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LOWER CENTRAL WORDS IN FINITE p-GROUPS

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Abstract: It is well known that the set of values of a lower central word in a group G need not be a subgroup. For a fixed lower central word γ_r and for $p \geq 5$, Guralnick showed that if G is a finite p-group such that the verbal subgroup $\gamma_r(G)$ is abelian and 2-generator, then $\gamma_r(G)$ consists only of γ_r-values. In this paper we extend this result, showing that the assumption that $\gamma_r(G)$ is abelian can be dropped. Moreover, we show that the result remains true even if $p=3$. Finally, we prove that the analogous result for pro-p groups is true.

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1. Introduction

A word w in k variables is an element of the free group F_k with k generators. For any group G, this word can be seen as a map from the Cartesian product of k copies of G to the group G itself by substituting group elements for the variables. The image of this map is called the set of w-values of G and is denoted by G_w. The subgroup generated by this set is called the verbal subgroup of w in G and is denoted by $w(G)$.

In this paper we will focus on the lower central words. These words are defined recursively by the rule $\gamma_1(x_1) = x_1$ and

$$\gamma_r(x_1, \dots, x_r) = [\gamma_{r-1}(x_1, \dots, x_{r-1}), x_r]$$

for $r \geq 2$. Thus, the verbal subgroup $\gamma_r(G)$ of the word γ_r in a group G coincides with the r-th term of the lower central series of G. In this context, it is well known that the set of γ_r-values need not be a subgroup. In other words, G_{γ_r} may be a proper subset of $\gamma_r(G)$.

However, several families of groups have been found for which the equality $\gamma_r(G) = G_{\gamma_r}$ holds. The study of this property started with the case $r = 2$, that is, when the word γ_r is the common commutator word

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and its verbal subgroup is just the derived subgroup of the group. One of the main results in this case is the proof by Liebeck, O'Brien, Shalev, and Tiep in [**13**] of the so-called Ore Conjecture, according to which every finite simple group G satisfies the condition $G' = G_{\gamma_2}$.

In the opposite direction, still in the case $r = 2$, the result is also true for nilpotent groups with cyclic derived subgroup, as proved by Rodney in [**17**]. If, instead, we drop the nilpotency assumption, the result fails to hold. Namely, in [**14**], Macdonald provides some examples of groups G with G' cyclic and $G' \neq G_{\gamma_2}$. For finite nilpotent groups or, equivalently, for finite p-groups, Rodney addressed the simplest cases, showing that $G' = G_{\gamma_2}$ if G' is 3-generator and central or if G' is elementary abelian of rank 3 ([**18**]). Guralnick extended Rodney's results proving that if G' is abelian, then $G' = G_{\gamma_2}$ whenever G' can be generated by 2 elements ([**8**, Theorem A]) or whenever G' can be generated by 3 elements and $p \geq 5$ ([**8**, Theorem B]). In addition, Guralnick himself showed that the result is no longer true if G' is 3-generator and $p = 2$ or $p = 3$ ([**8**, Example 3.5 and Example 3.6]).

On this basis, G. A. Fernández-Alcober and the first author in [**7**] and [**5**] improved Guralnick's results, showing that the condition that G' is abelian can be removed. Moreover, Macdonald ([**15**, Exercise 5, p. 78]) and Kappe and Morse ([**11**, Example 5.4]) had already shown that for every prime p there exist finite p-groups with 4-generator abelian derived subgroup such that $G' \neq G_{\gamma_2}$. Therefore, for $r = 2$, the study of this property for finite p-groups in terms of the number of generators of the derived subgroup is already completed.

For the case $r > 2$, however, much less is known. The first results were due to Dark and Newell in [**4**], where they generalized Macdonald's and Rodney's results in [**14**] and [**17**] to lower central words. So far, the main results in this context were proved by Guralnick: he showed in [**9**] and [**10**] that if G is a finite p-group, $p \geq 5$, such that $\gamma_r(G)$ is 2-generator and abelian, then $\gamma_r(G) = G_{\gamma_r}$. In addition, he found an example of a 2-group such that $\gamma_r(G) \neq G_{\gamma_r}$, but the case $p = 3$ remained unknown.

The goal of this paper is to generalize again Guralnick's result, showing that the condition that $\gamma_r(G)$ is abelian is not necessary. Moreover, we prove that the result is also true if $p = 3$, closing in that way the gap between the primes 2 and 5.

Theorem A. *Let G be a finite p-group and let $r \geq 2$. If $\gamma_r(G)$ is cyclic or if p is odd and $\gamma_r(G)$ can be generated with 2 elements, then there exist $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r \in G$ with $1 \leq j \leq r$ such that*

$$\gamma_r(G) = \{[x_1, \ldots, x_{j-1}, g, x_{j+1}, \ldots, x_r] \mid g \in G\}.$$

As in [**7**] and [**5**], we will also prove the analogous version of Theorem A for pro-p groups. In the case of a pro-p-group G, $\gamma_r(G)$ denotes the topological closure of the subgroup generated by the set of all γ_r-values.

Theorem B. *Let G be a pro-p group and let $r \geq 2$. If $\gamma_r(G)$ is procyclic or if p is odd and $\gamma_r(G)$ can be topologically generated with 2 elements, then there exist $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r \in G$ with $1 \leq j \leq r$ such that*

$$\gamma_r(G) = \{[x_1, \ldots, x_{j-1}, g, x_{j+1}, \ldots, x_r] \mid g \in G\}.$$

Notation and organization. Let G be a group. If L is a normal subgroup of G, then $[L,_1 G] = [L, G]$ denotes the subgroup generated by all commutators $[x, y]$ with $x \in L$ and $y \in G$, and we define recursively $[L,_{n+1} G] = [[L,_n G], G]$ for all $n \geq 1$. If $H \leq G$ and $x \in G$, then we set $[x, H] = \langle [x, h] \mid h \in H \rangle$. Moreover, H^n will denote the subgroup generated by all n-th powers of elements of H. We denote the Frattini subgroup of G by $\Phi(G)$ and if G is finitely generated, $d(G)$ stands for the minimum number of generators of G. Finally, if G is a topological group, we write $\mathrm{Cl}_G(H)$ to refer to the topological closure of H in G and we write $H \trianglelefteq_{\mathrm{o}} G$ to denote that H is an open normal subgroup of G.

We start with some general preliminary results in Section 2 that will be used frequently along the paper. Then we split the proof of Theorem A into three sections, dealing separately with two different cases: first, in Section 3 we prove the result when $\gamma_r(G)$ is cyclic, and then, in Section 5 and Section 6 we prove it when $d(\gamma_r(G)) = 2$ and p is odd, making an additional distinction on the position of a certain subgroup inside the group. However, the proof for the non-cyclic case in Sections 5 and 6 will require further preliminaries that will be developed in Section 4. Finally, we prove Theorem B in Section 7.

2. Preliminaries

Throughout the paper we will use freely the following well-known commutator identities (see, for instance, [**16**, 5.1.5]).

Lemma 2.1. *Let x, y, z be elements of a group. Then*

(i) $[x, y] = [y, x]^{-1}$.

(ii) $[xy, z] = [x, z]^y [y, z]$, *and* $[x, yz] = [x, z][x, y]^z$.

(iii) $[x, y^{-1}] = [y, x]^{y^{-1}}$, *and* $[x^{-1}, y] = [y, x]^{x^{-1}}$.

(iv) $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$ *(the Hall–Witt Identity).*

The next standard properties are consequences of the identities above and for the reader convenience we collect them in a lemma that will be often used without mentioning.

Lemma 2.2. *Let G be a group. Then*

(i) *If L and N are two normal subgroups of G and $n \in \mathbb{N}$, then $[L^n, N] \leq [L,N]^n[L,N,L]$.*

(ii) *If L is a normal subgroup of G, then $[L, \gamma_i(G)] \leq [L,_i G]$ for every $i \in \mathbb{N}$.*

We will also use without mentioning the fact that if $N \leq L$ are two normal subgroups of G such that $[L : N| = p^2$, then $[L, G, G] \leq N$, while if L/N is cyclic, then $[L,_i G] \leq L^{p^i}N$ for each $i \in \mathbb{N}$.

The following lemma is essentially the well-known Hall–Petresco Identity (see [**2**, Appendix A.1]).

Lemma 2.3. *Let x, y be elements of a group and let $n \in \mathbb{N}$. Then for each $i = 2, \ldots, n$ there exists $c_i \in \gamma_i(\langle y, [x,y]\rangle)$ such that*

$$[x,y]^n = [x,y^n]c_2^{\binom{n}{2}}c_3^{\binom{n}{3}} \cdots c_n^{\binom{n}{n}}.$$

Outer commutator words, also known under the name of multilinear commutator words, are words obtained by nesting commutators, but using always different variables. More formally, the word $w(x) = x$ in one variable is an outer commutator word; if α and β are outer commutator words involving different variables, then the word $w = [\alpha, \beta]$ is an outer commutator, and all outer commutator words are obtained in this way. Thus, lower central words are particular instances of outer commutator words, and as Lemma 2.5 below shows, the verbal subgroup of such words in finite p-groups is powerful whenever it can be generated by 2 elements (a finite p-group is said to be powerful if $G^p \leq G'$ for odd p or if $G^4 \leq G'$ for $p = 2$). Hence, the theory of powerful p-groups will be essential in this paper. These groups are usually seen as a generalization of abelian groups since they satisfy, among others, the following properties:

(i) $\Phi(G) = G^p$. In particular, $|G : G^p| = p^{d(G)}$.

(ii) $d(H) \leq d(G)$ for every $H \leq G$.

(iii) $G^p = \{g^p \mid g \in G\}$.

