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RELATING SECOND ORDER GEOMETRY OF MANIFOLDS THROUGH PROJECTIONS AND NORMAL SECTIONS

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Abstract: We use normal sections to relate the curvature locus of regular (resp. singular corank 1) 3-manifolds in \mathbb{R}^6 (resp. \mathbb{R}^5) with regular (resp. singular corank 1) surfaces in \mathbb{R}^5 (resp. \mathbb{R}^4). For example, we show how to generate a Roman surface by a family of ellipses different to Steiner's way. We also study the relations between the regular and singular cases through projections. We show that there is a commutative diagram of projections and normal sections which relates the curvature loci of the different types of manifolds, and therefore, that the second order geometry of all of them is related. In particular, we define asymptotic directions for singular corank 1 3-manifolds in \mathbb{R}^5 and relate them to asymptotic directions of regular 3-manifolds in \mathbb{R}^6 and singular corank 1 surfaces in \mathbb{R}^4 .

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Key words: projections, normal sections, curvature locus, immersed surfaces, immersed 3-manifolds, singular corank 1 manifolds.

1. Introduction

The study of second order geometry of manifolds in Euclidean spaces dates as far back as Gauss. By second order geometry we refer to any geometrical aspects which can be captured by the second fundamental form or, in modern terminology, by the 2-jet of a parametrisation of the manifold. Concepts such as elliptic/parabolic/hyperbolic points, normal curvature, asymptotic directions, and some aspects of the contacts with hyperplanes and spheres are included in the study of second order geometry.

In his seminal paper [15], Little studied second order geometry of immersed manifolds in Euclidean spaces of dimensions greater than 3. In particular, special attention was given to immersed surfaces in \mathbb{R}^4 .

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Little defined the second fundamental form and the curvature locus, which is an ellipse in this case. The curvature locus is the image in the normal space by the second fundamental form of the unitary tangent vectors. It can also be seen as the curvature vectors of normal hyperplane sections of the surface. The curvature ellipse is not an affine invariant but its topological type and its position with respect to the origin is an affine invariant. Besides, all the second order geometry is captured by this object.

The introduction of Singularity Theory techniques to study the differential geometry of manifolds in Euclidean spaces has given a great impulse to this subject in the last twenty years. There are many papers devoted to regular surfaces in \mathbb{R}^4 such as [9, 10, 13, 17, 18, 22, 23, 24], amongst others. For surfaces in \mathbb{R}^5 , [11, 20, 25] are good examples. In fact, there is a recent book which covers these topics ([14]). The study of regular 3-manifolds in \mathbb{R}^6 is also very recent. Here, the curvature locus is a Veronese surface with many different topological types (see [7, 8]).

The interest, however, both for singularists and differential geometers has turned to the study of singular manifolds ([26]). In this paper we deal with manifolds in Euclidean spaces with corank 1 singularities, i.e. the corank of the differential of the parametrisation at any point is at most 1. For singular corank 1 surfaces in \mathbb{R}^N , $N = 3, 4$, we can cite [3, 4, 5, 16], and for singular corank 1 3-manifolds in \mathbb{R}^5 , [6]. Here, the curvature locus is a parabola or a parabolic version of a Veronese surface. Curvature loci in general have been studied in [21], for example.

Our results are mostly local, so we consider germs of manifolds at a point p , that is, a k -manifold in \mathbb{R}^N is parametrised by $f: (\mathbb{R}^k, 0) \rightarrow (\mathbb{R}^N, p)$.

The aim of this paper is to relate the geometry of all these objects which have traditionally been studied separately. There is a natural relationship between regular and singular objects. When projecting orthogonally along a tangent direction a regular k -manifold in \mathbb{R}^N one obtains a singular k -manifold in \mathbb{R}^{N-1} . On the other hand, taking normal hyperplane sections of the k -manifold gives a family of $(k - 1)$ -manifolds in one dimension less. In Section 4 we establish a commutative diagram using projections and normal sections which induces a commutative diagram amongst the curvature loci with immersions and blow-ups. As a result of this we prove our main result, that the second order geometry of all these objects is related. This justifies known relations for projections when $k = 2$ and $N = 4$, for example, and motivates to look for further relations between the geometries of different manifolds, both regular and singular, in different Euclidean spaces.

Section 2 is devoted to preliminary results on the geometry of all the different objects appearing throughout the paper. In Section 3 we study normal sections of 3-manifolds, both for the regular and singular cases, and show that the curvature locus of a 3-manifold can be generated by the curvature loci of the surfaces obtained by normal sections. In particular, we show how a Roman Steiner surface or a cross-cap surface can be generated by ellipses. In [2] it is shown how using these normal sections some geometry of the 3-manifold can be recovered by the topological types of the curvature loci of the sections.

In Section 5, inspired by the commutative diagram of Section 4, we define asymptotic directions for singular 3-manifolds in \mathbb{R}^5 and relate them to asymptotic directions of regular 3-manifolds in \mathbb{R}^6 and singular surfaces in \mathbb{R}^4 . We prove that if the direction of projection is asymptotic, then it becomes a null tangent direction in the singular projection and so justify the existence of infinite asymptotic directions in the singular case, which was not fully understood until now.

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2. The geometry of regular and singular corank 1 manifolds in Euclidean spaces

Given a smooth k -dimensional manifold $M_{\text{reg}}^k \subset \mathbb{R}^N$, $N > k$, and $f: U \rightarrow \mathbb{R}^N$ a local parametrisation of M_{reg}^k with $U \subset \mathbb{R}^k$ an open subset, let $\{e_1, \dots, e_N\}$ be an orthonormal frame of \mathbb{R}^N such that at any $u \in U$, $\{e_1(u), \dots, e_k(u)\}$ is a basis for $T_p M_{\text{reg}}^k$ and $\{e_{k+1}(u), \dots, e_N(u)\}$ is a basis for $N_p M_{\text{reg}}^k$ at $p = f(u)$.

The second fundamental form of M_{reg}^k at a point p is a symmetric bilinear map $II_p: T_p M_{\text{reg}}^k \times T_p M_{\text{reg}}^k \rightarrow N_p M_{\text{reg}}^k$, given by $II_p(v, w) = \pi_2(d^2 f(v, w))$, where $\pi_2: T_p \mathbb{R}^N \rightarrow N_p M_{\text{reg}}^k$ is the canonical projection.

Furthermore, the second fundamental form of M_{reg}^k at p along a normal vector field ν is the bilinear map $II_p^\nu: T_p M_{\text{reg}}^k \times T_p M_{\text{reg}}^k \rightarrow \mathbb{R}$, given by $II_p^\nu(v, w) = \langle \nu, II_p(v, w) \rangle$.

