

SUMMABILITY IN A MONOMIAL FOR SOME CLASSES OF SINGULARLY PERTURBED PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: The aim of this paper is to continue the study of asymptotic expansions and summability in a monomial in any number of variables, as introduced in [3, 15]. In particular, we characterize these expansions in terms of bounded derivatives and we develop Tauberian theorems for the summability processes involved. Furthermore, we develop and apply the Borel–Laplace analysis in this framework to prove the monomial summability of solutions of a specific class of singularly perturbed PDEs.

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1. Introduction

The theory of asymptotic expansions in one complex variable is a well established and widely used branch of Analysis. It provides the adequate setting to treat solutions of analytic problems at singular points and opens naturally a way to study (Borel) summability of formal solutions, for instance, power series or exponential series. It finds its applications in some classes of ordinary and partial differential equations, analytic classification of formal objects, and some other classes of functional equations. In this framework we have at our disposal two main tools to approach such problems: the Ramis–Sibuya theorem and the Borel–Laplace analysis.

There are different notions of asymptotic expansions in several variables available in the literature. We can mention Majima’s strong asymptotic expansions which allows us to work with each variable independently, although the problem of identifying singular directions for summability persists; see e.g., [18]. In the Gevrey case this notion can also be approached through a Borel–Laplace analysis. Let us mention that in [10] this has been used as a regularization process to prove the

existence and “summability” of solutions for classes of nonlinear evolution partial differential equations.

On the other hand, monomial asymptotic expansions and monomial summability lie in between the theory of one variable and the one of Majima. The concept emerged naturally in several problems, including the study of certain systems of differential equations by H. Majima and W. Wasow, and in the analytic classification of resonant singularities of holomorphic foliations in two variables given by J. Martinet and J.-P. Ramis; see [15, Introduction] for more details. Later, these notions were formalized and developed systematically in [3] for the case of two variables, and then in [15] for higher dimensions. Furthermore, they were applied successfully to doubly singular ordinary differential equations, i.e., that are both singularly perturbed and exhibit an irregular singular point; see equation (33). In particular, the monomial involved identifies singular directions for summability of the formal solution, and thus helps to find large domains where actual holomorphic solutions of the problem exist.

In simple terms and leaving technical details for later, monomial summability formalizes the idea of studying summability – Borel, k -Borel, or k -summability defined using Gevrey asymptotic expansions – of formal power series $\hat{f}(x_1, \dots, x_d)$ but with respect to a variable $t = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, for some positive integers α_j . In other words, the source of the divergence of \hat{f} can be treated by putting $\hat{f}(x_1, \dots, x_d) = \hat{g}(x_1^{\alpha_1} \cdots x_d^{\alpha_d})(x_1, \dots, x_d)$, where $\hat{g}(t)$ is a formal power series in t with coefficients holomorphic functions in x_1, \dots, x_d in a certain Banach space. Then, the summability of \hat{f} is addressed as the summability of $\hat{g}(t)$ with respect to t . In this way, if $g(t)$ is a sum for $\hat{g}(t)$, we associate to \hat{f} the sum $f(x_1, \dots, x_d) = g(x_1^{\alpha_1} \cdots x_d^{\alpha_d})(x_1, \dots, x_d)$ which will represent \hat{f} asymptotically in adequate domains. This approach provides a way to understand more intricate divergence phenomena which are inaccessible when considering each variable x_j separately.

Our goal in this paper is to follow the sketch we present in Section 2 on the theory of asymptotic expansions in one variable to provide analogous results for the monomial case. The main theoretical results we obtain here are: a characterization of having a monomial asymptotic expansion in terms of bounded derivatives (Theorem 3.8), equivalent methods to establish monomial summability based on integral transformations without passing to the variable t (Theorem 4.10), and Tauberian theorems comparing such summability processes (Theorem 5.5).

For the two-dimensional case the last two issues were treated in [5, 6]. The key idea was to weigh the variables adequately to obtain integral

transformations, as the ones introduced in [2], and then to have at hand a direct Borel–Laplace analysis. One of the improvements we obtain here is that some of the weights can be zero. At first look, it might seem that the zero weighted variables act as regular parameters. The main difference with the parametric case is that the domains where the asymptotic expansions take place also depend on them. In fact, the summability methods involved are incompatible (Theorem 5.8).

As an application, we establish in Theorem 6.1 the $\mathbf{x}^\alpha \varepsilon^{\alpha'}$ -1-summability of the unique formal power series solution of the partial differential equation

$$(1) \quad \mathbf{x}^\alpha \varepsilon^{\alpha'} \left(\mu_1 x_1 \frac{\partial \mathbf{y}}{\partial x_1} + \cdots + \mu_n x_n \frac{\partial \mathbf{y}}{\partial x_n} \right) = \mathbf{G}(\mathbf{x}, \varepsilon, \mathbf{y}),$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ are complex variables, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha' = (\alpha'_1, \dots, \alpha'_m)$ are tuples of positive integers, (μ_1, \dots, μ_n) is, up to a non-zero multiple scalar, an n -tuple of positive real numbers, \mathbf{G} is a \mathbb{C}^N -valued holomorphic function in a neighborhood of the origin, and $\frac{\partial \mathbf{G}}{\partial \mathbf{y}}(0, 0, \mathbf{0})$ is an invertible matrix. In this way, we have generalized the results in [3, 6] corresponding to the case $n = m = 1$ by using directly the appropriate Borel–Laplace analysis.

Asymptotic expansions and summability have been recently generalized by J. Mozo and R. Schäfke in [15] from monomials to germs of analytic functions. Integral transformations for the corresponding summability methods are still not available and it is an interesting problem to determine whether it is possible to extend our results to that setting. It is worth mentioning that after our current results, we have recently extended these Tauberian theorems for k -summability with respect to analytic germs; see [7].

The plan for the paper is as follows: in Sections 2 and 3 we recall the basic results on asymptotic expansions and summability in one variable and for monomials, respectively. Section 4 is devoted to introduce and develop integral transformations to characterize monomial summability, and then in Section 5 these tools are applied to prove Tauberian theorems for these summability methods. Finally, Section 6 contains the proof of the monomial summability of the formal solution of the singularly perturbed partial differential equation mentioned above.

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2. Asymptotics and summability in one variable

We start by introducing some notation: Let \mathbb{N} denote the set of natural numbers including 0 and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Domains in the complex plane \mathbb{C} where holomorphic maps admit an asymptotic expansion are sectors with vertex at some fixed point, e.g., the origin. In this paper we denote them by $S = S(\theta, b - a, r) = V(a, b, r) = \{x \in \mathbb{C} : 0 < |x| < r, a < \arg(x) < b\}$ emphasizing on its *bisecting direction* $\theta = (b + a)/2$, *opening* $b - a > 0$, and *radius* $r > 0$. For unbounded sectors we simply write $S = S(\theta, b - a)$. For subsectors $S' = S(\theta', b' - a', r')$, $a < a' < b' < b$, $0 < r' < r$, we write $S' \Subset S$. We also denote by D_r the disc of radius r centered at the origin.

Let $(E, \|\cdot\|)$ be a complex Banach space. In most applications E is \mathbb{C}^d , for some $d \geq 1$, or a suitable space of functions. We will use the notation $\mathcal{C}(U, E)$ (resp. $\mathcal{O}(U, E)$, $\mathcal{O}_b(U, E)$) for the space of continuous (resp. holomorphic, holomorphic and bounded) E -valued maps defined on an open set $U \subseteq \mathbb{C}^d$. If $E = \mathbb{C}$, we will simply write $\mathcal{O}(U)$. We also denote by $E[[x]]$ (resp. $E\{x\}$) the space of formal (resp. convergent) power series in the variable x with coefficients in E .

Consider $f \in \mathcal{O}(S, E)$ and assume it has $\hat{f} = \sum_{n=0}^{\infty} a_n x^n \in E[[x]]$ as *asymptotic expansion* at the origin on S (denoted by $f \sim \hat{f}$ on S), i.e., for each $S' \Subset S$ and $N \in \mathbb{N}$, there exists $C_N(S') > 0$ such that

$$(2) \quad \left\| f(x) - \sum_{n=0}^{N-1} a_n x^n \right\| \leq C_N(S') |x|^N \quad \text{on } S'.$$

To check that $f \sim \hat{f}$ on S , it is actually sufficient to have inequality (2) only for the values $N = Mp$, where $p \in \mathbb{N}^+$ is fixed. The asymptotic expansion also holds if instead of the partial sums of \hat{f} we consider a sequence $(f_N)_{N \in \mathbb{N}} \subset \mathcal{O}_b(D_R, E)$ satisfying that for each $S' \Subset S$ and $N \in \mathbb{N}$, there are constants $A_N(S') > 0$ such that

$$\|f(x) - f_N(x)\| \leq A_N(S') |x|^N \quad \text{on } S' \cap D_R.$$

The series \hat{f} is completely determined by f since $a_n = \lim_{S' \ni x \rightarrow 0} \frac{f^{(n)}(x)}{n!}$ for any $S' \Subset S$. The series \hat{f} is also given by the limit of the Taylor series at the origin of the f_n , in the \mathfrak{m} -topology of $E[[x]]$, $\mathfrak{m} = (x)$. On the other hand, a map has an asymptotic expansion on S if and only if it has bounded derivatives of all orders on every $S' \Subset S$, a fact that follows by using Taylor's formula.

When no restrictions on the coefficients $C_N(S')$ or on the sector S are imposed, the map $f \mapsto \hat{f}$ is not injective. In applications to differential equations the types of asymptotic expansions that appear are of s -Gevrey kind (denoted by $f \sim_s \hat{f}$ on S), for some $s > 0$. This means that we can choose $C_n(S') = C(S')A(S')^n n!^s$, for some $C(S'), A(S') > 0$ independent of n . It follows from (2) that $\|a_n\| \leq C_n(S')$ for all $n \in \mathbb{N}$. Then, in the s -Gevrey case, we conclude that \hat{f} is an s -Gevrey series. The space of such series will be denoted by $E[[x]]_s$. The cornerstone to define k -summability is Watson's lemma: if $f \sim_s 0$ on $S(\theta, b - a, r)$ and $b - a > s\pi$, then $f \equiv 0$.

Given $\hat{f} \in E[[x]]$, $k > 0$, and a direction $\theta \in \mathbb{R}$, we say that:

- (1) The series \hat{f} is k -summable on $S = S(\theta, b - a, r)$ with sum $f \in \mathcal{O}(S, E)$ if $b - a > \pi/k$ and $f \sim_{1/k} \hat{f}$ on S . We also say that \hat{f} is k -summable in direction θ . The corresponding space of such series is denoted by $E\{x\}_{1/k, \theta}$.
- (2) The series \hat{f} is k -summable if it is k -summable in all directions, except possibly for a finite number of them modulo 2π (the singular directions). The corresponding space is denoted by $E\{x\}_{1/k}$.

Due to Watson's lemma, the k -sum of a k -summable series is unique. We have at our disposal integral transformations to compute these sums. Among the kernels of order k for moment summability, see e.g., [1, Section 6.5], it is common to consider:

- (1) *The k -Borel transform*, defined by $\mathcal{B}_k f(\xi) = \frac{k}{2\pi i} \int_\gamma f(x) e^{(\xi/x)^k} \frac{dx}{x^{k+1}}$, where $f \in \mathcal{O}_b(S, E)$, $S = S(\theta, \pi/k + 2\epsilon, R_0)$, $0 < 2\epsilon < \pi/k$, and γ is the boundary, oriented positively, of a subsector of S of opening larger than π/k . Its formal counterpart $\widehat{\mathcal{B}}_k$ acts on monomials by the formula $\widehat{\mathcal{B}}_k(x^\lambda)(\xi) = \frac{\xi^{\lambda-k}}{\Gamma(\lambda/k)}$, $\lambda \in \mathbb{C}$.
- (2) *The k -Laplace transform in direction θ* , defined by $\mathcal{L}_{k, \theta}(g)(x) = \int_0^{e^{i\theta} \infty} g(\xi) e^{-(\xi/x)^k} d\xi^k$, where g is continuous and has exponential growth of order at most k on the domain of integration. If g is defined on an unbounded sector, we obtain a map $\mathcal{L}_k(g)$ through analytic continuation by moving θ .

Using these transformations, and due to their compatibility with asymptotic expansions, a $1/k$ -Gevrey series $\hat{f} = \sum_{n=0}^{\infty} a_n x^n$ is called k -Borel summable in direction θ if $\widehat{\mathcal{B}}_k(\hat{f} - \sum_{n \leq k} a_n x^n)$ can be analytically continued, say as φ , to an unbounded sector S' containing θ , and having exponential growth of order at most k in S' , i.e., we can find constants $L, M > 0$ such that $\|\varphi(\xi)\| \leq L e^{M|\xi|^k}$ for all $\xi \in S'$. The

k -Borel sum of \hat{f} is defined by $f(x) = \sum_{n \leq k} a_n x^n + \mathcal{L}_k(\varphi)(x)$. It is well-known that a power series $\hat{f} \in E[[x]]$ is k -Borel summable in direction θ if and only if it is k -summable in direction θ and both sums coincide; see e.g., [16]. This equivalence is useful also to prove Tauberian theorems for k -summability. In particular, we know that if a series is k -summable for two different values of the parameter k , then it is convergent.

The Borel–Laplace analysis has been applied as a regularization process in differential equations to prove the summability of formal solutions in generic situations. It exploits the isomorphism between the following structures

$$(3) \quad (E[[x]]_{1/k}, +, \times, x^{k+1} d/dx) \xrightarrow{\widehat{\mathcal{B}}_k} (\xi^{-k} E\{\xi\}, +, *_k, k\xi^k(\cdot)),$$

where \times denotes the usual product and $*_k$ stands for the k -convolution product. For holomorphic maps, it is given by $(f *_k g)(\xi) = \xi^k \int_0^1 f(\xi\tau^{1/k}) g(\xi(1-\tau)^{1/k}) d\tau$. For more morphisms of this nature see, e.g., [13].

3. Asymptotic expansions in a monomial

In this section we recall the concepts of asymptotic expansions and k -summability in a monomial, and their main properties. In particular, we prove Theorem 3.8 that characterizes maps admitting a monomial asymptotic expansion in terms of bounds on their derivatives.

We introduce the remaining notation we will use along the text. For a fixed $d \in \mathbb{N}^+$, we will write $[1, d]$ for the set $\{1, 2, \dots, d\}$, $\mathbf{e}_1, \dots, \mathbf{e}_d$ will denote the canonical basis of \mathbb{C}^d , $\sigma_d = \{(t_1, \dots, t_d) \in \mathbb{R}_{>0}^d : t_1 + \dots + t_d = 1\}$ will be the standard d -simplex they generate, and $\overline{\sigma_d}$ will denote its topological closure. We will also write $\langle \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle = \lambda_1 \mu_1 + \dots + \lambda_d \mu_d$ for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{C}^d$.

We use complex coordinates $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{C}^d$. If $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}_{\geq 0}^d$, we use the multi-index notation $|\boldsymbol{\beta}| = \beta_1 + \dots + \beta_d$, $\boldsymbol{\beta}!^{\mathbf{s}} = \beta_1!^{s_1} \dots \beta_d!^{s_d}$, $\mathbf{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} \dots x_d^{\beta_d}$, and $\frac{\partial^{\boldsymbol{\beta}}}{\partial \mathbf{x}^{\boldsymbol{\beta}}} = \partial^{|\boldsymbol{\beta}|} / \partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}$. If $J \subseteq [1, d]$, we denote by $J^c = \{i \in [1, d] : i \notin J\}$ its complement, $\#J$ its cardinal, and we write $\mathbb{N}^J = \{(\beta_j)_{j \in J} \mid \beta_j \in \mathbb{N}, j \in J\}$, $\mathbf{x}_J = (x_j)_{j \in J}$, $\boldsymbol{\beta}_J = (\beta_j)_{j \in J}$, and $\mathbf{x}_J^{\boldsymbol{\beta}_J} = \prod_{j \in J} x_j^{\beta_j}$. Along the text we work with the partial order on \mathbb{N}^d defined by $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ if and only if $\beta_j \leq \alpha_j$ for all $j \in [1, d]$. We will also write $\boldsymbol{\beta} < \boldsymbol{\alpha}$ if $\beta_j < \alpha_j$ for all $j \in [1, d]$. Note that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$ if and only if there is $j \in [1, d]$ such that $\beta_j > \alpha_j$.

Given a complex Banach space $(E, \|\cdot\|)$, $E[[\mathbf{x}]]$ (resp. $E\{\mathbf{x}\}$) will denote the space of formal (resp. convergent) power series in the variables \mathbf{x} with coefficients in E . If $\mathbf{s} \in \mathbb{R}_{\geq 0}^d$, we denote by $E[[\mathbf{x}]]_{\mathbf{s}}$ the space

of s -Gevrey series in the variable \mathbf{x} , i.e., $\sum_{\beta \in \mathbb{N}^d} a_\beta \mathbf{x}^\beta$ is s -Gevrey if there exist constants $C, A > 0$ such that $\|a_\beta\| \leq CA^{|\beta|} \beta!^s$ for all $\beta \in \mathbb{N}^d$.

Given any $\hat{f} = \sum_{\beta \in \mathbb{N}^d} a_\beta \mathbf{x}^\beta \in E[[\mathbf{x}]]$, we can write \hat{f} uniquely for every nonempty subset $J \subsetneq [1, d]$ as

$$(4) \quad \hat{f} = \sum_{\beta_J \in \mathbb{N}^J} \hat{f}_{J, \beta_J}(\mathbf{x}_{J^c}) \mathbf{x}_{J^c}^{\beta_J}, \quad \hat{f}_{J, \beta_J}(\mathbf{x}_{J^c}) = \sum_{\beta_{J^c} \in \mathbb{N}^{J^c}} a_{\beta_{J \cup J^c}} \mathbf{x}_{J^c}^{\beta_{J^c}}.$$

Furthermore, if $\alpha \in (\mathbb{N}^+)^d$ is given and we consider the monomial \mathbf{x}^α , \hat{f} can be also written uniquely as

$$(5) \quad \hat{f} = \sum_{n=0}^{\infty} \hat{f}_{\alpha, n}(\mathbf{x}) \mathbf{x}^{n\alpha}, \quad \hat{f}_{\alpha, n} = \sum_{\alpha \leq \beta} a_{n\alpha + \beta} \mathbf{x}^\beta.$$

To ensure that each $\hat{f}_{\alpha, n} = f_{\alpha, n}$ gives rise to a holomorphic map, defined in a common polydisc at the origin for all $n \in \mathbb{N}$, it is necessary and sufficient that $\hat{f} \in \widehat{\mathcal{O}}'_d(E) := \bigcup_{r>0} \widehat{\mathcal{O}}'_d(r, E)$, where $\widehat{\mathcal{O}}'_d(r, E) := \bigcap_{j=1}^d \mathcal{O}_b(D_r^{d-1}, E)[[x_j]]$. If this is the case, then $\hat{f}_{J, \beta_J} = f_{J, \beta_J} \in E\{\mathbf{x}_{J^c}\}$ for all $\beta_J \in \mathbb{N}^J$ and $J \subsetneq [1, d]$, and they are defined in a common polydisc at the origin. Besides $f_{\alpha, n} \in \mathcal{E}^\alpha := \bigcup_{r>0} \mathcal{E}_r^\alpha$, where \mathcal{E}_r^α is the space of holomorphic maps $g \in \mathcal{O}_b(D_r^d, E)$ such that $\frac{\partial^\beta g}{\partial \mathbf{x}^\beta}(\mathbf{0}) = 0$ for all $\alpha \leq \beta$. Also, each \mathcal{E}_r^α becomes a Banach space with the norm $\|g\|_r := \sup_{|x_1|, \dots, |x_d| \leq r} \|g(\mathbf{x})\|$.

For any $\hat{f} \in \widehat{\mathcal{O}}'_d(E)$ and $\gamma \in \mathbb{N}^d$, we will use the notation

$$\text{App}_\gamma(\hat{f})(\mathbf{x}) := \sum_{\gamma \leq \beta} a_\beta \mathbf{x}^\beta \in E\{\mathbf{x}\}$$

for the *formal approximate of \hat{f} of order γ* . In particular, if $\gamma = N\alpha$, we have

$$(6) \quad \text{App}_{N\alpha}(\hat{f})(\mathbf{x}) = \sum_{n=0}^{N-1} f_{\alpha, n}(\mathbf{x}) \mathbf{x}^{n\alpha}.$$

Remark 3.1. For further use, we remark the following bounds on factorials. First, it is elementary to show that

$$(7) \quad n!^k \leq (kn)! \leq k^{kn} n!^k, \quad n, k \in \mathbb{N}^+.$$

Let us fix $\alpha \in (\mathbb{N}^+)^d$ and consider $\gamma \in \mathbb{N}^d \setminus \{\mathbf{0}\}$. If $N = \min_{1 \leq j \leq d} \lfloor \gamma_j / \alpha_j \rfloor = \lfloor \gamma_l / \alpha_l \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function, then $N \leq \gamma_l / \alpha_l < N + 1$ and $N \leq \gamma_j / \alpha_j$ for all $j = 1, \dots, d$. Then, using (7), the second inequality shows that $N!^{\alpha_j} \leq (\alpha_j N)! \leq \gamma_j!$. Also, the first inequality

shows that $\gamma_l^{1/\alpha_l} \leq (\alpha_l(N+1))^{1/\alpha_l} \leq \alpha_l^{N+1}(N+1)! \leq |\alpha|^{N+1}2^N N! = |\alpha|(2|\alpha|)^N N! \leq |\alpha|(2|\alpha|)^{|\gamma|} N!$, since $N \leq |\gamma|$. In conclusion,

$$|\alpha|^{-1}(2|\alpha|)^{-|\gamma|} \min_{1 \leq j \leq d} \gamma_j^{1/\alpha_j} \leq N! \leq \min_{1 \leq j \leq d} \gamma_j^{1/\alpha_j}.$$

Analogously, if we consider $N = \max_{1 \leq j \leq d} \lfloor \gamma_j / \alpha_j \rfloor + 1 = \lfloor \gamma_m / \alpha_m \rfloor + 1$ instead, now we have $\gamma_j / \alpha_j < N \leq \gamma_m / \alpha_m + 1$ for all $j = 1, \dots, d$. Then $N!^{\alpha_m} \leq (\alpha_m N)! \leq (\gamma_m + \alpha_m)! \leq 2^{\gamma_m + \alpha_m} \gamma_m! \alpha_m!$. Using that $N \leq 2|\gamma|$ and $\alpha_m^{1/\alpha_m} \leq \alpha_m \leq |\alpha|$, we can conclude as before that

$$|\alpha|^{-2|\gamma|} \max_{1 \leq j \leq d} \gamma_j^{1/\alpha_j} \leq N! \leq |\alpha|2^{2|\gamma|} \max_{1 \leq j \leq d} \gamma_j^{1/\alpha_j}.$$

For each $\alpha \in (\mathbb{N}^+)^d$, we consider the map $\widehat{T}_\alpha: \widehat{\mathcal{O}}'_d(E) \rightarrow \mathcal{E}^\alpha[[t]]$ given by $\widehat{T}_\alpha(\widehat{f}) = \sum_{n=0}^{\infty} f_{\alpha,n} t^n$, by using decomposition (5). We will say \widehat{f} is an *s-Gevrey series in the monomial \mathbf{x}^α* if for some $r > 0$, $\widehat{T}_\alpha(\widehat{f}) \in \mathcal{E}_r^\alpha[[t]]$, and it is an *s-Gevrey series in t* , i.e., there are constants $B, D > 0$ such that $\|f_{\alpha,n}\|_r \leq DB^n n!^s$ for all $n \in \mathbb{N}$. The space of *s-Gevrey series in \mathbf{x}^α* will be denoted by $E[[\mathbf{x}]]_s^\alpha$. Their elements admit another characterization, for which we need the following:

Lemma 3.2. *The following assertions are verified for a series $\widehat{f} = \sum a_\beta \mathbf{x}^\beta \in E[[\mathbf{x}]]$:*

(1) $\widehat{f} \in E[[\mathbf{x}]]_s^\alpha$ if and only if there are constants $C, A > 0$ satisfying

$$\|a_\beta\| \leq CA^{|\beta|} \min\{\beta_1!^{s/\alpha_1}, \dots, \beta_d!^{s/\alpha_d}\}, \quad \beta \in \mathbb{N}^d.$$

(2) If $\widehat{f} \in E[[\mathbf{x}]]_s^{\alpha'}$, then $\widehat{T}_\alpha(\widehat{f})$ is a $\max_{1 \leq j \leq d} \{\alpha_j / \alpha'_j\}$ -*s-Gevrey series*, in some \mathcal{E}_r^α .