(iv) If $G = \langle x_1, \ldots, x_n\rangle$, then $G^p = \langle x_1^p, \ldots, x_n^p\rangle$.

(v) The power map from $G^{p^{i-1}}/G^{p^i}$ to $G^{p^i}/G^{p^{i+1}}$ that sends gG^{p^i} to $g^pG^{p^{i+1}}$ is an epimorphism for every $i \geq 0$.

Background in such groups can be found, for instance, in [**6**, Chapter 2] or [**12**, Chapter 11].

In order to prove Lemma 2.5 we first need the following result, which is a basic fact about finite p-groups.

Lemma 2.4. *Let G be a finite p-group and N, K normal subgroups of G. If $N \le KN^p[N,G]$, then $N \le K$.*

Proof: Factor out K and just note that if N is non-trivial, then $N^p[N,G]$ is a proper subgroup of N, which is a contradiction. $\square$

Lemma 2.5. *Let G be a finite p-group and w an outer commutator word. If $d(w(G)) = 2$, then $w(G)' \le w(G)^{p^2}$. In particular, $w(G)$ is powerful.*

Proof: By Theorem 1 of [**3**] the result is true if w is the commutator word, so we assume $w(G) \le \gamma_3(G)$. In order to show that $w(G)' \le w(G)^{p^2}$ we may assume that $w(G)^{p^2} = 1$, and by Lemma 2.4 we can also assume $[w(G)', G] = (w(G)')^p = 1$.

Since $d(w(G)) = 2$, we have $|w(G) : \Phi(w(G))| = p^2$, and so $[w(G), G, G] \le \Phi(w(G))$. Observe first that

$$[\Phi(w(G)), w(G)] = [w(G)^p w(G)', w(G)] \le (w(G)')^p [w(G)', w(G)] = 1,$$

so, in particular, $\Phi(w(G))$ is abelian and $\Phi(w(G))^p = (w(G)^p)^p (w(G)')^p = w(G)^{p^2} = 1$. Moreover,

$$\begin{aligned} [\Phi(w(G)), G] &= [w(G)^p w(G)', G] = [w(G)^p, G][w(G)', G] \\ &\le [w(G), G]^p [w(G), G, w(G)]. \end{aligned} \tag{1}$$

We consider now two cases in turn: $[w(G), G] \le \Phi(w(G))$ and $[w(G), G] \not\le \Phi(w(G))$.

If $[w(G), G] \le \Phi(w(G))$, then by (1) we have

$$\begin{aligned} [\Phi(w(G)), G] &\le [w(G), G]^p [w(G), G, w(G)] \\ &\le \Phi(w(G))^p [\Phi(w(G)), w(G)] = 1. \end{aligned}$$

Hence,

$$\begin{aligned} w(G)' = [w(G), w(G)] &\le [w(G), \gamma_3(G)] \\ &\le [w(G), G, G, G] \le [\Phi(w(G)), G] = 1, \end{aligned}$$

as desired.

Suppose now $[w(G), G] \not\le \Phi(w(G))$. By (1), we have

$$\begin{aligned} [w(G), G, G, G, G] &\le [\Phi(w(G)), G, G] \\ &\le [[w(G), G]^p [w(G), G, w(G)], G] \\ &\le [w(G), G, G]^p [w(G), G, G, G, G, G] \\ &\le \Phi(w(G))^p [w(G), G, G, G, G, G] \\ &= [w(G), G, G, G, G, G], \end{aligned}$$

so $[w(G), G, G, G, G] = 1$. In addition, the quotient group

$$w(G)/[w(G), G]\Phi(w(G))$$

is cyclic. Hence,

$$\begin{aligned} w(G)' &= [w(G), [w(G), G]\Phi(w(G))] \\ &\leq [w(G), G, G, G, G] = 1, \end{aligned}$$

and the proof is complete. □

Therefore, as we will deal with 2-generator verbal subgroups, we will always assume that $\gamma_r(G)$ is powerful. Moreover, the next lemma, proved in Lemma 2.2 of [**7**], shows that actually all the subgroups of $\gamma_r(G)$ are also powerful.

Lemma 2.6. *Let G be a powerful p-group. If $d(G) = 2$, then every subgroup H of G is also powerful.*

The following result is a particular case of Lemma 3.1 of [**5**], where it is proved more generally for potent p-groups.

Lemma 2.7. *Let G be a powerful p-group with $p \geq 3$. If $N \leq L$ are two normal subgroups of G, then $|N : N^{p^i}| \leq |L : L^{p^i}|$ for all $i \geq 0$. In particular, $|L^{p^i} : N^{p^i}| \leq |L : N|$.*

In order to prove Theorem A we will construct a series of subgroups from $\gamma_r(G)$ to 1 with the property that every element of each factor group of two consecutive subgroups in the series can be written as a γ_r-value in a suitable way. Lemma 2.10 below will then allow us to go up in this series, proving that actually all the subgroups in the series consist of γ_r-values until we reach $\gamma_r(G)$. The key part of the proof is the following lemma, which is a generalization to outer commutator words of Lemma 2.1 in [**1**].

Lemma 2.8. *Let G be a group and let w be an outer commutator word in r variables. Let $y_1, \dots, y_{j-1}, h, y_{j+1}, \dots, y_r \in G$. Then there exist $h_1, \dots, h_r \in \langle h \rangle^G$ such that for every $g \in G$,*

$$\begin{aligned} &w(y_1, \dots, y_{j-1}, gh, y_{j+1}, \dots, y_r) \\ &\quad = w(y_1^{h_1}, \dots, y_{j-1}^{h_{j-1}}, g^{h_j}, y_{j+1}^{h_{j+1}}, \dots, y_r^{h_r}) w(y_1, \dots, y_{j-1}, h, y_{j+1}, \dots, y_r). \end{aligned}$$

Proof: We proceed by induction on the number of variables appearing in the outer commutator word w. If such number is 1, i.e. if $w = x$, then the result is obvious. Hence, assume $w = [\alpha, \beta]$, where α and β

are outer commutator words involving k and $r-k$ variables with $k<r$, respectively. Assume also that $j>k$, so that

$$w(y_1,\dots,y_{j-1},gh,y_{j+1},\dots,y_r) \\ = [\alpha(y_1,\dots,y_k),\beta(y_{k+1},\dots,y_{j-1},gh,y_{j+1},\dots,y_r)].$$

By induction, we have

$$\beta(y_{k+1},\dots,y_{j-1},gh,y_{j+1},\dots,y_r) \\ =\beta(y_{k+1}^{h_1},\dots,y_{j-1}^{h_{j-1}},g^{h_j},y_{j+1}^{h_{j+1}},\dots,y_r^{h_r})\beta(y_{k+1},\dots,y_{j-1},h,y_{j+1},\dots,y_r),$$

where $h_{k+1},\dots,h_r\in\langle h\rangle^G$.

For simplicity, write $z_1=\beta(y_{k+1}^{h_1},\dots,y_{j-1}^{h_{j-1}},g^{h_j},y_{j+1}^{h_{j+1}},\dots,y_r^{h_r})$, $z_2=\beta(y_{k+1},\dots,y_{j-1},h,y_{j+1},\dots,y_r)$, and notice that

$$[\alpha(y_1,\dots,y_k),z_1z_2]=[\alpha(y_1,\dots,y_k),z_2][\alpha(y_1,\dots,y_k),z_1]^{z_2} \\ =[\alpha(y_1,\dots,y_k),z_1]^{z_2^{\alpha(y_1,\dots,y_k)}}[\alpha(y_1,\dots,y_k),z_2].$$

Since clearly $z_2\in\langle h\rangle^G$, the result follows.

The case $j\le k$ is similar. $\square$

The following result is an easy consequence of Lemma 2.8; it is also proved in [**19**, Proposition 1.2.1].

Corollary 2.9. *Let G be a group. Then, for every $i=1,\dots,n$ and for every $g,g_1,\dots,g_{i-1},g_{i+1},\dots,g_n\in G$, $h\in\gamma_s(G)$, we have*

$$[g_1,\dots,g_{i-1},gh,g_{i+1},\dots,g_n] \\ \equiv[g_1,\dots,g_{i-1},g,g_{i+1},\dots,g_n][g_1,\dots,g_{i-1},h,g_{i+1},\dots,g_n]\ (\mathrm{mod}\ \gamma_{n+s}(G)).$$

In particular, if $h\in G'$, then

$$[g_1,\dots,g_{i-1},gh,g_{i+1},\dots,g_n] \\ \equiv[g_1,\dots,g_{i-1},g,g_{i+1},\dots,g_n]\quad(\mathrm{mod}\ \gamma_{n+1}(G)).$$

Lemma 2.10. *Let G be a group and w an outer commutator word on r variables. Let $N\le L\le G$ with N normal in G and suppose that for some $x_1,\dots,x_{j-1},x_{j+1},\dots,x_r\in G$, the following two conditions hold:*

(i) $L\subseteq\bigcup_{g\in G}Nw(y_1,\dots,y_{j-1},g,y_{j+1},\dots,y_r)$ *for every* $y_i\in x_i^G$.

(ii) $N\subseteq\{w(y_1,\dots,y_{j-1},g,y_{j+1},\dots,y_r)\mid g\in G\}$ *for every* $y_i\in x_i^G$.

Then, $L\subseteq\{w(y_1,\dots,y_{j-1},g,y_{j+1},\dots,y_r)\mid g\in G\}$ *for every* $y_i\in x_i^G$.