For singular k -dimensional manifolds with corank 1 singularities, we shall need the following construction, that can be found in [16]. A singular manifold $M_{\text{sing}}^k \subset \mathbb{R}^N$ at p is given by the image of a smooth map $g: \tilde{M} \rightarrow \mathbb{R}^N$, where \tilde{M} is a smooth regular k -dimensional manifold and $q \in \tilde{M}$ is a corank 1 point of g such that $g(q) = p$. Also,

consider $\phi: U \rightarrow \mathbb{R}^k$ a local coordinate system defined in an open neighbourhood U of q at \tilde{M} . Hence, we may consider a local parametrisation $f = g \circ \phi^{-1}$ of M_{sing}^k at p (see the diagram below).

$$\begin{array}{ccc} \mathbb{R}^k & \xleftarrow{\phi} & U \subset \tilde{M} \xrightarrow{g} M_{\text{sing}}^k \subset \mathbb{R}^N \\ & \searrow f & \nearrow \end{array}$$

The $(k - 1)$ -dimensional tangent space of M_{sing}^k at p , $T_p M_{\text{sing}}^k$, is given by $\text{Im } dg_q$, where $dg_q: T_q \tilde{M} \rightarrow T_p \mathbb{R}^N$ is the differential map of g at q . Thus, the $(N - k + 1)$ -dimensional normal space of M_{sing}^k at p , $N_p M_{\text{sing}}^k$, is the subspace orthogonal to $T_p M_{\text{sing}}^k$ satisfying $T_p M_{\text{sing}}^k \oplus N_p M_{\text{sing}}^k = T_p \mathbb{R}^N$.

The first fundamental form of M_{sing}^k at p , $I: T_q \tilde{M} \times T_q \tilde{M} \rightarrow \mathbb{R}$, is given by

$$I(u, v) = \langle dg_q(u), dg_q(v) \rangle \quad \forall u, v \in T_q \tilde{M}.$$

Since g has corank 1 at q , one can find $0 \neq u \in T_q \tilde{M}$ such that $dg_q(u) = 0$ and $I(u, u) = 0$. Therefore the first fundamental form is not a Riemannian metric on $T_q \tilde{M}$, but a pseudometric instead. Let (x_1, \dots, x_k) be the Cartesian coordinate system in \mathbb{R}^k . Taking the frame $\mathcal{B} = \{\partial x_1, \dots, \partial x_k\}$ of $T_q \tilde{M}$, the coefficients of the first fundamental form of M_{sing}^k at p with respect to ϕ are given by $E_{x_i x_j}(q) = I(\partial_{x_i}, \partial_{x_j}) = \langle f_{x_i}, f_{x_j} \rangle(\phi(q))$, $1 \leq i, j \leq k$, where $f_{x_i} = \frac{\partial f}{\partial x_i}$.

Following [16], consider the orthogonal projection $\pi_2: T_p \mathbb{R}^N \rightarrow N_p M_{\text{sing}}^k$. The second fundamental form of M_{sing}^k at p , $II: T_q \tilde{M} \times T_q \tilde{M} \rightarrow N_p M_{\text{sing}}^k$, in the basis \mathcal{B} of $T_q \tilde{M}$, is given by $II(\partial_{x_i}, \partial_{x_j}) = \pi_2(f_{x_i x_j}(\phi(q)))$, $1 \leq i, j \leq k$, and we extend it to the whole space in a unique way as a symmetric bilinear map. Given a normal vector $\nu \in N_p M_{\text{sing}}^k$, we define the second fundamental form along ν , $II_\nu: T_q \tilde{M} \times T_q \tilde{M} \rightarrow \mathbb{R}$, given by $II_\nu(u, v) = \langle II(u, v), \nu \rangle$ for all $u, v \in T_q \tilde{M}$.

2.1. The curvature loci. Given a k -dimensional manifold $M^k \subset \mathbb{R}^N$, $N > k$, the curvature locus at a point $p \in M^k$ is the set $\{II_p(u, u) : u \in T_p M^k, I(u, u)^{\frac{1}{2}} = 1\} \subset N_p M^k$. The curvature locus of a manifold contains all its second order geometry. Any isometric scalar invariant of the curvature locus is an isometric scalar invariant of the manifold, since rotations in $T_p M^k$ leave invariant the locus and rotations in $N_p M^k$ rotate the locus. The contact geometry of the manifold is affine invariant. The curvature locus is not affine invariant, but the position with respect to the origin and, in some cases, the topological type are affine invariant.

Regular case. For a regular manifold $M_{\text{reg}}^k \subset \mathbb{R}^N$, the curvature locus is also the image of the map $\eta: \mathbb{S}^{k-1} \subset T_p M_{\text{reg}}^k \rightarrow N_p M_{\text{reg}}^k$, where $\eta(u) = II_p(u, u)$.

When $k = 2$, the curvature locus is an ellipse denoted by Δ_e that may degenerate into a segment or a point. Taking $u = (\cos \theta)e_1 + (\sin \theta)e_2 \in \mathbb{S}^1$, Δ_e can be parametrised by

$$\eta(\theta) = \sum_{i=1}^{N-2} (l_i \cos^2 \theta + 2m_i \cos \theta \sin \theta + n_i \sin^2 \theta) e_{2+i},$$

where $l_i = \langle f_{xx}, e_{2+i} \rangle$, $m_i = \langle f_{xy}, e_{2+i} \rangle$, and $n_i = \langle f_{yy}, e_{2+i} \rangle$, $i = 1, \dots, N - 2$, are the coefficients of the second fundamental form, and (x, y) are the local coordinates of \mathbb{R}^2 .

For the case where $k = 3$ and $N = 6$, the curvature locus is denoted by Δ_v . The authors show in [7] that taking spherical coordinates in $\mathbb{S}^2 \subset T_p M_{\text{reg}}^3$, one can parametrise the curvature locus of M_{reg}^3 at p by $\eta: \mathbb{S}^2 \subset T_p M_{\text{reg}}^3 \rightarrow N_p M_{\text{reg}}^3$, $(\theta, \phi) \mapsto \eta(\theta, \phi)$, where

$$\begin{aligned} \eta(\theta, \phi) = & H + (1 + 3 \cos(2\phi))B_1 + \cos(2\theta) \sin^2 \theta B_2 \\ & + \sin(2\theta) \sin^2 \phi B_3 + \cos \theta \sin(2\phi) B_4 + \sin \theta \sin(2\phi) B_5 \end{aligned}$$

with

$$\begin{aligned} H &= \frac{1}{3}(f_{xx} + f_{yy} + f_{zz}), & B_1 &= \frac{1}{12}(-f_{xx} - f_{yy} + 2f_{zz}), \\ B_2 &= \frac{1}{2}(f_{xx} - f_{yy}), & B_3 &= f_{xy}, & B_4 &= f_{xz}, & B_5 &= f_{yz}. \end{aligned}$$

The first normal space is $N_p^1 M_{\text{reg}}^3 = \langle H, B_1, B_2, B_3, B_4, B_5 \rangle_{(p)}$. The affine hull of the curvature locus is denoted by Aff_p and the linear subspace of $N_p^1 M_{\text{reg}}^3$ parallel to Aff_p by E_p . The curvature locus of a regular 3-manifold in \mathbb{R}^N can be seen as the image of the classical Veronese surface of order 2 via a convenient linear map.

