HIGH ENERGY POSITIVE SOLUTIONS FOR A COUPLED HARTREE SYSTEM WITH HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENTS

FASHUN GAO[†], HAIDONG LIU[‡], VITALY MOROZ[⋆], AND MINBO YANG[∗⋆]

ABSTRACT. We study the coupled Hartree system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1\big(|x|^{-4} * u^2\big)u + \beta\big(|x|^{-4} * v^2\big)u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2\big(|x|^{-4} * v^2\big)v + \beta\big(|x|^{-4} * u^2\big)v & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geq 5$, $\beta > \max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$, and V_1, $V_2 \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative potentials. This system is critical in the sense of the Hardy-Littlewood-Sobolev inequality. For the system with $V_1 = V_2 = 0$ we employ moving sphere arguments in integral form to classify positive solutions and to prove the uniqueness of positive solutions up to translation and dilation, which is of independent interest. Then using the uniqueness property, we establish a nonlocal version of the global compactness lemma and prove the existence of a high energy positive solution for the system assuming that $|V_1|_{L^{N/2}(\mathbb{R}^N)} + |V_2|_{L^{N/2}(\mathbb{R}^N)} > 0$ is suitably small.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

The two-component coupled Hartree system

$$\begin{cases} i\partial_t \mathbf{\Psi}_1 = -\Delta \mathbf{\Psi}_1 + W_1(x)\mathbf{\Psi}_1 - \alpha_1\big(K(x) * |\mathbf{\Psi}_1|^2\big)\mathbf{\Psi}_1 - \beta\big(K(x) * |\mathbf{\Psi}_2|^2\big)\mathbf{\Psi}_1, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ i\partial_t \mathbf{\Psi}_2 = -\Delta \mathbf{\Psi}_2 + W_2(x)\mathbf{\Psi}_2 - \alpha_2\big(K(x) * |\mathbf{\Psi}_2|^2\big)\mathbf{\Psi}_2 - \beta\big(K(x) * |\mathbf{\Psi}_1|^2\big)\mathbf{\Psi}_2, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \end{cases} \tag{1.1}$$

appears in several physical models, such as in the nonlinear optics [38] and in the study of a two-component Bose-Einstein Condensate [17, 21]. Here, $\mathbf{\Psi}_i : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, W_i are the external potentials, K is a nonnegative response function which possesses information about the self-interaction between the particles and α_i measures the strength of the self-interactions in each component: $\alpha_i > 0$ corresponds to the attractive (focusing) and $\alpha_i < 0$ to the repulsive (defocusing) self-interactions. The coupling constant $\beta > 0$ corresponds to the attraction (cooperation) and $\beta < 0$ to the repulsion (competition) between the two components in the system. In this work we are interested in the purely attractive case $\alpha_i, \beta > 0$. We refer the reader to [25, 26] and references therein for the physical background and mathematical derivation of Hartree theory in the case of a single equation.

When the response function is a delta function, i.e., $K(x) = \delta(x)$, the nonlinear response is local and the problem has been intensively studied in the past twenty years. In this case, via the ansatz

2010 *Mathematics Subject Classification.* 35A15, 35J20, 35J60.
Key words and phrases. Coupled Hartree system; Critical nonlocal nonlinearity; Hardy-Littlewood-Sobolev inequality.
[†]Fashun Gao is partially supported by NSFC (11901155, 11671364).
[‡]Haidong Liu is partially supported by NSFC (11701220, 11926334, 11926335).
[⋆]Vitaly Moroz and Minbo Yang are partially supported by the Royal Society IEC\NSFC\191022.
[∗]Minbo Yang is partially supported by NSFC (11571317, 11971436, 12011530199) and ZJNSF(LD19A010001).

$\mathbf{\Psi}_1(t,x) = e^{-iE_1t}u(x)$ and $\mathbf{\Psi}_2(t,x) = e^{-iE_2t}v(x)$, (1.1) is transformed into a coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $V_i(x) = W_i(x) - E_i$ for $i = 1,2$. Existence, multiplicity and properties of weak solutions of (1.2) have been investigated by many authors. See, for example, [5, 6, 12–14, 18, 31, 33–35, 37, 43, 45, 46, 49, 52] and references therein. Clearly, (1.2) may have semitrivial solutions of the form $(u,0)$ for some $u \neq 0$ or $(0,v)$ for some $v \neq 0$. Looking for nontrivial solutions of (1.2) with both components being nonzero is more complicated and requires new techniques and ideas. Here we only recall some results closely related to the current paper. Chen and Zou [13, 14] investigated the nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \alpha_1 u^{2p-1} + \beta u^{p-1}v^p & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \alpha_2 v^{2p-1} + \beta u^p v^{p-1} & \text{in } \Omega, \\ u,\, v \geq 0 \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $2p = \frac{2N}{N-2}$ is the Sobolev critical exponent, $-\lambda_1(\Omega) < \lambda_1,\, \lambda_2 < 0$, $\alpha_1,\, \alpha_2 > 0$, and $\beta \neq 0$. They established the existence, uniqueness and limit behaviour of positive least energy solution. It turned out that results in the higher dimensions are quite different from those in $N = 4$. In [15], considering the functional constrained on a subset of the Nehari manifold consisting of functions invariant with respect to a subgroup of $O(N+1)$, the authors obtained infinitely many positive solutions. In [45], the authors showed that the Palais-Smale condition holds at any levels in a small right neighbourhood of the least energy and then proved via a contradiction argument that there is a positive solution with critical value in this small neighbourhood. Using positive solutions of the corresponding scalar equation as building blocks, the authors of [44] constructed positive solutions for systems via the Lyapunov-Schmidt reduction argument, revealing concentration and blow-up features as well as a tower shape of the solutions. Recently, Liu and Liu [34] considered the nonlinear Schrödinger system with critical nonlinearities

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^4, \\ -\Delta v + V_2(x)v = \alpha_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^4, \\ u,\, v \geq 0 \text{ in } \mathbb{R}^4, \quad u, v \in D^{1,2}(\mathbb{R}^4), \end{cases} \tag{1.3}$$

where $V_1,\, V_2 \in L^2(\mathbb{R}^4) \cap L^\infty_{\text{loc}}(\mathbb{R}^4)$ are nonnegative potential functions. If $\beta > \max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$ and $|V_1|_{L^2(\mathbb{R}^4)} + |V_2|_{L^2(\mathbb{R}^4)} > 0$ is suitably small, they proved that (1.3) has at least a positive solution with a high energy level. This generalizes the well known result for semilinear Schrödinger equation by Benci and Cerami [7] to the coupled nonlinear Schrödinger system.

In this work we study standing wave solutions of (1.1) in the purely attractive case $\alpha_i, \beta > 0$ with a Riesz potential response function, i.e., $K(x) = |x|^{-\mu}$ where $\mu \in (0, N)$. Clearly, $\mathbf{\Psi}_1(t,x) = e^{-iE_1t}u(x)$ and $\mathbf{\Psi}_2(t,x) = e^{-iE_2t}v(x)$ solve (1.1) if and only if $(u(x), v(x))$ is a solution of the system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1\big(|x|^{-\mu} * u^2\big)u + \beta\big(|x|^{-\mu} * v^2\big)u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2\big(|x|^{-\mu} * v^2\big)v + \beta\big(|x|^{-\mu} * u^2\big)v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where again $V_i(x) = W_i(x) - E_i$ for $i = 1,2$. There are very few results available on the coupled Hartree system of type (1.4). The first attempt is due to Yang, Wei and Ding [54], there the authors considered a singular perturbed problem related to (1.4) and proved the existence of a ground state solution when the coupling constant β is large. In [50], Wang and Shi studied (1.4) with positive constant potentials and proved the existence and nonexistence of positive ground state solutions. For the critical case, the authors of [55] considered a critical coupled Hartree system with a fractional Laplacian operator and proved the existence of a ground state solution via the Dirichlet-to-Neumann map.