Proof: Take an arbitrary coset $Nw(y_1,\dots,y_{j-1},h,y_{j+1},\dots,y_r)$ of N in L, with $y_i \in x_i^G$ and $h \in G$. Take $h_1,\dots,h_r$ as in Lemma 2.8 and let z be an arbitrary element of N. By assumption, there exists $u \in G$ such that $z = w(y_1^{h_1},\dots,y_{j-1}^{h_{j-1}},u,y_{j+1}^{h_{j+1}},\dots,y_r^{h_r})$ and we may also assume that u is of the form $u = g^{h_j}$ with $g \in G$.

So, by Lemma 2.8 our arbitrary element $zw(y_1,\dots,y_{j-1},h,y_{j+1},\dots,y_r)$ of the above coset can be written as

$$\begin{aligned} w(y_1^{h_1},\dots,y_{j-1}^{h_{j-1}},g^{h_j},y_{j+1}^{h_{j+1}},\dots,y_r^{h_r})&w(y_1,\dots,y_{j-1},h,y_{j+1},\dots,y_r)\\ &= w(y_1,\dots,y_{j-1},gh,y_{j+1},\dots,y_r), \end{aligned}$$

as desired. □

We end this section with the following three technical lemmas, which will be basically used to introduce powers inside commutators in the factor groups of the series of $\gamma_r(G)$ mentioned before Lemma 2.8. In particular, Lemma 2.13 will be especially useful to prove that these factor groups consist only of some suitable γ_r-values.

Lemma 2.11. *Let G be a finite p-group such that for some $r \geq 2$ we have $d(\gamma_r(G)) \leq 2$ if p is odd or $d(\gamma_r(G)) = 1$ if $p = 2$. Then*

$$[x_1,\dots,x_r]^{p^k} \equiv [[x_1,\dots,x_j]^{p^k},x_{j+1},\dots,x_r] \pmod{\gamma_r(G)^{p^{k+1}}}$$

for every $x_1,\dots,x_r \in G$, $k \geq 0$, and $2 \leq j \leq r$. Moreover, if $[x_1,\dots,x_i] \in R$ for some normal subgroup R of G and $1 \leq i \leq j$, then

$$[x_1,\dots,x_r]^{p^k} \equiv [[x_1,\dots,x_j]^{p^k},x_{j+1},\dots,x_r] \pmod{[R,_{r-i}\,G]^{p^{k+1}}}.$$

Proof: The first assertion follows immediately from the second one. We fix r, and we will prove by induction on $r-j$ that the assertion holds for all k. Thus, assume $[x_1,\dots,x_i] \in R$ for some normal subgroup R of G and some $1 \leq i \leq j$. For $r = j$ the result is clear, so assume $j < r$ and

$$[x_1,\dots,x_r]^{p^k} \equiv [[x_1,\dots,x_{j+1}]^{p^k},x_{j+2},\dots,x_r] \pmod{[R,_{r-i}\,G]^{p^{k+1}}}.$$

By the Hall–Petresco Identity, we have

$$[x_1,\dots,x_{j+1}]^{p^k} = [[x_1,\dots,x_j]^{p^k},x_{j+1}]c_2^{\binom{p^k}{2}}\cdots c_{p^k}$$

with $c_n \in \gamma_n(\langle [x_1,\dots,x_{j+1}],[x_1,\dots,x_j]\rangle)$ for $2 \leq n \leq p^k$. Since $j \geq 2$, it follows that

$$c_n \in [R,_{nj-i+1}\,G] \leq [R,_{j-i+2(n-1)+1}\,G]$$

for every n. Note that $p^{k-(n-2)} | \binom{p^k}{n}$ if p is odd and $p^{k-(n-1)} | \binom{p^k}{n}$ if $p = 2$. We denote with $\lceil s \rceil$ the smallest integer which is greater or equal to s. So, if p is odd, we get

$$c_n^{\binom{p^k}{n}} \in [R,_{j-i+2(n-1)+1} G]^{\lceil p^{k-(n-2)} \rceil},$$

and if $p = 2$, we get

$$c_n^{\binom{2^k}{n}} \in [R,_{j-i+2(n-1)+1} G]^{\lceil 2^{k-(n-1)} \rceil}.$$

Since $d(\gamma_r(G)) \leq 2$, it follows by Lemma 2.5 that $\gamma_r(G)$ is powerful. By Lemma 2.6 we then obtain that for all $m \geq 0$, $[R,_{j-i} G]^{p^m}$ is also powerful and $d([R,_{j-i} G]^{p^m}) \leq 2$, so

$$|[R,_{j-i} G]^{p^m} : [R,_{j-i} G]^{p^{m+1}}| \leq p^2$$

for all $m \geq 0$. This implies, in particular, that

$$[[R,_{j-i} G]^{p^m}, G, G] \leq [R,_{j-i} G]^{p^{m+1}},$$

for all $m \geq 0$, and therefore,

$$[R,_{j-i+2(n-1)+1} G]^{\lceil p^{k-(n-2)} \rceil} \leq [R,_{j-i+1} G]^{p^{k+1}}.$$

Now, if p is odd, using the inductive hypothesis with $k+1$ in place of k, we have

$$\begin{aligned}[][[R,_{j-i+2(n-1)+1} G]^{\lceil p^{k-(n-2)} \rceil},_{r-j-1} G] &\leq [[R,_{j-i+1} G]^{p^{k+1}}],_{r-j-1} G] \\ &\leq [R,_{r-i} G]^{p^{k+1}}.\end{aligned}$$

If $p = 2$, the result follows arguing in the same way, taking into account the fact that, in this case, $\gamma_r(G)$ is cyclic and hence

$$[[R,_{r-i} G]^{2^m}, G] \leq [R,_{r-i} G]^{2^{m+1}}. \qquad \square$$

Lemma 2.12. *Let G be a finite p-group such that for some $r \geq 2$ we have $d(\gamma_r(G)) \leq 2$ if p is odd and $d(\gamma_r(G)) = 1$ if $p = 2$. Assume that H and K are normal subgroups of G, with K generated by γ_{j-1}-values. Then for every $k \geq 0$ and for every j with $1 \leq j \leq r$, we have*

$$[K, H^{p^k},_{r-j} G] \leq [K, H,_{r-j} G]^{p^k}.$$

Proof: We use induction on k. The case $k = 0$ is trivial, so assume $k = 1$ first, and suppose $p \geq 3$ (if $p = 2$, the proof follows in the same way).

As p divides $\binom{p}{i}$ for $2 \leq i < p$ and $\gamma_3(\langle [K, H], H\rangle) \leq [K, H, H, H]$, the Hall–Petresco Identity yields

$$[K, H^p] \leq [K, H]^p [K, H, H, H].$$

Note that $[K, H]$ is generated by elements of the type $[x_1, \ldots, x_{j-1}, x_j]$, where $x_1, \ldots, x_{j-1} \in G$ and $x_j \in H$, so by Lemma 2.11, we have

$$[[K, H]^p,_{r-j} G] \leq [K, H,_{r-j} G]^p.$$

On the other hand, $\gamma_r(G)$ is powerful by Lemma 2.5. Thus, it follows from Lemma 2.6 that

$$|[K, H,_{r-j} G] : [K, H,_{r-j} G]^p| \leq p^2,$$

so we get

$$\begin{aligned}[K, H, H, H,_{r-j} G] &\leq [[K, H,_{r-j} G], G, G] \\ &\leq [K, H,_{r-j} G]^p.\end{aligned}$$

Hence,

$$[K, H^p,_{r-j} G] \leq [[K, H]^p [K, H, H, H]],_{r-j} G] \leq [K, H,_{r-j} G]^p,$$

as desired.

Assume now $k \geq 2$. Then, by induction,

$$\begin{aligned}[K, H^{p^k},_{r-j} G] &\leq [K, (H^p)^{p^{k-1}},_{r-j} G] \\ &\leq [K, H^p,_{r-j} G]^{p^{k-1}} \\ &\leq ([K, H,_{r-j} G]^p)^{p^{k-1}},\end{aligned}$$

and since $[K, H,_{r-j} G]$ is powerful by Lemma 2.6, we have

$$([K, H,_{r-j} G]^p)^{p^{k-1}} = [K, H,_{r-j} G]^{p^k}.$$

□

Lemma 2.13. *Let G be a finite p-group and let N, L be normal subgroups of G such that $\gamma_r(G)^p \leq N \leq L \leq \gamma_r(G)$ with $r \geq 2$ and $|L : N| = p$. Assume that there exist some j with $1 \leq j \leq r$ and $x_1, \ldots, x_{j-1}, h, x_{j+1}, \ldots, x_r \in G$ such that*

$$L = \langle [x_1, \ldots, x_{j-1}, h, x_{j+1}, \ldots, x_r]\rangle N.$$

Let H be the normal closure of $\langle h\rangle$ in G and assume also that one of the following conditions holds:

(i) *p is odd, $d(\gamma_r(G)) \leq 2$, and the subgroup*

$$[\gamma_j(G), H, H,_{r-j} G]$$

is central of exponent p modulo N^p.

(ii) *$p = 2$, the subgroup $\gamma_r(G)$ is cyclic, and*

$$[x_1, \dots, x_{j-1}, h, x_{j+1}, \dots, x_r]^2 \equiv [x_1, \dots, x_{j-1}, h^2, x_{j+1}, \dots, x_r] \pmod{N^2}.$$

Then

$$[x_1, \dots, x_{j-1}, h, x_{j+1}, \dots, x_r]^{p^k} \equiv [x_1, \dots, x_{j-1}, h^{p^k}, x_{j+1}, \dots, x_r] \pmod{N^{p^k}}$$

for every $k \geq 0$. In particular,

$$L^{p^k} = \langle [x_1, \dots, x_{j-1}, h^{p^k}, x_{j+1}, \dots, x_r] \rangle N^{p^k}.$$

Proof: We use induction on k. If $k = 0$, there is nothing to prove and, if $p = 2$ and $k = 1$, then the result follows from the hypothesis. Thus, assume $k \geq 1$ if p is odd or $k \geq 2$ if $p = 2$ and suppose, by induction, that

$$[x_1, \dots, x_{j-1}, h, x_{j+1}, \dots, x_r]^{p^{k-1}} = [x_1, \dots, x_{j-1}, h^{p^{k-1}}, x_{j+1}, \dots, x_r]y$$

for some $y \in N^{p^{k-1}}$.