In [7] it is shown that the curvature locus at a point p where $\dim(N_p^1 M_{\text{reg}}^3) = 3$ in a 3-manifold $M_{\text{reg}}^3 \subset \mathbb{R}^6$ has one of the following shapes: a Roman Steiner surface (Figure 1), a cross-cap surface (Figure 2), a Steiner surface of type 5 (Figure 3), a cross-cup surface (Figure 4), an ellipsoid, a (compact) cone, or a planar region.

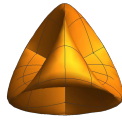


FIGURE 1. Roman Steiner surface.

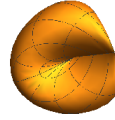


FIGURE 2. Cross-cap surface.

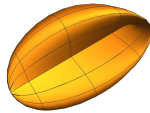


FIGURE 3. Steiner surface of type 5.

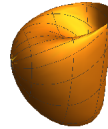


FIGURE 4. Cross-cup surface.

Singular case. The curvature locus at a singular corank 1 point p of a k -dimensional manifold $M_{\text{sing}}^k \subset \mathbb{R}^N$, $N > k$, is also given by the image of the map $\eta: C_q \rightarrow N_p M$ defined by $\eta(u) = II(u, u)$, where $C_q \subset T_q \tilde{M}$ is the subset of unit tangent vectors (i.e. vectors $u \in T_q \tilde{M}$ such that $I(u, u)^{\frac{1}{2}} = 1$).

When $k = 2$, the curvature locus denoted by Δ_p is a parabola that may degenerate into a half-line, a line, or a point. In the same way as the curvature ellipse does for regular surfaces, it contains information about the second order geometry of the surface. For $N = 3$, the normal space is a plane. However, if $N > 3$, the normal space has higher dimension and the plane that contains Δ_p must be defined, specially when Δ_p is degenerate. The minimal affine space that contains the curvature parabola is denoted by Aff_p . The plane denoted by E_p is the vector space: parallel to Aff_p when Δ_p is a non degenerate parabola, the plane through p that contains Aff_p when Δ_p is a non radial half-line or a non radial line, and any plane through p that contains Aff_p when Δ_p is a radial half-line, a radial line, or a point.

When $k = 3$, the curvature locus is denoted by Δ_{cv} . Some examples of topological types of Δ_{cv} can be found in [6].

It is possible to take a coordinate system ϕ and make rotations in the target in order to obtain a local parametrisation for M_{sing}^k at p given by

$$f(x_1, \dots, x_k) = (x_1, \dots, x_{k-1}, f_k(x_1, \dots, x_k), \dots, f_N(x_1, \dots, x_k)),$$

where $\frac{\partial f_i}{\partial x_j}(\phi(q)) = 0$ for $i = k, \dots, N$ and $1 \leq j \leq k$. Hence, the subset of unit tangent vectors $C_q \in T_q \tilde{M}$ is the cylinder given by $\{(a_1, \dots, a_k) \in T_q \tilde{M} : a_1^2 + \dots + a_k^2 = 1\}$. Taking an orthonormal frame $\{\nu_1, \dots, \nu_{N-k+1}\}$ of $N_p M_{\text{sing}}^k$, the curvature locus Δ_p can be parametrised by:

(i) If $k = 2$, $C_q = \{(a, b) \in T_q\tilde{M} : a^2 = 1\}$, and the parametrisation is

$$\eta(b) = \sum_{i=1}^{N-1} (l_{\nu_i} + 2m_{\nu_i}b + n_{\nu_i}b^2)\nu_i,$$

where $l_{\nu_i}(q) = \langle \pi_2(f_{xx}), \nu_i \rangle(\phi(q))$, $m_{\nu_i}(q) = \langle \pi_2(f_{xy}), \nu_i \rangle(\phi(q))$, and $n_{\nu_i}(q) = \langle \pi_2(f_{yy}), \nu_i \rangle(\phi(q))$, $1 \leq i \leq N-1$, are the coefficients of the second fundamental form, (x, y) are local coordinates of \mathbb{R}^2 , and the parameter $b \in \mathbb{R}$ corresponds to a unit tangent direction $u = \partial_x + b\partial_y = (1, b) \in C_q$.

(ii) If $k = 3$, $C_q = \{(a, b, c) \in T_q\tilde{M} : a^2 + b^2 = 1\}$, and the parametrisation is

$$\eta(a, b, c) = \sum_{i=1}^{N-2} (a^2l_{\nu_i} + 2abm_{\nu_i} + b^2n_{\nu_i} + c^2p_{\nu_i} + 2acq_{\nu_i} + 2bcr_{\nu_i})\nu_i,$$

where $a^2 + b^2 = 1$,

$$\begin{aligned} l_{\nu}(q) &= \langle \pi_2(f_{xx}), \nu \rangle, & m_{\nu}(q) &= \langle \pi_2(f_{xy}), \nu \rangle, & n_{\nu}(q) &= \langle \pi_2(f_{yy}), \nu \rangle, \\ p_{\nu}(q) &= \langle \pi_2(f_{zz}), \nu \rangle, & q_{\nu}(q) &= \langle \pi_2(f_{xz}), \nu \rangle, & r_{\nu}(q) &= \langle \pi_2(f_{yz}), \nu \rangle, \end{aligned}$$

are the coefficients of the second fundamental form with all the partial derivatives evaluated at $\phi(q)$, and (x, y, z) are local coordinates of \mathbb{R}^3 .

For singular corank 1 surfaces, $k = 2$, a non zero direction $u \in T_q\tilde{M}$ is called asymptotic if there is a non zero vector $\nu \in N_pM_{\text{sing}}^k$ (for $N = 3$) or $\nu \in E_p$ (for $N = 4$) such that $II_{\nu}(u, \nu) = \langle II(u, \nu), \nu \rangle = 0$ for all $v \in T_q\tilde{M}$. Moreover, in such case, we say that ν is a binormal direction.

For the case $N = 4$, the normal directions $\nu \in N_pM_{\text{sing}}^2$ which are not in the plane E_p , but also satisfy the condition $II_{\nu}(u, \nu) = \langle II(u, \nu), \nu \rangle = 0$, are called degenerate directions. The subset of degenerate directions in $N_pM_{\text{sing}}^2$ is a cone and the binormal directions are those in the intersection of this cone with E_p .