Clearly, semitrivial solutions of (1.4) correspond to solutions of the Choquard type equation

$$-\Delta u + V_i(x)u = \alpha_i\big(|x|^{-\mu} * u^2\big)u \quad \text{in } \mathbb{R}^N.$$

The Choquard type equation goes back to the description of the quantum theory of a polaron at rest by Pekar in 1954 [42] and the modelling of an electron trapped in its own hole in 1976 in the work of Choquard [29]. They also appear in the study of boson stars [20]. Mathematically, Lieb [29] and Lions [32] studied the existence of solutions to the Choquard's equation. Lieb [29] also established the uniqueness of the ground state solution when $N = 3$ and $\mu = -1$. Ma and Zhao [36] studied symmetry and uniqueness of positive solutions. The existence of radial ground states with nonlinearities more general than u^2 was studied in [39, 40]. The authors of [22] considered the Choquard equation

$$-\Delta u + V(x)u = \big(|x|^{-\mu} * |u|^{2^*_\mu}\big)|u|^{2^*_\mu-2}u \ \text{ in } \mathbb{R}^N, \tag{1.5}$$

where $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical exponent. They established a global compactness result and proved that (1.5) has at least a positive solution if $|V|_{L^{N/2}(\mathbb{R}^N)} > 0$ is suitable small. This extended to the nonlocal Choquard equation the well known result for semilinear Schrödinger equation by Benci and Cerami [7]. The existence of multiple solutions for (1.5) was established in [2]. Lei [24], Du and Yang [19] studied positive solutions of the critical equation

$$-\Delta u = \big(|x|^{-\mu} * |u|^{2^*_\mu}\big)|u|^{2^*_\mu-2}u \ \text{ in } \mathbb{R}^N, \tag{1.6}$$

and proved that every positive solution of (1.6) must assume the form

$$u(x) = c\Big(\frac{\delta}{\delta^2 + |x-z|^2}\Big)^{\frac{N-2}{2}}.$$

They also established the nondegeneracy result when μ is close to N. We also refer the readers to [1, 3, 4, 22, 36, 51] and a survey [41] for recent progress on the topic of Choquard equation.

Inspired by the work in [7, 33] for local nonlinear Schrödinger equations and [10] for a Schrödinger–Poisson system, in the present paper we aim to study the existence of nontrivial solutions of the critical Hartree system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1\big(|x|^{-4} * u^2\big)u + \beta\big(|x|^{-4} * v^2\big)u & \text{in } \mathbb{R}^N,\\ -\Delta v + V_2(x)v = \alpha_2\big(|x|^{-4} * v^2\big)v + \beta\big(|x|^{-4} * u^2\big)v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $N \geq 5$, $\beta > \max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$, and $V_1, V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ are nonnegative potential functions. Our goal is to find a nontrivial positive solution of (1.7) at a higher energy level when L^2-norm of V_i are both suitably small.

To state the main results, we first recall the Hardy-Littlewood-Sobolev inequality (see [30, Theorem 4.3]) to clarify the meaning of "critical" for the nonlocal Hartree equation.

Proposition 1.1. *Let t, $r > 1$ and $0 < \mu < N$ be such that $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$. Then there is a sharp constant $C(N, \mu, t)$ such that, for $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$,*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}dxdy\Big| \leq C(N, \mu, t)|f|_{L^t(\mathbb{R}^N)}|h|_{L^r(\mathbb{R}^N)}. \tag{1.8}$$

In particular, if $t = r = \frac{2N}{2N-\mu}$ then

$$C(N, \mu, t) = C(N, \mu) = \pi^{\frac{\mu}{2}}\frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})}\Big(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big)^{-1+\frac{\mu}{N}},$$

where $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}\,dx$, $s > 0$. In this case, equality in (1.8) holds if and only if $f \equiv (const.)\,h$ and

$$h(x) = c\big(\delta^2 + |x-z|^2\big)^{-\frac{2N-\mu}{2}}$$

for some $c \in \mathbb{R}$, $\delta > 0$ and $z \in \mathbb{R}^N$.

According to Proposition 1.1, the functional

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|v(y)|^p}{|x-y|^\mu}dxdy$$

is well defined in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ if $\frac{2N-\mu}{N}\le p\le\frac{2N-\mu}{N-2}$. Here the constant $2^*_\mu=\frac{2N-\mu}{N-2}$ is called the upper Hardy-Littlewood-Sobolev critical exponent. In this sense, (1.7) is said to be a critical Hartree system.

The authors of [23] investigated a critical Choquard type equation on a bounded domain and extended the well known results in [9]. In particular, it was proved in [23] that the infimum

$$S_{H,L}:=\inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u|^2dx}{\left(\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^4}dxdy\right)^{\frac12}}$$

is achieved if and only if

$$u(x)=c\Big(\frac{\delta}{\delta^2+|x-z|^2}\Big)^{\frac{N-2}{2}},$$

where $c>0$, $\delta>0$ and $z\in\mathbb{R}^N$. Moreover,

$$S_{H,L}=\frac{S}{\sqrt{C(N,4)}},\tag{1.9}$$

where S is the optimal constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$.

Our first step in this work is to establish a classification of positive solutions for the critical Hartree system

$$\begin{cases}-\Delta u=\alpha_1\big(|x|^{-4}*u^2\big)u+\beta\big(|x|^{-4}*v^2\big)u & \text{in } \mathbb{R}^N,\\ -\Delta v=\alpha_2\big(|x|^{-4}*v^2\big)v+\beta\big(|x|^{-4}*u^2\big)v & \text{in } \mathbb{R}^N,\end{cases}\tag{1.10}$$

which plays a role of the limit system for (1.7). To formulate our result denote $k_0=\frac{\beta-\alpha_2}{\beta^2-\alpha_1\alpha_2}$, $l_0=\frac{\beta-\alpha_1}{\beta^2-\alpha_1\alpha_2}$ and $R_N=\frac14\pi^{-\frac N2}\Gamma(\frac{N-2}{2})$. For $0<s<\frac N2$ we set

$$I(s)=\frac{\pi^{\frac N2}\Gamma(\frac{N-2s}{2})}{\Gamma(N-s)}.$$

Using moving sphere arguments in integral form inspired by [11, 19, 24] we establish the uniqueness of positive solutions of (1.10) up to translation and dilation.

Theorem 1.2. *Let $\beta>\max\{\alpha_1,\alpha_2\}\ge\min\{\alpha_1,\alpha_2\}>0$. If $(u,v)\in H:=D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ is a positive classical solution of (1.10), then*

$$u(x)=C_1\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^{\frac{N-2}{2}},\quad v(x)=C_2\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^{\frac{N-2}{2}}$$

for some $\tau>0$ and $\overline{x}\in\mathbb{R}^N$, where

$$C_1=\frac{\sqrt{k_0}}{\sqrt{R_NI(2)I\big(\frac{N-2}{2}\big)}},\quad C_2=\frac{\sqrt{l_0}}{\sqrt{R_NI(2)I\big(\frac{N-2}{2}\big)}}.$$

To study (1.7) using variational methods we introduce the energy functional

$$\begin{aligned}\mathcal{J}(u,v)=&\frac12\int_{\mathbb{R}^N}\big(|\nabla u|^2+|\nabla v|^2+V_1(x)u^2+V_2(x)v^2\big)dx\\&-\frac14\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\alpha_1|u(x)|^2|u(y)|^2+\alpha_2|v(x)|^2|v(y)|^2+2\beta|u(x)|^2|v(y)|^2}{|x-y|^4}dxdy,\end{aligned}$$

In view of the Hardy-Littlewood-Sobolev inequality, the functional $\mathcal{J}$ is of class $\mathcal{C}^1$ on the Hilbert space $H:=D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$. Critical points of $\mathcal{J}$ are weak solutions of (1.7). Consider the infimum

$$c=\inf_{(u,v)\in\tilde{\mathcal{N}}}\mathcal{J}(u,v),\tag{1.11}$$

where $\tilde{\mathcal{N}}=\{(u,v)\in H:(u,v)\ne(0,0),\ \langle\mathcal{J}'(u,v),(u,v)\rangle=0\}$ is the Nehari manifold of $\mathcal{J}$. A ground state solution of (1.7) is *by definition* a minimizer of (1.11).