Let $u = [x_1, \dots, x_{j-1}, h^{p^{k-1}}, x_{j+1}, \dots, x_r] \in \gamma_r(G)$. Note that $(uy)^p = u^p y^p c$ where $c \in [N^{p^{k-1}}, \gamma_r(G)] \leq [N^{p^{k-1}}, G, G] \leq (N^{p^{k-1}})^p = N^{p^k}$. Thus,

$$([x_1, \dots, x_{j-1}, h^{p^{k-1}}, x_{j+1}, \dots, x_r]y)^p \equiv [x_1, \dots, x_{j-1}, h^{p^{k-1}}, x_{j+1}, \dots, x_r]^p \pmod{N^{p^k}}.$$

Moreover, by Lemma 2.12, we have

$$\begin{aligned}[\gamma_{j-1}(G), H^{p^{k-1}},_{r-j} G]^{p^2} &\leq [\gamma_{j-1}(G), H,_{r-j} G]^{p^{k+1}} \\ &\leq \gamma_r(G)^{p^{k+1}} \leq N^{p^k},\end{aligned}$$

so using Lemma 2.11 with $R = [\gamma_{j-1}(G), H^{p^{k-1}}]$ we obtain

$$\begin{aligned}[x_1, \dots, x_{j-1}, h, x_{j+1}, \dots, x_r]^{p^k} &\equiv [x_1, \dots, x_{j-1}, h^{p^{k-1}}, x_{j+1}, \dots, x_r]^p \\ &\equiv [[x_1, \dots, x_{j-1}, h^{p^{k-1}}]^p, x_{j+1}, \dots, x_r] \pmod{N^{p^k}}.\end{aligned}$$

Suppose now p is odd. We first prove that

(2) $[\gamma_{j-1}(G), H^{p^{k-1}}, H^{p^{k-1}},_{r-j} G]$ is central of exponent p modulo N^{p^k}.

If $k = 1$, the claim follows from the hypothesis, so we may assume $k \geq 2$. Recall that L, N, and $[\gamma_{j-1}(G), H, H,_{r-j} G]$ are powerful by Lemmas 2.5 and 2.6. From Lemma 2.12 we then get

$$\begin{aligned}[\gamma_{j-1}(G), H^{p^{k-1}}, H^{p^{k-1}},_{r-j+1} G] &\leq [\gamma_{j-1}(G), H, H,_{r-j+1} G]^{p^{k-1}} \\ &\leq (N^p)^{p^{k-1}} \leq N^{p^k}\end{aligned}$$

and

$$\begin{aligned}[\gamma_{j-1}(G), H^{p^{k-1}}, H^{p^{k-1}},_{r-j} G]^p &\leq ([\gamma_{j-1}(G), H, H,_{r-j} G]^{p^{k-1}})^p \\ &\leq (N^p)^{p^{k-1}} \leq N^{p^k}.\end{aligned}$$

This proves (2).

By the Hall–Petresco Identity, since $p \geq 3$, we get

$$[x_1, \ldots, x_{j-1}, h^{p^{k-1}}]^p = [x_1, \ldots, x_{j-1}, h^{p^k}] z_2^p z_3,$$

where $z_i \in \gamma_i(\langle [x_1, \ldots, x_{j-1}, H^{p^{k-1}}], H^{p^{k-1}} \rangle)$ for $i = 2, 3$. Write

$$R = [\gamma_{j-1}(G), H^{p^{k-1}}, H^{p^{k-1}}],$$

so that $z_2 \in R$ and $z_3 \in [R, G]$.

On the one hand, by (2) we have

$$[z_3, x_{j+1}, \ldots, x_r] \in [R,_{r-j+1} G] \leq N^{p^k}.$$

On the other hand, it follows from Lemma 2.12 with $H = R$ and $K = G$ and from (2) that

$$[z_2^p, x_{j+1}, \ldots, x_r] \in [R,_{r-j} G]^p \leq N^{p^k}.$$

Therefore,

$$\begin{aligned}&[x_1, \ldots, x_{j-1}, h, x_{j+1}, \ldots, x_r]^{p^k} \\ &\qquad \equiv [[x_1, \ldots, x_{j-1}, h^{p^k}] z_2^p z_3, x_{j+1}, \ldots, x_r] \\ &\qquad \equiv [x_1, \ldots, x_{j-1}, h^{p^k}, x_{j+1}, \ldots, x_r] \pmod{N^{p^k}}\end{aligned}$$

as we wanted.

If $p = 2$, since $\gamma_r(G)$ is cyclic, we have $L = \gamma_r(G)$, $N = \gamma_r(G)^p$, and the inductive step easily follows from the Hall–Petresco Identity. Namely,

$$[x_1, \ldots, x_{j-1}, h^{2^{k-1}}]^2 = [x_1, \ldots, x_{j-1}, h^{2^k}] z_2,$$

where $z_2 \in [\gamma_{j-1}(G), G^{2^{k-1}}, G^{2^{k-1}}]$. By Lemma 2.12 we have

$$[\gamma_{j-1}(G), G^{2^{k-1}}, G^{2^{k-1}},_{r-j} G] \leq \gamma_{r+1}(G)^{2^{2k-2}} \leq \gamma_r(G)^{2^{k+1}} = N^{2^k},$$

so the result follows as above. □

3. Proof of Theorem A when $\gamma_r(G)$ is cyclic

Dark and Newell already proved Theorem A when $\gamma_r(G)$ is cyclic in [**4**], but we will give an alternative simpler proof in Theorem 3.4 below. In addition, we will also prove the case $p = 2$, which was omitted in [**4**] since it was pointed out to be very technical. Moreover, even if Theorem 3.4 can be modified so that it works for all primes, we will prove the case in which p is odd separately in Theorem 3.3, since in this case the proof turns out to be much shorter. First, however, we need the following simple but very helpful lemma.

Lemma 3.1. *Let N be a cyclic normal subgroup of a group G. Then $[N, G'] = 1$.*

Proof: Since N is cyclic, the automorphism group $\mathrm{Aut}(N)$ of N is abelian. Hence, $G/C_G(N)$ is also abelian, which means that $G' \leq C_G(N)$. □

We will also need the following result, which is Lemma 2.3 of [**7**].

Lemma 3.2. *Let G be a group and let $N \leq L \leq G$, with N normal in G. Suppose that for some $x \in G$ the following two conditions hold:*
(i) *$L/N \subseteq \{N[x, g] \mid g \in G\}$.*
(ii) *$N \subseteq \{[x, g] \mid g \in G\}$.*
Then $L \subseteq \{[x, g] \mid g \in G\}$.

Theorem 3.3. *Let G be a finite p-group with p odd and $\gamma_r(G)$ cyclic. Then*

$$\gamma_r(G) = \{[g_1, \ldots, g_r] \mid g_1, \ldots, g_r \in G\}.$$

Proof: Let $\gamma_r(G) = \langle [x_1, \ldots, x_r] \rangle$ with $x_1, \ldots, x_r \in G$. Then

$$\gamma_r(G)^{p^k} = \langle [x_1, \ldots, x_r]^{p^k} \rangle$$

for every $k \geq 1$. By the Hall–Petresco Identity, we have

$$[x_1, \ldots, x_r]^{p^k} = [x_1, \ldots, x_r^{p^k}] c_2^{\binom{p^k}{2}} \cdots c_{p^k}$$

with $c_i \in \gamma_i(\langle [x_1, \ldots, x_r], x_r \rangle)$. When $i < p^k$, we have $c_i \in \gamma_{r+i-1}(G) \leq \gamma_r(G)^{p^{i-1}}$, and so $c_i^{\binom{p^k}{i}} \in \gamma_r(G)^{p^{k+1}}$ since $p \geq 3$. If $i = p^k$, then $c_{p^k} \in \gamma_{r+p^k-1}(G) \leq \gamma_r(G)^{p^{p^k-1}} \leq \gamma_r(G)^{p^{k+1}}$. Therefore,

$$\gamma_r(G)^{p^k} = \langle [x_1, \ldots, x_r^{p^k}] \rangle$$

for every $k \geq 0$. Moreover, since $[x_1, \ldots, x_r^{p^k}, G] \leq \gamma_r(G)^{p^{k+1}}$, we have

$$[x_1, \ldots, x_r^{p^k}]^i \equiv [x_1, \ldots, x_r^{ip^k}] \pmod{\gamma_r(G)^{p^{k+1}}}$$

for every $i \geq 0$, so the result follows from Lemma 3.2. □

Theorem 3.4. *Let G be a finite 2-group with $\gamma_r(G)$ cyclic. Then*

$$\gamma_r(G) = \{[g_1,\dots,g_r] \mid g_1,\dots,g_r \in G\}.$$

Proof: Define $C = C_G(\gamma_r(G)/\gamma_r(G)^4)$. Since $\gamma_r(G)$ is cyclic, the quotient group $\gamma_r(G)/\gamma_r(G)^4$ has order 4, so that $|G : C| \le 2$. Let $\gamma_r(G) = \langle [x_1,\dots,x_r]\rangle$ with $x_1,\dots,x_r \in G$ and let j be the maximum number such that $x_j \in C$. Assume, in addition, that $[x_1,\dots,x_r]$ is, among all γ_r-values which are generators of $\gamma_r(G)$, the one with maximum j (observe that $j \ge 2$ since $G' = [G, C]$).