3. Normal sections

Consider $M_{\text{reg}}^3 \subset \mathbb{R}^{3+k}$, $k \geq 1$, a regular 3-manifold (resp. $M_{\text{sing}}^3 \subset \mathbb{R}^5$ a singular corank 1 3-manifold). Let u be a tangent direction in $T_pM_{\text{reg}}^3$ (resp. $T_pM_{\text{sing}}^3$) and $\{u = 0\}$ the hyperplane in \mathbb{R}^{3+k} (resp. \mathbb{R}^5) orthogonal to u . The normal section of M_{reg}^3 along u is a regular surface $M_{\text{reg}}^2 = M_{\text{reg}}^3 \cap \{u = 0\}$ contained in $\mathbb{R}^{3+k} \cap \{u = 0\} \cong \mathbb{R}^{2+k}$ (resp. the normal section M_{sing}^2 along u is a singular corank 1 surface $M_{\text{sing}}^2 = M_{\text{sing}}^3 \cap \{u = 0\}$ contained in $\mathbb{R}^5 \cap \{u = 0\} \cong \mathbb{R}^4$).

There is a relation between the curvature locus of $M_{\text{reg}}^3 \subset \mathbb{R}^{3+k}$ at p and the curvature ellipse of M_{reg}^2 at p (resp. the curvature locus of $M_{\text{sing}}^3 \subset \mathbb{R}^5$ at p and the curvature parabola of M_{sing}^2 at the same point).

Theorem 3.1. *Let $M_{\text{reg}}^3 \subset \mathbb{R}^{3+k}$, $k \geq 1$, be a regular 3-manifold and $p \in M_{\text{reg}}^3$. The curvature locus of M_{reg}^3 at p is generated by the union of the curvature ellipses at p of the regular surfaces in \mathbb{R}^{2+k} given by the normal sections along the tangent directions of M_{reg}^3 .*

Proof: A normal section induces a hyperplane section of $T_p M_{\text{reg}}^3$. This hyperplane section intersects the 2-sphere of unit tangent vectors in an S^1 . The second fundamental form restricted to this S^1 coincides with the second fundamental form of the surface M_{reg}^2 obtained as a normal section. Therefore, the curvature ellipses of the normal sections generate the curvature locus of M_{reg}^3 at p . □

Example 3.2. (i) Let $M_{\text{reg}}^3 \subset \mathbb{R}^6$ be given by $f: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^6, 0)$,

$$f(x, y, z) = \left(x, y, z, \frac{\sqrt{2}}{2}xy, \frac{\sqrt{2}}{2}xz, \frac{\sqrt{2}}{2}yz \right).$$

At the origin p , its curvature locus is a Roman Steiner surface. Taking coordinates (X, Y, Z, W, T, S) in the target, the normal sections given by $\{X = 0\}$, $\{Y = 0\}$, and $\{Z = 0\}$ are regular surfaces whose curvature ellipses at p are, respectively:

$$\begin{aligned} \eta_X(\theta) &= (0, 0, \sqrt{2} \sin \theta \cos \theta), & \eta_Y(\theta) &= (0, \sqrt{2} \sin \theta \cos \theta, 0), \\ \eta_Z(\theta) &= (\sqrt{2} \sin \theta \cos \theta, 0, 0), \end{aligned}$$

where $\theta \in [0, 2\pi]$. These curvature ellipses are segments which correspond to the double point curves of the Roman Steiner surface. The normal sections $\{X = Y\}$, $\{X = Z\}$, and $\{Y = Z\}$, after changes of coordinates in the source and rotations in the tangent spaces of the surfaces in \mathbb{R}^5 , yield the following curvature ellipses:

$$\begin{aligned} \eta_{XY}(\theta) &= \left(\frac{\sqrt{2}}{2} \sin^2 \theta, \sin \theta \cos \theta, \sin \theta \cos \theta \right), \\ \eta_{XZ}(\theta) &= \left(\sin \theta \cos \theta, \frac{\sqrt{2}}{2} \sin^2 \theta, \sin \theta \cos \theta \right), \\ \eta_{YZ}(\theta) &= \left(\sin \theta \cos \theta, \sin \theta \cos \theta, \frac{\sqrt{2}}{2} \sin^2 \theta \right), \end{aligned}$$

where $\theta \in [0, 2\pi]$. These three curves are non degenerate ellipses. Figure 5 shows the curvature ellipses on the Roman Steiner surface. It appears Steiner himself already knew how to generate the Roman surface by

ellipses (see [1]). However, all his ellipses pass through a “pole” whereas all of the ellipses obtained here pass through the triple point.

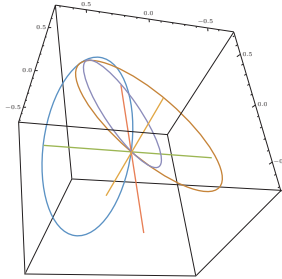


FIGURE 5. Curvature ellipses on the Roman Steiner surface.

(ii) Let $M_{\text{reg}}^3 \subset \mathbb{R}^5$ be given by $f(x, y, z) = (x, y, z, x^2 + z^2, xy)$. Taking coordinates (X, Y, Z, W, T) in \mathbb{R}^5 , its curvature locus at the origin p is an elliptic region contained in the normal plane $\{W, T\}$, with center at $(1, 0)$ and radius 1. Table 1 shows some curvature ellipses of regular surfaces given by normal sections. Here, $\theta \in [0, 2\pi]$. Figure 6 shows the curves in Table 1.

Normal section	Parametrisation of the curvature ellipse	Type
$\{X = 0\}$	$(2 \sin^2 \theta, 0)$	segment
$\{Y = 0\}$	$(2, 0)$	point
$\{Z = 0\}$	$(2 \sin^2 \theta, 2 \sin \theta \cos \theta)$	circle
$\{X = Z\}$	$(2 \sin^2 \theta, \frac{2}{\sqrt{2}} \sin \theta \cos \theta)$	ellipse
$\{Y = Z\}$	$(2 \sin^2 \theta + \cos^2 \theta, \frac{2}{\sqrt{2}} \sin \theta \cos \theta)$	ellipse
$\{X = Y\}$	$(\sin^2 \theta + 2 \cos^2 \theta, \sin^2 \theta)$	segment

TABLE 1.

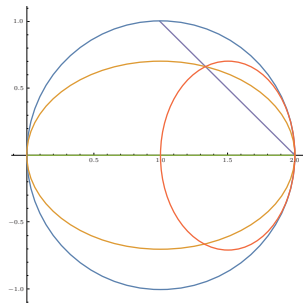


FIGURE 6. Curvature ellipses on the elliptic region.