Proof: (1) Assume there are constants $B, D > 0$ such that $\|f_{\alpha,n}\|_r \leq DB^n n!^s$ for all $n \in \mathbb{N}$. Given $\gamma \in \mathbb{N}^d$, let $N = \min_{1 \leq j \leq d} \lfloor \gamma_j / \alpha_j \rfloor$. Thus $\gamma = N\alpha + \beta$ with $\beta_l < \alpha_l$, for some l . Then, by Cauchy's inequalities, we see that

$$\|a_\gamma\| = \|a_{N\alpha + \beta}\| = \left\| \frac{1}{\beta!} \frac{\partial^\beta f_{\alpha,N}}{\partial \mathbf{x}^\beta}(\mathbf{0}) \right\| \leq \frac{DB^N}{r^{|\beta|}} N!^s,$$

which yields one implication with the aid of Remark 3.1. The converse follows by the same argument as in (2) below.

(2) If $\|a_\beta\| \leq CA^{|\beta|} \min_{1 \leq j \leq d} \{\beta_j!^{s/\alpha'_j}\}$, for all $\beta \in \mathbb{N}^d$, we can directly estimate the growth of $f_{\alpha,n}$ by means of equation (5): if $|\mathbf{x}| < r$ and $rA < 1$, we obtain

$$\begin{aligned} \|f_{\alpha,n}(\mathbf{x})\| &= \left\| \sum_{\alpha \preceq \gamma} a_{n\alpha+\gamma} \mathbf{x}^\gamma \right\| \\ &\leq \sum_{j=1}^d \sum_{\beta_j=0}^{\alpha_j-1} \sum_{\gamma \in \mathbb{N}^d, \gamma_j=\beta_j} CA^{n|\alpha|+|\gamma|} r^{|\gamma|} \min_{1 \leq l \leq d} (n\alpha_l + \gamma_l)!^{s/\alpha'_l} \\ &\leq \frac{CA^{n|\alpha|}}{(1-rA)^{d-1}} \sum_{j=1}^d \sum_{\beta_j=0}^{\alpha_j-1} (n\alpha_j + \beta_j)!^{s/\alpha'_j} (rA)^{\beta_j}. \end{aligned}$$

If we write $s' = \max_{1 \leq j \leq d} \{\alpha_j/\alpha'_j\}s$, by using inequalities (7), we find that

$$\begin{aligned} (n\alpha_j + \beta_j)!^{s/\alpha'_j} &\leq (\alpha_j(n+1))!^{s/\alpha'_j} \\ &\leq \alpha_j^{\alpha_j(n+1)} (n+1)!^{s\alpha_j/\alpha'_j} \leq \alpha_j^{\alpha_j(n+1)} (n+1)!^{s'}, \end{aligned}$$

for all $n \in \mathbb{N}$. Then it is clear that we can find constants $K, M > 0$ such that $\|f_{\alpha,n}(\mathbf{x})\| \leq KM^n n!^{s'}$ for all $|\mathbf{x}| < r$ and all $n \in \mathbb{N}$, as we wanted to show. \square

The previous lemma implies that $E[[\mathbf{x}]]_{M_s}^{M\alpha} = E[[\mathbf{x}]]_s^\alpha$ for all $M \in \mathbb{N}^+$. It also shows that $\hat{f} \in E\{\mathbf{x}\}$ if and only if $\hat{T}_\alpha(\hat{f}) \in \mathcal{E}_r^\alpha\{t\}$ for some $r > 0$, by taking $s = 0$. Moreover, Lemma 3.2 (1) shows that

$$(8) \quad E[[\mathbf{x}]]_s^\alpha = \bigcap_{j=1}^d E[[\mathbf{x}]]_{\frac{s}{\alpha_j} e_j} \subseteq E[[\mathbf{x}]]_s,$$

for any \mathbf{s} in the convex hull of $\{s/\alpha_1 \mathbf{e}_1, \dots, s/\alpha_d \mathbf{e}_d\}$. This inclusion follows from the first inequality in

$$(9) \quad \min\{a_1, \dots, a_d\} \leq a_1^{t_1} \cdots a_d^{t_d} \leq \max\{a_1, \dots, a_d\},$$

valid for any $a_1, \dots, a_d > 0$ and $(t_1, \dots, t_d) \in \overline{\sigma_d}$.

In the analytic setting, we use *sectors in the monomial* \mathbf{x}^α , i.e., sets of the form

$$\Pi_\alpha = \Pi_\alpha(a, b, r) = \{\mathbf{x} \in \mathbb{C}^d : a < \arg(\mathbf{x}^\alpha) < b, 0 < |x_j|^{\alpha_j} < r, j \in [1, d]\}.$$

Here any convenient branch of arg may be used. The number $r > 0$ denotes the *radius*, $b - a > 0$ the *opening*, and $\theta = (b + a)/2$ the *bisecting direction* of the monomial sector. We will also use the notation $\Pi_\alpha(a, b, r) = S_\alpha(\theta, b - a, r) = S_\alpha$. In the case $r = +\infty$ we will simply write $\Pi_\alpha(a, b) = S_\alpha(\theta, b - a)$, and we will refer to it as an *unbounded sector*. The definition of subsector in a monomial is clear.

Remark 3.3. Given two monomial sectors $\Pi'_\alpha = \Pi_\alpha(a', b', r) \subset \Pi''_\alpha = \Pi_\alpha(a'', b'', r)$, we can always cover the first one by polysectors, i.e., Cartesian products of sectors, of constant opening contained in the second one. In particular, we can check that $\Pi'_\alpha \subset U \subset \Pi''_\alpha$, where U is given by

$$U = \bigcup_{\mu_1, \dots, \mu_{d-1} \in \mathbb{R}} \prod_{j=1}^{d-1} V \left(\mu_j, \mu_j + \frac{b' - a'}{(d-1)\alpha_j}, r^{1/\alpha_j} \right) \\ \times V \left(\frac{a'' - \sum_{k=1}^{d-1} \alpha_k \mu_k}{\alpha_d}, \frac{b'' - (b' - a') - \sum_{k=1}^{d-1} \alpha_k \mu_k}{\alpha_d}, r^{1/\alpha_d} \right).$$

Indeed, if $\mathbf{x}_0 = (x_{1,0}, \dots, x_{d,0}) \in \Pi'_\alpha$, then \mathbf{x}_0 belongs, for instance, to the polysector with μ_j given by

$$\mu_j = \arg(x_{j,0}) - \frac{\phi}{\alpha_j}, \\ \max \left\{ 0, \frac{\arg(\mathbf{x}_0^\alpha) - b'' + b' - a'}{d-1} \right\} < \phi < \min \left\{ \frac{b' - a'}{d-1}, \frac{\arg(\mathbf{x}_0^\alpha) - a''}{d-1} \right\}.$$

If $f \in \mathcal{O}_b(\Pi_\alpha, E)$, $\Pi_\alpha = \Pi_\alpha(a, b, r)$, then we can construct an operator $T_\alpha(f)_\rho: V \rightarrow \mathcal{E}_\rho^\alpha$, where $V = V(a, b, \rho^d)$ and $0 < \rho < r$, as it is done in the formal case, such that

$$T_\alpha(f)_\rho(\mathbf{x}^\alpha)(\mathbf{x}) = f(\mathbf{x}).$$

We recall this construction by following [15]. We start with the case $\alpha = \mathbf{1} := (1, \dots, 1)$. Define the map $g(t, x_2, \dots, x_d) := f\left(\frac{t}{x_2 \cdots x_d}, x_2, \dots, x_d\right)$ for $|x_2|, \dots, |x_d| < r$ and $|t|/r < |x_2 \cdots x_d|$. The map g admits a Laurent series expansion in this domain, $g(t, x_2, \dots, x_d) = \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} g_{\mathbf{m}}(t) \mathbf{x}'^{\mathbf{m}}$, where $g_{\mathbf{m}} \in \mathcal{O}(V, E)$.

If $\mathbf{m} \in \mathbb{Z}^{d-1}$, we use the notation $\mu(\mathbf{m}) = \min\{0, m_2, \dots, m_d\} \leq 0$, and $\phi: \mathbb{Z}^{d-1} \rightarrow \mathcal{M}_d$ for the bijection $\phi(\mathbf{m}) = (-\mu(\mathbf{m}), m_2 - \mu(\mathbf{m}), \dots, m_d - \mu(\mathbf{m}))$, where $\mathcal{M}_d \subset \mathbb{N}^d$ is the set of all $(n_1, \dots, n_d) \in \mathbb{N}^d$ such that at least one of the n_j vanishes. Then, by definition,

$$T_{\mathbf{1}}(f)_\rho(t)(\mathbf{x}) := \sum_{\mathbf{m} \in \mathbb{Z}^d} t^{\mu(\mathbf{m})} g_{\mathbf{m}}(t) \mathbf{x}^{\phi(\mathbf{m})}.$$

With these considerations, we guarantee that all the exponents in \mathbf{x} are non-negative. To check that this expression is well-defined and satisfies what it is required, note that since f is bounded, say by some constant C , Cauchy's inequalities yield $\|g_{\mathbf{m}}(t)\| \leq C r_2^{-m_2} \cdots r_d^{-m_d}$. If $m_l = \mu(\mathbf{m})$, choose $r_j = r$ for all $j \neq l$ and r_l such that $r_2 \cdots r_d = |t|/r$, to deduce that

$$\|g_{\mathbf{m}}(t)\| \leq C |t|^{-\mu(\mathbf{m})} r^{d\mu(\mathbf{m}) - (m_2 + \cdots + m_d)}.$$

Thus each $t^{\mu(\mathbf{m})}g_{\mathbf{m}}(t)$ is holomorphic and bounded on V . It is also clear that $\mathbf{x}^{\phi(\mathbf{m})} \in \mathcal{E}_{\rho}^{\mathbf{1}}$, since $\phi(\mathbf{m}) \in \mathcal{M}_d$, and then the map defined through the previous series also belongs to the same space.

More generally, if there is a function $K : (0, r^d) \rightarrow \mathbb{R}$ such that $\|f(\mathbf{x})\| \leq K(|x_1 \cdots x_d|)$, $\mathbf{x} \in \Pi_{\mathbf{1}}$, then $\|t^{\mu(\mathbf{m})}g_{\mathbf{m}}(t)\| \leq K(|t|)^{r^{d\mu(\mathbf{m})} - (m_2 + \cdots + m_d)}$. Thus $T_{\mathbf{1}}(f)_{\rho}(t)(\mathbf{x})$ is bounded by

$$K(|t|) \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} \left(\frac{1}{r}\mathbf{x}\right)^{\phi(\mathbf{m})} = K(|t|) \sum_{\mathbf{n} \in \mathcal{M}_d} \left(\frac{1}{r}\mathbf{x}\right)^{\mathbf{n}} \leq \frac{K(|t|)}{\prod_{j=1}^d (1 - \frac{|x_j|}{r})}.$$

We conclude that

$$(10) \quad \|T_{\mathbf{1}}(f)_{\rho}(t)(\mathbf{x})\| \leq \frac{K(|t|)}{\prod_{j=1}^d (1 - \frac{|x_j|}{r})}, \quad t \in V(a, b, \rho^d), \mathbf{x} \in \Pi_{\mathbf{1}}(a, b, r).$$

In the general case, for an arbitrary monomial \mathbf{x}^{α} , we can write

$$(11) \quad f(\mathbf{x}) = \sum_{\mathbf{0} \leq \beta < \alpha} \mathbf{x}^{\beta} f_{\beta}(x_1^{\alpha_1}, \dots, x_d^{\alpha_d}),$$

with $f_{\beta} \in \mathcal{O}(\Pi_{\mathbf{1}}(a, b, r), E)$. In fact, if for each j , ω_j is a α_j -th primitive root of unity, then

$$(12) \quad \mathbf{x}^{\beta} f_{\beta}(x_1^{\alpha_1}, \dots, x_d^{\alpha_d}) = \frac{1}{\alpha_1 \cdots \alpha_d} \times \sum_{\mathbf{0} \leq \delta < \alpha} \omega_1^{-\delta_1 \beta_1} \cdots \omega_d^{-\delta_d \beta_d} f(\omega_1^{\delta_1} x_1, \dots, \omega_d^{\delta_d} x_d).$$

Then, we define

$$(13) \quad T_{\alpha}(f)_{\rho}(t)(\mathbf{x}) := \sum_{\mathbf{0} \leq \beta < \alpha} \mathbf{x}^{\beta} T_{\mathbf{1}}(f_{\beta})_{\rho}(t)(x_1^{\alpha_1}, \dots, x_d^{\alpha_d}).$$

To see that $T_{\alpha}(f)_{\rho}$ is well-defined and thus holomorphic, we can actually show that if f satisfies $\|f(\mathbf{x})\| \leq K(|\mathbf{x}^{\alpha}|)$ on Π_{α} , for some function $K : (0, r^d) \rightarrow \mathbb{R}$, then

$$(14) \quad \|T_{\alpha}(f)_{\rho}(t)(\mathbf{x})\| \leq R_{\alpha}(|x_1|^{\alpha_1}, \dots, |x_d|^{\alpha_d}, r) \frac{K(|t|)}{|t|},$$

for $t \in V(a, b, \rho^d)$, $\mathbf{x} \in \Pi_{\alpha}(a, b, r)$, where

$$R_{\alpha}(\rho_1, \dots, \rho_d, r) := \frac{r^d}{\prod_{j=1}^d (1 - (\frac{\rho_j}{r})^{1/\alpha_j})}, \quad 0 \leq \rho_1, \dots, \rho_d < r.$$

Indeed, by using equation (12) we get

$$\|\mathbf{x}^{\alpha} f_{\beta}(x_1^{\alpha_1}, \dots, x_d^{\alpha_d})\| \leq |x_1|^{\alpha_1 - \beta_1} \cdots |x_d|^{\alpha_d - \beta_d} K(|\mathbf{x}^{\alpha}|).$$

In the variables $u_j = x_j^{\alpha_j}$ these inequalities take the form

$$\|f_{\beta}(\mathbf{u})\| \leq r^{1-\beta_1/\alpha_1} \dots r^{1-\beta_d/\alpha_d} \frac{K(|u_1 \cdots u_d|)}{|u_1 \cdots u_d|}.$$

We can thus apply inequality (10) to each summand in (13) to finally obtain

$$\|T_{\alpha}(f)_{\rho}(t)(\mathbf{x})\| \leq \sum_{\mathbf{0} \leq \beta < \alpha} \left(\frac{|x_1|}{r^{1/\alpha_1}} \right)^{\beta_1} \cdots \left(\frac{|x_d|}{r^{1/\alpha_d}} \right)^{\beta_d} \frac{r^d K(|t|)}{\prod_{j=1}^d (1 - \frac{|x_j|^{\alpha_j}}{r}) |t|}.$$

Then (14) follows by noticing that $|x_j|^{\alpha_j} < \rho < r$, and also from the identity $\sum_{\mathbf{0} \leq \beta < \alpha} a_1^{\beta_1} \cdots a_d^{\beta_d} = \frac{(1-a_1^{\alpha_1}) \cdots (1-a_d^{\alpha_d})}{(1-a_1) \cdots (1-a_d)}$.

In the case $d = 2$ and $\alpha = \mathbf{1} = (1, 1)$, if $f \in \mathcal{O}(\Pi_1(a, b, r))$, the previous construction consists of writing $f(t/x_2, x_2) = \sum_{m \in \mathbb{Z}} f_m(t) x_2^m$, as a Laurent series on $|t|/r < |x_2| < r$ and setting

$$T_1(f)_{\rho}(t)(x_1, x_2) = \sum_{m=0}^{\infty} \frac{f_{-m}(t)}{t^m} x_1^m + \sum_{m=1}^{\infty} f_m(t) x_2^m.$$

As an application of this construction we can prove the following proposition on the dependence and growing of a holomorphic map in a monomial. There is also an interesting approach in [14] for the case of entire functions with polynomial bounds.

Proposition 3.4. *Let $d \geq 2$ and $f \in \mathcal{O}(\Pi_{\alpha}, E)$ be holomorphic, where $\Pi_{\alpha} = \Pi_{\alpha}(a, b)$. If $\|f(\mathbf{x})\| \leq K(|\mathbf{x}^{\alpha}|)$ on Π_{α} , for some function $K: (0, +\infty) \rightarrow (0, +\infty)$, then there is $g \in \mathcal{O}(V, E)$, $V = V(a, b)$, such that $f(\mathbf{x}) = g(\mathbf{x}^{\alpha})$.*

Proof: We proceed by induction on d . The case $d = 2$ is proved in [6], but we repeat the proof here for the sake of completeness. First, note that we can assume $\text{g.c.d.}(\alpha_1, \alpha_2) = 1$ by changing K adequately. Given $\alpha = (\alpha_1, \alpha_2)$ and $f \in \mathcal{O}(\Pi_{\alpha}, E)$, we write f as in equation (11), and if $f_{\beta_1, \beta_2}(t/u_2, u_2) = \sum_{m \in \mathbb{Z}} f_{\beta_1, \beta_2, m}(t) u_2^m$ for $|t|/r < |u_2| < r$, equation (12) shows that $\|f_{\beta_1, \beta_2}(u_1, u_2)\| \leq |u_1|^{-\beta_1/\alpha_1} |u_2|^{-\beta_2/\alpha_2} K(|u_1 u_2|)$, $(u_1, u_2) \in \Pi_1$. Using Cauchy's formulas we obtain

$$\|f_{\beta_1, \beta_2, m}(t)\| \leq \frac{|t|^{-\beta_1/\alpha_1} K(|t|)}{r^{m+\beta_2/\alpha_2-\beta_1/\alpha_1}},$$

$$\|f_{\beta_1, \beta_2, -m}(t)\| \leq \frac{|t|^{m-\beta_2/\alpha_2} K(|t|)}{r^{m+\beta_1/\alpha_1-\beta_2/\alpha_2}}, \quad m \in \mathbb{N}, 0 \leq \beta_j < \alpha_j, j = 1, 2.$$

If $m \geq 1$, the exponents of r in the previous inequalities are positive. Since r was arbitrary, we can take $r \rightarrow +\infty$ and conclude that $f_{\beta_1, \beta_2, m} \equiv 0$. If $m = 0$ and $(\beta_1, \beta_2) \neq (0, 0)$, by considering one of the preceding inequalities, according to $\beta_2/\alpha_2 > \beta_1/\alpha_1$ or $\beta_2/\alpha_2 < \beta_1/\alpha_1$, the same conclusion follows. Thus $f(x_1, x_2) = f_{0,0,0}(x_1^{\alpha_1} x_2^{\alpha_2})$.

Now assume that the result is valid for some d and let us prove it for $d+1$. To simplify notations we name our coordinates $(\mathbf{x}, z) \in \mathbb{C}^d \times \mathbb{C}$ and $(\boldsymbol{\alpha}, p) \in (\mathbb{N}^+)^d \times \mathbb{N}^+$. If we decompose f as

$$f(\mathbf{x}, z) = \sum_{j=0}^{p-1} z^j f_j(\mathbf{x}, z^p), \quad f_j \in \mathcal{O}(\Pi_{\boldsymbol{\alpha}, 1}, E),$$

the bounds for f show that $\|f_j(\mathbf{x}, \zeta)\| \leq |\zeta|^{-j/p} K(|\mathbf{x}^\alpha \zeta|)$, $j = 0, \dots, p-1$. For a fixed $\zeta \in \mathbb{C}^*$ let us write $\Pi_{\boldsymbol{\alpha}}^\zeta = \{\mathbf{x} \in (\mathbb{C}^*)^d : a < \arg(\mathbf{x}^\alpha) + \arg(\zeta) < b\}$ and $V^\zeta = \{\xi \in \mathbb{C}^* : a < \arg(\xi) + \arg(\zeta) < b\}$. Applying the induction hypothesis to each $f_j(\cdot, \zeta) \in \mathcal{O}(\Pi_{\boldsymbol{\alpha}}^\zeta, E)$, we can conclude that there are maps $g_j(\cdot, \zeta) \in \mathcal{O}(V^\zeta, E)$ such that $f_j(\mathbf{x}, \zeta) = g_j(\mathbf{x}^\alpha, \zeta)$.