For every $i = 1,\dots,r$ consider an arbitrary element $y_i \in x_i^G$, so that $y_i = x_i[x_i, g]$ for some $g \in G$. Since $\gamma_{r+1}(G) \le \gamma_r(G)^2$, it follows from Corollary 2.9 that

$$[y_1,\dots,y_r] \equiv [x_1,\dots,x_r] \pmod{\gamma_r(G)^2},$$

and since $\gamma_r(G)^2 = \Phi(\gamma_r(G))$, we have

$$\gamma_r(G) = \langle [y_1,\dots,y_r]\rangle.$$

Therefore,

$$\gamma_r(G)^{2^k} = \langle [y_1,\dots,y_r]^{2^k}\rangle$$

for every $k \ge 1$. We claim that

$$[y_1,\dots,y_r]^{2^k} \equiv [y_1,\dots,y_j^{2^k},\dots,y_r] \pmod{\gamma_r(G)^{2^{k+1}}}$$

for every $y_i \in x_i^G$ and $k \ge 1$. Take $k = 1$ first. By Lemma 2.11 we have

$$[y_1,\dots,y_r]^2 \equiv [[y_1,\dots,y_j]^2, y_{j+1},\dots,y_r] \pmod{\gamma_r(G)^4},$$

and observe that

$$[y_1,\dots,y_j^2,\dots,y_r] = [[y_1,\dots,y_j]^2[y_1,\dots,y_j,y_j], y_{j+1},\dots,y_r].$$

If

$$[y_1,\dots,y_j,y_j,y_{j+1},\dots,y_r] \notin \gamma_r(G)^4,$$

then

$$\gamma_{r+1}(G) = \gamma_r(G)^2 = \langle [y_1,\dots,y_j,y_j,y_{j+1},\dots,y_r]\rangle,$$

and so

$$\gamma_r(G) = \langle [y_1,\dots,y_j,y_j,y_{j+1},\dots,y_{r-1}]\rangle,$$

which contradicts the maximality of j in the choice of the generator $[x_1,\dots,x_r]$.

Hence,

$$[y_1,\dots,y_j,y_j,y_{j+1},\dots,y_r] \in \gamma_r(G)^4,$$

so that

$$[y_1,\dots,y_r]^2 \equiv [y_1,\dots,y_j^2,\dots,y_r] \pmod{\gamma_r(G)^4}.$$

The claim follows now from Lemma 2.13 with $L = \gamma_r(G)$, $N = \gamma_r(G)^2$.

Now, we can conclude our proof. Let 2^m be the order of $\gamma_r(G)$. We will prove by induction on $m-k$ that

$$\gamma_r(G)^{2^k} \subseteq \{[g_1,\dots,g_r] \mid g_1,\dots,g_r \in G\}.$$

The result is true when $k=m$, so assume $k<m$ and

$$\gamma_r(G)^{2^{k+1}} \subseteq \{[g_1,\dots,g_r] \mid g_1,\dots,g_r \in G\}.$$

We apply Lemma 2.10 with $L=\gamma_r(G)^{2^{k-1}}$ and $N=\gamma_r(G)^{2^k}$. As

$$L=[y_1,\dots,y_j^{2^k},\dots,y_r]N \cup N \subseteq \bigcup_{g\in G} \gamma_r(y_1,\dots,y_{j-1},g,y_{j+1},\dots,y_r)N$$

for every $y_i \in x_i^G$, by Lemma 2.10 we get

$$\gamma_r(G)^{2^k} \subseteq \{[g_1,\dots,g_r] \mid g_1,\dots,g_r \in G\}.$$

In particular, when $k=0$ we obtain

$$\gamma_r(G) \subseteq \{[g_1,\dots,g_r] \mid g_1,\dots,g_r \in G\},$$

as we wanted. □

Thus, combining Theorems 3.3 and 3.4 we get the result for all primes when $\gamma_r(G)$ is cyclic.

4. Preliminaries for the proof of Theorem A when $\gamma_r(G)$ is generated by 2 elements

We will use the following notation: if H, K are subgroups of a group G, by $U \max_H K$ we mean that U is maximal among the proper subgroups of K which are normalized by H, while $U \max K$ simply means that U is a maximal subgroup of K.

The subgroups defined in Definitions 4.1 and 4.2 will be essential in our proof.

Definition 4.1. Let G be a finite p-group and let $U \max_G \gamma_r(G)$ for some $r \geq 2$. We define

$$D_r(U) = C_{\gamma_{r-1}(G)}(G/U).$$

In other words, for $x \in \gamma_{r-1}(G)$ we have $x \in D_r(U)$ if and only if $[x,G] \leq U$.

Definition 4.2. Let G be a finite p-group and let $U \max_{\gamma_{r-1}(G)} \gamma_r(G)$ for some $r \geq 2$. We define

$$E_r(U) = C_G(\gamma_{r-1}(G)/U).$$

In other words, $x \in E_r(U)$ if and only if $[x,\gamma_{r-1}(G)] \leq U$.

Remark 4.3. The subset $E(U)$ may not be a subgroup of G if U is not normal in G.

The significance of these subgroups becomes clear in the following lemma.

Lemma 4.4. *Let G be a finite p-group and let $r \geq 2$. Then, for $x \in \gamma_{r-1}(G)$, we have $\gamma_r(G) = [x, G]$ if and only if*

$$x \notin \cup\{D_r(U) \mid U \max_G \gamma_r(G)\}.$$

Similarly, $\gamma_r(G) = [\gamma_{r-1}(G), y]$ if and only if

$$y \notin \cup\{E_r(U) \mid U \max_{\gamma_{r-1}(G)} \gamma_r(G)\}.$$

Proof: The proof is essentially the same as the one of Lemma 2.9 of [**7**]. Let $x \in \gamma_{r-1}(G)$. Since $[x, G]$ is a normal subgroup of G, we have $[x, G] < \gamma_r(G)$ if and only if $x \in D_r(U)$ for some $U \max_G \gamma_r(G)$, and the first assertion follows. Similarly, since $[\gamma_{r-1}(G), y]$ is normalized by $\gamma_{r-1}(G)$, we have $[\gamma_{r-1}(G), y] < \gamma_r(G)$ if and only if $y \in E_r(U)$ for some $U \max_{\gamma_{r-1}(G)} \gamma_r(G)$. □

Lemma 4.5. *Let G be a finite p-group with $d(\gamma_r(G)) = 2$ for some $r \geq 2$. Let $U, V, W \max_G \gamma_r(G)$ with $V \neq W$ and $R, S, T \max_{\gamma_{r-1}(G)} \gamma_r(G)$ with $S \neq T$. Then*

(i) *$D_r(U) \neq \gamma_{r-1}(G)$ and $E_r(R) \neq G$.*

(ii) *$D_r(V) \cap D_r(W) \leq D_r(U)$ and $E_r(S) \cap E_r(T) \subseteq E_r(R)$.*

(iii) *If $U \neq R$, then $[D_r(U), E_r(R)] \leq \gamma_r(G)^p$.*

Proof: (i) is obvious, since $D_r(U) = \gamma_{r-1}(G)$ implies that $\gamma_r(G) \leq U$ and similarly $E_r(R) = G$ implies that $\gamma_r(G) \leq R$, and in both cases we have a contradiction.

We now prove (ii). As $d(\gamma_r(G)) = 2$, the subgroup $\gamma_r(G)$ is powerful by Lemma 2.5, so $\gamma_r(G)^p = \Phi(\gamma_r(G))$. Hence, $V \cap W \leq \gamma_r(G)^p \leq U$ and $S \cap T \leq \gamma_r(G)^p \leq R$. Then, the result follows from the fact that $x \in D_r(V) \cap D_r(W)$ if and only if $[x, G] \leq V \cap W$ and $y \in E_r(S) \cap E_r(T)$ if and only if $[y, \gamma_{r-1}(G)] \leq S \cap T$.

(iii) is true because $[D_r(U), E_r(R)] \leq U \cap R \leq \gamma_r(G)^p$. □

The following subgroup plays a fundamental role in [**8**], [**7**], and [**5**], and so does in our proof.

Definition 4.6. Let G be a finite p-group. We define

$$C_r(G) = C_G(\gamma_r(G)/\gamma_r(G)^p).$$

Lemma 4.7. *Let G be a finite p-group with $d(\gamma_r(G)) = 2$ for some $r \geq 2$. Then*

(i) $|G : C_r(G)| \leq p$.

(ii) *We have $G = C_r(G)$ if and only if $\gamma_{r+1}(G) \leq \gamma_r(G)^p$. In this case, all subgroups U such that $\gamma_r(G)^p < U < \gamma_r(G)$ are normal in G. Otherwise, $C_r(G) \neq G$ and there is only one normal subgroup U of G such that $\gamma_r(G)^p < U < \gamma_r(G)$, namely $U = \gamma_{r+1}(G)\gamma_r(G)^p$.*

(iii) *We have $[\gamma_r(G)^{p^k}, C_r(G)] \leq \gamma_r(G)^{p^{k+1}}$ for all $k \geq 0$.*

Proof: By Lemma 2.5 the subgroup $\gamma_r(G)$ is powerful, so $\gamma_r(G)/\gamma_r(G)^p$ is an elementary abelian p-group of rank 2. Now, (i) follows from the fact that the quotient group $G/C_r(G)$ embeds in a Sylow p-subgroup of the automorphism group of $\gamma_r(G)/\gamma_r(G)^p$.