Although it was known that the Roman Steiner surface could be generated by ellipses, geometrically speaking this is not so obvious for the cross-cap, the Steiner type 5, or the cross-cup surfaces.

Theorem 3.3. *Let $M_{\text{sing}}^3 \subset \mathbb{R}^5$ be a singular corank 1 3-manifold. The curvature locus of M_{sing}^3 at p is generated by the union of the curvature parabolas at p of the singular surfaces in \mathbb{R}^4 given by the normal sections along the tangent directions of M_{sing}^3 .*

Proof: The proof follows as in the regular case. The only different consideration is that $T_p M_{\text{sing}}^3$ is a plane and if $w \in T_p M_{\text{sing}}^3$ is a non zero vector, then $(dg_q)^{-1}(w) \subset T_q \tilde{M}$ is a plane which contains the subset $\ker(dg_q)$, where g is the corank 1 map at q used in the initial construction and $g(q) = p$.

Hence, the subset $C'_q = (dg_q)^{-1}(w) \cap C_q$ is a pair of lines contained in the unit cylinder C_q and such that $\eta_q(C'_q)$ is the curvature parabola at p of the singular surface contained in the 4-space given by the normal section $\{w = 0\}$. Figure 7 shows the previous construction. □

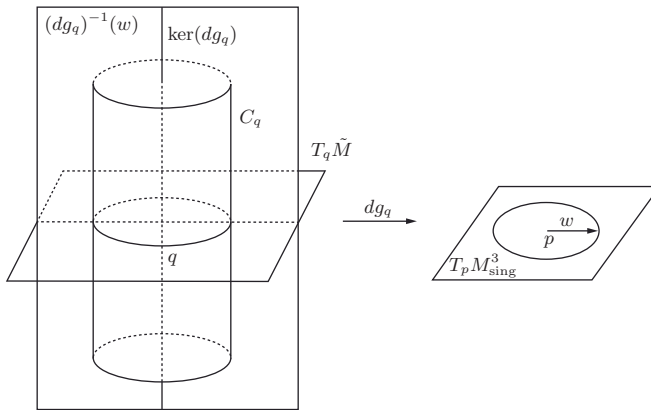


FIGURE 7. Theorem 3.3.

Example 3.4. Let $M_{\text{sing}}^3 \subset \mathbb{R}^5$ be the singular 3-manifold at the origin p locally given by $f(x, y, z) = (x, y, x^2 - 2yz, y^2 - 2xz, z^2 - 2xy)$, whose curvature locus Δ_{cv} at p is

$$\{(2\alpha^2 - 4\beta\gamma, 2\beta^2 - 4\alpha\gamma, 2\gamma^2 - 4\alpha\beta) \in N_p M_{\text{sing}}^3 : \alpha^2 + \beta^2 = 1\};$$



FIGURE 8. Side and top views of Δ_{cv} .

see Figure 8. The normal section given by $\{Y = 0\}$ is parametrised by $\tilde{f}(x, z) = (x, x^2, -2xz, z^2)$ and its curvature parabola is a non degenerate parabola, $\tilde{\eta}(z) = (2, -4z, 2z^2)$. Taking the normal section $\{X + aY = 0\}$, $a \in \mathbb{R}$, after changes of coordinates at the source and isometries at the target, we obtain the singular surface given by

$$\left(0, y, \frac{a^2\sqrt{a^2+1}y^2 - 2(a^2+1)yz}{(a^2+1)^{3/2}}, \frac{\sqrt{a^2+1}y^2 + 2a(a^2+1)yz}{(a^2+1)^{3/2}}, \frac{(a^2+1)z^2 + 2ay^2}{a^2+1}\right),$$

and Δ_{cv} is parametrised by

$$\begin{aligned} &\eta_a(z) \\ &= \left(\frac{2a^2\sqrt{a^2+1} - 4(a^2+1)z}{(a^2+1)^{3/2}}, \frac{2\sqrt{a^2+1} + 4a(a^2+1)z}{(a^2+1)^{3/2}}, \frac{4a + 2(a^2+1)z^2}{a^2+1}\right), \end{aligned}$$

a non degenerate parabola for $a \in \mathbb{R}$. Figure 9 shows some of the curvature parabolas in the curvature locus.

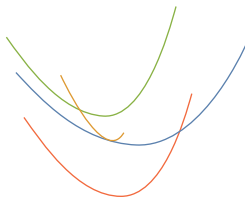


FIGURE 9. Curvature parabolas.

4. Relating second order geometry through projections and normal sections

When projecting a regular k -manifold in \mathbb{R}^N along a tangent direction we obtain a singular k -manifold in \mathbb{R}^{N-1} . It is natural to expect certain relations between the curvature loci in each case. For example, in [3] we showed the relation between the curvature ellipse of $M_{reg}^2 \subset \mathbb{R}^4$ and the curvature parabola of the projection $M_{sing}^2 \subset \mathbb{R}^3$ and obtained some relations between their second order geometry. It is also known that in the

previous case, the tangent direction is asymptotic if and only if the singularity of the projection is worse than a cross-cap ([9, 19]). Similarly, for projections from $M_{\text{reg}}^2 \subset \mathbb{R}^5$ to $M_{\text{sing}}^2 \subset \mathbb{R}^4$, the direction is asymptotic if and only if the singularity is worse than an I_1 -singularity ([25]).

In the previous section, geometrical relations between manifolds are obtained by normal sections. The following result justifies why relations through projections and normal sections are possible.

Theorem 4.1. *Let $M_{\text{reg}}^3 \subset \mathbb{R}^6$ be a regular 3-manifold, $v \in T_p M_{\text{reg}}^3$, and π_v the projection along the direction v . Consider a normal section Σ such that it contains v . Let i_1, i_2 be the immersions of the normal sections in \mathbb{R}^6 and \mathbb{R}^5 , respectively. Let $v' = i_{1*}^{-1}(v) \in T_{i_1^{-1}(p)} M_{\text{reg}}^2$. We have a commutative diagram*

$$\begin{array}{ccc} M_{\text{reg}}^3 \subset \mathbb{R}^6 & \xrightarrow{\pi_v} & M_{\text{sing}}^3 \subset \mathbb{R}^5 \\ i_1 \uparrow & & \uparrow i_2 \\ M_{\text{reg}}^2 \subset \mathbb{R}^5 & \xrightarrow{\pi_{v'}} & M_{\text{sing}}^2 \subset \mathbb{R}^4 \end{array}$$

where $M_{\text{reg}}^2 = M_{\text{reg}}^3 \cap \Sigma$ and $M_{\text{sing}}^3, M_{\text{sing}}^2$ are the corresponding singular projections, which induces a commutative diagram amongst the curvature loci of the four manifolds.