We can now define g_j on $\Pi_{(1,1)} = \Pi_{(1,1)}(a, b)$ in such a way that $g_j \in \mathcal{O}(\Pi_{(1,1)}, E)$. Indeed, if $(\xi, \zeta) \in \Pi_{(1,1)}$, then $\xi \in V^\zeta$ and $g_j(\xi, \zeta)$ is already defined. To show that g_j is holomorphic, by using Hartog's theorem (see e.g., [19, p. 28]) it is sufficient to show that g_j is holomorphic at any point $(\xi_0, \zeta_0) \in \Pi_{(1,1)}$ with respect to each of the variables. It only remains to prove this for the second one: choosing $\mathbf{x}_0 \in \Pi_{\boldsymbol{\alpha}}^{\zeta_0}$ such that $\mathbf{x}_0^\alpha = \xi_0$, we know that $g_j(\xi_0, \zeta_0) = f_j(\mathbf{x}_0^\alpha, \zeta_0)$, that depends holomorphically on the second variable. The functions g_j satisfy $\|g_j(\xi, \zeta)\| \leq |\zeta|^{-j/p} K(|\xi \zeta|)$ for $(\xi, \zeta) \in \Pi_{(1,1)}$. The same argument used in the case $d = 2$ shows that $g_j \equiv 0$ for $j \neq 0$ and $g_0(\xi, \zeta) = g_{0,0}(\xi \zeta)$ for some $g_{0,0} \in \mathcal{O}(V, E)$. In conclusion, $f(\mathbf{x}, z) = g_{0,0}(\mathbf{x}^\alpha z^p)$ as we wanted to show. The induction principle allows us to conclude the proof. \square

Definition 3.5. Let $f \in \mathcal{O}(\Pi_{\boldsymbol{\alpha}}, E)$, $\Pi_{\boldsymbol{\alpha}} = \Pi_{\boldsymbol{\alpha}}(a, b, r)$ and $\hat{f} \in \widehat{\mathcal{O}}'_d(E)$ be given. We will say that f has \hat{f} as asymptotic expansion at the origin in \mathbf{x}^α (denoted by $f \sim^\alpha \hat{f}$ on $\Pi_{\boldsymbol{\alpha}}$) if there is $0 < r' \leq r$ such that $\widehat{T}_\alpha(\hat{f}) = \sum f_{\boldsymbol{\alpha}, n} t^n \in \mathcal{E}_{r'}^\alpha[[t]]$, and for every proper subsector $\Pi'_\alpha = \Pi_\alpha(a', b', \rho)$, $0 < \rho < r'$, and $N \in \mathbb{N}$, there exists $C_N(\Pi'_\alpha) > 0$ such that

$$(15) \quad \left\| f(\mathbf{x}) - \sum_{n=0}^{N-1} f_{\boldsymbol{\alpha}, n}(\mathbf{x}) \mathbf{x}^{n\alpha} \right\| \leq C_N(\Pi'_\alpha) |\mathbf{x}^{N\alpha}| \quad \text{on } \Pi'_\alpha.$$

The asymptotic expansion is said to be of *s-Gevrey type* (denoted by $f \sim_s^\alpha \hat{f}$ on $\Pi_{\boldsymbol{\alpha}}$) if it is possible to choose $C_n(\Pi'_\alpha) = C(\Pi'_\alpha) A(\Pi'_\alpha)^n n!^s$, for some $C(\Pi'_\alpha), A(\Pi'_\alpha) > 0$ independent of n .

From the very definition of $f \sim^\alpha \hat{f} = \sum a_\beta \mathbf{x}^\beta$ on Π_α we can deduce, by using (15) for $N = 1$, that

$$(16) \quad a_0 = \lim_{\substack{\Pi'_\alpha \ni \mathbf{x} \rightarrow 0 \\ \mathbf{x} \in \Pi'_\alpha}} f(\mathbf{x}), \quad f_{J,0_J}(\mathbf{x}_{J^c}) = \lim_{\substack{\mathbf{x}_J \rightarrow 0 \\ \mathbf{x} \in \Pi'_\alpha}} f(\mathbf{x}), \quad J \subsetneq [1, d], \quad \Pi'_\alpha \subset \Pi_\alpha.$$

Monomial asymptotic expansions can be reduced to the case of one variable by using the operators T_α and \hat{T}_α . Indeed, direct estimates using (14) show that if $f \in \mathcal{O}(\Pi_\alpha(a, b, r), E)$ and $\hat{f} \in \hat{\mathcal{O}}'_d(r', E)$, $r' \leq r$, then $f \sim^\alpha \hat{f}$ on $\Pi_\alpha(a, b, r)$ if and only if for every $0 < \rho < r'$, $T_\alpha(f)_\rho \sim \hat{T}_\alpha(\hat{f})$ on $V(a, b, \rho^d)$. The same statement is valid in the Gevrey case, and in this case it follows that $\hat{f} \in E[[\mathbf{x}]]_s^\alpha$; see [15, Proposition 3.11] for details.

Another characterization of monomial asymptotic expansions is obtained by approximating by holomorphic functions.

Proposition 3.6. *Let $f \in \mathcal{O}(\Pi_\alpha, E)$, $\Pi_\alpha = \Pi_\alpha(a, b, r)$, and $\hat{f} \in \hat{\mathcal{O}}'_d(r', E)$, $r' \leq r$, be given. The following assertions are equivalent:*

- (1) $f \sim^\alpha \hat{f}$ on Π_α .
- (2) *There is $R > 0$ and a sequence $(F_N)_{N \in \mathbb{N}} \subset \mathcal{O}_b(D_R^d, E)$, $F_0 = 0$, such that for each subsector Π'_α of Π_α and $N \in \mathbb{N}$, there is $A_N(\Pi'_\alpha) > 0$ such that*

$$(17) \quad \|f(\mathbf{x}) - F_N(\mathbf{x})\| \leq A_N(\Pi'_\alpha) |\mathbf{x}^{N\alpha}| \quad \text{on } \Pi'_\alpha \cap D_R^d.$$

If $s > 0$, $f \sim_s^\alpha \hat{f}$ on Π_α if and only if inequality (17) is satisfied with $A_n(\Pi'_\alpha) = C(\Pi'_\alpha) A(\Pi'_\alpha)^n n!^s$ for some $C(\Pi'_\alpha), A(\Pi'_\alpha) > 0$ independent of n , and there are $B, D > 0$ such that $\|F_n\|_R \leq DB^n n!^s$ for all $n \in \mathbb{N}$. In any case, \hat{f} is given by the limit of the Taylor series at the origin of the F_n , in the \mathfrak{m} -topology of $E[[\mathbf{x}]]$, $\mathfrak{m} = (\mathbf{x})$.

Proof: If $f \sim^\alpha \hat{f}$ on Π_α and $\hat{T}_\alpha(\hat{f})(t) = \sum f_{\alpha,n} t^n$, then $F_N(\mathbf{x}) = \sum_{n < N} f_{\alpha,n}(\mathbf{x}) \mathbf{x}^{n\alpha}$ satisfies the requirements. Conversely, suppose we have such a sequence $(F_N)_{N \in \mathbb{N}}$. Note that each $T_\alpha(F_N)_R$ is holomorphic on D_R and has $\hat{T}_\alpha(F_N)$ as Taylor series at the origin. Let $g_{N+1} = T_\alpha(F_N)_R$. Applying inequalities (14) for $N + 1$ to inequality (17) with $K(u) = u^{N+1}$, it follows that

$$\|T_\alpha(f)_{\rho'}(t) - g_{N+1}\|_{\rho'} \leq R_\alpha(\rho', \dots, \rho', \rho) A_{N+1}(\Pi'_\alpha) |t|^N,$$

in the corresponding sector, where $0 < |t| < \rho' < \min\{\rho, R\}$. Thus we obtain that $T_\alpha(f)_{\rho'}$ has $\hat{T}_\alpha(\hat{f})$ as asymptotic expansion. But $\hat{T}_\alpha(\hat{f})$ is given by the limit of the series $\hat{T}_\alpha(F_N)$ in the \mathfrak{m} -topology of $\mathcal{E}_r^\alpha[[t]]$,

$\mathbf{m} = (t)$, thus $f \sim^\alpha \hat{f}$ on Π_α as we wanted to show. The s -Gevrey case also follows since the sequence $(g_{N+1})_{N \in \mathbb{N}}$ has bounds of s -Gevrey type if the sequence $(F_N)_{N \in \mathbb{N}}$ does. \square

These characterizations allow us to prove that monomial asymptotic expansions are stable under sums, products, and partial derivatives. In particular, it follows from the relations (16) that if $f \sim^\alpha \hat{f}$ on Π_α , then

$$(18) \quad \begin{aligned} a_\beta &= \lim_{\Pi'_\alpha \ni \mathbf{x} \rightarrow \mathbf{0}} \frac{1}{\beta!} \frac{\partial^\beta f}{\partial \mathbf{x}^\beta}(\mathbf{x}), \\ f_{J, \beta_J}(\mathbf{x}_{J^c}) &= \lim_{\substack{\mathbf{x}_J \rightarrow \mathbf{0} \\ \mathbf{x} \in \Pi'_\alpha}} \frac{1}{\beta_J!} \frac{\partial^{\beta_J} f}{\partial \mathbf{x}_J^{\beta_J}}(\mathbf{x}), \quad \beta \in \mathbb{N}^d, J \subsetneq [1, d]. \end{aligned}$$

In particular, \hat{f} is completely determined by f .

Remark 3.7. Assume that $f \sim_s^\alpha \hat{f} = \sum a_\beta \mathbf{x}^\beta$ on Π_α , and take $D, B > 0$ such that $\|a_\beta\| \leq DB^{|\beta|} \min_{1 \leq j \leq d} \{\beta_j!^{s/\alpha_j}\}$ for all $\beta \in \mathbb{N}^d$. We can also consider how \hat{f} approximates f for any index γ , other than $N\alpha$ as in Definition 3.5. Indeed, given $\gamma \in \mathbb{N}^d$, if we consider $N = \max_{1 \leq j \leq d} \lfloor \gamma_j / \alpha_j \rfloor + 1$, then $\mathbf{0} \leq N\alpha - \gamma$. Thus, we find that in any subsector Π'_α of radius $\rho < \min_{1 \leq j \leq d} 1/B^{\alpha_j}$,

$$\|f(\mathbf{x}) - \text{App}_\gamma(\hat{f})(\mathbf{x})\| \leq CA^N N!^s |\mathbf{x}^{N\alpha}| + \|\text{App}_{N\alpha}(\hat{f})(\mathbf{x}) - \text{App}_\gamma(\hat{f})(\mathbf{x})\|,$$

but the second term is bounded by

$$\begin{aligned} &\sum_{N\alpha - \gamma \not\leq \delta} \|a_{\gamma+\delta}\| |\mathbf{x}^{\gamma+\delta}| \\ &\leq DB^{|\gamma|} |\mathbf{x}^\gamma| \sum_{j=1}^d \sum_{\beta_j=0}^{N\alpha_j - \gamma_j - 1} \sum_{\delta \in \mathbb{N}^d, \delta_j = \beta_j} B^{|\delta|} |\mathbf{x}^\delta| \min_{1 \leq l \leq d} (\gamma_l + \delta_l)^{s/\alpha_l} \\ &\leq DB^{|\gamma|} \left[\sum_{j=1}^d \sum_{\beta_j=0}^{N\alpha_j - \gamma_j - 1} \frac{(\gamma_j + \beta_j)!^{s/\alpha_j}}{\prod_{k \neq j} (1 - B\rho^{1/\alpha_k})} \right] |\mathbf{x}^\gamma|. \end{aligned}$$

Since $(\gamma_j + \beta_j)! < (\alpha_j N)!$, by taking into account Remark 3.1 we can conclude that

$$\|f(\mathbf{x}) - \text{App}_\gamma(\hat{f})(\mathbf{x})\| \leq \tilde{C} \tilde{A}^{|\gamma|} \left(\max_{1 \leq j \leq d} \gamma_j!^{s/\alpha_j} \right) |\mathbf{x}^\gamma|,$$

for adequate constants $\tilde{C}, \tilde{A} > 0$ depending only on Π'_α .

We can finally give a new, but expected, characterization of monomial asymptotic expansions in terms of bounded derivatives. We follow the proof of Proposition 3 in [11].

Theorem 3.8. *Let $f \in \mathcal{O}(\Pi_\alpha, E)$, $\Pi_\alpha = \Pi_\alpha(a, b, r)$, and $\hat{f} \in \widehat{\mathcal{O}}'_d(r', E)$, $r' \leq r$. Then $f \sim^\alpha \hat{f}$ on Π_α if and only if for each $\Pi'_\alpha \subset \Pi_\alpha$, $\sup_{\mathbf{x} \in \Pi'_\alpha} \left\| \frac{1}{\beta!} \frac{\partial^\beta f}{\partial \mathbf{x}^\beta}(\mathbf{x}) \right\|$ is finite for all $\beta \in \mathbb{N}^d$. More precisely, if $s > 0$, the following assertions are equivalent:*

- (1) $f \sim_s^\alpha \hat{f}$ on Π_α .
- (2) For each $\Pi'_\alpha \subset \Pi_\alpha$, there are constants $C, A > 0$ such that

$$\sup_{\mathbf{x} \in \Pi'_\alpha} \left\| \frac{1}{(N\alpha)!} \frac{\partial^{N\alpha} f}{\partial \mathbf{x}^{N\alpha}}(\mathbf{x}) \right\| \leq CA^N N!^s, \quad N \in \mathbb{N}.$$

- (3) For each $\Pi'_\alpha \subset \Pi_\alpha$, there are constants $C, A > 0$ such that

$$\sup_{\mathbf{x} \in \Pi'_\alpha} \left\| \frac{1}{\beta!} \frac{\partial^\beta f}{\partial \mathbf{x}^\beta}(\mathbf{x}) \right\| \leq CA^{|\beta|} \max\{\beta_1!^{s/\alpha_1}, \dots, \beta_d!^{s/\alpha_d}\}, \quad \beta \in \mathbb{N}^d.$$

Proof: We only consider the statement for the s -Gevrey case. It is clear that (3) implies (2). To prove that (2) implies (1), we use Taylor's formula and equations (6) and (18) to write

$$\begin{aligned} \left\| f(\mathbf{x}) - \sum_{n=0}^{N-1} f_{\alpha, n}(\mathbf{x}) \mathbf{x}^{n\alpha} \right\| &= \|f(\mathbf{x}) - \text{App}_{N\alpha}(\hat{f})(\mathbf{x})\| \\ &= \left\| \int_0^{x_1} \dots \int_0^{x_d} \frac{(x_1 - w_1)^{N\alpha_1 - 1}}{(N\alpha_1 - 1)!} \dots \frac{(x_d - w_d)^{N\alpha_d - 1}}{(N\alpha_d - 1)!} \frac{\partial^{N\alpha} f}{\partial \mathbf{x}^{N\alpha}}(\mathbf{w}) d\mathbf{w} \right\| \\ &\leq \sup_{\mathbf{w} \in \Pi'_\alpha} \left\| \frac{1}{(N\alpha)!} \frac{\partial^{N\alpha} f}{\partial \mathbf{x}^{N\alpha}}(\mathbf{w}) \right\| |\mathbf{x}^{N\alpha}| \leq CA^N N!^s |\mathbf{x}^{N\alpha}|, \end{aligned}$$

and then we can conclude the desired bounds. Note that the paths of integration are the segments $w_j = \rho_j x_j$, $0 < \rho_j \leq 1$, and $\mathbf{w} = (\rho_1 x_1, \dots, \rho_d x_d) \in \Pi'_\alpha$ if $\mathbf{x} \in \Pi'_\alpha$.

Finally, to show that (1) implies (3), take $\mathbf{x} \in \Pi'_\alpha \subset \Pi''_\alpha \subset \Pi_\alpha$, where $\Pi'_\alpha = \Pi'_\alpha(a', b', \rho)$ and $\Pi''_\alpha = \Pi_\alpha(a'', b'', \rho)$. Using Remark 3.3, we can find a polysector $P = \prod_{j=1}^d V_j$, with V_j of opening $(b' - a')/(d-1)\alpha_j$ for $j = 1, \dots, d-1$, and $b'' - a'' - (b' - a')$ for $j = d$, such that $\mathbf{x} \in P \subset \Pi''_\alpha$. Note that these openings are independent of the point $\mathbf{x} \in \Pi'_\alpha$. Thus we can find numbers $\sigma_1, \dots, \sigma_d > 0$ depending only on Π'_α and Π''_α such

that the circles C_j given by $|w_j - x_j| = \sigma_j |x_j|$ are contained in V_j for all $j = 1, \dots, d$. Using Cauchy's formula and Remark 3.7 we find that

$$\begin{aligned} \left\| \frac{1}{\beta!} \frac{\partial^\beta f}{\partial \mathbf{x}^\beta}(\mathbf{x}) \right\| &= \left\| \frac{1}{\beta!} \frac{\partial^\beta}{\partial \mathbf{x}^\beta} (f - \text{App}_\beta(\hat{f}))(\mathbf{x}) \right\| \\ &= \left\| \frac{1}{(2\pi i)^d} \int_{C_1} \cdots \int_{C_d} \frac{f(\mathbf{w}) - \text{App}_\beta(\hat{f})(\mathbf{w})}{(w_1 - x_1)^{\beta_1+1} \cdots (w_d - x_d)^{\beta_d+1}} dw_d \cdots dw_1 \right\| \\ &\leq \left(\frac{\sigma_1 + 1}{\sigma_1} \right)^{\beta_1+1} \cdots \left(\frac{\sigma_d + 1}{\sigma_d} \right)^{\beta_d+1} \tilde{C}(\Pi''_\alpha) \tilde{A}(\Pi''_\alpha)^{|\beta|} \cdot \max_{1 \leq j \leq d} \beta_j!^{s/\alpha_j}, \end{aligned}$$

for some constants $\tilde{C}(\Pi''_\alpha), \tilde{A}(\Pi''_\alpha) > 0$ independent of β . Then (3) follows. \square

Remark 3.9. The previous theorem shows that if $f \sim^\alpha \hat{f} \in \mathcal{O}'_d(E)$ on Π_α , then f also has \hat{f} as strong asymptotic expansion in Majima's sense, i.e., f is *strongly asymptotically developable* as $\mathbf{x} \rightarrow 0$ in any polysector properly contained in Π_α . Moreover, if $f \sim_s^\alpha \hat{f}$ on Π_α , then the strong asymptotic expansion is $(s/\alpha_1, \dots, s/\alpha_d)$ -Gevrey, since $\max_{1 \leq j \leq d} \beta_j!^{s/\alpha_j} \leq \beta_1!^{s/\alpha_1} \cdots \beta_d!^{s/\alpha_d}$; see, e.g., [11, Definition 3].

When we fix some of the variables in the monomial asymptotic expansion of a map, the expansion still holds in the remaining variables. We state this result when we only fix one variable. The proof is an immediate consequence of Proposition 3.6.

Proposition 3.10. *Consider $f \in \mathcal{O}(\Pi_{(\alpha,p)}, E)$, $\Pi_{(\alpha,p)} = \Pi_{(\alpha,p)}(a, b, r)$, and $\hat{f} \in \hat{\mathcal{O}}'_{d+1}(E)$ such that $f \sim_s^{(\alpha,p)} \hat{f}$ on $\Pi_{(\alpha,p)}$. Then there is $r' > 0$ such that, for all $z_0 \in D_{r'}$, we have $f(\mathbf{x}, z_0) \sim_s^\alpha \hat{f}(\mathbf{x}, z_0)$ on $\Pi_\alpha(a', b', r')$, $a' = a - \arg(z_0^p)$, $b' = b - \arg(z_0^p)$.*

We can characterize maps with null s -Gevrey asymptotic expansion in a monomial: $f \sim_s^\alpha 0$ on Π_α if and only if for every subsector $\Pi'_\alpha \subset \Pi_\alpha$, there are constants $C, A > 0$ such that

$$\|f(\mathbf{x})\| \leq C \exp(-A/|\mathbf{x}^\alpha|^{1/s}), \quad \mathbf{x} \in \Pi'_\alpha.$$

In this framework Watson's lemma also holds: if $f \sim_s^\alpha \hat{0}$ on $\Pi_\alpha(a, b, r)$ and $b - a > s\pi$, then $f \equiv 0$. As in the case of one variable it is natural to consider the following definition.

Definition 3.11. Let $\hat{f} \in \hat{\mathcal{O}}'_d(E)$, $k > 0$, and $\theta \in \mathbb{R}$ be a direction.

- (1) The series \hat{f} is called \mathbf{x}^α - k -summable on $S_\alpha = S_\alpha(\theta, b - a, r)$ with sum $f \in \mathcal{O}(S_\alpha, E)$ if $b - a > \pi/k$ and $f \sim_{1/k}^\alpha \hat{f}$ on S_α . We also say that \hat{f} is \mathbf{x}^α - k -summable in direction θ . The space of \mathbf{x}^α - k -summable series in direction θ will be denoted by $E\{\mathbf{x}\}_{1/k, \theta}^\alpha$.
- (2) The series \hat{f} is called \mathbf{x}^α - k -summable if it is \mathbf{x}^α - k -summable in all directions, except possibly for a finite number of them modulo 2π (the singular directions). The corresponding space is denoted by $E\{\mathbf{x}\}_{1/k}^\alpha$.

Note that both $E\{\mathbf{x}\}_{1/k, \theta}^\alpha$ and $E\{\mathbf{x}\}_{1/k}^\alpha$ are vector spaces, stable by partial derivatives, and they inherit naturally a structure of algebra when E is a Banach algebra.

Remark 3.12. Given $\alpha \in (\mathbb{N}^+)^d$, we note that formulas (11) and (12) can also be applied to formal power series. In particular, $\hat{f} \in \widehat{\mathcal{O}}_d'(E)$ and $\hat{f}(\mathbf{x}) = \sum_{\mathbf{0} \leq \beta < \alpha} \mathbf{x}^\beta \hat{f}_\beta(x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$, and it is straightforward to show that \hat{f} is \mathbf{x}^α - k -summable in direction θ if and only if \hat{f}_β is \mathbf{z}^1 - k -summable in direction θ , for all $\mathbf{0} \leq \beta < \alpha$, where $\mathbf{1} = (1, \dots, 1)$ and $\mathbf{z} = (z_1, \dots, z_d) = (x_1^{\alpha_1}, \dots, x_d^{\alpha_d})$.