To prove (ii), we may assume that $\gamma_r(G)^p = 1$. There are precisely $p + 1$ non-trivial proper subgroups of $\gamma_r(G)$, all cyclic of order p, and each of them is normal in G if and only if it is central. In addition, all such subgroups are central if and only $G = C_r(G)$, which is equivalent to $\gamma_{r+1}(G) = 1$. If there exists a non central subgroup U of G with $1 \neq U < \gamma_r(G)$, then the conjugacy class of U has size p, $C_r(G) \neq G$, and $\gamma_{r+1}(G) \neq 1$ is the only non-trivial normal subgroup of G properly contained in $\gamma_r(G)$. This proves (ii).

The proof of (iii) is an easy induction on k. The base of the induction is given by the definition of $C_r(G)$, and if $k > 0$, then

$$\begin{aligned}[][\gamma_r(G)^{p^k}, C_r(G)] &\leq [\gamma_r(G)^{p^{k-1}}, C_r(G)]^p[\gamma_r(G)^{p^{k-1}}, C_r(G), \gamma_r(G)^{p^{k-1}}] \\ &\leq \gamma_r(G)^{p^{k+1}}[\gamma_r(G)^{p^k}, \gamma_r(G)] \leq \gamma_r(G)^{p^{k+1}}\end{aligned}$$

by using the inductive hypothesis and the fact that $\gamma_r(G)$ is powerful. $\square$

In the case $r = 2$, i.e. when we deal with the common commutator word, we will also need the next lemma, which is just Lemma 2.9 (i) of [**7**].

Lemma 4.8. *If G is a non-abelian finite p-group with $d(G') \leq 2$, then for every $U \max_G G'$, we have $D_2(U) \leq C_2(G)$.*

5. Proof of Theorem A when $C_r(G) = G$

In order to apply Lemma 2.13 we will first find in Lemma 5.1 suitable generators for the verbal subgroup $\gamma_r(G)$. Then, as mentioned before, we will conclude by applying Lemma 2.10.

Lemma 5.1. *Let G be a finite p-group with $d(\gamma_r(G)) = 2$ for some $r \geq 2$. If $C_r(G) = G$, then there exist an integer j with $1 \leq j \leq r$ and $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_r \in G$ such that*

$$\gamma_r(G) = \langle [y_1, \dots, y_{j-1}, g, y_{j+1}, \dots, y_r] \mid g \in G \rangle$$

for every $y_i \in x_i^G$.

Proof: We may assume that $\Phi(\gamma_r(G)) = 1$, so using Lemma 4.7 (ii) we also have $\gamma_{r+1}(G) \leq \gamma_r(G)^p = 1$. Notice that it suffices to find an integer j and $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_r \in G$ such that

$$\gamma_r(G) = \langle [x_1, \dots, x_{j-1}, g, x_{j+1}, \dots, x_r] \mid g \in G \rangle,$$

since if $y_i \in x_i^G$, then $y_i = x_i h_i$ for some $h_i \in G'$, so it follows from Corollary 2.9 that $[y_1, \dots, y_{j-1}, g, y_{j+1}, \dots, y_r] = [x_1, \dots, x_{j-1}, g, x_{j+1}, \dots, x_r]$.

We will proceed by induction on r. If $r = 2$, then the result is true by the aforementioned Theorem A of [**7**].

Now, if there exists $x \in G_{\gamma_{r-1}}$ such that $\gamma_r(G) = [x, G]$, then we are done. Hence, suppose $[x, G] < \gamma_r(G)$ for every $x \in G_{\gamma_{r-1}}$. Observe that all subgroups U such that $\gamma_r(G)^p \leq U \leq \gamma_r(G)$ are normal in G by Lemma 4.7 (ii), so we have

$$U \max_G \gamma_r(G) \quad \text{for every} \quad U \max \gamma_r(G).$$

If

$$D = \prod_{V \max \gamma_r(G)} D_r(V) < \gamma_{r-1}(G),$$

then we could choose a γ_{r-1}-value not belonging to D, which contradicts Lemma 4.4. Therefore, assume

$$\prod_{V \max \gamma_r(G)} D_r(V) = \gamma_{r-1}(G).$$

Thus, by (i) and (ii) of Lemma 4.5, there exists $U \max \gamma_r(G)$ such that $D_r(U)$ properly contains $\cap\{D_r(V) \mid V \max \gamma_r(G)\}$, and therefore $[D_r(U), G] = U$. Now, by Lemma 4.5 (iii), we have $[D_r(U), E_r(V)] = 1$ for all $V \neq U$, and so

$$\prod_{\substack{V \max \gamma_r(G) \\ U \neq V}} E_r(V) \neq G.$$

Hence, as G can not be the union of two proper subgroups, we can choose

$$x_r \in G \setminus \Bigg(E_r(U) \cup \prod_{\substack{V \max \gamma_r(G) \\ U \neq V}} E_r(V) \Bigg),$$

and observe that by Lemma 4.4 we have

$$\gamma_r(G) = [\gamma_{r-1}(G), x_r].$$

Define now $C_{x_r} = C_{\gamma_{r-1}(G)}(x_r)$ and notice that C_{x_r} is normal in G since

$$[C_{x_r}, G, x_r] \leq [\gamma_{r-1}(G), G, x_r] \leq \gamma_{r+1}(G) = 1.$$

Thus, we consider the quotient group G/C_{x_r}. Since $\gamma_{r+1}(G) = 1$, the map

$$\begin{aligned} \eta\colon \gamma_{r-1}(G) &\longrightarrow \gamma_r(G) \\ g &\longmapsto [g, x_r] \end{aligned}$$

is a group epimorphism whose kernel is C_{x_r}, so

$$|\gamma_{r-1}(G/C_{x_r})| = p^2.$$

Furthermore, since $\gamma_{r+1}(G) = 1$, we have $C_{r-1}(G/C_{x_r}) = G/C_{x_r}$. By inductive hypothesis, there exist an integer j with $1 \leq j \leq r-1$ and $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{r-1} \in G$ such that

$$\gamma_{r-1}(G) = \langle [x_1, \dots, x_{j-1}, g, x_{j+1}, \dots, x_{r-1}] \mid g \in G \rangle C_{x_r}.$$

Finally,

$$\begin{aligned} \gamma_r(G) &= [\gamma_{r-1}(G), x_r] \\ &= [\langle [x_1, \dots, x_{j-1}, g, x_{j+1}, \dots, x_{r-1}] \mid g \in G \rangle C_{x_r}, x_r] \\ &= \langle [x_1, \dots, x_{j-1}, g, x_{j+1}, \dots, x_{r-1}, x_r] \mid g \in G \rangle, \end{aligned}$$

and this concludes the proof. □

Theorem 5.2. *Let G be a finite p-group with p odd and $d(\gamma_r(G)) = 2$. If $C_r(G) = G$, then there exist an integer j with $1 \leq j \leq r$ and $x_1, \dots, x_{j-1}$, $x_{j+1}, \dots, x_r \in G$ such that*

$$\gamma_r(G) = \{[x_1, \dots, x_{j-1}, g, x_{j+1}, \dots, x_r] \mid g \in G\}.$$

Proof: By Lemma 5.1, there exist an integer j with $1 \leq j \leq r$ and $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_r \in G$ such that

$$\gamma_r(G) = \langle [y_1, \dots, y_{j-1}, g, y_{j+1}, \dots, y_r] \mid g \in G \rangle$$

for every $y_i \in x_i^G$. Choose arbitrary $y_i \in x_i^G$ for all i. We have

$$\gamma_r(G) = \langle [y_1, \dots, y_{j-1}, g_1, y_{j+1}, \dots, y_r], [y_1, \dots, y_{j-1}, g_2, y_{j+1}, \dots, y_r] \rangle$$

for some $g_1, g_2 \in G$. Let

$$U = \langle [y_1, \dots, y_{j-1}, g_2, y_{j+1}, \dots, y_r] \rangle \gamma_r(G)^p,$$

and notice that it is normal in G since $C_r(G)=G$. Observe that $\gamma_{r+1}(G)\leq \gamma_r(G)^p$, and $\gamma_r(G)^p$ is central of exponent p modulo $\gamma_r(G)^{p^2}$ by (iii) of Lemma 4.7. Therefore, we apply Lemma 2.13 to both quotients

$$\gamma_r(G)/U \quad \text{and} \quad U/\gamma_r(G)^p$$

and we get

$$\gamma_r(G)^{p^k} = \langle [y_1,\ldots,y_{j-1},g_1^{p^k},y_{j+1},\ldots,y_r]\rangle U^{p^k}$$

and

$$U^{p^k} = \langle [y_1,\ldots,y_{j-1},g_2^{p^k},y_{j+1},\ldots,y_r]\rangle \gamma_r(G)^{p^{k+1}}$$

for every $k\geq 0$. Furthermore, as $\gamma_{r+1}(G)\leq \gamma_r(G)^p$, it follows from Corollary 2.9 that

$$[y_1,\ldots,y_{j-1},g_1,y_{j+1},\ldots,y_r]^s \equiv [y_1,\ldots,y_{j-1},g_1^s,y_{j+1},\ldots,y_r] \pmod U$$

and

$$\begin{aligned}&[y_1,\ldots,y_{j-1},g_2,y_{j+1},\ldots,y_r]^s\\ &\qquad\equiv [y_1,\ldots,y_{j-1},g_2^s,y_{j+1},\ldots,y_r] \pmod{\gamma_r(G)^p}\end{aligned}$$

for each integer s. Thus, using Lemma 2.13 and the aforementioned property (v) of powerful p-groups it can be easily proved that

$$\begin{aligned}&[y_1,\ldots,y_{j-1},g_1^{p^k},y_{j+1},\ldots,y_r]^s\\ &\qquad\equiv [y_1,\ldots,y_{j-1},g_1,y_{j+1},\ldots,y_r]^{sp^k}\\ &\qquad\equiv ([y_1,\ldots,y_{j-1},g_1^s,y_{j+1},\ldots,y_r]u)^{p^k}\\ &\qquad\equiv [y_1,\ldots,y_{j-1},g_1^{sp^k},y_{j+1},\ldots,y_r] \pmod{U^{p^k}},\end{aligned}$$

where $u\in U$, and similarly

$$\begin{aligned}&[y_1,\ldots,y_{j-1},g_2^{p^k},y_{j+1},\ldots,y_r]^s\\ &\qquad\equiv [y_1,\ldots,y_{j-1},g_2^{sp^k},y_{j+1},\ldots,y_r] \pmod{\gamma_r(G)^{p^{k+1}}}.\end{aligned}$$

Hence, for each $k\geq 0$ we have

$$\gamma_r(G)^{p^k} \subseteq \bigcup_{g\in G} \gamma_r(y_1,\ldots,y_{j-1},g,y_{j+1},\ldots,y_r)U^{p^k}$$

for every $y_i\in x_i^G$, and similarly

$$U^{p^k} \subseteq \bigcup_{g\in G} \gamma_r(y_1,\ldots,y_{j-1},g,y_{j+1},\ldots,y_r)\gamma_r(G)^{p^{k+1}}$$

for every $y_i\in x_i^G$.