Proof: For simplicity, we take M_{reg}^3 given in Monge form by

$$(x, y, z) \mapsto (x, y, z, f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)),$$

and fix the direction of projection $v = (0, 0, 1)$ and the normal section $\Sigma = \{Y = 0\}$. In this setting $v' = (0, 1)$. Consider (X, Y, Z, W, T, S) to be the coordinates of \mathbb{R}^6 . Then i_1 and i_2 are given by $i_1(X, Z, W, T, S) = (X, 0, Z, W, T, S)$ and $i_2(X, W, T, S) = (X, 0, W, T, S)$. We have that M_{reg}^2 is given by $(x, z, f_1(x, 0, z), f_2(x, 0, z), f_3(x, 0, z))$ and clearly $\pi_v \circ i_1(M_{\text{reg}}^2) = i_2 \circ \pi_{v'}(M_{\text{reg}}^2)$.

Now, the curvature locus of M_{reg}^3 is the image by II of the unit tangent vectors in $T_p M_{\text{reg}}^3$. We can parametrise the sphere \mathbb{S}^2 of unit tangent vectors in spherical coordinates by (θ, ϕ) , where $\theta \in [0, 2\pi]$ is the azimuth (i.e. the angle from the X -axis in a plane of constant height) and $\phi \in [0, \pi]$ is the polar angle (i.e. the angle from the Z -axis). When projecting along the tangent direction $v = (0, 0, 1)$ we obtain a singular 3-manifold. The unit tangent vectors in $T_{\pi_v(p)} \tilde{M}_{\text{sing}}^3$ form a cylinder C which is obtained by blowing up the north and south poles of \mathbb{S}^2 . There is a natural map from \mathbb{S}^2 to C which takes the spherical coordinates $(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$ to the cylindrical coordinates $(\cos \theta, \sin \theta, \frac{\cos \phi}{\sin \phi})$ by dividing each component by $\sin \phi$ (i.e. it maps

the point of intersection with \mathbb{S}^2 of a ray from the origin to the point of intersection with C , the north and south poles go to infinity). This map induces a relation between the parametrisations of the curvature locus of M_{reg}^3 and M_{sing}^3 . In fact, since the loci are the image of II and the coefficients of these second fundamental forms are the same in the regular and singular cases, the fact of II being a quadratic homogeneous map means that if $\eta_e(\theta, \phi)$ is the parametrisation of the curvature locus of M_{reg}^3 , then

$$\eta_p(\theta, \phi) = \frac{1}{\sin^2 \phi} \eta_e(\theta, \phi)$$

is the parametrisation of the curvature locus of M_{sing}^3 .

On the other hand, the section $\{Y = 0\}$ induces a section in $T_p M_{\text{reg}}^3$. In spherical coordinates, this gives the section $\{\theta = 0\}$ of \mathbb{S}^2 . So, by Theorem 3.1 the curvature ellipse of M_{reg}^2 is given by $\eta_e(0, \phi)$. Similarly, by Theorem 3.3 the curvature parabola of M_{sing}^2 is given by $\eta_p(0, \phi)$.

It remains to see that to pass from the curvature ellipse to the curvature parabola we must divide each component of the parametrisation by $\sin^2 \phi$. This follows from the geometrical interpretation of $\cot \phi = \frac{\cos \phi}{\sin \phi}$, which again shows that we must divide the components $(\sin \phi, \cos \phi)$ of \mathbb{S}^1 by $\sin \phi$ to get the components of the unit tangent vectors in $T_p \tilde{M}_{\text{sing}}^2$, and the fact that the second fundamental form is a homogeneous quadratic map. \square

Example 4.2. Consider M_{reg}^3 given by $f(x, y, z) = (x, y, z, x^2 + \frac{1}{2}z^2, xz, yz)$. The projection along the tangent vector $(0, 0, 1)$ is M_{sing}^3 , given by $(x, y, x^2 + \frac{1}{2}z^2, xz, yz)$, and the normal section $\{Y = 0\}$ gives the regular surface M_{reg}^2 by $(x, z, x^2 + \frac{1}{2}z^2, xz, 0)$. The normal section of M_{sing}^3 , which coincides with the projection of M_{reg}^2 along the tangent vector $(0, 1)$, is given by $(x, x^2 + \frac{1}{2}z^2, xz, 0)$. The curvature locus of M_{reg}^3 is a Roman Steiner surface parametrised by

$$\eta_e(\theta, \phi) = (1 + \sin^2 \phi \cos(2\theta), \cos \theta \sin(2\phi), \sin \theta \sin(2\phi)),$$

and the curvature locus of M_{sing}^3 is a surface given by

$$\begin{aligned} \eta_p(\theta, \phi) &= \frac{1}{\sin^2 \phi} (1 + \sin^2 \phi \cos(2\theta), \cos \theta \sin(2\phi), \sin \theta \sin(2\phi)) \\ &= (2a^2 + c^2, 2ac, 2bc), \end{aligned}$$

where $a = \cos \theta$, $b = \sin \theta$, and $c = \frac{\cos \phi}{\sin \phi}$, so $a^2 + b^2 = 1$. The normal section of M_{sing}^3 is given by $\{\theta = 0\} = \{a = 1, b = 0\}$, so we get a curvature parabola $(2 + c^2, 2c, 0)$.

On the other hand, the curvature ellipse of M_{reg}^2 is parametrised by $(1 + \sin^2 \phi, \sin(2\phi), 0)$. Dividing by $\sin^2 \phi$ and changing $\frac{\cos \phi}{\sin \phi} = c$ we again obtain the curvature parabola $(2 + c^2, 2c, 0)$.

Remark 4.3. Since the curvature locus contains all the second order geometry of the manifold, the previous theorem proves that the second order geometries of $M_{\text{reg}}^3 \subset \mathbb{R}^6$, $M_{\text{reg}}^2 \subset \mathbb{R}^5$, $M_{\text{sing}}^3 \subset \mathbb{R}^5$, and $M_{\text{sing}}^2 \subset \mathbb{R}^4$ are all related with one another.

5. Asymptotic directions of singular 3-manifolds in \mathbb{R}^5

Let M be a corank 1 singular 3-manifold in \mathbb{R}^5 , $p \in M$, and take M as the image of a smooth map $g: \tilde{M} \rightarrow \mathbb{R}^5$, where \tilde{M} is a regular 3-manifold and $q \in \tilde{M}$ is a corank 1 point of g such that $g(q) = p$.