4. Borel–Laplace analysis for monomial summability

The goal of this section is to generalize the Borel and Laplace transformations for monomial asymptotic expansions contained in [6] to any number of variables, and develop their main properties. We will prove that monomial summability is equivalent to Borel-summability in this framework. It is worth to remark that, in contrast with the approach in [6], we have improved these results, since now we can allow some of the weights we use to be zero. This will be crucial in the application we present in Section 6.

From now on, if $\mathbf{c} \in \mathbb{R}_{\geq 0}^d$, we will write $J_{\mathbf{c}} := \{j \in [1, d] : c_j \neq 0\}$ for the set of indices where \mathbf{c} has nonzero entries.

Definition 4.1. Consider $\alpha \in (\mathbb{N}^+)^d$, $k > 0$, and $\mathbf{s} \in \overline{\sigma_d}$. The \mathbf{x}^α - k - \mathbf{s} -Borel transform of a map f is defined by the formula

$$\mathcal{B}_\lambda(f)(\xi) = \frac{(\xi^{k\alpha}_{J_{\mathbf{s}}})^{-1}}{2\pi i} \int_\gamma f(\xi_1 u^{-\frac{s_1}{\alpha_1 k}}, \dots, \xi_d u^{-\frac{s_d}{\alpha_d k}}) e^u du,$$

where γ denotes a Hankel path as we will explain below. Along this section we will write

$$(19) \quad \lambda = \left(\frac{s_1}{\alpha_1 k}, \dots, \frac{s_d}{\alpha_d k} \right), \quad \lambda' = \left(\frac{\alpha_1 k}{s_1}, \dots, \frac{\alpha_d k}{s_d} \right).$$

If $s_j = 0$ for some j , we interpret the j th entry of λ' as 0, and the variable x_j remains unchanged. In this situation, to avoid cumbersome notation, we call it ξ_j nevertheless. This convention will be used further on without explicit mention. Note that the factor outside the integral only includes the variables ξ_j such that $s_j \neq 0$. The ambient space \mathbb{C}^d with coordinates ξ will be referred to as the ξ -Borel plane.

Since we admit that some weights can be zero, it is necessary to consider monomial sectors where some of the variables are bounded. Thus if $f \in \mathcal{O}_b(S_\alpha, E)$, $S_\alpha = S_\alpha(\theta, \pi/k + 2\epsilon, R_0)$, $0 < 2\epsilon < \pi/k$, then $\mathcal{B}_\lambda(f)$ will be defined and holomorphic on

$$S_\alpha^s(\theta, 2\epsilon, R_0) := S_\alpha(\theta, 2\epsilon) \cap \{\xi \in \mathbb{C}^d : |\xi_j|^{\alpha_j} < R_0, j \notin J_s\}.$$

We will use the same notation for these sectors, for every $\mathbf{c} \in \mathbb{R}_{\geq 0}^d$ other than \mathbf{s} . Note that if all entries of \mathbf{s} are different from zero, then $S_\alpha^s(\theta, 2\epsilon, R_0)$ is simply $S_\alpha(\theta, 2\epsilon)$.

If $\xi \in S_\alpha^s(\theta, 2\epsilon', R_0)$, $0 < \epsilon' < \epsilon$, we take γ oriented positively and given by the arc of a circle centered at 0 and radius

$$R > \max_{j \in J_s} (|\xi_j|^{\alpha_j} / R_0)^{\frac{k}{s_j}},$$

with endpoints in the directions $-\pi/2 - k(\epsilon - \epsilon')$ and $\pi/2 + k(\epsilon - \epsilon')$, and the half-lines with those directions from this arc to ∞ . If u goes along this path, the integrand is evaluated on S_α and the integral converges absolutely, since f is bounded and the exponential term tends to 0 on those directions. The result is independent of ϵ' and R due to Cauchy's theorem.

Using Hankel's formula for the Gamma function, we obtain the formula

$$\mathcal{B}_\lambda(x^\mu)(\xi) = \frac{\xi^\mu \xi_{J_s}^{-k\alpha_{J_s}}}{\Gamma(\langle \mu, \lambda \rangle)}, \quad \mu \in \mathbb{C}^d,$$

and thus $\widehat{\mathcal{B}}_\lambda$, the *formal x^α - k - \mathbf{s} -Borel transform*, can be defined for formal power series term by term.

By looking at the derivative with respect to u of the integrand defining \mathcal{B}_λ , it is natural to consider the vector field X_λ and its flow ϕ_z^λ (at time z) given by

$$X_\lambda := \frac{\mathbf{x}_{J_s}^{k\alpha_{J_s}}}{k} \left(\frac{s_1}{\alpha_1} x_1 \frac{\partial}{\partial x_1} + \cdots + \frac{s_d}{\alpha_d} x_d \frac{\partial}{\partial x_d} \right),$$

$$\phi_z^\lambda(\mathbf{x}) = \sum_{j=1}^d \frac{x_j}{(1 - z \mathbf{x}_{J_s}^{k\alpha_{J_s}})^{s_j/\alpha_j k}} \mathbf{e}_j.$$

If $f \in \mathcal{O}_b(S_\alpha, E)$ is as before, it follows that

$$(20) \quad \begin{aligned} \mathcal{B}_\lambda(X_\lambda(f))(\xi) &= \xi_{J_s}^{k\alpha_{J_s}} \mathcal{B}_\lambda(f)(\xi), \\ \mathcal{B}_\lambda(f \circ \phi_z^\lambda)(\xi) &:= \exp(z\xi_{J_s}^{k\alpha_{J_s}}) \mathcal{B}_\lambda(f)(\xi). \end{aligned}$$

Both formulas are naturally related since the first one is the linearization of the second one at $z = 0$. In the variable $t = \mathbf{x}^\alpha$, the vector field X_λ reduces to $\frac{t^{k+1}}{k} \frac{\partial}{\partial t}$, and the first formula is just the one contained in the isomorphism (3).

We will say that f has *exponential growth of order at most $\mathbf{c} \in \mathbb{R}_{\geq 0}^d$* on $S_\alpha^c(\theta, b-a, R)$ if for every subsector $S_\alpha^c(\theta', b'-a', R')$ ($a < a' < b' < b$, $R' < R$) there are constants $C, M > 0$ such that

$$(21) \quad \|f(\xi)\| \leq C \exp(MR_c(\xi)), \quad R_c(\xi) := \max_{j \in J_c} |\xi_j|^{c_j},$$

on $S_\alpha^c(\theta', b' - a', R')$. Note we can also work with the term $\sum_{j \in J_c} |\xi_j|^{c_j}$ in this bound, since $R_c(\xi) \leq \sum_{j \in J_c} |\xi_j|^{c_j} \leq dR_c(\xi)$.

If f is holomorphic on $\{\xi \in \mathbb{C}^d : |\xi_j| < r_j, j \notin J_c\}$ for some fixed $r_j > 0$ (resp. entire if $\mathbf{c} \in \mathbb{R}_{> 0}^d$), and $\sum_{\beta \in \mathbb{N}^d} a_\beta \mathbf{x}^\beta$ is its Taylor series at the origin, condition (21) is equivalent to the existence of constants $D, B_1, \dots, B_d > 0$ such that

$$\|a_\beta\| \leq \frac{DB_1^{\beta_1} \dots B_d^{\beta_d}}{\Gamma(1 + \sum_{j \in J_c} \frac{\beta_j}{c_j})} \quad \text{for all } \beta \in \mathbb{N}^d.$$

This statement is standard and it can be deduced from Cauchy's integral formulas for the coefficients, Stirling's formula, and the inequalities

$$(22) \quad \Gamma(1+a)\Gamma(1+b) \leq \Gamma(1+a+b) \leq 2^{a+b}\Gamma(1+a)\Gamma(1+b), \quad a, b > 0,$$

satisfied by the Gamma function.

Remark 4.2. Consider $\hat{f} \in \widehat{\mathcal{O}}'_d(E)$, with $\widehat{T}_\alpha(\hat{f}) = \sum f_{\alpha,n} t^n$ and λ as in (19). If we write $\widehat{\phi}_\lambda = \widehat{B}_\lambda(\hat{f} - \sum_{k\alpha_{J_s} \leq \beta_{J_s}} a_\beta \mathbf{x}^\beta)$ and $\widehat{T}_\alpha(\xi_{J_s}^{k\alpha_{J_s}} \widehat{\phi}_\lambda) = \sum \varphi_{\alpha,n} \tau^n$, then

$$(23) \quad \varphi_{\alpha,n}(\xi) = \sum_{\alpha \leq \beta, k\alpha_{J_s} \leq \beta_{J_s}} \frac{a_{n\alpha+\beta}}{\Gamma\left(\frac{n}{k} + \langle \beta, \lambda \rangle\right)} \xi^\beta.$$

If $n \geq k$, then $k\alpha_l \leq n\alpha_l \leq n\alpha_l + \beta_l$ for all l . In particular, the condition $k\alpha_{J_s} \leq \beta_{J_s}$ in the previous sum is satisfied and we can conclude that $f_{\alpha,n}$ and $\varphi_{\alpha,n}$ are related by the formula $\xi_{J_s}^{-k\alpha_{J_s}} \xi^{n\alpha} \varphi_{\alpha,n} = \mathcal{B}_\lambda(\mathbf{x}^{n\alpha} f_{\alpha,n})$, $n \geq k$. Since we can find $\rho > 0$ such that $f_{\alpha,n} \in \mathcal{O}(D_\rho^d, E)$,

for all $n \in \mathbb{N}$, we see that the $\varphi_{\alpha,n}$ are holomorphic maps on $\{\xi \in \mathbb{C}^d : |\xi_j| < \rho, j \notin J_s\}$, and a direct estimate using expansion (23) shows that they satisfy bounds of the type

$$\|\varphi_{\alpha,n}(\xi)\| \leq \frac{L\|f_{\alpha,n}\|_{\rho}}{\Gamma(1 + \frac{n}{k})} \exp(MR_{\lambda'}(\xi)),$$

where $L, M > 0$ are some constants independent of n but depending on ρ .

The behavior of the Borel transform with respect to monomial asymptotic expansions is presented in the next proposition. It is based on estimates included in [18] for the Borel transform in several variables.

Proposition 4.3. *Assume $f \sim_s^{\alpha} \hat{f}$ on $S_{\alpha}(\theta, \pi/k + 2\epsilon, R_0)$, where $0 < 2\epsilon < \pi/k$ and $\hat{f} = \sum_{k\alpha_{J_s} \leq \beta_{J_s}} a_{\beta} x^{\beta}$. If $s > 0$, $s \in \overline{\sigma}_d$, and λ, λ' are given by (19), the following statements are verified:*

- (1) *If $g = \mathcal{B}_{\lambda}(f)$ and $\hat{g} = \widehat{\mathcal{B}}_{\lambda}(\hat{f})$, then $\xi_{J_s}^{k\alpha_{J_s}} g \sim_{s'}^{\alpha} \xi_{J_s}^{k\alpha_{J_s}} \hat{g}$ on $S_{\alpha}^s(\theta, 2\epsilon, R_0)$, where $s' = \max\{s - \frac{1}{k}, 0\}$.*
- (2) *For every unbounded subsector S''_{α} of $S_{\alpha}^s(\theta, 2\epsilon, R_0)$ there are $B, D, M > 0$ such that*

$$\left\| g(\xi) - \sum_{n=0}^{N-1} g_{\alpha,n}(\xi) \xi^{n\alpha} \right\| \leq DB^N \Gamma(1 + Ns') |\xi^{N\alpha}| |\xi_{J_s}^{-k\alpha_{J_s}}| \exp(MR_{\lambda'}(\xi)) \text{ on } S''_{\alpha},$$

where $\widehat{T}_{\alpha}(\xi_{J_s}^{k\alpha_{J_s}} \hat{g}) = \sum (\xi_{J_s}^{k\alpha_{J_s}} g_{\alpha,n}) t^n$. If $N = 0$, this inequality means that $\xi_{J_s}^{k\alpha_{J_s}} g$ has exponential growth of order at most λ' on $S_{\alpha}^s(\theta, 2\epsilon, R_0)$.

Proof: It is sufficient to prove (2). Thus, we have to establish those bounds for sectors of the form $S'_{\alpha} = S_{\alpha}^s(\theta, 2\epsilon', R_0)$ with $0 < \epsilon' < \epsilon$. The proof relies on choosing adequately the radius of the arc of the path γ in the definition. Write $\gamma = \gamma_1 + \gamma_2 - \gamma_3$, where γ_1, γ_3 denote the half-lines and γ_2 denotes the circular part, parameterized by $\gamma_2(\phi) = \text{Re}^{i\phi}$, $|\phi| \leq \pi/2 + k(\epsilon'' - \epsilon')/2$, where $0 < \epsilon' < \epsilon'' < \epsilon$ and R will be chosen so that $(\xi_1 u^{-\frac{s_1}{\alpha_1 k}}, \dots, \xi_d u^{-\frac{s_d}{\alpha_d k}}) \in S_{\alpha} = S_{\alpha}(\theta, \pi/k + 2\epsilon'', R_0/2)$ for all u on γ and $\xi \in S'_{\alpha}$.

We may assume that $\widehat{T}_{\alpha}(\hat{f}) = \sum f_{\alpha,n} t^n$, with $f_{\alpha,n} \in \mathcal{O}_b(D_{R_0}^d, E)$ by reducing R_0 if necessary. By hypothesis, inequality (15) holds with $C_N = CA^N \Gamma(1 + Ns)$ on S_{α} . Setting $a = \sin(\frac{k}{2}(\epsilon'' - \epsilon')) > 0$, taking

$R > 1$ to be chosen, and by using the relation between $f_{\alpha,n}$ and $g_{\alpha,n}$ explained in Remark 4.2, a direct estimate shows that

$$(24) \quad \left\| g(\boldsymbol{\xi}) - \sum_{n=0}^{N-1} g_{\alpha,n}(\boldsymbol{\xi}) \boldsymbol{\xi}^{n\alpha} \right\| \leq \frac{C}{a} A^N \Gamma(1 + Ns) \frac{|\boldsymbol{\xi}^{N\alpha}| |\boldsymbol{\xi}_{J_s}^{-k\alpha_{J_s}}|}{R^{N/k-1}} \left(\frac{e^{-aR}}{R} + e^R \right) \leq \frac{2C}{a} A^N \Gamma(1 + Ns) |\boldsymbol{\xi}^{N\alpha}| |\boldsymbol{\xi}_{J_s}^{-k\alpha_{J_s}}| \frac{e^R}{R^{N/k-1}} \quad \text{on } S'_\alpha,$$

for all $N \in \mathbb{N}$. For $N = 0$ we are denoting $C = \sup_{\mathbf{x} \in S_\alpha} \|f(\mathbf{x})\|$ (note that f is bounded here, as we have reduced the radius and the opening of the sector).

To prove (2) we divide our sector in two parts. First of all, consider $\boldsymbol{\xi} \in S_\alpha(\theta, 2\epsilon', r)$, with $r > R_0/2$ fixed. In this case choose $R \geq \max_{j \in J_s} (2r/R_0)^{k/s_j} > 1$. Since it is enough to establish the bounds for large N we can suppose N is large enough and take $R = N/k$. Then the bound follows using Stirling's formula, since asymptotically

$$\frac{e^R}{R^{N/k-1}} = \frac{N/k e^{N/k}}{(N/k)^{N/k}} \sim \frac{\sqrt{2\pi} (N/k)^{3/2}}{\Gamma(1 + N/k)},$$

and also because $\Gamma(1 + Ns) \leq 2^{Ns} \Gamma(1 + N/k) \Gamma(1 + N(s - 1/k))$ if $s \geq 1/k$, due to the second inequality in (22).

Now, we establish the bound in the complementary region, i.e., for $\boldsymbol{\xi} \in S'_\alpha \setminus S_\alpha(\theta, 2\epsilon', r)$. For each $j \in J_s$, choose $R_j < R_0/2$ and let us take $R = R(\boldsymbol{\xi}) = \max_{j \in J_s} (|\xi_j|^{\alpha_j}/R_j)^{k/s_j} > 1$. Note that $R(\boldsymbol{\xi}) \leq \sum_{j \in J_s} (|\xi_j|^{\alpha_j}/R_j)^{k/s_j}$ and $\frac{|\boldsymbol{\xi}_{J_s}^{\alpha_{J_s}}|^k}{\prod_{j \in J_s} R_j^k} \leq R(\boldsymbol{\xi})$ (second inequality of (9)). Then we can bound (24) by

$$\frac{2C}{a} A^N \Gamma(1 + Ns) |\boldsymbol{\xi}_{J_s}^{N\alpha_{J_s^c}}| \prod_{j \in J_s} R_j^{N-k} \exp((|\xi_j|^{\alpha_j}/R_j)^{k/s_j}).$$

Let $M = \max_{j \in J_s} (4/R_0)^{k/s_j} > 0$. For each $j \in J_s$, we consider two cases: choose $R_j = |\xi_j|^{\alpha_j}/(s_j N/k)^{s_j/k} < R_0/2$ as long as this inequality holds. Then

$$R_j^{N-k} \exp((|\xi_j|^{\alpha_j}/R_j)^{k/s_j}) = |\xi_j|^{(N-k)\alpha_j} \left(\frac{s_j N}{k} \right)^{s_j} \frac{e^{s_j N/k}}{(s_j N/k)^{s_j N/k}}.$$

In the second case, choose $R_j = R_0/4 < R_0/2 \leq |\xi_j|^{\alpha_j}/(s_j N/k)^{s_j/k}$ and then

$$R_j^{N-k} \exp((|\xi_j|^{\alpha_j}/R_j)^{k/s_j}) < |\xi_j|^{(N-k)\alpha_j} \left(\frac{s_j N}{k}\right)^{s_j} \frac{\exp(M|\xi_j|^{\alpha_j k/s_j})}{(s_j N/k)^{s_j N/k}}.$$

In both cases, we conclude that

$$\begin{aligned} R_j^{N-k} \exp((|\xi_j|^{\alpha_j}/R_j)^{k/s_j}) \\ \leq |\xi_j|^{(N-k)\alpha_j} \left(\frac{s_j N}{k}\right)^{s_j} \frac{e^{s_j N/k}}{(s_j N/k)^{s_j N/k}} \exp(M|\xi_j|^{\alpha_j k/s_j}). \end{aligned}$$

Using Stirling's formula we conclude that there are constants $L, K > 0$ such that

$$\begin{aligned} \left\| g(\xi) - \sum_{n=0}^{N-1} g_{\alpha, n}(\xi) \xi^{n\alpha} \right\| \\ \leq LK^N \frac{\Gamma(1+Ns)}{\prod_{j \in J_s} \Gamma(1 + \frac{s_j N}{k})} |\xi^{N\alpha}| |\xi_{J_s}^{-k\alpha_{J_s}}| \exp(MR_{\lambda'}(\xi)), \end{aligned}$$

on $S'_\alpha \setminus S_\alpha(\theta, 2\epsilon', r)$. Finally, we can use the second inequality in (22) to conclude the proof. \square

Now we move to the study of the Laplace transform, which will turn out to be the inverse of the Borel transformation introduced above.

Definition 4.4. Consider $\alpha \in (\mathbb{N}^+)^d$, $k > 0$, and $s \in \overline{\sigma}_d$. The α - k - s -Laplace transform in direction ϕ , $|\phi| < \pi/2$, of a map f is defined by the formula

$$\mathcal{L}_{\lambda, \phi}(f)(\mathbf{x}) = \mathbf{x}_{J_s}^{k\alpha_{J_s}} \int_0^{e^{i\phi}\infty} f(x_1 u^{\frac{s_1}{\alpha_1 k}}, \dots, x_d u^{\frac{s_d}{\alpha_d k}}) e^{-u} du.$$

As in the case of \mathcal{B}_λ , if $s_j = 0$ for some j , the variable ξ_j is not affected by the transformation, although we will still call it x_j .

If the map $f \in \mathcal{O}(S_\alpha, E)$, $S_\alpha = S_\alpha(\theta, b-a)$, had exponential growth of order k in the monomial ξ^α , i.e., $\|f(\xi)\| \leq C \exp(M|\xi^{k\alpha}|)$ on S_α , then by Proposition 3.4 f would be a map depending only on ξ^α . Instead, in view of Proposition 4.3 we assume that f is defined and has exponential growth of order at most λ' on $S_\alpha^s(\theta, b-a, R)$. If f satisfies (21) with $\mathbf{c} = \lambda'$, then $\mathcal{L}_{\lambda, \phi}(f)$ converges if \mathbf{x} satisfies

$$a - \phi/k < \arg(\mathbf{x}^\alpha) < b - \phi/k, \quad MR_{\lambda'}(\mathbf{x}) < \cos \phi, \quad |x_j|^{\alpha_j} < R, \quad j \notin J_s.$$

The domain in \mathbb{C}^d defined by these conditions will be denoted by $D_\alpha^s(\theta - \phi/k, b - a; M, R)$, indicating its bisecting direction $(b + a)/2 - \phi/k = \theta - \phi/k$ and opening $b - a$. We will also denote

$$D_\alpha^s(\theta, b - a + \pi/k; M, R) := \bigcup_{|\phi| < \pi/2} D_\alpha^s(\theta - \phi/k, b - a; M, R),$$

with bisecting direction θ and opening $b - a + \pi/k$. Note that given $a - \pi/2k < a' < b' < b + \pi/2k$, there is $r > 0$ such that $S_\alpha^s(\theta', b' - a', r) \subset D_\alpha^s(\theta, b - a + \pi/k; M, R)$.

It follows that $\mathcal{L}_{\lambda, \phi}(f)$ is holomorphic on $D_\alpha^s(\theta - \phi/k, b - a; M, R)$. Furthermore, if we change direction ϕ by ϕ' , we obtain an analytic continuation of $\mathcal{L}_{\lambda, \phi}(f)$ when $|\phi' - \phi| < k(b - a)$, a fact that follows directly from Cauchy's theorem. This process leads to a holomorphic map $\mathcal{L}_\lambda(f)$ defined on $D_\alpha^s(\theta, b - a + \pi/k; M, R)$.