In the case $V_1 = V_2 = 0$, the analogues of $\mathcal{J}$ and c are denoted by $\mathcal{J}_\infty$ and c_∞ respectively. We prove in Lemma 2.3 that if $\beta > \max\{\alpha_1, \alpha_2\}$, then

$$c_\infty = \frac{1}{4}(k_0 + l_0)S^2_{H,L}.$$

This estimate, combined with the uniqueness result of Theorem 1.2 implies that *every finite energy positive classical solution of the limit system* (1.10) *is a ground state solution* (see Corollary 3.5).

We also prove in Lemma 2.4 that if $V_j \neq 0$ for some $j \in \{1,2\}$ and $\beta > \max\{\alpha_1, \alpha_2\}$ then

$$c = c_\infty$$

and the infimum in (1.11) is not attained, i.e. (1.7) does not have ground state solutions.

The main result of this paper is the following.

Theorem 1.3. *Let $N \geq 5$ and $\beta > \max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$. If V_1, $V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative functions satisfying*

$$\begin{aligned} 0 &< \frac{\beta - \alpha_2}{2\beta - \alpha_1 - \alpha_2} C(N,4)^{-\frac{1}{2}} |V_1|_{L^{N/2}(\mathbb{R}^N)} + \frac{\beta - \alpha_1}{2\beta - \alpha_1 - \alpha_2} C(N,4)^{-\frac{1}{2}} |V_2|_{L^{N/2}(\mathbb{R}^N)} \\ &< \min\left\{ \sqrt{\frac{\beta^2 - \alpha_1\alpha_2}{\alpha_1(2\beta - \alpha_1 - \alpha_2)}}, \sqrt{\frac{\beta^2 - \alpha_1\alpha_2}{\alpha_2(2\beta - \alpha_1 - \alpha_2)}}, \sqrt{2} \right\} S_{H,L} - S_{H,L}, \end{aligned} \tag{1.12}$$

then system (1.7) *has a positive solution $(u, v) \in H$ such that*

$$c_\infty = c < \mathcal{J}(u,v) \leq \min\left\{ \frac{S^2_{H,L}}{4\alpha_1}, \frac{S^2_{H,L}}{4\alpha_2}, 2c_\infty \right\}.$$

The main difficulties in the proof of Theorem 1.3 are the loss of compactness and the need to distinguish the solutions from the semitrivial ones. To overcome the loss of compactness we will follow the idea of Struwe [47] to establish a novel nonlocal version of the global compactness lemma for the critical Hartree system (Lemma 4.2). This result is more delicate than the lemma obtained in [22] for the scalar equation, since the proofs in this work rely heavily on the classification of positive solutions of the limit system established in Theorem 1.2.

Remark 1.4. The proofs in this paper can also be adapted to the general form of the critical Hartree system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1\big(|x|^{-\mu} * |u|^{2^*_\mu}\big)|u|^{2^*_\mu - 2}u + \beta\big(|x|^{-\mu} * |v|^{2^*_\mu}\big)|u|^{2^*_\mu - 2}u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2\big(|x|^{-\mu} * |v|^{2^*_\mu}\big)|v|^{2^*_\mu - 2}v + \beta\big(|x|^{-\mu} * u^{2^*_\mu}\big)|v|^{2^*_\mu - 2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.13}$$

where $N \geq 3$, $N > \mu > 0$, $\beta > \frac{N-\mu+2}{N-2}\max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$, and V_1, $V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ are nonnegative potential functions with small $L^{\frac{N}{2}}$ norms.

Throughout this paper, we will use the following notations.

- The standard norm in the Sobolev space $D^{1,2}(\mathbb{R}^N)$ is given by
$$\|u\| := \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx \Big)^{\frac{1}{2}}.$$
- Set $H := D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ equipped with the norm
$$\|(u,v)\| := \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx \Big)^{\frac{1}{2}}.$$
- The standard norm in $L^q(\Omega)$ is denoted by $|\cdot|_{q,\Omega}$ and by $|\cdot|_q$ if $\Omega = \mathbb{R}^N$.
- $o(1)$ means a quantity which tends to 0.
- $c, C_j, C_N, C(\cdot)$ stand for various positive constants whose exact values are irrelevant.

The paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we prove a uniqueness result for limit system by the method of moving spheres. Section 4 is devoted to the proof of a nonlocal global compactness lemma and Theorem 1.3 is proved in Section 5.

2. Preliminaries

To prove the existence of positive solutions of (1.7), we will study the modified system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1\big(|x|^{-4} * |u^+|^2\big)u^+ + \beta\big(|x|^{-4} * |v^+|^2\big)u^+ & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2\big(|x|^{-4} * |v^+|^2\big)v^+ + \beta\big(|x|^{-4} * |u^+|^2\big)v^+ & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $u^+ = \max\{u, 0\}$ and $v^+ = \max\{v, 0\}$. In fact, if (u, v) is a nontrivial solution of (2.1), then $u(x) > 0$ and $v(x) > 0$ for all $x \in \mathbb{R}^N$ by the strong maximum principle, which implies that (u, v) is a positive solution of (1.7). Therefore, we only need to prove the existence of a nontrivial solution of (2.1).

Semitrivial solutions of (2.1) are closely related to solutions of the single elliptic equation

$$-\Delta u + V_i(x)u = \alpha_i\big(|x|^{-4} * |u^+|^2\big)u^+ \quad \text{in } \mathbb{R}^N,$$

of which the associated functional $I_i : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$I_i(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V_i(x)u^2\big)dx - \frac{\alpha_i}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^+(x)|^2|u^+(y)|^2}{|x-y|^4}dxdy.$$

Consider the infimum

$$c_i = \inf_{u\in\mathcal{M}_i} I_i(u),$$

where $\mathcal{M}_i = \{u \in D^{1,2}(\mathbb{R}^N) : u \neq 0, \langle I_i'(u), u\rangle = 0\}$. In the case $V_i = 0$, we denote the analogues of I_i, c_i, $\mathcal{M}_i$ by $I_{i\infty}$, $c_{i\infty}$, $\mathcal{M}_{i\infty}$ respectively.

For $\delta > 0$ and $z \in \mathbb{R}^N$, we denote

$$U_{\delta,z}(x) = C_N\Big(\frac{\delta}{\delta^2 + |x-z|^2}\Big)^{\frac{N-2}{2}}$$

where $C_N = S^{-\frac{N-4}{4}}C(N,4)^{-\frac{1}{2}}[N(N-2)]^{\frac{N-2}{4}}$. Then we have

$$\int_{\mathbb{R}^N} |\nabla U_{\delta,z}|^2dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{U^2_{\delta,z}(x)U^2_{\delta,z}(y)}{|x-y|^4}dxdy = S^2_{H,L}$$

and, according to [19, Theorem 1.3], the set $\{U_{\delta,z} : \delta > 0, z \in \mathbb{R}^N\}$ contains all positive solutions of

$$-\Delta u = \big(|x|^{-4} * u^2\big)u \quad \text{in } \mathbb{R}^N.$$

It is easy to verify that $c_{i\infty} = \frac{1}{4\alpha_i}S^2_{H,L}$.