The result now follows by repeatedly applying Lemma 2.10 to the subgroups of the series

$$1 = \gamma_r(G)^{p^s} \leq U^{p^{s-1}} \leq \gamma_r(G)^{p^{s-1}} \leq \cdots \leq \gamma_r(G)^{p^i} \leq U^{p^{i-1}} \leq \cdots \leq \gamma_r(G),$$

where p^s is the exponent of $\gamma_r(G)$. □

6. Proof of Theorem A when $C_r(G) \neq G$

To end the proof of Theorem A, we need a further technical definition.

Definition 6.1. Let G be a finite p-group and let $r \geq 2$. We define $C_r^r(G) = \gamma_r(G)^p$ and

$$C_i^r(G) = C_{\gamma_i(G)}(G/C_{i+1}^r(G))$$

for all $2 \leq i \leq r-1$.

As in Section 5, we start finding suitable generators for $\gamma_r(G)$.

Lemma 6.2. *Let G be a finite p-group with $d(\gamma_r(G)) = 2$ for some $r \geq 2$ and $C_r(G) \neq G$. Let $U = \gamma_{r+1}(G)\gamma_r(G)^p$. Then, there exist an integer j with $2 \leq j \leq r$, $x_1, \ldots, x_{j-1} \in G$, and $c \in C_r(G)$ such that*

$$\gamma_r(G) = \langle [y_1, \ldots, y_{j-1}, c, g_{j+1}, \ldots, g_r] \rangle U$$

for every $y_k \in x_k^G$ with $k = 1, \ldots, j-1$ and every $g_{j+1}, \ldots, g_r \in G \backslash C_r(G)$. Moreover, $[\gamma_i(G), C_r(G)] \leq C_i^r(G)$ for every $j \leq i \leq r$.

Proof: We proceed by induction on r. Suppose first $r = 2$ and take an arbitrary $x \in G \backslash C_2(G)$. Since $C_2(G)$ is maximal in G by Lemma 4.7 (i), we have $G = \langle x \rangle C_2(G)$. Also, as $D_2(U) \leq C_2(G)$ by Lemma 4.8, we have $x \notin D_2(U)$. Moreover, by Lemma 4.7 (ii), U is the unique subgroup such that $U \max_G \gamma_r(G)$, so by Lemma 4.4 we have $G = [x, G']$. Thus we get

$$G' = [x, G] = [x, \langle x \rangle C_2(G)] = [x, C_2(G)].$$

In addition, $[G', C_2(G)] \leq (G')^p = C_2^2(G)$, as desired.

Take then $r \geq 3$ and write $C = C_r(G)$ for simplicity. We may assume $\gamma_r(G)^p = C_r^r(G) = 1$. Suppose first there exist $x_1, \ldots, x_{r-1} \in G$ such that $\gamma_r(G) = [x_1, \ldots, x_{r-1}, C]$. Since $[\gamma_r(G), C] = 1$ and since $x_i^g = x_i[x_i, g]$ for every $g \in G$, it follows from Corollary 2.9 that

$$\gamma_r(G) = [y_1, \ldots, y_{r-1}, C]$$

for all $y_i \in x_i^G$. Hence, we may assume there is no such an element. In other words, if $x \in G_{\gamma_{r-1}}$, then $[x, C] \neq \gamma_r(G)$. Note, however, that $[x, C]$ is normal in G since, as above, $[x, C]^g = [x^g, C] = [x, C]$. Since U is the only non-trivial normal subgroup of G properly contained in $\gamma_r(G)$, we get $[x, C] \leq U$ for every γ_{r-1}-value x. Since $\gamma_{r-1}(G)$ is generated by all γ_{r-1}-values, we have $[\gamma_{r-1}(G), C] \leq U$. This, in particular, implies that

$C \leq E_r(U)$, and since $E_r(U) \neq G$ by Lemma 4.5, we have $C = E_r(U)$. Note that we have $V \max_{\gamma_{r-1}(G)} \gamma_r(G)$ for every $V \max \gamma_r(G)$ since

$$[\gamma_r(G), \gamma_{r-1}(G)] \leq [\gamma_r(G), G' \leq [\gamma_r(G), G, G] = 1.$$

On the other hand, $U = \gamma_{r+1}(G)$, so for every $V \max \gamma_r(G)$ with $V \neq U$ we have $[\gamma_r(G), E_r(V)] \leq U \cap V = 1$, and then $E_r(V) \leq C$. Therefore,

$$\cup\{E_r(V) \mid V \max \gamma_r(G)\} \subseteq C$$

and then, by Lemma 4.4, we get

$$\gamma_r(G) = [\gamma_{r-1}(G), g]$$

for every $g \in G \setminus C$.

As $[\gamma_r(G), \gamma_{r-1}(G)] = 1$, the map

$$\begin{aligned} \eta_g \colon \gamma_{r-1}(G) &\longrightarrow \gamma_r(G) \\ x &\longmapsto [x, g] \end{aligned}$$

is a group epimorphism for every $g \in G \setminus C$ whose kernel is $C_{\gamma_{r-1}(G)}(g)$. Choose an arbitrary $g \in G \setminus C$, write $C_g = C_{\gamma_{r-1}(G)}(g)$ for simplicity, and note that

$$[C_g, G] = [C_g, \langle g \rangle C] = [C_g, C] \leq [\gamma_{r-1}(G), C] \leq U \leq C_g,$$

where the last equality holds since $U \leq Z(G)$. Thus, the subgroups C_g are all normal in G, and we can consider the groups G/C_g. Now, $\gamma_{r-1}(G/C_g) = \gamma_{r-1}(G)/C_g$ is isomorphic to $\gamma_r(G)$, so it has order p^2 and exponent p. In addition $\gamma_r(G) \not\leq C_g$ since otherwise $[\gamma_r(G), g] = 1$, which contradicts the fact that $g \notin C$. Thus,

$$G/C_g \neq C_{r-1}(G/C_g).$$

Moreover, since $[\gamma_{r-1}(G), C] \leq U \leq C_g$, it follows that

$$C_{r-1}(G/C_g) = C/C_g$$

for all $g \in G \setminus C$. By Lemma 4.7 (ii), there is only one normal subgroup R of G with $C_g < R < \gamma_{r-1}(G)$, so $R = C_g \gamma_r(G)$.

We apply now the inductive hypothesis to all groups G/C_g. It follows that for each $g \in G \setminus C$, there exist $j_g \geq 1$, $x_{1,g}, \dots, x_{j_g-1,g} \in G$, and $c_g \in C$ such that

$$\gamma_{r-1}(G) = \langle [y_{1,g}, \dots, y_{j_g-1,g}, c_g, g_{j_g+1}, \dots, g_{r-1}] \rangle C_g \gamma_r(G)$$

for every $y_{i,g} \in x_{i,g}^G$, $i = 1, \dots, j_g - 1$, and every $g_{j_g+1}, \dots, g_{r-1} \in G \setminus C$. Moreover, if we define

$$C_{i,g}/C_g = C_i^{r-1}(G/C_g),$$

then we have $[\gamma_i(G), C] \leq C_{i,g}$ for all $j_g \leq i \leq r-1$.

Define now

$$U^* = \gamma_r(G) \prod_{g \in G \setminus C} C_g,$$

which is, of course, normal in G.

We claim that $U^* = C_g\gamma_r(G)$ for all $g \in G \setminus C$. For that purpose, fix $g \in G \setminus C$ and take $h \in G \setminus C$ arbitrary. Then C_gC_h is normal in G, so either $C_gC_h = \gamma_{r-1}(G)$ or $C_h \le C_g\gamma_r(G)$. In the first case we would have

$$\gamma_r(G) = [\gamma_{r-1}(G), h] = [C_hC_g, h] = [C_g, h] \le C_g,$$

which is a contradiction since $[\gamma_r(G), g] \ne 1$. Hence, $C_h \le C_g\gamma_r(G)$, and so $C_g\gamma_r(G) = C_hC_g\gamma_r(G)$. Since this holds for all $h \in G \setminus C$, it follows that $C_g\gamma_r(G) = U^*$, and the claim is proved.