Definition 5.1. A direction $u \in T_q \tilde{M}$ is called an *asymptotic direction* of M at p if there is a non zero vector $\nu \in N_p M$ such that

$$II_\nu(u, v) = \langle II(u, v), \nu \rangle = 0 \quad \forall v \in T_q \tilde{M}.$$

Moreover, in this case, we say that ν is a *binormal direction*.

In Theorem 2.2 of [12] several equivalent notions are given to define asymptotic directions in the regular case. We will prove a similar result for the singular case, but before proving this we need the following definition due to Dreibelbis (adapted to the singular case here).

Definition 5.2. Let $\{e_1, e_2, e_3\}$ be a basis for $T_q \tilde{M}$ and $\{n_1, n_2, n_3\}$ be a basis for $N_p M$. For any vector $u \in T_q \tilde{M}$, define $A(u)$ as the 3×3 matrix with $A(u)_{ij} = II_{n_i}(e_j, u) = \langle II(e_j, u), n_i \rangle$.

Given a unit direction $\nu \in \mathbb{S}^4$, the height function of $M \subset \mathbb{R}^5$ along ν is given by $h_\nu: M \rightarrow \mathbb{R}$, $h_\nu(p) = \langle p, \nu \rangle$.

Proposition 5.3. *Given $u \in T_q \tilde{M}$, the following are equivalent:*

- (1) u is an asymptotic direction.
- (2) $\det A(u) = 0$.
- (3) *There exists $\nu \in N_p M$ such that the height function h_ν has a degenerate singularity and $u \in \ker \text{Hess } h_\nu$.*

Proof: Let $u = \alpha \partial_x + \beta \partial_y + \gamma \partial_z$, $v = \bar{\alpha} \partial_x + \bar{\beta} \partial_y + \bar{\gamma} \partial_z$, and $\nu = \nu_1 n_1 + \nu_2 n_2 + \nu_3 n_3$. If $u \in T_q \tilde{M}$ is an asymptotic direction we have

$$\begin{aligned}
 II_\nu(u, v) = & \bar{\alpha} \left[\sum_{i=1}^3 \nu_i (\alpha l_{n_i} + \beta m_{n_i} + \gamma q_{n_i}) \right] + \bar{\beta} \left[\sum_{i=1}^3 \nu_i (\alpha m_{n_i} + \beta n_{n_i} + \gamma r_{n_i}) \right] \\
 & + \bar{\gamma} \left[\sum_{i=1}^3 \nu_i (\alpha p_{n_i} + \beta q_{n_i} + \gamma r_{n_i}) \right] = 0.
 \end{aligned}$$

This must be satisfied for all $v = (\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in T_q \tilde{M}$, so we get a homogeneous system of three equations in variables ν_1, ν_2, ν_3 whose coefficient matrix is precisely $A(u)$. This proves the first equivalence.

Rewriting this system we can see that $u \in \ker \text{Hess } h_\nu$ and that $\det \text{Hess } h_\nu$ must be 0, which proves the second equivalence. \square

Let $M_{\text{sing}}^3 \subset \mathbb{R}^5$ be the projection under a tangent direction of $M_{\text{reg}}^3 \subset \mathbb{R}^6$ and $p' \in M_{\text{reg}}^3$ the point which is projected to p . Since the coefficients of the second fundamental form are the same for both manifolds, the height functions are the same. Therefore, by (3) in Proposition 5.3 and its regular counterpart (Theorem 2.2 in [12]), the binormal directions are the same and $u \in T_{p'} M_{\text{reg}}^3$ is asymptotic if and only if $u \in T_q \tilde{M}$ is asymptotic.

On the other hand, by direct calculation of the coefficients of the first fundamental form for the singular projection in terms of the coefficients of the first fundamental form for the regular manifold, we can see that the direction of projection u becomes the null tangent direction of the singular projection (this explains the concept of infinite asymptotic directions appearing in [16] or [5]).

By the above discussion we get the following:

Theorem 5.4. *Let $M_{\text{sing}}^3 \subset \mathbb{R}^5$ be the projection under a tangent direction of $M_{\text{reg}}^3 \subset \mathbb{R}^6$. The direction of projection is asymptotic for $M_{\text{reg}}^3 \subset \mathbb{R}^6$ if and only if the null tangent direction is asymptotic for $M_{\text{sing}}^3 \subset \mathbb{R}^5$.*

Definition 5.5. When the null tangent direction $u \in T_p M_{\text{sing}}^n$ is asymptotic we call it *infinite asymptotic direction* and denote it by u_∞ .

The idea of an infinite asymptotic direction is as follows. Any singular manifold can be seen as the projection of a regular manifold along a tangent direction. A singular manifold with an infinite asymptotic direction necessarily comes from a regular manifold projected along an asymptotic direction (which has become the infinite asymptotic direction).

In the case of $M_{\text{reg}}^2 \subset \mathbb{R}^4$ projected to $M_{\text{sing}}^2 \subset \mathbb{R}^3$ the following are equivalent:

- (i) The direction of projection is an asymptotic direction.
- (ii) $M_{\text{sing}}^2 \subset \mathbb{R}^3$ has a singularity worse than a cross-cap.
- (iii) The curvature parabola Δ_p of $M_{\text{sing}}^2 \subset \mathbb{R}^3$ is degenerate.

(The equivalence between (i) and (ii) can be found in [19, 9] and the equivalence between (ii) and (iii) is shown in [16].) Therefore, by Theorem 5.4 adapted to these dimensions (the proof is the same), for $M_{\text{sing}}^2 \subset \mathbb{R}^3$ there is an infinite asymptotic direction only if Δ_p is degenerate.

For $M_{\text{reg}}^3 \subset \mathbb{R}^6$ projected to $M_{\text{sing}}^3 \subset \mathbb{R}^5$, an analogous result to the equivalence between (i) and (ii) can be proved using results from [6]. However, the equivalence between (i) and (iii) for 3-manifolds is not true in general (see [6] for a partial result):

Example 5.6. Let $M_{\text{reg}}^3 \subset \mathbb{R}^6$ be given by $(x, y, z, x^2+z^2, xy+xz, y^2)$. The direction $u = (0, 0, 1) \in \ker \text{Hess } h_\nu$ for the binormal direction $\nu = (0, 0, 1)$, so u is an asymptotic direction. Projection along u yields $(x, y, x^2 + z^2, xy + xz, y^2)$ and the curvature locus is given by

$$\left(\cos^2 \theta + \frac{\cos^2 \phi}{\sin^2 \phi}, \cos \theta \sin \theta + \cos \theta \frac{\cos \phi}{\sin \phi}, \sin^2 \theta \right)$$

which is not contained in a plane, i.e. it is not a degenerate curvature locus.

The previous example shows that in some cases a non-degenerate curvature locus has an infinite asymptotic direction, so we must define the image of u_∞ and its tangent space for some $M_{\text{sing}}^3 \subset \mathbb{R}^5$ with non-degenerate curvature locus.

