\infty(\Sigma^n) : \int_{\Sigma^n} u\, d\sigma = 0 \right\}. \tag{5.1}$$

Just as in [**5**], we can establish the following criterion of f-stability: a hypersurface $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ is f-stable if and only if $\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) \geq 0$ for all $u \in \mathcal{G}$.

In what follows, associated with a hypersurface $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ we will consider a particular smooth function, namely, the *angle function*

$$\begin{aligned} \Theta\colon \Sigma^n &\to \mathbb{R} \\ p &\mapsto \Theta(p) = \langle N(p), Y(p) \rangle, \end{aligned} \tag{5.2}$$

where N is the normal vector field on Σ^n that determines its orientation and Y is the Killing vector field on $M^n_f \times_\rho \mathbb{R}$. In this setting, we get the following key lemma, which provides sufficient conditions to obtain a eigenfunction of the drift Laplacian Δ_f on Σ^n. Let us denote by $\overline{\nabla}$, ∇, and $\widetilde{\nabla}$ the Levi–Civita connections of $M^n_f \times_\rho \mathbb{R}$, Σ^n and M^n, respectively.

Proposition 3. *Let $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ be a hypersurface immersed into weighted Killing warped product $M^n_f \times_\rho \mathbb{R}$. If $\Theta \in C^\infty(\Sigma)$ is the function defined in* (5.2)*, then*

$$\begin{aligned} \Delta_f \Theta + \left\{ \widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*, N^*) - \Theta^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2 \right\} \Theta \\ = -nY^\top(H_f), \end{aligned}$$

where we are using the same notations of Proposition 2. In addition, if Σ^n is closed and both H_f and

$$\mu = \widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*, N^*) - \Theta^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2$$

are constants, then μ is an eigenvalue of Δ_f on Σ^n with eigenfunction Θ.

Proof: Firstly, from (2.3) we note that

$$\begin{aligned} -nY^\top(H) &= -\,Y^\top(nH_f - \langle \overline{\nabla} f, N \rangle) \\ &= -\,nY^\top(H_f) + Y^\top \langle \overline{\nabla} f, N \rangle \\ &= -\,nY^\top(H_f) + \overline{\mathrm{Hess}}\, f(Y, N) \\ &\quad -\Theta \overline{\mathrm{Hess}}\, f(N, N) - \langle AY^\top, \overline{\nabla} f \rangle. \end{aligned} \tag{5.3}$$

Moreover, with a straightforward computation we can show that

$$\nabla\Theta = -AY^\top - (\overline{\nabla}_N Y)^\top$$

and, since f is invariant along the flow determined by Y, we get

$$\begin{aligned}\langle\nabla\Theta, \overline{\nabla} f\rangle &= -\langle AY^\top + (\overline{\nabla}_N Y)^\top, \overline{\nabla} f\rangle \\ &= -\langle AY^\top, \overline{\nabla} f\rangle - \langle\overline{\nabla}_N Y, \overline{\nabla} f\rangle \\ &= -\langle AY^\top, \overline{\nabla} f\rangle + \langle Y, \overline{\nabla}_N \overline{\nabla} f\rangle \\ &= -\langle AY^\top, \overline{\nabla} f\rangle + \overline{\operatorname{Hess}} f(Y, N).\end{aligned} \tag{5.4}$$

Taking into account equations (5.3) and (5.4) we get

$$-nY^\top(H) = -nY^\top(H_f) - \Theta\overline{\operatorname{Hess}} f(N,N) + \langle\nabla\Theta, \overline{\nabla} f\rangle. \tag{5.5}$$

On the other hand, from Proposition 2.12 of [**6**] we have

$$\Delta\Theta = -nY^\top(H) - \Theta(\overline{\operatorname{Ric}}(N,N) + |A|^2). \tag{5.6}$$

Therefore, from (2.2), (2.4), (3.18), (5.6), and (5.5) we obtain the result. □

Our stability result stated in Theorem 3 gives us a characterization of f-stable H_f-hypersurfaces in $M^n_f \times_\rho \mathbb{R}$ through the first eigenvalue of the drift Laplacian Δ_f, which extends a classic result of Barbosa, do Carmo, and Eschenburg (see Proposition 2.13 of [**6**]).

Proof of Theorem 3: Since μ is constant, Proposition 3 guarantees that μ is in the spectrum of the drift Laplacian Δ_f. So, let μ_1 be the first eigenvalue of Δ_f on Σ^n. If $\mu = \mu_1$, then the variational characterization of λ_1 (see, for instance, Section 1 of [**7**]) gives

$$\mu = \min_{u\in\mathcal{G}\setminus\{0\}} \frac{-\displaystyle\int_{\Sigma^n} u\Delta_f(u)\,d\sigma}{\displaystyle\int_{\Sigma^n} u^2\,d\sigma},$$

where $\mathcal{G}$ is defined in (5.1). Then, from (3.13) and (3.14) we obtain

$$\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) = \int_{\Sigma^n}\{-u\Delta_f(u) - \mu u^2\}\,d\sigma \ge (\mu - \mu)\int_{\Sigma^n} u^2\,d\sigma = 0,$$

for any $u \in \mathcal{G}$ and, according to Remark 8, $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ is f-stable.

Now suppose that $x\colon \Sigma^n \hookrightarrow M^n_f \times_\rho \mathbb{R}$ is f-stable, which according to Remark 8 is equivalent to $\frac{d^2}{ds^2}\mathcal{F}_f(0)(u) \ge 0$ for all $u \in \mathcal{G}$. Let u

be an eigenfunction associated to the first eigenvalue μ_1 of the drift Laplacian Δ_f on Σ^n. Consequently, by (3.13) and (3.14) we get

$$0 \leq \frac{d^2}{ds^2}\mathcal{F}_f(0)(u) = (\mu_1 - \mu)\int_{\Sigma^n} u^2\, d\sigma.$$

Therefore, since $\mu_1 \leq \mu$, we must have $\mu_1 = \mu$. □

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