Lemma 2.1. *If $V_i \in L^{\frac{N}{2}}(\mathbb{R}^N)$ is nonnegative, then $c_i = c_{i\infty} = \frac{1}{4\alpha_i}S^2_{H,L}$.*

Proof. For $u \in \mathcal{M}_i$, let $t_u > 0$ be such that $t_u u \in \mathcal{M}_{i\infty}$. Since $V_i(x) \geq 0$ for $x \in \mathbb{R}^N$, we have

$$t_u^2 = \frac{\|u\|^2}{\alpha_i\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u^+(x)|^2|u^+(y)|^2}{|x-y|^4}dxdy} \leq \frac{\|u\|^2 + \displaystyle\int_{\mathbb{R}^N} V_i(x)u^2dx}{\alpha_i\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u^+(x)|^2|u^+(y)|^2}{|x-y|^4}dxdy} = 1.$$

Then

$$c_{i\infty} \leq I_{i\infty}(t_u u) = \frac{1}{4}t_u^2\|u\|^2 \leq \frac{1}{4}\Big(\|u\|^2 + \int_{\mathbb{R}^N} V_i(x)u^2dx\Big) = I_i(u),$$

which implies that $c_i \geq c_{i\infty}$.

To prove the reverse inequality, let $t_n > 0$ be given by

$$t_n^2 = \frac{\|U_{1,z_n}\|^2 + \displaystyle\int_{\mathbb{R}^N} V_i(x)U^2_{1,z_n}dx}{\alpha_i\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{U^2_{1,z_n}(x)U^2_{1,z_n}(y)}{|x-y|^4}dxdy},$$

then we have $t_nU_{1,z_n} \in \mathcal{M}_i$. Therefore

$$c_i \leq I_i(t_nU_{1,z_n}) = \frac{1}{4}t_n^2\Big(\|U_{1,z_n}\|^2 + \int_{\mathbb{R}^N} V_i(x)U^2_{1,z_n}dx\Big).$$

Since $V_i \in L^{\frac{N}{2}}(\mathbb{R}^N)$, for any $\varepsilon > 0$ there exists a number $r = r(\varepsilon) > 0$ such that

$$\Big(\int_{\mathbb{R}^N\setminus B_r(0)} |V_i|^{\frac{N}{2}}dx\Big)^{\frac{2}{N}} < \varepsilon.$$

For such r, let $\{z_n\} \subset \mathbb{R}^N$ be a sequence such that $\lim_{n\to\infty} |z_n| = +\infty$, then we can find $n_0 \in \mathbb{N}$ such that

$$\Big(\int_{B_r(0)} U_{1,z_n}^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{N}} = \Big(\int_{B_r(-z_n)} U_{1,0}^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{N}} < \varepsilon, \quad \text{for } n \geq n_0.$$

Then the Hölder inequality leads to

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} V_i(x)U_{1,z_n}^2 dx\Big| &= \Big|\int_{B_r(0)} V_i(x)U_{1,z_n}^2 dx + \int_{\mathbb{R}^N\setminus B_r(0)} V_i(x)U_{1,z_n}^2 dx\Big| \\ &\leq |V_i|_{\frac{N}{2}}\Big(\int_{B_r(0)} U_{1,z_n}^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{N}} + \Big(\int_{\mathbb{R}^N\setminus B_r(0)} |V_i|^{\frac{N}{2}}dx\Big)^{\frac{2}{N}}\Big(\int_{\mathbb{R}^N} U_{1,0}^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{N}} \\ &< C\varepsilon\end{aligned}$$

for $n \geq n_0$, which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_i(x)U_{1,z_n}^2 dx = 0. \tag{2.2}$$

Moreover, by the definition of t_n, we know $t_n^2 = \frac{1}{\alpha_i} + o_n(1)$ as $n \to \infty$. Then

$$c_i \leq I_i(t_n U_{1,z_n}) = \frac{1}{4}\Big(\frac{1}{\alpha_i} + o_n(1)\Big)\big(S_{H,L}^2 + o_n(1)\big) = \frac{1}{4\alpha_i}S_{H,L}^2 + o_n(1)$$

and so $c_i \leq \frac{1}{4\alpha_i}S_{H,L}^2 = c_{i\infty}$. Combining this with $c_i \geq c_{i\infty}$, we get the desired conclusion. □

To study (2.1) by variational methods, we define the energy functional $J : H \to \mathbb{R}$ by

$$\begin{aligned}J(u,v) = &\frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla u|^2 + |\nabla v|^2 + V_1(x)u^2 + V_2(x)v^2\big)dx \\ &- \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\alpha_1|u^+(x)|^2|u^+(y)|^2 + \alpha_2|v^+(x)|^2|v^+(y)|^2 + 2\beta|u^+(x)|^2|v^+(y)|^2}{|x-y|^4}dxdy.\end{aligned}$$

In view of the Hardy-Littlewood-Sobolev inequality, the functional J is well defined and belongs to $\mathcal{C}^1(H,\mathbb{R})$. Then we see that (u,v) is a weak solution of (2.1) if and only if (u,v) is a critical point of the functional J.

Consider the infimum

$$c = \inf_{(u,v)\in\mathcal{N}} J(u,v),$$

where

$$\mathcal{N} = \{(u,v) \in H : (u,v) \neq (0,0),\ \langle J'(u,v),(u,v)\rangle = 0\}.$$

In the case $V_1 = V_2 = 0$, the analogues of J, c, $\mathcal{N}$ will be denoted by J_∞, c_∞, $\mathcal{N}_\infty$ respectively. As is known, critical points of the functional J_∞ correspond to weak solutions of the coupled system

$$\begin{cases} -\Delta u = \alpha_1\big(|x|^{-4} * |u^+|^2\big)u^+ + \beta\big(|x|^{-4} * |v^+|^2\big)u^+ & \text{in } \mathbb{R}^N, \\ -\Delta v = \alpha_2\big(|x|^{-4} * |v^+|^2\big)v^+ + \beta\big(|x|^{-4} * |u^+|^2\big)v^+ & \text{in } \mathbb{R}^N. \end{cases} \tag{2.3}$$

The next lemma is proved in [34].

Lemma 2.2. *If $\beta > \max\{\alpha_1,\alpha_2\}$ and $f : [0,+\infty) \to \mathbb{R}$ is defined by*

$$f(t) = \frac{(t+1)^2}{\alpha_1 t^2 + 2\beta t + \alpha_2},$$

then $\min_{t\geq 0} f(t) = k_0 + l_0$, where $k_0 = \frac{\beta-\alpha_2}{\beta^2-\alpha_1\alpha_2}$ and $l_0 = \frac{\beta-\alpha_1}{\beta^2-\alpha_1\alpha_2}$.

Lemma 2.3. *If $\beta > \max\{\alpha_1, \alpha_2\}$, then we have*

$$c_\infty = \frac{1}{4}(k_0 + l_0)S_{H,L}^2$$

and any least energy solution of (2.3) *must be of the form*

$$(\sqrt{k_0}U_{\delta,z}, \sqrt{l_0}U_{\delta,z})$$

for some $\delta > 0$ and $z \in \mathbb{R}^N$, where again $k_0 = \frac{\beta-\alpha_2}{\beta^2-\alpha_1\alpha_2}$ and $l_0 = \frac{\beta-\alpha_1}{\beta^2-\alpha_1\alpha_2}$.