An adequate choice of the path γ in the definition of \mathcal{B}_λ , a limiting process and the residue theorem, as it is done for the case of one variable (see e.g. [1, Theorem 24, p. 82]) show that if $f \in \mathcal{O}_b(S_\alpha^s(\theta, \pi/k + 2\epsilon, R_0), E)$, $0 < 2\epsilon < \pi/k$, then

$$\mathcal{L}_\lambda \mathcal{B}_\lambda(f) = f \text{ on the intersection of their domains.}$$

The operator \mathcal{L}_λ is also injective as the usual Laplace transform. Thus, if g is of exponential growth of order at most λ' , then

$$\mathcal{B}_\lambda \mathcal{L}_\lambda(g) = g \text{ on the intersection of their domains.}$$

We then define $\widehat{\mathcal{L}}_\lambda$, the formal \mathbf{x}^α - k - \mathbf{s} -Laplace transform, as the inverse of $\widehat{\mathcal{B}}_\lambda$. When we write a series as a series in a monomial, it is natural to ask what is the relation between its Laplace transform and the transform of its components. That is the content of the next remark.

Remark 4.5. Let $\hat{f} = \sum_{\beta \in \mathbb{N}^d} a_\beta \boldsymbol{\xi}^\beta \boldsymbol{\xi}_{J_s}^{-k\alpha_{J_s}}$ be a formal power series and $\widehat{\mathcal{T}}_\alpha(\boldsymbol{\xi}_{J_s}^{k\alpha_{J_s}} \hat{f}) = \sum f_{\alpha, n} \tau^n$. A necessary and sufficient condition for $\widehat{\mathcal{L}}_\lambda(\hat{f})$ to be convergent is that $\boldsymbol{\xi}_{J_s}^{k\alpha_{J_s}} \hat{f}$ defines a holomorphic map f of exponential growth of order at most λ' on $\{\boldsymbol{\xi} \in \mathbb{C}^d : |\xi_j| < r_j, j \notin J_s\}$ for some $r_j > 0$ (resp. on \mathbb{C}^d if \mathbf{s} has all nonzero entries). Then $\mathcal{L}_\lambda(f)$ is holomorphic in a polydisc at the origin and $\widehat{\mathcal{L}}_\lambda(\hat{f})$ is its Taylor series.

Now assume that there are constants $s, B, D, M > 0$ such that the family of maps $f_{\alpha, n}$ are holomorphic and satisfy the bounds

$$\|f_{\alpha, n}(\boldsymbol{\xi})\| \leq DB^n \Gamma(1 + ns) \exp(MR_{\lambda'}(\boldsymbol{\xi})) \text{ on } \{\boldsymbol{\xi} \in \mathbb{C}^d : |\xi_j| < r_j, j \notin J_s\}.$$

In particular, $\boldsymbol{\xi}_{J_s}^{k\alpha_{J_s}} \hat{f} \in E[[\boldsymbol{\xi}]]_s^\alpha$. Then all the maps $\mathcal{L}_\lambda(\boldsymbol{\xi}_{J_s}^{-k\alpha_{J_s}} f_{\alpha, n})$ are holomorphic in a common polydisc centered at the origin. Furthermore,

if we write $\widehat{T}_\alpha(\widehat{\mathcal{L}}_\lambda(\widehat{f})) = \sum h_{\alpha,n} t^n$, then $f_{\alpha,n}$ and $h_{\alpha,n}$ are related by the formula $h_{\alpha,n} \mathbf{x}^{n\alpha} = \mathcal{L}_\lambda(\xi_{J_s}^{-k\alpha_{J_s}} \xi^{n\alpha} f_{\alpha,n})$. A direct estimate shows that

$$\|h_{\alpha,n}(\mathbf{x})\| \leq DB^n \Gamma(1 + ns) \Gamma(n/k) (\cos \phi - MR_{\lambda'}(\mathbf{x}))^{-1},$$

if $MR_{\lambda'}(\mathbf{x}) < \cos \phi$. Since $\Gamma(n/k) \leq n\Gamma(n/k) = k\Gamma(1 + n/k)$, an application of the first inequality in (22) leads us to conclude that $\widehat{\mathcal{L}}_\lambda(\widehat{f}) \in E[[\mathbf{x}]]_{s+1/k}^\alpha$.

The next technical proposition explains the behavior of the Laplace transform with respect to monomial asymptotic expansions. The hypotheses, although restrictive, are natural when compared to Proposition 4.3.

Proposition 4.6. *Consider $\mathbf{s} \in \overline{\sigma_d}$, $f \in \mathcal{O}(S_\alpha^s, E)$, $S_\alpha^s = S_\alpha^s(\theta, b-a, R)$, $\xi_{J_s}^{k\alpha_{J_s}} \widehat{f} \in \widehat{\mathcal{O}}'_d(E)$ with $\widehat{T}_\alpha(\xi_{J_s}^{k\alpha_{J_s}} \widehat{f}) = \sum f_{\alpha,n} t^n$, and $s \geq 0$. Assume that:*

- (1) *There are constants $B, D, K > 0$ and $0 < r < R$ such that $f_{\alpha,n}$ are holomorphic and satisfy bounds of the type*

$$\|f_{\alpha,n}(\xi)\| \leq DB^n \Gamma(1 + ns) \exp(KR_{\lambda'}(\xi)) \text{ on } \{\xi \in \mathbb{C}^d \mid |\xi_j|^{\alpha_j} < r, j \notin J_s\}.$$

- (2) *For every unbounded subsector S'_α of S_α^s there are constants $C, A, M > 0$ such that*

$$\left\| \xi_{J_s}^{k\alpha_{J_s}} f(\xi) - \sum_{n=0}^{N-1} f_{\alpha,n}(\xi) \xi^{n\alpha} \right\| \leq CA^N \Gamma(1 + Ns) |\xi^{N\alpha}| \exp(MR_{\lambda'}(\xi)) \text{ on } S'_\alpha,$$

for all $N \in \mathbb{N}$. In particular, $\xi_{J_s}^{k\alpha_{J_s}} f \sim_s^\alpha \xi_{J_s}^{k\alpha_{J_s}} \widehat{f}$ on S_α^s .

Then, $\mathcal{L}_\lambda(f) \sim_{s'}^\alpha \widehat{\mathcal{L}}_\lambda(\widehat{f})$ on any sector in \mathbf{x}^α contained in $D_\alpha^s(\theta, b-a + \pi/k; M, R)$, where $s' = s + 1/k$.

Proof: If $N = 0$, assertion (2) means that f has exponential growth of order at most λ' on S_α^s . Write $h = \mathcal{L}_\lambda(f)$ and $\widehat{T}_\alpha(\widehat{\mathcal{L}}_\lambda(\widehat{f})) = \sum h_{\alpha,n} t^n$ as in Remark 4.5. For a fixed $|\phi| < \pi/2$, it is enough to prove the estimates for subsectors S''_α contained in $D_\alpha^s(\theta - \phi/k, b-a; M, r)$. We can find $\delta > 0$ small enough such that $MR_{\lambda'}(\mathbf{x}) < \cos \phi - \delta$ on S''_α . Now, let S'_α be a subsector of S_α^s such that $(x_1 u^{\frac{s_1}{\alpha_1 k}}, \dots, x_d u^{\frac{s_d}{\alpha_d k}}) \in S'_\alpha$ if $\mathbf{x} \in S''_\alpha$ and u is on the half-line from 0 to ∞ in direction ϕ . Applying the hypothesis (2) for S'_α we see that

$$\left\| h(\mathbf{x}) - \sum_{n=0}^{N-1} h_{\alpha,n}(\mathbf{x}) \mathbf{x}^{n\alpha} \right\| \leq \frac{CA^N}{\delta^{N/k}} \Gamma(1 + Ns) \Gamma(N/k) |\mathbf{x}^{N\alpha}| \text{ on } S''_\alpha,$$

as we wanted to show. \square

Finally, we introduce a convolution product that shares similar properties with the classical one. Indeed, the \mathbf{x}^α - k - s -convolution between f and g , $\mathbf{s} \in \overline{\sigma}_d$, is defined by

$$(f *_{\lambda} g)(\mathbf{x}) = \mathbf{x}_{J_s}^{k\alpha_{J_s}} \int_0^1 f(x_1 \tau^{\frac{s_1}{\alpha_1 k}}, \dots, x_d \tau^{\frac{s_d}{\alpha_d k}}) \times g(x_1(1 - \tau)^{\frac{s_1}{\alpha_1 k}}, \dots, x_d(1 - \tau)^{\frac{s_d}{\alpha_d k}}) d\tau,$$

where λ is given by (19). As an example we can compute, with the aid of the Beta function, the convolution between two monomials

$$\frac{\mathbf{x}^\mu}{\Gamma(\langle \mu, \lambda \rangle + 1)} *_{\lambda} \frac{\mathbf{x}^\eta}{\Gamma(\langle \eta, \lambda \rangle + 1)} = \frac{\mathbf{x}^{\mu+\eta} \mathbf{x}_{J_s}^{k\alpha_{J_s}}}{\Gamma(\langle \mu + \eta, \lambda \rangle + 2)},$$

a formula valid for $\mu, \eta \in \mathbb{C}^d$ with entries of positive real part.

If f and g have exponential growth of order at most λ' (resp. they belong to $\mathcal{O}_b(S_\alpha, E)$), then the same is valid for $f *_{\lambda} g$ and

$$(25) \quad \mathcal{L}_\lambda(f *_{\lambda} g) = \mathcal{L}_\lambda(f)\mathcal{L}_\lambda(g) \quad (\text{resp. } \mathcal{B}_\lambda(fg) = \mathcal{B}_\lambda(f) *_{\lambda} \mathcal{B}_\lambda(g)),$$

as in the classical case. This shows in particular that $*_{\lambda}$ is a bilinear, commutative, and associative binary operation, which is distributive over addition.

Remark 4.7. In analogy with the isomorphism explained in (3), for each $\alpha \in (\mathbb{N}^+)^d$, $k > 0$, and $\mathbf{s} \in \overline{\sigma}_d$, we have the following monomorphism between the structures

$$(E[[\mathbf{x}]]_{1/k}^\alpha, +, \times, X_\lambda) \xrightarrow{\widehat{\mathcal{B}}_\lambda} (\xi^{-k\alpha} E\{\xi\}, +, *_{\lambda}, \xi^{k\alpha}(\cdot)),$$

by taking into account (8) for the image, and also (20) and (25).

Remark 4.8. We remark that all transformations introduced here reduce to their counterparts in one variable for maps depending only on the corresponding monomial.

At this point we are ready to define the summation methods based on the above Borel and Laplace transforms. We will see that they turn out to be equivalent to monomial summability.

Definition 4.9. Consider $\alpha \in (\mathbb{N}^+)^d$, $k > 0$, $\mathbf{s} \in \overline{\sigma}_d$, and let λ, λ' be given by (19). We will say that $\hat{f} = \sum_{\beta \in \mathbb{N}^d} a_\beta \mathbf{x}^\beta \in E[[\mathbf{x}]]_{1/k}^\alpha$ is \mathbf{x}^α - k - s -Borel summable in direction θ if $\hat{\varphi}_s = \widehat{\mathcal{B}}_\lambda(\hat{f} - \sum_{k\alpha_{J_s} \leq \beta_{J_s}} a_\beta \mathbf{x}^\beta)$ can be analytically continued, say as φ_s , to a domain of the form $S_\alpha^s(\theta, 2\epsilon, R)$

and having exponential growth of order at most λ' there. In this case, the \mathbf{x}^α - k - s -Borel sum of \hat{f} in direction θ is defined by

$$f(\mathbf{x}) = \sum_{k\alpha_{J_s} \not\leq \beta_{J_s}} a_\beta \mathbf{x}^\beta + \mathcal{L}_\lambda(\varphi_s)(\mathbf{x}).$$

Note that the terms we have removed from \hat{f} give an analytic map at the origin since $\hat{f} \in \widehat{\mathcal{O}}'_d(E)$. Another way to avoid the use of power series with non-integer exponents is to consider $\widehat{\mathcal{B}}_\lambda(\mathbf{x}_{J_s}^{k\alpha_{J_s}} \hat{f})$, attempt analytic continuation with adequate exponential growth, and multiply by $\mathbf{x}_{J_s}^{-k\alpha_{J_s}}$ the corresponding Laplace transformation. We have not followed this equivalent approach since the introduction of such factor does not adapt well for the non-linear terms in the PDEs we consider in Section 6.

Theorem 4.10. *The following statements are equivalent for a series $\hat{f} \in E[[\mathbf{x}]]_{1/k}^\alpha$:*

- (1) $\hat{f} \in E\{\mathbf{x}\}_{1/k, \theta}^\alpha$, i.e., \hat{f} is \mathbf{x}^α - k -summable in direction θ .
- (2) There is $\mathbf{s} \in \overline{\sigma}_d$ such that \hat{f} is \mathbf{x}^α - k - \mathbf{s} -Borel summable in direction θ .
- (3) For all $\mathbf{s} \in \overline{\sigma}_d$, \hat{f} is \mathbf{x}^α - k - \mathbf{s} -Borel summable in direction θ .

In all cases the corresponding sums coincide.

Proof: We may restrict our attention to the case $\hat{f} = \sum_{k\alpha_{J_s} \leq \beta_{J_s}} a_\beta \mathbf{x}^\beta$. To show that (1) implies (3), assume $\hat{f} \in E\{\mathbf{x}\}_{1/k, \theta}^\alpha$ and let $f \in \mathcal{O}(S_\alpha(\theta, \pi/k + 2\epsilon, R_0), E)$ be its \mathbf{x}^α - k -sum in direction θ . For a fixed $\mathbf{s} \in \overline{\sigma}_d$ and λ as usual, set $\varphi_s = \mathcal{B}_\lambda(f)$ and $\hat{\varphi}_s = \widehat{\mathcal{B}}_\lambda(\hat{f})$, convergent in some D_r^d . We can apply Proposition 4.3 to $s = 1/k$ to conclude that $\xi_{J_s}^{k\alpha_{J_s}} \varphi_s \sim_0 \xi_{J_s}^{k\alpha_{J_s}} \hat{\varphi}_s$ on $S_\alpha^s(\theta, 2\epsilon, R_0)$. These two properties imply that φ_s coincides with the sum of $\hat{\varphi}_s$ on $S_\alpha(d, 2\epsilon) \cap D_r^d$. In other words, $\hat{\varphi}_s$ can be analytically continued and having exponential growth of order at most λ' on $S_\alpha^s(\theta, 2\epsilon, R_0)$. Therefore, \hat{f} is \mathbf{x}^α - k - \mathbf{s} -Borel summable in direction θ . Since \mathcal{B}_λ and \mathcal{L}_λ are inverses of each other, the \mathbf{x}^α - k - \mathbf{s} -Borel sum of \hat{f} is f .

The implication (3) to (2) is clear. Assuming (2), fix $\mathbf{s} \in \overline{\sigma}_d$ such that \hat{f} is \mathbf{x}^α - k - \mathbf{s} -Borel summable in direction θ , and let $\varphi_s \in \mathcal{O}(S_\alpha^s(\theta, 2\epsilon, R_0), E)$ and $\hat{\varphi}_s$ be as in Definition 4.9. Also write $\widehat{T}_\alpha(\xi_{J_s}^{k\alpha_{J_s}} \hat{\varphi}_s) = \sum \varphi_{\alpha, n} \tau^n$. Then we can find constants $B, D > 0$ such that

$$(26) \quad \begin{aligned} \|\xi_{J_s}^{k\alpha_{J_s}} \varphi_s(\xi)\| &\leq D \exp(MR_{\lambda'}(\xi)), \\ \|\varphi_{\alpha, n}(\xi)\| &\leq DB^n \exp(MR_{\lambda'}(\xi)), \end{aligned}$$

the first on $S_\alpha^s(d, 2\epsilon', R')$, $0 < \epsilon' < \epsilon$, $0 < R' < R_0$, by our hypothesis, and the second on $\{\boldsymbol{\xi} \in \mathbb{C}^d : |\xi_j|^{\alpha_j} < \rho, j \notin J_s\}$ for some $0 < \rho$, by using Remark 4.2. We may assume $R_0 < \rho$ by reducing R_0 if necessary.

To apply Proposition 4.6 we have to show that there are constants $C, A > 0$ such that, for all $N \in \mathbb{N}$, we have

$$(27) \quad \left\| \boldsymbol{\xi}_{J_s}^{k\alpha_{J_s}} \varphi_s(\boldsymbol{\xi}) - \sum_{n=0}^{N-1} \varphi_{\alpha,n}(\boldsymbol{\xi}) \boldsymbol{\xi}^{n\alpha} \right\| \leq CA^N |\boldsymbol{\xi}^{N\alpha}| \exp(MR_{\lambda'}(\boldsymbol{\xi}))$$

on $S_\alpha^s(\theta, 2\epsilon', R')$.

Since $\boldsymbol{\xi}_{J_s}^{k\alpha_{J_s}} \hat{\varphi}_s$ is the convergent Taylor series of $\boldsymbol{\xi}_{J_s}^{k\alpha_{J_s}} \varphi_s$ at the origin, then (27) is satisfied for all $|\xi_1|, \dots, |\xi_d| \leq R$ for some $R > 0$. Due to (26), the series of maps $\sum \varphi_{\alpha,n}(\boldsymbol{\xi}) \boldsymbol{\xi}^{n\alpha}$ converges absolutely in compact subsets of $\{\boldsymbol{\xi} \in \mathbb{C}^d : B|\boldsymbol{\xi}^\alpha| < 1\}$ and therefore, $\boldsymbol{\xi}_{J_s}^{k\alpha_{J_s}} \varphi_s$ can be analytically continued on this domain through this series. Thus, if $B|\boldsymbol{\xi}^\alpha| < 1/2$, inequality (27) is also satisfied. On the other hand, using again (26), the left hand side of (27) is bounded by

$$\left(D + \sum_{n=0}^{N-1} DB^n |\boldsymbol{\xi}^{n\alpha}| \right) \exp(MR_{\lambda'}(\boldsymbol{\xi})) \quad \text{on } S_\alpha^s(\theta, 2\epsilon', R').$$

If $1/2 \leq B|\boldsymbol{\xi}^\alpha| \leq 2$, the previous term is bounded by $D2^N e^{MR_{\lambda'}(\boldsymbol{\xi})} \leq D(4B)^N |\boldsymbol{\xi}^{N\alpha}| e^{MR_{\lambda'}(\boldsymbol{\xi})}$. If $B|\boldsymbol{\xi}^\alpha| > 2$, we can bound it by

$$\left(D + D \frac{B^N |\boldsymbol{\xi}^{N\alpha}| - 1}{B|\boldsymbol{\xi}^\alpha| - 1} \right) e^{MR_{\lambda'}(\boldsymbol{\xi})} < DB^N |\boldsymbol{\xi}^{N\alpha}| e^{MR_{\lambda'}(\boldsymbol{\xi})},$$

and thus (27) is valid in all cases with C, A large enough. By applying Proposition 4.6 to $\varphi_s, \hat{\varphi}_s$, and $s = 0$, we conclude that $f = \mathcal{L}_\lambda(\varphi_s) \sim_{1/k}^\alpha \hat{\mathcal{L}}_\lambda(\hat{\varphi}_s) = \hat{f}$ on $D_\alpha^s(\theta, 2\epsilon + \pi/k; M, R_0)$. In conclusion, \hat{f} is $\boldsymbol{x}^{\alpha-k}$ -summable in direction θ and its sum can be found through the $\boldsymbol{x}^{\alpha-k}$ - \boldsymbol{s} -Laplace transform of φ_s . \square

As an immediate corollary we can relate monomial summability for different powers of a monomial. The proof follows from Theorem 4.10, by noticing that $\boldsymbol{\lambda} = \left(\frac{s_1}{(N\alpha_1)\frac{k}{N}}, \dots, \frac{s_d}{(N\alpha_d)\frac{k}{N}} \right)$ for all $N \in \mathbb{N}^+$.

Corollary 4.11. *Consider $\boldsymbol{\alpha} \in (\mathbb{N}^+)^d$, $k > 0$, and $\theta \in \mathbb{R}$. Then $E\{\boldsymbol{x}\}_{1/k, \theta}^\alpha = E\{\boldsymbol{x}\}_{N/k, N\theta}^{N\alpha}$ and $E\{\boldsymbol{x}\}_{1/k}^\alpha = E\{\boldsymbol{x}\}_{N/k}^{N\alpha}$ for all $N \in \mathbb{N}^+$.*

5. Tauberian properties for monomial summability

The goal of this section is to recover Tauberian theorems for monomial summability. For instance, the relation between different levels of summability for distinct monomials, and the comparison of summability in one variable with holomorphic coefficients in the remaining ones. The main tool we use to treat these situations are blow-ups with centers of codimension two. We also establish the behavior of Borel–Laplace transformations under these blow-ups.

In one variable we have the following two statements that provide Tauberian properties for k -summability; see e.g., [1, Section 6.4].

Theorem 5.1. *The following statements are true for $0 < k < l$:*

- (1) *If $\hat{f} \in E\{x\}_{1/k}$ has no singular directions, then it is convergent.*
- (2) *$E[[x]]_{1/l} \cap E\{x\}_{1/k} = E\{x\}_{1/l} \cap E\{x\}_{1/k} = E\{x\}$.*

Our goal is to extend this theorem for monomial summability. We know that a series \hat{f} is \mathbf{x}^α - k -summable in some direction θ if and only if there exists $r = r_\theta > 0$ such that $\widehat{T}_\alpha(\hat{f})$ is k -summable in direction θ in $\mathcal{E}_{r_\theta}^\alpha$ in the classical sense. Unfortunately, r_θ might tend to 0 when θ tends to a singular direction. Therefore, \mathbf{x}^α - k -summability of a series \hat{f} does not imply that $\widehat{T}_\alpha(\hat{f})$ is k -summable in \mathcal{E}_r^α for some fixed $r > 0$. For a counterexample, see [3, Section 6]. However, we still can recover Theorem 5.1. We start with the following proposition.