Take now $j = \max\{j_g \mid g \in G\setminus C\}$. Then, there exist $x_1, \dots, x_{j-1} \in G$ and $c \in C$ such that

$$\gamma_{r-1}(G) = \langle [y_1, \dots, y_{j-1}, c, g_{j+1}, \dots, g_{r-1}] \rangle U^*$$

for every $y_i \in x_i^G$, $i = 1, \dots, j-1$, and every $g_{j+1}, \dots, g_{r-1} \in G \setminus C$. Moreover, because of the choice of j, we have

$$[\gamma_i(G), C] \le \bigcap_{g \in G\setminus C} C_{i,g}$$

for all $j \le i \le r-1$. Let us prove that

$$\bigcap_{g \in G\setminus C} C_{i,g} \le C_i^r(G) \text{ for every } i \text{ such that } j \le i \le r-1.$$

We proceed by induction on $r - i$. If $r - i = 1$, that is, if $i = r-1$, then $C_{r-1,g} = C_g = C_{\gamma_{r-1}(G)}(g)$, and since $G = \langle G \setminus C \rangle$, it follows that

$$\bigcap_{g \in G\setminus C} C_g = C_{\gamma_{r-1}(G)}(G) = C_{r-1}^r(G).$$

Assume now $i \le r-2$. Then

$$\left[\bigcap_{g \in G\setminus C} C_{i,g}, G\right] \le \bigcap_{g \in G\setminus C} C_{i+1,g} \le C_{i+1}^r(G),$$

by the inductive hypothesis, and so

$$\bigcap_{g \in G\setminus C} C_{i,g} \le C_i^r(G)$$

as claimed.

Since $[\gamma_r(G), C] = 1 = C_r^r(G)$, we have $[\gamma_i(G), C] \le C_i^r(G)$ for every i such that $j \le i \le r$.

Finally, take $g_r \in G \setminus C$ arbitrary. Observe that

$$[U^*, g_r] = [C_{g_r}\gamma_r(G), g_r] = [\gamma_r(G), g_r] = U,$$

where the last equality holds since $1 \neq [\gamma_r(G), g_r] \leq \gamma_{r+1}(G)$. Hence,

$$\begin{aligned}\gamma_r(G) &= [\gamma_{r-1}(G), g_r] \\ &= [\langle [y_1, \dots, y_{j-1}, c, g_{j+1}, \dots, g_{r-1}]\rangle U^*, g_r] \\ &= [\langle [y_1, \dots, y_{j-1}, c, g_{j+1}, \dots, g_{r-1}]\rangle, g_r]U \\ &= \langle [y_1, \dots, y_{j-1}, c, g_{j+1}, \dots, g_r]\rangle U,\end{aligned}$$

and the proof is complete. □

Theorem 6.3. *Let G be a finite p-group with p odd and $d(\gamma_r(G)) = 2$ for some $r \geq 2$. If $C_r(G) \neq G$, then there exist an integer j with $1 \leq j \leq r$ and $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_r$ such that*

$$\gamma_r(G) = \{[x_1, \dots, x_{j-1}, c, x_{j+1}, \dots, x_r] \mid c \in C_r(G)\}.$$

Proof: Let $U = \gamma_{r+1}(G)\gamma_r(G)^p$ and write $C = C_r(G)$ for simplicity. By Lemma 6.2, there exist an integer j with $1 \leq j \leq r$ and $x_1, \dots, x_{j-1} \in G$, $c \in C$, such that

$$\gamma_r(G) = \langle [y_1, \dots, y_{j-1}, c, g_{j+1}, \dots, g_r]\rangle U$$

for every $y_i \in x_i^G$, $i = 1, \dots, j-1$, and every $g_{j+1}, \dots, g_r \in G \setminus C$. Moreover, $[\gamma_i(G), C] \leq C_i^r(G)$ for every $j \leq i \leq r$.

Write $x = [y_1, \dots, y_{j-1}]$. It follows from the Hall–Witt Identity and standard commutator calculus that

$$[x, c, g_{j+1}] = [c, g_{j+1}, x]^{-1}[g_{j+1}, x, c]^{-1}z$$

for some $z \in \gamma_{j+2}(G)$. On the one hand, we have

$$[z, g_{j+2}, \dots, g_r] \in \gamma_{r+1}(G) \leq U.$$

On the other hand,

$$[g_{j+1}, x, c] \in [\gamma_j(G), C] \leq C_j^r(G) \cap \gamma_{j+1}(G),$$

and since $[C_i^r(G), G] \leq C_{i+1}^r(G)$ for every $i \leq r-1$, we have

$$[C_j^r(G) \cap \gamma_{j+1}(G), g_{j+2}, \dots, g_r] \leq C_{r-1}^r(G) \cap \gamma_r(G) \leq U,$$

where the last inequality holds since $C_{r-1}^r(G) \cap \gamma_r(G)$ is normal in G but $\gamma_r(G) \nleq C_{r-1}^r(G)$. Thus,

$$[x, c, g_{j+1}, \dots, g_r] \equiv [x, [c, g_{j+1}], g_{j+2}, \dots, g_r] \pmod{U},$$

so, in particular,

$$\gamma_r(G) = \langle [x, [c, g_{j+1}], g_{j+2}, \dots, g_r]\rangle U.$$

Take now $g_{r+1} \in G \setminus C$ arbitrary. Since, clearly, we have $[U, g_{r+1}] \leq \gamma_r(G)^p$, it follows that

$$U = \langle [x, [c, g_{j+1}], g_{j+2}, \ldots, g_{r+1}] \rangle \gamma_r(G)^p.$$

Now observe that, on the one hand, we have

$$\begin{aligned}[\gamma_{j-1}(G), C, C,_{r-j} G] &\leq [\gamma_j(G), C,_{r-j} G] \\ &\leq [C_j^r(G),_{r-j} G] \\ &\leq C_r^r(G) = \gamma_r(G)^p,\end{aligned}$$

which is central of exponent p modulo U^p and, on the other hand, we have

$$[\gamma_{j-1}(G), G', G',_{r-j} G] \leq \gamma_{r+3}(G) \leq U^p,$$

which is central of exponent p modulo $\gamma_r(G)^{p^2}$. Therefore, we can apply Lemma 2.13 to both quotients

$$\gamma_r(G)/U \quad \text{and} \quad U/\gamma_r(G)^p$$

and we conclude in the same way as in the proof of Theorem 5.2. $\square$

7. Proof of Theorem B

Now, we prove Theorem B using a similar idea as in Theorem B of [**7**] and Theorem A′ and Theorem B′ of [**5**].

Proof of Theorem B: We first claim that there exists $1 \leq j \leq r$ such that for every $N \trianglelefteq_{\mathrm{o}} G$ there exist $g_{N,1}, \ldots, g_{N,j-1}, g_{N,j+1}, \ldots, g_{N,r} \in G$ such that

$$\gamma_r(G)N/N = \{[g_{N,1}, \ldots, g_{N,j-1}, g, g_{N,j+1}, \ldots, g_{N,r}]N \mid g \in G\}.$$

For every $N \trianglelefteq_{\mathrm{o}} G$, write j_N for the smallest integer such that there exist $g_{N,1}, \ldots, g_{N,j_N-1}, g_{N,j_N+1}, \ldots, g_{N,r} \in G$ such that

$$\gamma_r(G)N/N = \{[g_{N,1}, \ldots, g_{N,j_N-1}, g, g_{N,j_N+1}, \ldots, g_{N,r}]N \mid g \in G\}.$$

Note that the existence of j_N is guaranteed by Theorem A.

Let M be an open normal subgroup of G for which j_M is maximal in the set $\{j_N \mid N \trianglelefteq_{\mathrm{o}} G\}$. We will prove that $j = j_M$ has the required property. Indeed, take $N \trianglelefteq_{\mathrm{o}} G$ arbitrary and consider the intersection $N \cap M$, which is also open and normal in G. Now, as $N \cap M \leq M$, we have $j_M \leq j_{N\cap M}$ and, by maximality, it follows that $j_M = j_{N\cap M}$. Again, since $N \cap M \leq N$, we have

$$\gamma_r(G)N/N = \{[g_{N,1}, \ldots, g_{N,j_M-1}, g, g_{N,j_M+1}, \ldots, g_{N,r}]N \mid g \in G\},$$

and the claim is proved.

Now, for every $N \trianglelefteq_{\mathrm{o}} G$, write

$$X_N = \big\{(g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_r) \in G \times \overset{r-1}{\cdots} \times G \mid \\ \gamma_r(G)N/N = \{[g_1, \ldots, g_{j-1}, g, g_{j+1}, \ldots, g_r]N \mid g \in G\}\big\}.$$

Clearly, the family $\{X_N\}_{N \trianglelefteq_{\mathrm{o}} G}$ has the finite intersection property, and since $G \times \overset{r-1}{\cdots} \times G$ is compact,

$$\bigcap_{N \trianglelefteq_{\mathrm{o}} G} X_N \neq \varnothing.$$

Thus, if $(g_1, \ldots, g_{j-1}, g_{j+1}, g_r)$ belongs to this intersection, write

$$\mathcal{K}(G) = \{[g_1, \ldots, g_{j-1}, g, g_{j+1}, \ldots, g_r] \mid g \in G\},$$

so that we have

$$\gamma_r(G)N/N = \mathcal{K}(G)N/N$$

for all $N \trianglelefteq_{\mathrm{o}} G$.

Now, note that $\mathcal{K}(G)$ is closed in G, being the image of a continuous function from G to G. Thus,

$$\gamma_r(G) = \bigcap_{N \trianglelefteq_{\mathrm{o}} G} \gamma_r(G)N = \bigcap_{N \trianglelefteq_{\mathrm{o}} G} \mathcal{K}(G)N = \mathrm{Cl}_G(\mathcal{K}(G)) = \mathcal{K}(G)$$

and the proof is complete. □

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