Proof. Firstly, we show that $c_\infty = \frac{1}{4}(k_0 + l_0)S_{H,L}^2$. Since $(\sqrt{k_0}U_{\delta,z}, \sqrt{l_0}U_{\delta,z}) \in \mathcal{N}_\infty$, we have

$$c_\infty \leq J_\infty(\sqrt{k_0}U_{\delta,z}, \sqrt{l_0}U_{\delta,z}) = \frac{1}{4}(k_0 + l_0)S_{H,L}^2.$$

To prove the reverse inequality, let $(u, v) \in \mathcal{N}_\infty$ and assume without loss of generality that $v \neq 0$. Set

$$t = \frac{|u|_{2^*}^2}{|v|_{2^*}^2} \geq 0,$$

where $2^* = \frac{2N}{N-2}$. Then, by the Sobolev inequality and Proposition 1.1,

$$\begin{aligned}
S(t+1)|v|_{2^*}^2 &= S|u|_{2^*}^2 + S|v|_{2^*}^2 \\
&\leq \int_{\mathbb{R}^N} \big(|\nabla u|^2 + |\nabla v|^2\big)dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\alpha_1|u^+(x)|^2|u^+(y)|^2 + \alpha_2|v^+(x)|^2|v^+(y)|^2 + 2\beta|u^+(x)|^2|v^+(y)|^2}{|x-y|^4}dxdy \\
&\leq C(N,4)\big(\alpha_1|u|_{2^*}^4 + \alpha_2|v|_{2^*}^4 + 2\beta|u|_{2^*}^2|v|_{2^*}^2\big) \\
&= C(N,4)\big(\alpha_1 t^2 + 2\beta t + \alpha_2\big)|v|_{2^*}^4
\end{aligned}$$

and so

$$|v|_{2^*}^2 \geq \frac{S(t+1)}{C(N,4)\big(\alpha_1 t^2 + 2\beta t + \alpha_2\big)}.$$

Noting that $(u, v) \in \mathcal{N}_\infty$, by Lemma 2.2 and (1.9), we obtain

$$\begin{aligned}
J_\infty(u,v) &= \frac{1}{4}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + |\nabla v|^2\big)dx \\
&\geq \frac{1}{4}S(t+1)|v|_{2^*}^2 \\
&\geq \frac{S^2(t+1)^2}{4C(N,4)\big(\alpha_1 t^2 + 2\beta t + \alpha_2\big)} \\
&\geq \frac{1}{4}(k_0 + l_0)S_{H,L}^2.
\end{aligned} \tag{2.4}$$

Then $c_\infty \geq \frac{1}{4}(k_0 + l_0)S_{H,L}^2$ and we conclude that $c_\infty = \frac{1}{4}(k_0 + l_0)S_{H,L}^2$.

Secondly, we prove the uniqueness of least energy solutions up to translation and dilation. On one hand, if $(u, v) \in \mathcal{N}_\infty$ and either $u \neq a_1 U_{\delta,z}$ or $v \neq a_2 U_{\delta,z}$, where $a_1 \neq 0$, $a_2 \neq 0$, $\delta > 0$ and $z \in \mathbb{R}^N$ are parameters, then we see from (2.4) and

$$c_{i\infty} = \frac{1}{4\alpha_i}S_{H,L}^2 > \frac{1}{4}(k_0 + l_0)S_{H,L}^2, \quad i = 1, 2$$

that $J_\infty(u, v) > \frac{1}{4}(k_0 + l_0)S_{H,L}^2$. On the other hand, if $(u, v) = (a_1 U_{\delta,z}, a_2 U_{\delta,z}) \in \mathcal{N}_\infty$ and $J_\infty(u, v) = c_\infty$, then there must be $a_1 = \sqrt{k_0}$ and $a_2 = \sqrt{l_0}$. Therefore, any least energy solution of (2.3) must be of the form

$$(\sqrt{k_0}U_{\delta,z}, \sqrt{l_0}U_{\delta,z})$$

for some $\delta > 0$ and $z \in \mathbb{R}^N$. The proof is complete. □

Lemma 2.4. *If $\beta > \max\{\alpha_1, \alpha_2\}$ and V_1, $V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N)$ are nonnegative functions satisfying*

$$|V_1|_{\frac{N}{2}} + |V_2|_{\frac{N}{2}} > 0, \tag{2.5}$$

then $c = c_\infty$ and c is not achieved.

Proof. Using the arguments in the proof of Lemma 2.1, one can prove $c \geq c_\infty = \frac{1}{4}(k_0 + l_0)S^2_{H,L}$. We shall show that the equality holds indeed. Let us consider the sequence $\{z_n\} \subset \mathbb{R}^N$ satisfying $|z_n| \to +\infty$ as $n \to \infty$. By the claim in the proof of Lemma 2.1, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big(k_0 V_1(x) U^2_{1,z_n} + l_0 V_2(x) U^2_{1,z_n}\big) dx = 0.$$

For $t_n > 0$ defined by

$$t_n^2 = \frac{\|(\sqrt{k_0}U_{1,z_n}, \sqrt{l_0}U_{1,z_n})\|^2 + \int_{\mathbb{R}^N} \big(k_0 V_1(x) U^2_{1,z_n} + l_0 V_2(x) U^2_{1,z_n}\big) dx}{(k_0 + l_0) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{U^2_{1,z_n}(x) U^2_{1,z_n}(y)}{|x-y|^4} dxdy},$$

we have $(t_n\sqrt{k_0}U_{1,z_n}, t_n\sqrt{l_0}U_{1,z_n}) \in \mathcal{N}$ and $t_n^2 = 1 + o_n(1)$ as $n \to \infty$. Then

$$\begin{aligned}
c &\leq J(t_n\sqrt{k_0}U_{1,z_n}, t_n\sqrt{l_0}U_{1,z_n}) \\
&= \frac{1}{4} t_n^2 \Big(\|(\sqrt{k_0}U_{1,z_n}, \sqrt{l_0}U_{1,z_n})\|^2 + \int_{\mathbb{R}^N} \big(k_0 V_1(x) U^2_{1,z_n} + l_0 V_2(x) U^2_{1,z_n}\big) dx \Big) \\
&= \frac{1}{4}\big(1 + o_n(1)\big)\big((k_0 + l_0)S^2_{H,L} + o_n(1)\big) \\
&= \frac{1}{4}(k_0 + l_0) S^2_{H,L} + o_n(1),
\end{aligned}$$

which implies $c \leq \frac{1}{4}(k_0 + l_0)S^2_{H,L} = c_\infty$. Therefore, we conclude that $c = c_\infty$.

We use an argument of contradiction to prove the nonexistence result. Assume that $(u, v) \in \mathcal{N}$ satisfies

$$J(u, v) = \frac{1}{4}\Big(\|(u, v)\|^2 + \int_{\mathbb{R}^N} \big(V_1(x)u^2 + V_2(x)v^2\big) dx \Big) = c.$$

Let $t_{(u,v)} > 0$ be such that $(t_{(u,v)}u, t_{(u,v)}v) \in \mathcal{N}_\infty$. It is easy to verify that $t_{(u,v)} \leq 1$. Then

$$c_\infty \leq J_\infty(t_{(u,v)}u, t_{(u,v)}v) = \frac{1}{4} t^2_{(u,v)} \|(u, v)\|^2 \leq \frac{1}{4}\Big(\|(u, v)\|^2 + \int_{\mathbb{R}^N} \big(V_1(x)u^2 + V_2(x)v^2\big) dx \Big) = c = c_\infty,$$

which indicates that $t_{(u,v)} = 1$ and

$$\int_{\mathbb{R}^N} \big(V_1(x)u^2 + V_2(x)v^2\big) dx = 0. \tag{2.6}$$

This means that (u, v) is a least energy solution of (2.3). By Lemma 2.3, we know $u(x) > 0$ and $v(x) > 0$ for $x \in \mathbb{R}^N$. Combining this with (2.6), we have $V_1 = V_2 = 0$ almost everywhere in $\mathbb{R}^N$, which contradicts (2.5). Therefore, the infimum c is not attained. □

By Lemma 2.4, we know that (2.1) does not have a ground state solution. Therefore, nontrivial solutions of (2.1) only at high energy levels can be expected.