Proposition 5.2. *If $\hat{f} \in E\{\mathbf{x}\}_{1/k}^\alpha$ has no singular directions, then it is convergent.*

Proof: Let us fix $\mathbf{s} \in \sigma_d$ and write $\hat{f} = \sum_{\beta \in \mathbb{N}^d} a_\beta \mathbf{x}^\beta$. We use \mathbf{x}^α - k - \mathbf{s} -Borel summability as explained below Definition 4.9, thus we consider $\varphi_\lambda = \widehat{\mathcal{B}}_\lambda(\mathbf{x}^{k\alpha} \hat{f})$. If \hat{f} has no singular directions for \mathbf{x}^α - k -summability, Theorem 4.10 shows that for each direction $\theta \in [0, 2\pi]$, there are constants $\delta_\theta, C_\theta, M_\theta > 0$ such that $\|\varphi_\lambda(\boldsymbol{\xi})\| \leq C_\theta \exp(M_\theta R_{\lambda'}(\boldsymbol{\xi}))$ for all $\boldsymbol{\xi} \in S_\alpha(\theta, 2\delta_\theta)$. Since the interval $[0, 2\pi]$ is compact, we can choose a finite number of directions $\theta_1, \dots, \theta_n$ such that $[0, 2\pi] \subseteq \bigcup_{j=1}^n (\theta_j - \delta_{\theta_j}, \theta_j + \delta_{\theta_j})$. Then, the sectors $S_\alpha(\theta_j, 2\delta_{\theta_j})$, $j = 1, \dots, n$, cover $\mathbb{C}^d \setminus \{x_1 \cdots x_d = 0\}$, and if we write $C = \max_{1 \leq j \leq n} C_{\theta_j}$ and $M = \max_{1 \leq j \leq n} M_{\theta_j}$, we find that

$$\|\varphi_\lambda(\boldsymbol{\xi})\| \leq C \exp(MR_{\lambda'}(\boldsymbol{\xi})) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{C}^d \setminus \{x_1 \cdots x_d = 0\}.$$

Applying Cauchy’s estimates we see that

$$\frac{\|a_\beta\|}{\Gamma(1 + \sum_{j=1}^d \frac{\beta_j s_j}{\alpha_j k})} \leq C \prod_{j=1}^d \frac{e^{MR_j^{\alpha_j k/s_j}}}{R_j^{\beta_j}} \quad \text{for all } R_j > 0.$$

Since the map $x \mapsto \exp(Mx^l)/x^n$, $l > 0$, $n \in \mathbb{N}$, attains a minimum at $x = (n/Ml)^{1/l}$, if we choose $R_j = (\beta_j s_j / M \alpha_j k)^{\frac{s_j}{\alpha_j k}}$, $j = 1, \dots, d$, we conclude that

$$\|a_\beta\| \leq C \prod_{j=1}^d \left[\left(\frac{2^d M e \alpha_j k}{\beta_j s_j} \right)^{\beta_j s_j / \alpha_j k} \Gamma \left(1 + \frac{\beta_j s_j}{\alpha_j k} \right) \right].$$

Note we have used inequality (22) repeatedly. An application of Stirling’s formula in each factor leads to the existence of constants $A, K > 0$ such that $\|a_\beta\| \leq KA^{|\beta|}$ for all $\beta \in \mathbb{N}^d$, i.e., \hat{f} is a convergent power series. \square

To generalize Theorem 5.1 (2), we consider the monomial transformations

$$\pi_{ij}(x_1, \dots, x_d) = (x_1, \dots, \underbrace{x_i x_j}_{j\text{th entry}}, \dots, x_d),$$

where $i, j = 1, \dots, d$, $i \neq j$. Note that the maps π_{ij} , π_{ji} correspond to the usual charts of the blow-up with the center of codimension two given by $\{x_i = x_j = 0\}$. At the formal level we need the following lemma, whose proof is identical as the one of Lemma 3.6 in [5].

Lemma 5.3. *Let $\hat{f} \in E[[\mathbf{x}]]$ be a formal power series. Then the following assertions are true:*

- (1) $\hat{f} \in E\{\mathbf{x}\}$ if and only if $\hat{f} \circ \pi_{ij} \in E\{\mathbf{x}\}$ for some $i, j = 1, \dots, d$.
- (2) $\hat{f} \in E[[\mathbf{x}]]_s^\alpha$ if and only if there are $i, j = 1, \dots, d$, $i \neq j$, such that $\hat{f} \circ \pi_{ij} \in E[[\mathbf{x}]]_s^{\alpha + \alpha_j e_i}$ and $\hat{f} \circ \pi_{ji} \in E[[\mathbf{x}]]_s^{\alpha + \alpha_i e_j}$.

To establish Lemma 5.3 (2) for summability we will use the following interesting relation between the monomial Borel and Laplace transformations and the blow-up maps π_{ij} . Fix $\alpha \in (\mathbb{N}^+)^d$, $k > 0$, and $\mathbf{s} \in \overline{\sigma}_d$. If for some indices $i \neq j$ we have $s_j \alpha_i \geq s_i \alpha_j$ ($s_i = 0$ if $s_j = 0$), then $\mathbf{s}' = (s'_1, \dots, s'_d) \in \overline{\sigma}_d$, where $s'_l = s_l$ if $l \neq i, j$, $s'_i = s_i + \frac{\alpha_j}{\alpha_i} s_i$, and $s'_j = s_j - \frac{\alpha_j}{\alpha_i} s_i$. Furthermore, if $\lambda = (\frac{s_1}{\alpha_1 k}, \dots, \frac{s_d}{\alpha_d k})$, then $\lambda - \frac{s_i}{\alpha_i k} e_j = (\frac{s'_1}{\alpha'_1 k}, \dots, \frac{s'_d}{\alpha'_d k})$, where $\alpha' = \alpha + \alpha_j e_i$, and the $\mathbf{x}^{\alpha-k-\mathbf{s}}$ -Borel (resp. Laplace) and $\mathbf{x}^{\alpha'-k-\mathbf{s}'}$ -Borel (resp. Laplace) transformations are related by the formulas

$$(28) \quad \mathcal{B}_\lambda(f) \circ \pi_{ij}(\xi) = \mathcal{B}_{\lambda - \frac{s_i}{\alpha_i k} e_j}(f \circ \pi_{ij})(\xi),$$

$$(29) \quad \mathcal{L}_\lambda(f) \circ \pi_{ij}(\xi) = \mathcal{L}_{\lambda - \frac{s_i}{\alpha_i k} e_j}(f \circ \pi_{ij})(\xi),$$

whenever the functions are defined. The same relations hold for their formal counterparts.

The next theorem corresponds to Theorem 7.24 in [15] for monomial summability. Although our approach follows the Borel–Laplace analysis

developed in the previous section, the idea of the proof is based on the same arguments.

Theorem 5.4. $\hat{f} \in E\{\mathbf{x}\}_{1/k, \theta}^\alpha$ if and only if there exist $i \neq j$ such that $\hat{f} \circ \pi_{ij} \in E\{\mathbf{x}\}_{1/k, \theta}^{\alpha + \alpha_j e_i}$ and $\hat{f} \circ \pi_{ji} \in E\{\mathbf{x}\}_{1/k, \theta}^{\alpha + \alpha_i e_j}$.

Proof: Using Proposition 3.6 we see that if $(f_N)_{N \in \mathbb{N}}$ is a family of bounded analytic functions that provide the monomial asymptotic expansion of f , then $(f_N \circ \pi_{ij})_{N \in \mathbb{N}}$ will provide the asymptotic expansion of $f \circ \pi_{ij}$, $i, j = 1, \dots, d$, $i \neq j$.

Conversely, it is enough to do the proof for the case $\alpha = \mathbf{1} = (1, 1, \dots, 1)$; see Remark 3.12. To fix ideas, we take $i = 1$, $j = 2$. It is also not restrictive to assume that $\hat{f} = \sum_{k \leq \beta_1, \beta_2} a_{\beta} \mathbf{x}^{\beta}$. Applying Theorem 4.10, we know by hypothesis that $\hat{f} \circ \pi_{12}$ is $\mathbf{x}^{1+e_1} - k\mathbf{e}_1$ -summable and $\hat{f} \circ \pi_{21}$ is $\mathbf{x}^{1+e_2} - k\mathbf{e}_2$ -summable, both in direction θ . This means that we can find $\varepsilon > 0$ and $R_0 < 1$ such that the maps

$$\varphi_1(\boldsymbol{\xi}) = \widehat{\mathcal{B}}_{\frac{1}{2k}\mathbf{e}_1}(\hat{f} \circ \pi_{12}), \quad \varphi_2(\boldsymbol{\xi}) = \widehat{\mathcal{B}}_{\frac{1}{2k}\mathbf{e}_2}(\hat{f} \circ \pi_{21}),$$

can be analytically continued to the domains $S_{\mathbf{1}+e_1}^{\mathbf{e}_1}(\theta, 2\varepsilon, R_0)$ and $S_{\mathbf{1}+e_2}^{\mathbf{e}_2}(\theta, 2\varepsilon, R_0)$, and furthermore there are constants $C, M > 0$ such that

$$(30) \quad \|\varphi_1(\boldsymbol{\xi})\| \leq C \exp(M|\xi_1|^{2k}), \quad \|\varphi_2(\boldsymbol{\xi})\| \leq C \exp(M|\xi_2|^{2k}),$$

on their respective domains.

We will prove that $\hat{f} \in E\{\mathbf{x}\}_{1/k, \theta}^{\mathbf{1}}$ by showing that \hat{f} is $\mathbf{x}^{1-k}\mathbf{s}$ -summable in direction θ , where $\mathbf{s} = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$. In this case, $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ in (19) are given by $\boldsymbol{\lambda} = \frac{1}{2k}(\mathbf{e}_1 + \mathbf{e}_2)$ and $\boldsymbol{\lambda}' = 2k(\mathbf{e}_1 + \mathbf{e}_2)$. We know that $\varphi = \widehat{\mathcal{B}}_{\boldsymbol{\lambda}}(\hat{f})$ is analytic in a polydisk at the origin. By reducing R_0 if necessary, we assume that $\varphi \in \mathcal{O}(D_{R_0}^d, E)$.

To finish the proof we show that φ can be analytically continued to $S_{\mathbf{1}}^{\mathbf{s}}(\theta, 2\varepsilon, R_0)$ with exponential growth at most $\boldsymbol{\lambda}'$. First of all, we use formulas (28) to write

$$\begin{aligned} \widehat{\mathcal{B}}_{\boldsymbol{\lambda}}(\hat{f})(\xi_1, \xi_1 \xi_2, \boldsymbol{\xi}'') &= \widehat{\mathcal{B}}_{\frac{1}{2k}\mathbf{e}_1}(\hat{f} \circ \pi_{12})(\boldsymbol{\xi}), \\ \widehat{\mathcal{B}}_{\boldsymbol{\lambda}}(\hat{f})(\xi_1 \xi_2, \xi_2, \boldsymbol{\xi}'') &= \widehat{\mathcal{B}}_{\frac{1}{2k}\mathbf{e}_2}(\hat{f} \circ \pi_{21})(\boldsymbol{\xi}), \end{aligned}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \boldsymbol{\xi}'')$. Then, we can extend φ by the rules

$$(31) \quad \varphi(\boldsymbol{\xi}) = \begin{cases} \varphi_1(\xi_1, \xi_2/\xi_1, \boldsymbol{\xi}''), & \text{if } |\xi_2| < R_0|\xi_1|, \\ \varphi_2(\xi_1/\xi_2, \xi_2, \boldsymbol{\xi}''), & \text{if } |\xi_1| < R_0|\xi_2|, \end{cases}$$

as long as $\boldsymbol{\xi}$ satisfies $|\arg(\boldsymbol{\xi}^{\mathbf{1}}) - \theta| < \varepsilon$, $|\xi_j| < R_0$, $j \neq 1, 2$, and $1/R_0 < |\xi_2/\xi_1|$ or $|\xi_2/\xi_1| < R_0$. Note that φ has exponential growth at most $\boldsymbol{\lambda}'$ in

this domain due to the inequalities in (30). Finally, to remove the restrictions on the norms of ξ_1, ξ_2 , we use Cauchy's integral formula. To simplify notation, we use the auxiliary variable $w = \xi_3 \cdots \xi_d$ (the case $d = 2$ does not require w or ξ'' above). Since we are working with the monomial ξ^1 , we also introduce the variable $\tau = \xi^1 = \xi_1 \xi_2 w$. Then we define the map

$$(32) \quad G(\xi_1, \tau, \xi'') = \frac{1}{2\pi i} \left(\int_{|\zeta|=R} - \int_{|\zeta|=r} \right) \frac{\varphi\left(\zeta, \frac{\tau/w}{\zeta}, w\right)}{\zeta - \xi_1} d\zeta,$$

where φ is given by (31). Thus, we require that $R_0^{-1}r^2 < |\tau/w| < R_0R^2$ and $0 < r < R$. Note that the integral is independent of r and R as long as these constraints are satisfied. We will check that G defines an holomorphic map on $\Omega = \mathbb{C}^* \times S(\theta, 2\varepsilon) \times (D_{R_0}^{d-2} \setminus \{w = 0\}) = \mathbb{C}^* \times \Omega_1$. Then G will provide the required extension to $S_1^s(\theta, 2\varepsilon, R_0)$, since $G(\xi_1, \xi^1, \xi'') = \varphi(\xi)$ if $R_0^{-1}r^2 < |\xi_1 \xi_2| < R_0R^2$ and $r < |\xi_1| < R$, due to Cauchy's integral formula.

To prove the holomorphy of G , consider $U \subseteq K \subseteq \mathbb{C}^*$ and $U_1 \subseteq K_1 \subseteq \Omega_1$ where U, U_1 are open and K, K_1 are compact. Write $L_1 = \inf_{(\tau, \xi'') \in K_1} |\tau/w|$ and $L_2 = \sup_{(\tau, \xi'') \in K_1} |\tau/w|$, both finite positive numbers, since τ and w do not vanish on Ω_1 . Then choose $0 < r < R$ such that $r^2 < R_0L_1 < R_0^{-1}L_2 < R^2$ (recall that $R_0 < 1$) and $r < \inf_{\xi_1 \in K} |\xi_1| \leq \sup_{\xi_1 \in K} |\xi_1| < R$. Then G is defined at all points of $U \times U_1$ and it is clearly holomorphic there.

Finally, we need to show that the extension of φ has exponential growth of order at most λ' on $S_1^s(\theta, 2\varepsilon, R_0)$ for ξ_1, ξ_2 such that $R_0 \leq |\xi_2/\xi_1| \leq R_0^{-1}$. By calculating the values of φ in (32) for $|\zeta| = R$ using φ_1 and for $|\zeta| = r$ using φ_2 instead, we can employ the inequalities in (30) to find

$$\|\varphi(\xi)\| = \|G(\xi_1, \xi^1, \xi'')\| \leq \frac{CR \exp(MR^{2k})}{\text{dist}(\xi_1, \partial D_R)} + \frac{Cr \exp(M|\tau/rw|^{2k})}{\text{dist}(\xi_1, \partial D_r)}.$$

Since $\tau/w = \xi_1 \xi_2$, by taking $R^2 = 4|\xi_1 \xi_2|/R_0$ and $r^2 = R_0|\xi_1 \xi_2|/4$, we conclude that

$$\|\varphi(\xi)\| \leq \left(\frac{R}{|R - |\xi_1||} + \frac{r}{|r - |\xi_1||} \right) C \exp(M(4/R_0)^k |\xi_1 \xi_2|^k).$$

Note that the denominators do not vanish and are uniformly bounded due to the restriction $R_0 \leq |\xi_2/\xi_1| \leq R_0^{-1}$. The conclusion now follows by noting that $|\xi_1 \xi_2|^k \leq \max\{|\xi_1|^{2k}, |\xi_2|^{2k}\}$. \square

We are ready to state and prove the third main result so far, comparing summable series in different monomials. This result was obtained

in [5]. Although the content is correct, the proof given there is based on the false statement discussed above Proposition 5.2. This is repaired here.

Theorem 5.5. *Consider $\alpha, \alpha' \in (\mathbb{N}^+)^d$ and $k, k' > 0$. The following statements hold:*

- (1) *If $\hat{f} \in E\{\mathbf{x}\}_{1/k}^\alpha$ and $\widehat{T}_\alpha(\hat{f})$ is an s -Gevrey series with some $s < 1/k$, then \hat{f} is convergent. In particular, if $\max_{1 \leq j \leq d} \{\alpha_j / \alpha'_j\} < k'/k$, then $E\{\mathbf{x}\}_{1/k}^\alpha \cap E[[\mathbf{x}]]_{1/k'}^{\alpha'} = E\{\mathbf{x}\}$.*
- (2) *$E\{\mathbf{x}\}_{1/k}^\alpha \cap E\{\mathbf{x}\}_{1/k'}^{\alpha'} = E\{\mathbf{x}\}$ except in the case $k\alpha = k'\alpha'$, where $E\{\mathbf{x}\}_{1/k}^\alpha = E\{\mathbf{x}\}_{1/k'}^{\alpha'}$.*

Proof: (1) The proof of the first statement is based on the proof of Theorem 3.8.2 in [17]. We write $\widehat{T}_\alpha(\hat{f})(t) = \sum_{n=0}^{\infty} f_{\alpha,n} t^n$, with $f_{\alpha,n} \in \mathcal{E}_r^\alpha$, and use the k -Borel transform g of $\widehat{T}_\alpha(\hat{f})$ in the form $g(\mathbf{x}, \xi) = \sum_{n=0}^{\infty} \frac{f_{\alpha,n}(\mathbf{x})}{\Gamma(1+n/k)} \xi^n$. Since $\widehat{T}_\alpha(\hat{f})$ is s -Gevrey with some $s < 1/k$, we find constants $K, A > 0$ such that

$$\frac{\|f_{\alpha,n}(\mathbf{x})\|}{\Gamma(1+n/k)} \leq KA^n n!^{-1/\mu} \quad \text{for all } \mathbf{x} \in D_r^d, n \in \mathbb{N}, \quad 1/\mu := 1/k - s.$$

This implies that g defines a holomorphic function on $D_r^d \times \mathbb{C}$ and we can find constants $L, B > 0$ such that

$$\|g(\mathbf{x}, \xi)\| \leq L \exp(B|\xi|^\mu) \quad \text{for all } (\mathbf{x}, \xi) \in D_r^d \times \mathbb{C}.$$

We now show that \hat{f} has no singular directions for \mathbf{x}^α - k -summability. Thus, it is convergent due to Proposition 5.2. Indeed, arguing by contradiction, we assume θ is a singular direction of \hat{f} . We choose $0 < \delta < \frac{\pi}{2\mu}$ such that $\widehat{T}_\alpha(\hat{f})$ is k -Borel-summable in the directions $\theta_\pm = \theta \pm \delta$, in some $\mathcal{E}_{r'}^\alpha$, $0 < r' < r$. Then, there exist $0 < \rho < r'$ and $M, C > 0$ such that g satisfies

$$\|g(\mathbf{x}, \xi)\| \leq M \exp(C|\xi|^k) \quad \text{for all } \mathbf{x} \in D_\rho^d, \arg(\xi) = \theta_\pm.$$

We can use the Phragmén–Lindelöf principle (see e.g. [1, Theorem 70, p. 235]) to show that g also has exponential growth of order k on the sector $S(\theta, 2\delta)$. Indeed, consider the map $h(\mathbf{x}, \xi) = g(\mathbf{x}, \xi) \exp(-D(\xi e^{-i\theta})^k)$, where $D \cos(k\delta) = C$. By using the previous bounds, it follows that $\|h(\mathbf{x}, \xi)\| \leq M$ if $\arg(\xi) = \theta_\pm$ and $\|h(\mathbf{x}, \xi)\| \leq L \exp(B|\xi|^\mu)$ if $\arg(\xi) \in [\theta_-, \theta_+]$ for all $\mathbf{x} \in D_\rho^d$. Since the opening of $S(\theta, 2\delta)$ is smaller than π/μ ,

the Phragmén–Lindelöf principle yields that h is bounded on the full sector. Thus we can find constants $\widetilde{M}, \widetilde{C} > 0$ such that

$$\|g(\mathbf{x}, \xi)\| \leq \widetilde{M} \exp(\widetilde{C}|\xi|^k) \quad \text{for all } (\mathbf{x}, \xi) \in D_\rho^d \times S(\theta, 2\delta).$$

This means that $\widehat{T}_\alpha(\hat{f})$ is k -Borel-summable in direction θ in \mathcal{E}_ρ^α . Thus, \hat{f} is \mathbf{x}^α - k -summable in direction θ , which contradicts the assumption.

The second statement in (1) follows from the first one, since Lemma 3.2 (2) implies that if $\hat{f} \in E[[\mathbf{x}]]_{1/k}^{\alpha'}$, then the series $\widehat{T}_\alpha(\hat{f})$ is $\max_{1 \leq j \leq d} \{\alpha_j/\alpha'_j\}/k'$ -Gevrey in some \mathcal{E}_r^α .

(2) Consider $\hat{f} \in E\{\mathbf{x}\}_{1/k}^\alpha \cap E\{\mathbf{x}\}_{1/k'}^{\alpha'}$. If α_j/α'_j is independent of j , write this positive rational number as a/b , with $\text{g.c.d.}(a, b) = 1$. Then $\alpha_j = m_j a$, $\alpha'_j = m_j b$, for some $m_j \in \mathbb{N}^+$ and, by applying Corollary 4.11, we have $E\{\mathbf{x}\}_{1/k}^\alpha = E\{\mathbf{x}\}_{1/ak}^m$ and $E\{\mathbf{x}\}_{1/k'}^{\alpha'} = E\{\mathbf{x}\}_{1/bk'}^m$ where $\mathbf{m} = (m_1, \dots, m_d)$. Then, this case follows from (1). Furthermore, the cases $\max_{1 \leq j \leq d} \{\alpha_j/\alpha'_j\} < k'/k$ and $k'/k < \min_{1 \leq j \leq d} \{\alpha_j/\alpha'_j\}$ also follow from (1).