Definition 5.7. Let $\eta(\theta, \phi)$ denote the parametrisation of the curvature locus Δ_{cv} of M_{sing}^3 . For each topological type of the curvature locus we must define $\eta(u_\infty)$:

- (i) If Δ_{cv} is a point r , then $\eta(u_\infty) = r$ and $\frac{\partial \eta}{\partial \theta}(u_\infty) = \frac{\partial \eta}{\partial \phi}(u_\infty) = 0$.
- (ii) If Δ_{cv} is a line or a half-line, then $\eta(u_\infty) = \frac{\partial \eta}{\partial \theta}(u_\infty) = \frac{\partial \eta}{\partial \phi}(u_\infty) = \frac{\eta'(t)}{|\eta'(t)|}$ for any t such that $\eta'(t) \neq 0$, where t is the parameter of the line.
- (iii) If Δ_{cv} is a planar region or a plane, then $\eta(u_\infty) = \frac{\partial \eta}{\partial \theta}(u_\infty) = \frac{\partial \eta}{\partial \theta}(v)$ for any v such that $\eta(v)$ does not lie in the boundary of Δ_{cv} and $\frac{\partial \eta}{\partial \phi}(u_\infty) = \left(\frac{\partial \eta}{\partial \theta}(v)\right)^\perp$.
- (iv) If Δ_{cv} is non-degenerate such that u_∞ is an asymptotic direction of M_{reg}^3 , then $\eta(u_\infty) = \frac{\partial \eta}{\partial \theta}(u_\infty) = \frac{\partial \eta}{\partial \phi}(u_\infty) = \lim_{\phi \rightarrow 0} \frac{\eta(\theta, \phi)}{|\eta(\theta, \phi)|}$.

We can now add an equivalence to Proposition 5.3:

Theorem 5.8. *Let $u \in C_q \cup \{u_\infty\}$. Then u is asymptotic if and only if*

- (4) *The vector $\eta(u)$ is tangent to $\eta(C_q \cup \{u_\infty\})$ at $\eta(u)$ or $\Delta_{cv} = \eta(C_q)$ is singular at u .*

Proof: Suppose first that u is not the null tangent direction. In this case we proceed as in [12]. Let C_q be parametrised by (θ, ϕ) in cylindrical coordinates and let $u = u(\theta, \phi)$. Then $\frac{\partial \eta(u)}{\partial \theta} = II(u, u)_\theta = 2II(u, u_\theta)$

and $\frac{\partial \eta(u)}{\partial \phi} = II(u, u)_\phi = 2II(u, u_\phi)$. Since $\{u, u_\theta, u_\phi\}$ are linearly independent, having the tangency or a singularity means that $\{II(u, u), II(u, u)_\theta, II(u, u)_\phi\} = \{II(u, u), 2II(u, u_\theta), 2II(u, u_\phi)\}$ is linearly dependent, and this happens if and only if there exists a unit tangent vector $w \in C_q$ such that $II(u, w) = 0$. Consider now $g = II(u, \cdot): T_q\tilde{M} \rightarrow N_pM$. Since $w \in \ker g$, the image of g is contained in a plane in N_pM and so, what we have is equivalent to the fact that there exists $v \in N_pM$ such that $II_\nu(u, v) = 0$ for all $v \in T_q\tilde{M}$, i.e. u is asymptotic.

For u_∞ the tangency occurs by construction of $\frac{\partial \eta}{\partial \theta}(u_\infty)$ and $\frac{\partial \eta}{\partial \phi}(u_\infty)$. □

Theorem 5.9. *Let $M_{\text{sing}}^2 \subset \mathbb{R}^4$ be a normal section given by M_{sing}^3 , $p \in M_{\text{sing}}^3$, and suppose that $\text{Aff}_{i_2^{-1}(p)} = E_{i_2^{-1}(p)}$. Then $u \in T_q\tilde{M}$ is an asymptotic direction of M_{sing}^3 if and only if $(i_{2*})^{-1}(u)$ is an asymptotic direction of M_{sing}^2 .*

Proof: Lemma 4.10 in [5] is the equivalent result to Theorem 5.8 for singular surfaces in \mathbb{R}^4 . Taking a normal section is taking a hyperplane U in $T_q\tilde{M}$. This induces an intersection of Δ_{cv} with the plane $II(U)$. A direction $u \in T_q\tilde{M}$ is asymptotic if $\eta(u)$ is tangent to $\eta(C_q \cup \{u_\infty\})$ or $\eta(C_q)$ is singular at u . The curvature parabola of $M_{\text{sing}}^2 \subset \mathbb{R}^4$ is given by $\Delta_{cv} \cap II(U)$ and here $\eta(u)$ is tangent to $\Delta_{cv} \cap II(U)$ if and only if $II(U)$ is a plane that passes through the origin, i.e. $\text{Aff}_{i_2^{-1}(p)} = E_{i_2^{-1}(p)}$. Hence, $\eta|_{U \cap C_q}((i_{2*})^{-1}(u)) \in E_{i_2^{-1}(p)}$ and is also parallel to $\eta'|_{U \cap C_q}((i_{2*})^{-1}(u))$, that is, $(i_{2*})^{-1}(u)$ is an asymptotic direction of the normal section. □

Example 5.10. Consider the singular 3-manifold given by $(x, y, x^2 - 2yz, y^2 - 2xz, z^2 - 2xy)$. Its curvature locus is given by

$$\left(2 \cos^2 \theta - 4 \sin \theta \frac{\cos \phi}{\sin \phi}, 2 \sin^2 \theta - 4 \cos \theta \frac{\cos \phi}{\sin \phi}, 2 \frac{\cos^2 \phi}{\sin^2 \phi} - 4 \cos \theta \sin \theta \right).$$

We have that $(0, 1, -1)$ is an asymptotic direction associated to the binormal direction $(-1, 1, 1)$. Consider now the normal section given by $\{X = 0\}$ and parametrised by $(y, -2yz, y^2, z^2)$. The curvature parabola is given by $(-4y, 2, 2y^2)$. Here $(-1, 1, 1)$ is a degenerate direction, but it is not binormal since it is not in E_p , and therefore $(1, -1)$ is not an asymptotic direction of the singular surface.

Remark 5.11. Proposition 5.3 and Theorem 5.8 can be generalised to other dimensions. The setting in this section works well for $M_{\text{reg}}^k \subset \mathbb{R}^{2k}$ projected to $M_{\text{sing}}^k \subset \mathbb{R}^{2k-1}$ and taking normal sections of the singular manifold to get $M_{\text{sing}}^{k-1} \subset \mathbb{R}^{2(k-1)}$.

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