3. Uniqueness for a limit problem

This section is devoted to the classification of positive solutions for critical coupled Hartree system

$$\begin{cases} -\Delta u = \alpha_1\big(|x|^{-4} * u^2\big)u + \beta\big(|x|^{-4} * v^2\big)u & \text{in } \mathbb{R}^N, \\ -\Delta v = \alpha_2\big(|x|^{-4} * v^2\big)v + \beta\big(|x|^{-4} * u^2\big)v & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

We will employ the Kelvin transformation and the method of moving spheres in integral forms to complete the proof. See [11, 19, 24] and references therein for uniqueness results for a single elliptic equation.

Recall that $k_0 = \frac{\beta-\alpha_2}{\beta^2-\alpha_1\alpha_2}$ and $l_0 = \frac{\beta-\alpha_1}{\beta^2-\alpha_1\alpha_2}$ in Lemma 2.3. We also denote $R_N = \frac{1}{4}\pi^{-\frac{N}{2}}\Gamma(\frac{N-2}{2})$ and, for $0 < s < \frac{N}{2}$,

$$I(s) = \frac{\pi^{\frac{N}{2}}\Gamma(\frac{N-2s}{2})}{\Gamma(N-s)},$$

where $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}\,dx$, $s > 0$.

According to [11], (3.1) is equivalent to the following integral system in $\mathbb{R}^N$

$$\begin{cases} u(x) = \alpha_1 R_N \displaystyle\int_{\mathbb{R}^N} \frac{u(y)w(y)}{|x-y|^{N-2}}dy + \beta R_N \int_{\mathbb{R}^N} \frac{u(y)g(y)}{|x-y|^{N-2}}dy, \\ v(x) = \alpha_2 R_N \displaystyle\int_{\mathbb{R}^N} \frac{v(y)g(y)}{|x-y|^{N-2}}dy + \beta R_N \int_{\mathbb{R}^N} \frac{v(y)w(y)}{|x-y|^{N-2}}dy, \\ w(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^4}dy, \\ g(x) = \displaystyle\int_{\mathbb{R}^N} \frac{v^2(y)}{|x-y|^4}dy. \end{cases} \tag{3.2}$$

Let $x_0 \in \mathbb{R}^N$ and $\lambda > 0$. The inversion of $x \in \mathbb{R}^N\backslash\{x_0\}$ about the sphere $\partial B_\lambda(x_0)$ is given by

$$x_{x_0,\lambda} = \frac{\lambda^2(x-x_0)}{|x-x_0|^2} + x_0.$$

Assume (u, v, w, g) satisfies (3.2) and each component is positive. We define the Kelvin transform of u and v with respect to $\partial B_\lambda(x_0)$ by

$$u_{x_0,\lambda}(x) = \Big(\frac{\lambda}{|x-x_0|}\Big)^{N-2} u(x_{x_0,\lambda}), \quad v_{x_0,\lambda}(x) = \Big(\frac{\lambda}{|x-x_0|}\Big)^{N-2} v(x_{x_0,\lambda})$$

and the Kelvin transform of w and g with respect to $\partial B_\lambda(x_0)$ by

$$w_{x_0,\lambda}(x) = \Big(\frac{\lambda}{|x-x_0|}\Big)^{4} w(x_{x_0,\lambda}), \quad g_{x_0,\lambda}(x) = \Big(\frac{\lambda}{|x-x_0|}\Big)^{4} g(x_{x_0,\lambda}),$$

respectively. Set

$$U_{x_0,\lambda} = u_{x_0,\lambda} - u, \quad V_{x_0,\lambda} = v_{x_0,\lambda} - v, \quad W_{x_0,\lambda} = w_{x_0,\lambda} - w, \quad G_{x_0,\lambda} = g_{x_0,\lambda} - g.$$

When $x_0 = 0$, we will drop x_0 in the subscript of above notations and write, for example,

$$x_\lambda = \frac{\lambda^2 x}{|x|^2}, \quad u_\lambda(x) = \Big(\frac{\lambda}{|x|}\Big)^{N-2} u(x_\lambda), \quad U_\lambda = u_\lambda - u.$$

Denoting

$$B^u_\lambda = \{x \in B_\lambda\backslash\{0\} : U_\lambda(x) < 0\}, \qquad B^v_\lambda = \{x \in B_\lambda\backslash\{0\} : V_\lambda(x) < 0\},$$
$$B^w_\lambda = \{x \in B_\lambda\backslash\{0\} : W_\lambda(x) < 0\}, \quad B^g_\lambda = \{x \in B_\lambda\backslash\{0\} : G_\lambda(x) < 0\},$$

we have

Lemma 3.1. *There exists a positive constant C independent of λ such that*

$$\begin{aligned} |U_\lambda|_{2^*,B^u_\lambda} \le C\big(\alpha_1|u|^2_{2^*,B^u_\lambda} + \alpha_1|w|_{\frac{N}{2},B^u_\lambda} + \beta|g|_{\frac{N}{2},B^u_\lambda}\big)|U_\lambda|_{2^*,B^u_\lambda} \\ + C\beta|u|_{2^*,B^u_\lambda}|v|_{2^*,B^v_\lambda}|V_\lambda|_{2^*,B^v_\lambda} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |V_\lambda|_{2^*,B^v_\lambda} \le C\big(\alpha_2|v|^2_{2^*,B^v_\lambda} + \alpha_2|g|_{\frac{N}{2},B^v_\lambda} + \beta|w|_{\frac{N}{2},B^v_\lambda}\big)|V_\lambda|_{2^*,B^v_\lambda} \\ + C\beta|u|_{2^*,B^u_\lambda}|v|_{2^*,B^v_\lambda}|U_\lambda|_{2^*,B^u_\lambda}, \end{aligned} \tag{3.4}$$

where $2^* = \frac{2N}{N-2}$.