Finally, the case $\min_{1 \leq j \leq d} \alpha_j/\alpha'_j \leq k'/k \leq \max_{1 \leq j \leq d} \alpha_j/\alpha'_j$ can be reduced to the previous situations by using monomial transformations. To fix ideas, assume that $\alpha_1/\alpha'_1 \leq \alpha_2/\alpha'_2 \leq \dots \leq \alpha_d/\alpha'_d$ and $\alpha_1/\alpha'_1 < k'/k$. If j_0 is the smallest index such that $k'/k \leq \alpha_{j_0}/\alpha'_{j_0}$, for each $j_0 \leq j \leq d$ choose $N_j \in \mathbb{N}^+$ such that

$$\frac{k\alpha_j - k'\alpha'_j}{k'\alpha'_1 - k\alpha_1} < N_j.$$

Theorem 5.4 shows that

$$\hat{f}_1 = \hat{f} \circ \pi_{d1}^{N_d} \circ \dots \circ \pi_{j_0 1}^{N_{j_0}} \in E\{\mathbf{x}\}_{1/k}^{\alpha + \alpha_1 \sum_{j=j_0}^d N_j e_j} \cap E\{\mathbf{x}\}_{1/k'}^{\alpha' + \alpha'_1 \sum_{j=j_0}^d N_j e_j},$$

but the new monomials satisfy

$$\max_{\substack{1 \leq i < j_0 \\ j_0 \leq j \leq d}} \{\alpha_i/\alpha'_i, (N_j \alpha_1 + \alpha_j)/(N_j \alpha'_1 + \alpha'_j)\} < k'/k.$$

Thus \hat{f}_1 is convergent, and by Lemma 5.3 (1) so is \hat{f} . □

The same idea of proof can be generalized to construct series which are not \mathbf{x}^α - k -summable for any \mathbf{x}^α or $k > 0$. The following theorem is a version of [5, Theorem 3.9], which is incorrect as it is stated there. What is actually proved there, by induction on n , is the following.

Theorem 5.6. *Consider $\alpha_1, \dots, \alpha_n \in (\mathbb{N}^+)^d$ and $k_1, \dots, k_n > 0$. For each $j = 1, \dots, n$, take a series $\hat{f}_j \in E\{\mathbf{x}\}_{1/k_j}^{\alpha_j}$. If $k_i \alpha_i \neq k_j \alpha_j$, for all $i \neq j$ and $\hat{f}_1 + \dots + \hat{f}_n = 0$, then $\hat{f}_j \in E\{\mathbf{x}\}$ for all $j = 1, \dots, n$.*

Given $\hat{f} \in \mathcal{O}'_d(E)$, we can also consider the case where \hat{f} is k -summable in a monomial in some variables with coefficients which are holomorphic maps in the remaining ones and compare the summability phenomena we have at our disposal. As a matter of fact, in this situation the methods are again incompatible and the proofs can be reduced to Theorem 5.5 using monomial transformations. The key statement is the following.

Proposition 5.7. *Let $J \subsetneq [1, d]$ be non-empty, $n = \#J$, and let $\hat{f} \in \mathcal{O}_b(D_R^{d-n}, E)\{\mathbf{x}_J\}_{1/k, \theta}^{\alpha_J}$ with $\mathbf{x}_J^{\alpha_J}$ k -sum f in direction θ . Then $\hat{f} \circ \pi_{ij} \in \mathcal{O}_b(D_{R'}^{d-n}, E)\{\mathbf{x}_J\}_{1/k, \theta}^{\alpha_J}$ if $j \notin J$, and $\hat{f} \circ \pi_{ij} \in \mathcal{O}_b(D_{R'}^{d-n-1}, E)\{\mathbf{x}_{J \cup \{i\}}\}_{1/k, \theta}^{\alpha_{J \cup \{i\}}}$ if $j \in J$, where $\mathbf{x}_{J \cup \{i\}}^{\alpha_{J \cup \{i\}}} = x_i^{\alpha_j} \mathbf{x}_J^{\alpha_J}$ for small $0 < R' \leq R$. In both cases the corresponding sum is given by $f \circ \pi_{i,j}$.*

Proof: Let $\hat{f} = \sum_{\beta_J \in \mathbb{N}^J} f_{J, \beta_J}(\mathbf{x}_{J^c}) \mathbf{x}_J^{\beta_J}$ as in equation (4), where $f_{J, \beta_J} \in \mathcal{O}_b(D_R^{d-n}, E)$ and $\hat{T}_{\alpha_J}(\hat{f}) = \sum_{n=0}^{\infty} f_{\alpha_J, n}(\mathbf{x}) \mathbf{x}_J^{n\alpha_J}$. If $f \sim_{1/k}^{\alpha_J} \hat{f}$ on $S_{\alpha_J}(\theta, b-a, r)$, $b-a > \pi/k$, for every subsector S'_{α_J} of S_{α_J} we can find constants $C, A > 0$ such that for every $N \in \mathbb{N}$ we have

$$\left\| f(\mathbf{x}) - \sum_{n=0}^{N-1} f_{\alpha_J, n}(\mathbf{x}) \mathbf{x}_J^{n\alpha_J} \right\| \leq C A^N N!^{\frac{1}{k}} |\mathbf{x}_J^{N\alpha_J}|$$

on $\{\mathbf{x} \in \mathbb{C}^d \mid \mathbf{x}_J \in S'_{\alpha_J}, \mathbf{x}_{J^c} \in D_R^{d-n}\}$.

Note that $\mathbf{x}_J^{\alpha_J} \circ \pi_{i,j} = \mathbf{x}_J^{\alpha_J}$ if $j \notin J$ and $\mathbf{x}_J^{\alpha_J} \circ \pi_{i,j} = x_i^{\alpha_j} \mathbf{x}_J^{\alpha_J}$ if $j \in J$. Then both statements follow with the aid of Proposition 3.6 by replacing \mathbf{x} by $\pi_{ij}(\mathbf{x})$ in the previous inequality as long as we choose the radii r, R small enough such that $\pi_{ij}(\mathbf{x}) \in \{\mathbf{x} \in \mathbb{C}^d : \mathbf{x}_J \in S'_{\alpha_J}, \mathbf{x}_{J^c} \in D_R^{d-n}\}$. \square

Theorem 5.8. *Let $I, J \subseteq [1, d]$ be non-empty sets, $n = \#J$, $m = \#I$, and consider $\alpha_J \in (\mathbb{N}^+)^m$, $\alpha'_I \in (\mathbb{N}^+)^n$, and $k, k' > 0$. Then*

$$\mathcal{O}_b(D_R^{d-n}, E)\{\mathbf{x}_J\}_{1/k}^{\alpha_J} \cap \mathcal{O}_b(D_R^{d-m}, E)\{\mathbf{x}_I\}_{1/k'}^{\alpha'_I} = E\{\mathbf{x}\},$$

except in the case $J = I$ and $k\alpha_J = k'\alpha'_I$, where the spaces coincide.

Proof: We divide the proof into several cases. First, if $J = I$ the result follows from Theorem 5.5 applied to the space $\mathcal{O}_b(D_R^{d-m}, E)$ (resp. E if $m = d$). Second, assume $J \subsetneq I$. Changing the order of coordinates if necessary we can assume $J = \{1, \dots, n\}$ and $I = \{1, \dots, m\}$. Then $\hat{f}_1 = \hat{f} \circ \pi_{m,1} \circ \dots \circ \pi_{n+1,1}$ is $\mathbf{x}_J^{\alpha_J} (x_{n+1} \dots x_m)^{\alpha_1}$ - k -summable and $\mathbf{x}_I^{\alpha'_I} (x_{n+1} \dots x_m)^{\beta_1}$ - k' -summable, both with coefficients in $\mathcal{O}_b(D_R^{d-m}, E)$ (resp. E if $m = d$).

We can apply the previous case to conclude that \hat{f}_1 and thus \hat{f} are convergent since $\alpha_1/(\alpha'_m + \alpha'_1) = \alpha_1/\alpha'_1$ implies $\alpha'_m = 0$, which is not the case. Finally, we can assume there are $j_0 \in J \setminus I$ and $i_0 \in I \setminus J$, and we consider the series

$$\hat{f}_2 = \hat{f} \circ \circ \prod_{i \in I \setminus J} \pi_{i j_0} \circ \circ \prod_{j \in J \setminus I} \pi_{j i_0}.$$

Here $\circ \prod$ denotes the composition product, which in this case is independent of the order because the monomial transformations involved commute since $j_0 \neq i_0$. Then \hat{f}_2 is $\mathbf{x}_I^{\alpha'_I} (\prod_{j \in J \setminus I} x_j)^{\alpha'_{i_0}}$ - k' -summable and $\mathbf{x}_J^{\alpha_J} (\prod_{i \in I \setminus J} x_i \cdot \prod_{j \in J \setminus I} x_j)^{\alpha_{j_0}}$ - k -summable with coefficients in $\mathcal{O}_b(D_R^{d-\#(I \cup J)}, E)$ (resp. E if $J \cup I = [1, d]$). Since the i_0 -components (resp. j_0 -components) of these monomials are α'_{i_0} and α_{j_0} (resp. α'_{i_0} and $2\alpha_{j_0}$) respectively, then $\alpha_{j_0}/\alpha'_{i_0} \neq 2\alpha_{j_0}/\alpha'_{i_0}$, and therefore \hat{f}_2 and \hat{f} are convergent as we wanted to prove. \square

Having this result at hand it is possible to formulate and prove a statement similar to the one in Theorem 5.6. This provides more examples of series that cannot be summed with the methods we have studied along this work. In fact, this has been done recently in [7] in the more general setting of k -summability in analytic germs and we refer the reader to this work for a complete proof of these facts.

6. Monomial summability of a family of singular perturbed PDEs

Summability in a monomial is useful to study formal solutions of doubly singular equations, i.e., singularly perturbed ordinary differential equations of the form

$$(33) \quad \varepsilon^q x^{p+1} \frac{\partial \mathbf{y}}{\partial x} = \mathbf{F}(x, \varepsilon, \mathbf{y}),$$

where $p, q \in \mathbb{N}^+$ and \mathbf{F} is a \mathbb{C}^N -valued holomorphic map in some neighborhood of $(0, 0, \mathbf{0}) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^N$. If $\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(0, 0, \mathbf{0})$ is an invertible matrix, this system has a unique formal power series solution and it is $x^p \varepsilon^q$ -1-summable; see [3, Theorem 5.2]. The technique employed to prove this result in the case $p = q = 1$ is to apply the change of variables $t = x\varepsilon$ to obtain an equation involving t and ε . The new equation is then solved on large sectors and the differences of such solutions, in their common domains, are studied in order to apply Ramis–Sibuya’s theorem. The general case follows after rank reduction.

The goal of this section is to generalize the previous result by establishing the $\mathbf{x}^\alpha \varepsilon^{\alpha'}$ -1-summability of the unique formal power series solution of the singularly perturbed partial differential equation (1) explained in the introduction. By hypothesis, the n -tuple $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ has entries positive real numbers, up to a non-zero multiple scalar. Then, dividing equation (1) by $\langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle$, it is sufficient to study the singularly perturbed partial differential equation

$$(34) \quad \varepsilon^{\alpha'} X_{\boldsymbol{\lambda}}(\mathbf{y})(\mathbf{x}, \varepsilon) = \varepsilon^{\alpha'} \mathbf{x}^\alpha \left(\frac{s_1}{\alpha_1} x_1 \frac{\partial \mathbf{y}}{\partial x_1} + \dots + \frac{s_n}{\alpha_n} x_n \frac{\partial \mathbf{y}}{\partial x_n} \right) = \mathbf{F}(\mathbf{x}, \varepsilon, \mathbf{y}),$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$ are tuples of complex variables, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\boldsymbol{\alpha}' = (\alpha'_1, \dots, \alpha'_m)$ are tuples of positive integers, $\boldsymbol{\lambda} = \left(\frac{s_1}{\alpha_1}, \dots, \frac{s_n}{\alpha_n} \right)$ where the $s_j / \alpha_j = \mu_j / \langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle > 0$ satisfy $s_1 + \dots + s_n = 1$, $\mathbf{F} = \langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle^{-1} \mathbf{G}$ is a \mathbb{C}^N -valued holomorphic map in some neighborhood of $(\mathbf{0}, \mathbf{0}, \mathbf{0}) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}^N$, and $A_0 := \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{0}, \mathbf{0}, \mathbf{0})$ is an invertible matrix.

We will apply directly the Borel–Laplace analysis developed in Section 4, based on the methods of one variable; see e.g., [9]. The existence of the unique formal solution is a straightforward result. To determine the Gevrey type of this solution we can use a variant of Nagumo norms, as the ones used in [4]. For the summability, we will study the convolution equation obtained from (34) after applying the adequate Borel transformation. After introducing suitable spaces of analytic functions and norms, we will solve the convolution equation using the Banach fixed-point theorem.

Theorem 6.1. *Consider the singularly perturbed partial differential equation (1). If $\mathbf{G}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $B_0 = \frac{\partial \mathbf{G}}{\partial \mathbf{y}}(\mathbf{0}, \mathbf{0}, \mathbf{0})$ is an invertible matrix, then (1) has a unique formal power series solution $\hat{\mathbf{y}} \in \mathbb{C}[[\mathbf{x}, \boldsymbol{\varepsilon}]]^N$ and it is $\mathbf{x}^\alpha \varepsilon^{\alpha'}$ -1-summable, with possible singular directions determined by the equation*

$$\det \left(\langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle \boldsymbol{\xi}^\alpha \boldsymbol{\eta}^{\alpha'} I_N - B_0 \right) = 0,$$

in the $(\boldsymbol{\xi}, \boldsymbol{\eta})$ -Borel space. Here I_N denotes the identity matrix of size N .

Proof: We divide the proof into four main steps: existence and Gevrey type of the formal solution, establishment of the associated convolution equation, introduction of adequate Banach algebras and their properties, and finally the application of the fixed point theorem.

1. *The formal solution.* We will consider the norms $|\mathbf{y}| = \max_{1 \leq j \leq N} |y_j|$ on \mathbb{C}^N and $|A| = \max_{1 \leq i \leq N} \sum_{j=1}^N |A_{ij}|$ on $\mathbb{C}^{N \times N}$, where the notation should be clear from context.

Let us write

$$\mathbf{F}(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{y}) = c(\mathbf{x}, \boldsymbol{\varepsilon}) + A(\mathbf{x}, \boldsymbol{\varepsilon})\mathbf{y} + \sum_{|I| \geq 2} A_I(\mathbf{x}, \boldsymbol{\varepsilon})\mathbf{y}^I,$$

as a power series in \mathbf{y} , where $c, A_I \in \mathcal{O}_b(D_r^n \times D_R^m, \mathbb{C}^N)$, $A \in \mathcal{O}_b(D_r^n \times D_R^m, \mathbb{C}^{N \times N})$ for all $I = (i_1, \dots, i_N) \in \mathbb{N}^N$ and $\mathbf{y}^I = y_1^{i_1} \cdots y_N^{i_N}$. We also write $c(\mathbf{x}, \boldsymbol{\varepsilon}) = \sum_{\boldsymbol{\beta} \in \mathbb{N}^n} c_{\boldsymbol{\beta}}(\boldsymbol{\varepsilon})\mathbf{x}^{\boldsymbol{\beta}} = \sum_{\boldsymbol{\beta}' \in \mathbb{N}^m} c_{\boldsymbol{\beta}'}(\mathbf{x})\boldsymbol{\varepsilon}^{\boldsymbol{\beta}'}$ as a convergent power series in \mathbf{x} (resp. in $\boldsymbol{\varepsilon}$) with coefficients $c_{\boldsymbol{\beta}} \in \mathcal{O}_b(D_R^m, \mathbb{C}^N)$ (resp. $c_{\boldsymbol{\beta}'} \in \mathcal{O}_b(D_r^n, \mathbb{C}^N)$). Since \mathbf{F} is holomorphic we can find constants $K, \delta > 0$ such that

$$(35) \quad \|A_I\|_{r,R} := \sup_{(\mathbf{x}, \boldsymbol{\varepsilon}) \in D_r^n \times D_R^m} |A_I(\mathbf{x}, \boldsymbol{\varepsilon})| \leq K\delta^{|I|}.$$

The notation $\|\cdot\|_{r,R}$ will be also used for matrix valued functions.

To find the formal solution set $\hat{\mathbf{y}} = \sum_{\boldsymbol{\beta} \in \mathbb{N}^n} \mathbf{y}_{\boldsymbol{\beta}}(\boldsymbol{\varepsilon})\mathbf{x}^{\boldsymbol{\beta}}$. Since $\mathbf{F}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = 0$ and A_0 is invertible we can apply the implicit function theorem to find a unique $\mathbf{y}_0(\boldsymbol{\varepsilon}) \in \mathcal{O}_b(D_R^m, \mathbb{C}^N)$, $\mathbf{y}_0(\mathbf{0}) = \mathbf{0}$ (with R small enough) solving the equation $\mathbf{F}(\mathbf{0}, \boldsymbol{\varepsilon}, \mathbf{y}_0(\boldsymbol{\varepsilon})) = 0$. To determine the higher order terms we use the recurrence

$$\langle \boldsymbol{\lambda}, \boldsymbol{\beta} - \boldsymbol{\alpha} \rangle \boldsymbol{\varepsilon}^{\boldsymbol{\alpha}'} \mathbf{y}_{\boldsymbol{\beta} - \boldsymbol{\alpha}}(\boldsymbol{\varepsilon}) = c_{\boldsymbol{\beta}}(\boldsymbol{\varepsilon}) + A(\mathbf{0}, \boldsymbol{\varepsilon})\mathbf{y}_{\boldsymbol{\beta}}(\boldsymbol{\varepsilon}) + \text{known terms},$$

obtained by inserting $\hat{\mathbf{y}}$ in the differential equation and equating the coefficient of $\mathbf{x}^{\boldsymbol{\beta}}$. Since $A(\mathbf{0}, \mathbf{0}) = A_0$ is invertible, we may assume (by reducing R again if necessary) that $A(\mathbf{0}, \boldsymbol{\varepsilon})$ is invertible for all $\boldsymbol{\varepsilon} \in D_R^m$. Thus this recurrence determines $\mathbf{y}_{\boldsymbol{\beta}}$ uniquely, and the uniqueness of $\hat{\mathbf{y}}$ follows.

Similar computations as for the classical irregular singularities for ODEs show that $\hat{\mathbf{y}}$ is a 1-Gevrey series in $\mathbf{x}^{\boldsymbol{\alpha}}$. To determine the Gevrey order in $\boldsymbol{\varepsilon}$, we use the following variant of the Nagumo norms for higher dimensions: if $f \in \mathcal{O}(D_r^n \times D_R^m)$ and $l \in \mathbb{N}$, we define

$$\|f\|_l := \sup_{(\mathbf{x}, \boldsymbol{\varepsilon}) \in D_r^n \times D_R^m} |f(\mathbf{x}, \boldsymbol{\varepsilon})|(r - |x_1|)^l \cdots (r - |x_n|)^l.$$

They satisfy the majorant inequalities

$$\|f+g\|_l \leq \|f\|_l + \|g\|_l, \quad \|fg\|_{l+k} \leq \|f\|_l \|g\|_k, \quad \left\| \frac{\partial f}{\partial x_j} \right\|_{l+1} \leq e(l+1)r^{n-1} \|f\|_l,$$

for all $l, k \in \mathbb{N}$, $j = 1, \dots, n$. The proof of the last inequality can be done in the same way as it is proved in [4] for the Nagumo norms intro-

duced there. Then, applying the usual majorant series technique to the recurrence PDEs

$$\mathbf{x}^\alpha \sum_{j=1}^n \frac{s_j}{\alpha_j} x_j \frac{\partial}{\partial x_j} (\mathbf{y}_{\beta' - \alpha'}) = c_{\beta'}(\mathbf{x}) + A(\mathbf{x}, \mathbf{0}) \mathbf{y}_{\beta'}(\mathbf{x}) + \text{known terms},$$

obtained by replacing $\hat{\mathbf{y}} = \sum_{\beta' \in \mathbb{N}^m} \mathbf{y}_{\beta'}(\mathbf{x}) \varepsilon^{\beta'}$ into (34), we may conclude that $\hat{\mathbf{y}}$ is also 1-Gevrey in $\varepsilon^{\alpha'}$. Here we have also reduced r (if necessary) to guarantee that $A(\mathbf{x}, \mathbf{0})$ is invertible for all $\mathbf{x} \in D_r^n$.

2. *The convolution equation.* To simplify notation we will write $\widehat{\mathcal{B}} = \widehat{\mathcal{B}}_{(\boldsymbol{\lambda}, \mathbf{0})}$ and $*$ = $*_{(\boldsymbol{\lambda}, \mathbf{0})}$, where $(\boldsymbol{\lambda}, \mathbf{0}) \in \mathbb{R}_{>0}^n \times \mathbb{R}^m$. Also set $\mathbf{s} = (s_1, \dots, s_n)$ which by hypothesis belongs to σ_n .

Applying \mathcal{B} to (34) we obtain a convolution equation, that written as a fixed point equation, is given by

$$(36) \quad \begin{aligned} (\boldsymbol{\xi}^\alpha \boldsymbol{\eta}^{\alpha'} I_N - A_0) \mathbf{Y} &= \mathcal{B}(c) + (\mathcal{B}(A - A(\mathbf{0}, \boldsymbol{\eta}))) * \mathbf{Y} + (A(\mathbf{0}, \boldsymbol{\eta}) - A_0) \mathbf{Y} \\ &+ \sum_{|I| \geq 2} \mathcal{B}(A_I - A_I(\mathbf{0}, \boldsymbol{\eta})) * \mathbf{Y}^{*I} \\ &+ \sum_{|I| \geq 2} A_I(\mathbf{0}, \boldsymbol{\eta}) \mathbf{Y}^{*I}, \end{aligned}$$

in the $(\boldsymbol{\xi}, \boldsymbol{\eta})$ -Borel plane. Here we write $\mathbf{Y}^{*I} = Y_1^{*i_1} * \dots * Y_N^{*i_N}$ and $Y_j^{*i_j} = Y_j * \dots * Y_j$, i_j times.

Under the holomorphic change of variables $\mathbf{w} = \mathbf{y} - \sum_{\alpha \neq \beta} \mathbf{y}_\beta(\varepsilon) \mathbf{x}^\beta$, we may assume that $c(\mathbf{x}, \varepsilon) = \sum_{\beta > \alpha} c_\beta(\varepsilon) \mathbf{x}^\beta$, $\mathcal{B}(c)$ is holomorphic at the origin, and $\mathcal{B}(c)(\mathbf{0}, \boldsymbol{\eta}) = \mathbf{0}$, and so we will do it from now on.