Proof. Since $dy_\lambda = (\frac{\lambda}{|y|})^{2N} dy$, a direct computation shows that

$$
\begin{aligned}
u(x) &= \alpha_1 R_N \Big(\int_{B_\lambda} \frac{u(y)w(y)}{|x-y|^{N-2}} dy + \int_{\mathbb{R}^N \setminus B_\lambda} \frac{u(y)w(y)}{|x-y|^{N-2}} dy \Big) \\
&\quad + \beta R_N \Big(\int_{B_\lambda} \frac{u(y)g(y)}{|x-y|^{N-2}} dy + \int_{\mathbb{R}^N \setminus B_\lambda} \frac{u(y)g(y)}{|x-y|^{N-2}} dy \Big) \\
&= \alpha_1 R_N \Big(\int_{B_\lambda} \frac{u(y)w(y)}{|x-y|^{N-2}} dy + \int_{B_\lambda} \frac{u_\lambda(y)w_\lambda(y)}{\big|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\big|^{N-2}} dy \Big) \\
&\quad + \beta R_N \Big(\int_{B_\lambda} \frac{u(y)g(y)}{|x-y|^{N-2}} dy + \int_{B_\lambda} \frac{u_\lambda(y)g_\lambda(y)}{\big|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\big|^{N-2}} dy \Big)
\end{aligned}
$$

and

$$
\begin{aligned}
u_\lambda(x) &= \Big(\frac{\lambda}{|x|}\Big)^{N-2} u(x_\lambda) \\
&= \alpha_1 R_N \Big(\frac{\lambda}{|x|}\Big)^{N-2} \Big(\int_{B_\lambda} \frac{u(y)w(y)}{|x_\lambda - y|^{N-2}} dy + \int_{\mathbb{R}^N \setminus B_\lambda} \frac{u(y)w(y)}{|x_\lambda - y|^{N-2}} dy \Big) \\
&\quad + \beta R_N \Big(\frac{\lambda}{|x|}\Big)^{N-2} \Big(\int_{B_\lambda} \frac{u(y)g(y)}{|x_\lambda - y|^{N-2}} dy + \int_{\mathbb{R}^N \setminus B_\lambda} \frac{u(y)g(y)}{|x_\lambda - y|^{N-2}} dy \Big) \\
&= \alpha_1 R_N \Big(\int_{B_\lambda} \frac{u(y)w(y)}{\big|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\big|^{N-2}} dy + \int_{B_\lambda} \frac{u_\lambda(y)w_\lambda(y)}{|x-y|^{N-2}} dy \Big) \\
&\quad + \beta R_N \Big(\int_{B_\lambda} \frac{u(y)g(y)}{\big|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\big|^{N-2}} dy + \int_{B_\lambda} \frac{u_\lambda(y)g_\lambda(y)}{|x-y|^{N-2}} dy \Big),
\end{aligned}
$$

where we have used

$$
|x_\lambda - y| = \frac{\lambda^2 |x - y_\lambda|}{|x||y_\lambda|} = \frac{|y|}{|x|}\Big| x - \frac{\lambda^2}{|y|^2} y \Big| = \frac{\lambda}{|x|}\Big| \frac{|y|}{\lambda} x - \frac{\lambda}{|y|} y \Big|
$$

and

$$
|x_\lambda - y_\lambda| = \frac{\lambda^2 |x-y|}{|x||y|}
$$

for $x,\, y \in \mathbb{R}^N \setminus \{0\}$. Then it follows

$$
\begin{aligned}
U_\lambda(x) &= u_\lambda(x) - u(x) \\
&= \alpha_1 R_N \int_{B_\lambda} \Big(\frac{1}{|x-y|^{N-2}} - \frac{1}{\big|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\big|^{N-2}} \Big) \times (u_\lambda(y)w_\lambda(y) - u(y)w(y)) dy \\
&\quad + \beta R_N \int_{B_\lambda} \Big(\frac{1}{|x-y|^{N-2}} - \frac{1}{\big|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\big|^{N-2}} \Big) \times (u_\lambda(y)g_\lambda(y) - u(y)g(y)) dy.
\end{aligned} \tag{3.5}
$$

Denoting $a^- = \min\{a, 0\}$, we claim

$$
u_\lambda w_\lambda - uw \geq w U_\lambda^- + u W_\lambda^- \quad \text{and} \quad u_\lambda g_\lambda - ug \geq g U_\lambda^- + u G_\lambda^-. \tag{3.6}
$$

Since the argument is analogous, we only prove the first inequality and divide the discussion into four cases.
Case 1. If $u_\lambda(y) \geq u(y)$ and $w_\lambda(y) \geq w(y)$, then $U_\lambda^-(y) = W_\lambda^-(y) = 0$ and so

$$
u_\lambda(y)w_\lambda(y) - u(y)w(y) \geq 0 = w(y)U_\lambda^-(y) + u(y)W_\lambda^-(y).
$$

Case 2. If $u_\lambda(y) \geq u(y)$ and $w_\lambda(y) < w(y)$, then $U_\lambda^-(y) = 0$ and $W_\lambda^-(y) = W_\lambda(y)$, which implies that

$$
u_\lambda(y)w_\lambda(y) - u(y)w(y) \geq u(y)W_\lambda(y) = w(y)U_\lambda^-(y) + u(y)W_\lambda^-(y).
$$

Case 3. If $u_\lambda(y) < u(y)$ and $w_\lambda(y) \geq w(y)$, then $U_\lambda^-(y) = U_\lambda(y)$ and $W_\lambda^-(y) = 0$, which implies that

$$
u_\lambda(y)w_\lambda(y) - u(y)w(y) \geq w(y)U_\lambda(y) = w(y)U_\lambda^-(y) + u(y)W_\lambda^-(y).
$$

Case 4. If $u_\lambda(y) < u(y)$ and $w_\lambda(y) < w(y)$, then $U_\lambda^-(y) = U_\lambda(y) \le 0$ and $W_\lambda^-(y) = W_\lambda(y) \le 0$. Hence

$$\begin{aligned} u_\lambda(y)w_\lambda(y) - u(y)w(y) &= w_\lambda(y)U_\lambda(y) + u(y)W_\lambda(y) \\ &\ge w(y)U_\lambda(y) + u(y)W_\lambda(y) = w(y)U_\lambda^-(y) + u(y)W_\lambda^-(y). \end{aligned}$$

Using (3.5), (3.6) and the fact that

$$\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^2 - |x-y|^2 = \frac{\left(|x|^2-\lambda^2\right)\left(|y|^2-\lambda^2\right)}{\lambda^2} > 0, \quad \text{for } x,\, y \in B_\lambda \backslash \{0\}$$

leads to

$$\begin{aligned} U_\lambda(x) &= \alpha_1 R_N \int_{B_\lambda} \Big(\frac{1}{|x-y|^{N-2}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{N-2}}\Big) \times (u_\lambda(y)w_\lambda(y) - u(y)w(y))dy \\ &\quad + \beta R_N \int_{B_\lambda} \Big(\frac{1}{|x-y|^{N-2}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{N-2}}\Big) \times (u_\lambda(y)g_\lambda(y) - u(y)g(y))dy \\ &\ge \alpha_1 R_N \int_{B_\lambda} \Big(\frac{1}{|x-y|^{N-2}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{N-2}}\Big) \times (w(y)U_\lambda^-(y) + u(y)W_\lambda^-(y))dy \\ &\quad + \beta R_N \int_{B_\lambda} \Big(\frac{1}{|x-y|^{N-2}} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^{N-2}}\Big) \times (g(y)U_\lambda^-(y) + u(y)G_\lambda^-(y))dy \\ &\ge \alpha_1 R_N \Big(\int_{B_\lambda} \frac{w(y)U_\lambda^-(y)}{|x-y|^{N-2}}dy + \int_{B_\lambda} \frac{u(y)W_\lambda^-(y)}{|x-y|^{N-2}}dy\Big) \\ &\quad + \beta R_N \Big(\int_{B_\lambda} \frac{g(y)U_\lambda^-(y)}{|x-y|^{N-2}}dy + \int_{B_\lambda} \frac{u(y)G_\lambda^-(y)}{|x-y|^{N-2}}dy\Big). \end{aligned}$$