Equation (36) has a unique analytic solution at the origin given by $\mathbf{Y} = \mathcal{B}(\hat{\mathbf{y}})$. To solve (36) in larger domains, we ask $\boldsymbol{\xi}^\alpha \boldsymbol{\eta}^{\alpha'} I_N - A_0$ to be invertible. Let $\nu_1 = |\nu_1| e^{i\theta_1}, \dots, \nu_N = |\nu_N| e^{i\theta_N}$ be the eigenvalues of A_0 repeated according to their multiplicity, all different from zero by hypothesis. We will work on domains contained in

$$\Omega := \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{C}^n \times \mathbb{C}^m : \boldsymbol{\xi}^\alpha \boldsymbol{\eta}^{\alpha'} \neq \nu_j \text{ for all } j = 1, \dots, N\}.$$

3. *Some focusing spaces.* Consider

$$S := S_{r, R'} = S_{(\boldsymbol{\alpha}, \boldsymbol{\alpha}')}^{(s, \mathbf{0})}(\theta, 2\epsilon, R') \cup (D_r^n \times \overline{D}_{R'^{1/\alpha'_1}} \times \dots \times \overline{D}_{R'^{1/\alpha'_m}}),$$

where $\theta \neq \theta_j$ for all $j = 1, \dots, N$, $\epsilon > 0$ is chosen small such that $S \subset \Omega$, and $R' < R^{\alpha'_j}$ for all $j = 1, \dots, m$, to ensure that $S \subset \mathbb{C}^n \times D_R^m$. Note that S contains a polydisc around the origin.

If $\mu > 0$, we introduce the spaces of holomorphic maps

$$\mathcal{A}_\mu^N(S) := \{\mathbf{f} = (f_1, \dots, f_N) \in \mathcal{O}(S, \mathbb{C}^N) : \mathbf{f}(\mathbf{0}, \cdot) = \mathbf{0}, \\ \|\mathbf{f}\|_{N,\mu} := \max_{1 \leq j \leq N} \|f_j\|_\mu < +\infty\},$$

$$\|f\|_\mu := \|f\|_{1,\mu} = M_0 \sup_{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in S} |f(\boldsymbol{\xi}, \boldsymbol{\eta})| (1 + R(\boldsymbol{\xi})^2) e^{-\mu R(\boldsymbol{\xi})}, \quad f \in \mathcal{O}(S).$$

Here $R(\boldsymbol{\xi}) = \max_{1 \leq j \leq n} |\xi_j|^{\alpha_j/s_j}$ and $M_0 = \sup_{s>0} s(1+s^2)I(s) \approx 3.76$, where

$$I(s) := \int_0^1 \frac{d\tau}{(1+s^2\tau^2)(1+s^2(1-\tau)^2)} = \frac{2(\ln(1+s^2) + s \arctan(s))}{s^2(4+s^2)}.$$

This family of norms is an adaptation for monomials of the norms introduced in [9, Definition 4.1] for one variable, useful to treat non-linear partial PDEs; see e.g., [12]. We refer to [10] for similar norms in higher dimensions.

One particular feature is that

$$\mathcal{A}^N(S) := \bigcup_{\mu>0} \mathcal{A}_\mu^N(S)$$

constitutes a *focusing space* [8, p. 14]. In our case this means each $(\mathcal{A}_\mu^N(S), \|\cdot\|_{N,\mu})$ is a Banach space, and if any \mathbf{f} satisfies $\|\mathbf{f}\|_{N,\mu_0} < +\infty$, then $\|\mathbf{f}\|_{N,\mu} \rightarrow 0$ as $\mu \rightarrow +\infty$. This, and some other properties we need are described in the following three technical lemmas.

Lemma 6.2. *Let $\mu > 0$, R , and S be as before. The following statements hold:*

(1) *If $Q \in \mathcal{O}_b(D_R^m, \mathbb{C}^N)$, then*

$$\|fQ\|_{N,\mu} \leq \|Q\|_{r,R} \|f\|_\mu, \quad f \in \mathcal{A}_\mu^1(S).$$

(2) *$(\mathcal{A}_\mu^1(S), *, \|\cdot\|_\mu)$ is a Banach algebra. More precisely, if $f, g \in \mathcal{A}_\mu^1(S)$, then $f * g \in \mathcal{A}_\mu^1(S)$ and*

$$\|f * g\|_\mu \leq \|f\|_\mu \|g\|_\mu.$$

(3) *If $0 < \mu_0 < \mu$ and $\mathbf{f} \in \mathcal{A}_{\mu_0}^N(S)$, then $\mathbf{f} \in \mathcal{A}_\mu^N(S)$ and $\|\mathbf{f}\|_{N,\mu} \rightarrow 0$ as $\mu \rightarrow +\infty$.*

Proof: The inequality in (1) is an immediate consequence of the definition. To prove (2) note that $f * g$ is clearly analytic on S as long as f and g are analytic there. To establish the desired bound, we use that

$R(\xi_1 \tau^{\alpha_1/s_1}, \dots, \xi_d \tau^{\alpha_n/s_n}) = \tau R(\xi)$, for all $\tau > 0$, $|\xi^\alpha| \leq R(\xi)$ (second inequality in (9)), and also the definition of M_0 to get

$$|(f * g)(\xi, \eta)| \leq |\xi^\alpha| \frac{e^{\mu R(\xi)}}{M_0^2} \|f\|_\mu \|g\|_\mu I(R(\xi)) \leq \frac{e^{\mu R(\xi)}}{M_0(1 + R(\xi)^2)} \|f\|_\mu \|g\|_\mu.$$

To prove (3), it is sufficient to do it for $N = 1$. If $f \in \mathcal{A}_{\mu_0}^1(S)$ and $\mu_0 < \mu$, then it follows from the definition that $\|f\|_\mu \leq \|f\|_{\mu_0}$. To show that $\|f\|_\mu \rightarrow 0$ as $\mu \rightarrow +\infty$, let $\epsilon > 0$. We can find $\rho > 0$ small enough such that $|(1 + R(\xi)^2)f(\xi, \eta)| \leq \epsilon/2M_0$ if $(\xi, \eta) \in \bar{D}$, $D = D_\rho^n \times D_{R^{\rho/\alpha_1}} \times \dots \times D_{R^{\rho/\alpha_n}}$, since $f(\mathbf{0}, \eta) = 0$. Then

$$M_0 \sup_{(\xi, \eta) \in D} |f(\xi, \eta)|(1 + R(\xi)^2)e^{-\mu R(\xi)} \leq \frac{\epsilon}{2}.$$

If $(\xi, \eta) \in S \setminus D$, then $R(\xi) \geq \rho' = \min_{1 \leq j \leq n} \rho^{s_j/\alpha_j} > 0$ and

$$M_0 \sup_{(\xi, \eta) \in S \setminus D} |f(\xi, \eta)|(1 + R(\xi)^2)e^{-\mu R(\xi)} \leq e^{-\rho'(\mu - \mu_0)} \|f\|_{\mu_0}.$$

Taking a large μ such that $e^{-\rho'(\mu - \mu_0)} \|f\|_{\mu_0} < \epsilon/2$, we obtain $\|f\|_\mu < \epsilon$. This proves the claim. \square

Lemma 6.3. *Let $P \in \mathcal{O}_b(D_r^n \times D_R^m, \mathbb{C}^N)$ be a map such that $P(\mathbf{0}, \varepsilon) = \mathbf{0}$ for all $\varepsilon \in D_R^m$. Then for any $0 < \rho < r$ we have*

$$\|f * \mathcal{B}(P)\|_{N, \mu} \leq C_{\mu, \rho} \|P\|_{\rho, R} \|f\|_\mu, \quad f \in \mathcal{A}_\mu^1(S),$$

where $a := \min_{1 \leq j \leq n} s_j/\alpha_j$, $C_{\mu, \rho} := 3((1 - 2/\mu^a \rho)^n - 1)$, and $\mu > \max\{4\sqrt{2}, (2/\rho)^{1/a}\}$. The same inequality is valid for $P \in \mathcal{O}_b(D_r^n \times D_R^m, \mathbb{C}^{N \times N})$ and $\mathbf{f} \in \mathcal{A}_\mu^N(S)$.

Proof: Let us write $P(\mathbf{x}, \varepsilon) = \sum_{\beta \in \mathbb{N}^n \setminus \{\mathbf{0}\}} P_\beta(\varepsilon) \mathbf{x}^\beta$ as a convergent power series at the origin with coefficients in $\mathcal{O}_b(D_R^m, \mathbb{C}^N)$ and also $\mathcal{B}(P)(\xi, \eta) = \sum_{\beta \in \mathbb{N}^n \setminus \{\mathbf{0}\}} \frac{P_\beta(\eta)}{\Gamma(\langle \beta, \lambda \rangle)} \xi^{\beta - \alpha}$.

If $\beta \neq \mathbf{0}$, we have that

$$|\xi^{\beta - \alpha} * f(\xi, \eta)| \leq |\xi^\beta| \frac{e^{\mu R(\xi)}}{M_0} \|f\|_\mu \int_0^1 \frac{\tau^{\langle \beta, \lambda \rangle - 1} e^{-\mu R(\xi)\tau}}{1 + R(\xi)^2(1 - \tau)^2} d\tau.$$

To properly bound this integral expression we split it from 0 to 1/2 and from 1/2 to 1. In the first case, the integral is bounded by

$$\int_0^{1/2} \frac{\tau^{\langle \beta, \lambda \rangle - 1} e^{-\mu R(\xi)\tau}}{1 + R(\xi)^2/4} d\tau \leq \frac{4}{1 + R(\xi)^2} \frac{\Gamma(\langle \beta, \lambda \rangle)}{(\mu R(\xi))^{\langle \beta, \lambda \rangle}},$$

where we have used the integral representation of the Gamma function, which is possible since $\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle > 0$. In the second case, the integral is bounded by

$$\begin{aligned} \int_{1/2}^1 \tau^{\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle - 1} e^{-\mu R(\boldsymbol{\xi})\tau} d\tau &\leq e^{-\mu R(\boldsymbol{\xi})/4} \int_{1/2}^1 \tau^{\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle - 1} e^{-\mu R(\boldsymbol{\xi})\tau/2} d\tau \\ &\leq \left(\frac{2}{\mu R(\boldsymbol{\xi})} \right)^{\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle} \Gamma(\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle) e^{-\mu R(\boldsymbol{\xi})/4} \\ &\leq \left(\frac{2}{\mu R(\boldsymbol{\xi})} \right)^{\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle} \frac{\Gamma(\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle)}{1 + \mu^2/32R(\boldsymbol{\xi})^2} \\ &\leq \left(\frac{2}{\mu R(\boldsymbol{\xi})} \right)^{\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle} \frac{\Gamma(\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle)}{1 + R(\boldsymbol{\xi})^2}, \end{aligned}$$

as long as $\mu^2/32 > 1$. Now, if $\mu > 1$, by using the definition of a , $a|\boldsymbol{\beta}| \leq \langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle \leq |\boldsymbol{\beta}|$, the fact that $|\boldsymbol{\xi}^{\boldsymbol{\beta}}| \leq R(\boldsymbol{\xi})^{\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle}$, and $4 + 2^{|\boldsymbol{\beta}|} \leq 3 \cdot 2^{|\boldsymbol{\beta}|}$ for $|\boldsymbol{\beta}| \geq 1$, we can conclude that

$$|\boldsymbol{\xi}^{\boldsymbol{\beta}-\boldsymbol{\alpha}} * f(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq \frac{3 \cdot 2^{|\boldsymbol{\beta}|} \Gamma(\langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle)}{\mu^{a|\boldsymbol{\beta}|}} \frac{e^{\mu R(\boldsymbol{\xi})}}{M_0(1 + R(\boldsymbol{\xi})^2)} \|f\|_{\mu}.$$

By Cauchy's inequalities, if $0 < \rho < r$, then $|P_{\boldsymbol{\beta}}(\boldsymbol{\eta})| \leq \rho^{-|\boldsymbol{\beta}|} \|P\|_{\rho, R}$ for any $\boldsymbol{\eta} \in D_R^m$. Therefore

$$\begin{aligned} |f * \mathcal{B}(P)(\boldsymbol{\xi}, \boldsymbol{\eta})| &\leq \left(\sum_{\boldsymbol{\beta} \neq \mathbf{0}} \frac{3 \cdot 2^{|\boldsymbol{\beta}|} \|P\|_{\rho, R}}{(\mu^a \rho)^{|\boldsymbol{\beta}|}} \right) \frac{e^{\mu R(\boldsymbol{\xi})}}{M_0(1 + R(\boldsymbol{\xi})^2)} \|f\|_{\mu} \\ &= ((1 - 2/\mu^a \rho)^{-n} - 1) \frac{3 \|P\|_{\rho, R} e^{\mu R(\boldsymbol{\xi})}}{M_0(1 + R(\boldsymbol{\xi})^2)} \|f\|_{\mu}, \end{aligned}$$

as long as $\mu^a > 2/\rho$. This estimate allows us to conclude the proof. \square

Lemma 6.4. *For all $N \in \mathbb{N}^+$, $\mathbf{I} = (i_1, \dots, i_N) \in \mathbb{N}^N$, and $\mathbf{Y}, \mathbf{h} \in \mathcal{A}_{\mu}^N(S)$, we have*

$$\|(\mathbf{Y} + \mathbf{h})^{*\mathbf{I}} - \mathbf{Y}^{*\mathbf{I}}\|_{\mu} \leq |\mathbf{I}| (\|\mathbf{Y}\|_{N, \mu} + \|\mathbf{h}\|_{N, \mu})^{|\mathbf{I}|-1} \|\mathbf{h}\|_{N, \mu}.$$

The proof can be done by induction on \mathbf{I} ; see e.g., [8, p. 19].

4. *The application of the fixed point theorem.* Let $M = M(S_r, R') > 0$ such that $|(\boldsymbol{\xi}^{\boldsymbol{\alpha}} \boldsymbol{\eta}^{\boldsymbol{\alpha}'} I_N - A_0)^{-1}| < M$ on S_r, R' . Fix $0 < \rho < r$ and using the continuity of $A(\mathbf{0}, \cdot)$ take $0 < \rho' \leq R'$ small enough such that $\|A(\mathbf{0}, \cdot) - A_0\|_{\rho, \rho'} < 1/4M$.

We consider the operator \mathcal{H} given by solving \mathbf{Y} in equation (36). Recall that $K, \delta > 0$ in (35) are determined by \mathbf{F} and are fixed. Let

$\mu_0 > 0$ be such that $\mathcal{B}(c) \in \mathcal{A}_{\mu_0}^N(S_{\rho,\rho'})$ (which exists by Step 2). If $\mu \geq \max\{\mu_0, 4\sqrt{2}, (2/\rho)^{1/a}\}$ and $\|\mathbf{Y}\|_{N,\mu} < 1/\delta$, it follows using Lemmas 6.2, 6.3, and the fact that $\|\mathbf{Y}^{*\mathbf{I}}\|_{\mu} \leq \|\mathbf{Y}\|_{N,\mu}^{|\mathbf{I}|}$ (Lemma 6.2 (2)) that

$$\begin{aligned} M^{-1}\|\mathcal{H}(\mathbf{Y})\|_{N,\mu} &\leq \|\mathcal{B}(c)\|_{N,\mu} \\ &\quad + (C_{\mu,\rho}\|A - A(\mathbf{0}, \cdot)\|_{\rho,\rho'} + \|A(\mathbf{0}, \cdot) - A_0\|_{\rho,\rho'})\|\mathbf{Y}\|_{N,\mu} \\ &\quad + K(2C_{\mu,\rho} + 1)((1 - \delta\|\mathbf{Y}\|_{N,\mu})^{-N} - 1). \end{aligned}$$

We may conclude that \mathcal{H} maps $\bigcup_{\mu \geq \mu_0} B_{\mu}(1/2\delta)$ to $\mathcal{A}^N(S_{\rho,\rho'})$, where $B_{\mu}(1/2\delta)$ is the ball

$$B_{\mu}(1/2\delta) = \{\mathbf{Y} \in \mathcal{A}_{\mu}^N(S_{\rho,\rho'}) \mid \|\mathbf{Y}\|_{N,\mu} \leq 1/2\delta\}.$$

Now choose $0 < \epsilon < 1/\delta$ such that $(1 - \delta\epsilon)^{-N-1} - 1 < (2K\delta NM)^{-1}$. Using Lemma 6.4 we can conclude that if $\|\mathbf{Y}\|_{N,\mu} + \|\mathbf{h}\|_{N,\mu} \leq \epsilon$, then

$$\begin{aligned} M^{-1}\|\mathcal{H}(\mathbf{Y} + \mathbf{h}) - \mathcal{H}(\mathbf{Y})\|_{N,\mu} &\leq (C_{\mu,\rho}\|A - A(\mathbf{0}, \cdot)\|_{\rho,\rho'} + \|A(\mathbf{0}, \cdot) - A_0\|_{\rho,\rho'})\|\mathbf{h}\|_{N,\mu} \\ &\quad + K\delta N(2C_{\mu,\rho} + 1)((1 - \delta(\|\mathbf{Y}\|_{N,\mu} + \|\mathbf{h}\|_{N,\mu}))^{-N-1} - 1)\|\mathbf{h}\|_{N,\mu} \\ &\leq (C_{\mu,\rho}\|A - A(\mathbf{0}, \cdot)\|_{\rho,\rho'} + (4M)^{-1} + C_{\mu,\rho}M^{-1} + (2M)^{-1})\|\mathbf{h}\|_{N,\mu}, \end{aligned}$$

where we have used the identity $\sum_{|\mathbf{I}| \geq 2} |\mathbf{I}| \tau^{|\mathbf{I}|-1} = N((1 - \tau)^{-N-1} - 1)$, valid for all $|\tau| < 1$. In conclusion, we have obtained the inequality

$$\|\mathcal{H}(\mathbf{Y} + \mathbf{h}) - \mathcal{H}(\mathbf{Y})\|_{N,\mu} \leq \left(C_{\mu,\rho}M\|A - A(\mathbf{0}, \cdot)\|_{\rho,\rho'} + C_{\mu,\rho} + \frac{3}{4} \right) \|\mathbf{h}\|_{N,\mu}.$$

Since ρ has been fixed and $C_{\mu,\rho} \rightarrow 0$ as $\mu \rightarrow +\infty$, taking $\mu \geq \mu_0$ large enough, we conclude that \mathcal{H} is eventually a contraction, say

$$\|\mathcal{H}(\mathbf{Y} + \mathbf{h}) - \mathcal{H}(\mathbf{Y})\|_{N,\mu} \leq \frac{7}{8} \|\mathbf{h}\|_{N,\mu}.$$

If we also take μ large such that $\|\mathcal{B}(c)\|_{N,\mu} < \epsilon/8M$ (Lemma 6.2 (3)), then $\|\mathcal{H}(\mathbf{0})\|_{N,\mu} < \epsilon/8$ and the previous inequality shows that \mathcal{H} maps the ball $B_{\mu}(\epsilon)$ to itself and, being a contraction, it has a unique fixed point $\mathbf{Y}_0 \in \mathcal{O}(S, \mathbb{C}^N)$.

Since S contains a neighborhood of the origin and equation (36) has $\mathcal{B}(\hat{\mathbf{y}})$ as unique analytic solution there, then it coincides with the Taylor series expansion of \mathbf{Y}_0 at the origin. This means that $\mathcal{B}(\hat{\mathbf{y}})$ can be analytically continued to S with exponential growth of order at most $(\alpha_1/s_1, \dots, \alpha_n/s_n, \mathbf{0})$ there. Since this can be done for all θ , up to $\theta_1, \dots, \theta_N$, we conclude that $\hat{\mathbf{y}}$ is $\mathbf{x}^{\alpha}\boldsymbol{\varepsilon}^{\alpha'}$ -1-summable. Thus the possible singular directions of $\hat{\mathbf{y}}$ for $\mathbf{x}^{\alpha}\boldsymbol{\varepsilon}^{\alpha'}$ -1-summability are determined by the equation $\det(\boldsymbol{\xi}^{\alpha}\boldsymbol{\eta}^{\alpha'}I_N - A_0) = 0$. \square

Corollary 6.5. *Assuming the same hypotheses of the previous theorem, consider the vector field $X = \mathbf{x}^\alpha (\mu_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \mu_n x_n \frac{\partial}{\partial x_n})$. If $b_1, \dots, b_{l-1} \in \mathbb{C}$, the system of singularly perturbed PDEs*

$$\varepsilon^{l\alpha'} X^l(\mathbf{y}) + b_{l-1} \varepsilon^{(l-1)\alpha'} X^{l-1}(\mathbf{y}) + \cdots + b_1 \varepsilon^{\alpha'} X(\mathbf{y}) = \mathbf{G}(\mathbf{x}, \varepsilon, \mathbf{y})$$

has a unique formal solution $\hat{\mathbf{y}} \in \mathbb{C}[[\mathbf{x}, \varepsilon]]^N$ and it is $\mathbf{x}^\alpha \varepsilon^{\alpha'}$ -1-summable.

Proof: Dividing the given PDE by $\langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle^l$, it is enough to prove the statement for the equation

$$\varepsilon^{l\alpha'} X_\lambda^l(\mathbf{y}) + a_{l-1} \varepsilon^{(l-1)\alpha'} X_\lambda^{l-1}(\mathbf{y}) + \cdots + a_1 \varepsilon^{\alpha'} X_\lambda(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \varepsilon, \mathbf{y}),$$

where $a_j = \langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle^{j-l} b_j$, $\mathbf{F} = \langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle^{-l} \mathbf{G}$, and $\boldsymbol{\lambda} = (\frac{s_1}{\alpha_1}, \dots, \frac{s_n}{\alpha_n})$ and $s_j/\alpha_j = \mu_j/\langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle > 0$ are as before. If we put $(\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{l-1}) = (\mathbf{y}, \varepsilon^{\alpha'} X_\lambda(\mathbf{y}), \dots, \varepsilon^{(l-1)\alpha'} X_\lambda^{l-1}(\mathbf{y}))$, the result now follows by applying Theorem 6.1 to the corresponding system of PDEs of size lN given by

$$\varepsilon^{\alpha'} X_\lambda(\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{l-1}) = (\mathbf{y}_1, \dots, \mathbf{y}_{l-1}, \mathbf{F}(\mathbf{x}, \varepsilon, \mathbf{y}_0) - a_1 \mathbf{y}_0 - \cdots - a_{l-1} \mathbf{y}_{l-1}).$$

In this case, the possible singular directions in $t = \boldsymbol{\xi}^\alpha \boldsymbol{\eta}^{\alpha'}$ are determined by the arguments of the solutions of $t^l + a_{l-1} t^{l-1} + \cdots + a_1 t = \nu_j$, $j = 1, \dots, N$, where ν_1, \dots, ν_N are the eigenvalues of $A_0 = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{0}, \mathbf{0}, \mathbf{0})$. Note that zero is not a solution of these polynomials since A_0 is invertible. \square

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