Invoking the Hardy-Littlewood-Sobolev inequality and the Hölder inequality, we have

$$\begin{aligned} |U_\lambda|_{2^*,B_\lambda^u} &\le C\alpha_1 |wU_\lambda|_{\frac{2N}{N+2},B_\lambda^u} + C\alpha_1 |uW_\lambda|_{\frac{2N}{N+2},B_\lambda^u \cap B_\lambda^w} \\ &\quad + C\beta |gU_\lambda|_{\frac{2N}{N+2},B_\lambda^u} + C\beta |uG_\lambda|_{\frac{2N}{N+2},B_\lambda^u \cap B_\lambda^g} \\ &\le C\alpha_1 |w|_{\frac{N}{2},B_\lambda^u} |U_\lambda|_{2^*,B_\lambda^u} + C\alpha_1 |u|_{2^*,B_\lambda^u} |W_\lambda|_{\frac{N}{2},B_\lambda^w} \\ &\quad + C\beta |g|_{\frac{N}{2},B_\lambda^u} |U_\lambda|_{2^*,B_\lambda^u} + C\beta |u|_{2^*,B_\lambda^u} |G_\lambda|_{\frac{N}{2},B_\lambda^g}. \end{aligned} \tag{3.7}$$

By a similar argument, we can deduce from (3.2) that

$$W_\lambda(x) = \int_{B_\lambda} \Big(\frac{1}{|x-y|^4} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^4}\Big) \times (u_\lambda^2(y) - u^2(y))dy \ge C \int_{B_\lambda} \frac{u(y)U_\lambda^-(y)}{|x-y|^4}dy \tag{3.8}$$

and

$$G_\lambda(x) = \int_{B_\lambda} \Big(\frac{1}{|x-y|^4} - \frac{1}{\left|\frac{|y|}{\lambda}x - \frac{\lambda}{|y|}y\right|^4}\Big) \times (v_\lambda^2(y) - v^2(y))dy \ge C \int_{B_\lambda} \frac{v(y)V_\lambda^-(y)}{|x-y|^4}dy. \tag{3.9}$$

Using the Hardy-Littlewood-Sobolev inequality and the Hölder inequality again leads to

$$|W_\lambda|_{\frac{N}{2},B_\lambda^w} \le C|uU_\lambda|_{\frac{N}{N-2},B_\lambda^w \cap B_\lambda^u} \le C|u|_{2^*,B_\lambda^u}|U_\lambda|_{2^*,B_\lambda^u} \tag{3.10}$$

and

$$|G_\lambda|_{\frac{N}{2},B_\lambda^g} \le C|vV_\lambda|_{\frac{N}{N-2},B_\lambda^g \cap B_\lambda^v} \le C|v|_{2^*,B_\lambda^v}|V_\lambda|_{2^*,B_\lambda^v}. \tag{3.11}$$

Then (3.3) follows easily from (3.7), (3.10) and (3.11). Similarly, one can prove (3.4). □

Lemma 3.2. *For any $x_0 \in \mathbb{R}^N$, the sets*

$$\Gamma_{x_0}^u := \{\lambda > 0 : U_{x_0,\sigma} \ge 0 \ \text{in}\ B_\sigma(x_0)\backslash\{x_0\} \ \text{for all}\ \sigma \in (0,\lambda]\}$$

and

$$\Gamma^v_{x_0} := \{\lambda > 0 : V_{x_0,\sigma} \geq 0 \text{ in } B_\sigma(x_0)\backslash\{x_0\} \text{ for all } \sigma \in (0, \lambda]\}$$

are not empty.

Proof. Since (3.2) is invariant by translations, it suffices to consider the case $x_0 = 0$. Let $C > 0$ be the constant given in Lemma 3.1 and choose $\varepsilon_0 \in (0, 1)$ sufficiently small such that, for all $0 < \lambda \leq \varepsilon_0$,

$$\alpha_1|u|^2_{2^*,B^u_\lambda} + \alpha_1|w|_{\frac{N}{2},B^u_\lambda} + \beta|g|_{\frac{N}{2},B^u_\lambda} + \beta|u|_{2^*,B^u_\lambda}|v|_{2^*,B^v_\lambda} \leq \frac{1}{4C}$$

and

$$\alpha_2|v|^2_{2^*,B^v_\lambda} + \alpha_2|g|_{\frac{N}{2},B^v_\lambda} + \beta|w|_{\frac{N}{2},B^v_\lambda} + \beta|u|_{2^*,B^u_\lambda}|v|_{2^*,B^v_\lambda} \leq \frac{1}{4C}.$$

We see from Lemma 3.1 that

$$|U_\lambda|_{2^*,B^u_\lambda} = |V_\lambda|_{2^*,B^v_\lambda} = 0,$$

which implies meas(B^u_λ) = meas$(B^v_\lambda) = 0$. Then it follows from (3.5), (3.8) and (3.9) that $U_\lambda \geq 0$ in $B_\lambda\backslash\{0\}$. Similarly, we also have $V_\lambda \geq 0$ in $B_\lambda\backslash\{0\}$. Therefore, we have $\Gamma^u_0 \neq \emptyset \neq \Gamma^v_0$. □

According to Lemma 3.2, we define

$$\lambda^u_{x_0} := \sup \Gamma^u_{x_0} > 0, \quad \lambda^v_{x_0} := \sup \Gamma^v_{x_0} > 0, \quad \lambda_{x_0} := \min\{\lambda^u_{x_0}, \lambda^v_{x_0}\} > 0.$$

Lemma 3.3. *If* $\lambda_{x_0} < +\infty$, *then* $U_{x_0,\lambda_{x_0}} = V_{x_0,\lambda_{x_0}} = 0$ *in* $B_{\lambda_{x_0}}(x_0)\backslash\{x_0\}$.

Proof. As before, it is sufficient to consider the case $x_0 = 0$. Without loss of generality, we assume that $\lambda_0 = \lambda^u_0 < +\infty$. Since U_λ and V_λ are continuous with respect to λ, we have $U_{\lambda_0} \geq 0$ and $V_{\lambda_0} \geq 0$ in $B_{\lambda_0}\backslash\{0\}$. Then we see from (3.8) and (3.9) that

$$W_{\lambda_0}(x) = \int_{B_{\lambda_0}} \Big(\frac{1}{|x-y|^4} - \frac{1}{|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y|^4}\Big) \times (u^2_{\lambda_0}(y) - u^2(y))dy \geq 0$$

and

$$G_{\lambda_0}(x) = \int_{B_{\lambda_0}} \Big(\frac{1}{|x-y|^4} - \frac{1}{|\frac{|y|}{\lambda_0}x - \frac{\lambda_0}{|y|}y|^4}\Big) \times (v^2_{\lambda_0}(y) - v^2(y))dy \geq 0. \tag{3.12}$$

Observe that if $U_{\lambda_0} = 0$ in $B_{\lambda_0}\backslash\{0\}$, then it follows from (3.5) that $G_{\lambda_0} = 0$ in $B_{\lambda_0}\backslash\{0\}$, which combined with (3.12) implies $V_{\lambda_0} = 0$ in $B_{\lambda_0}\backslash\{0\}$. Analogously, if $V_{\lambda_0} = 0$ in $B_{\lambda_0}\backslash\{0\}$, then we also have $U_{\lambda_0} = 0$ in $B_{\lambda_0}\backslash\{0\}$.

Assume to the contrary that $U_{\lambda_0} \not\equiv 0$ and $V_{\lambda_0} \not\equiv 0$ in $B_{\lambda_0}\backslash\{0\}$. Then $W_{\lambda_0} > 0$ in $B_{\lambda_0}\backslash\{0\}$, which together with (3.5) indicates that $U_{\lambda_0} > 0$ in $B_{\lambda_0}\backslash\{0\}$. Similarly, there also holds $V_{\lambda_0} > 0$ in $B_{\lambda_0}\backslash\{0\}$. We claim that there exist two numbers $\rho > 0$ and $\delta > 0$ such that

$$U_{\lambda_0} \geq \delta \text{ and } V_{\lambda_0} \geq \delta \text{ in } B_\rho\backslash\{0\}. \tag{3.13}$$

Indeed, since $U_{\lambda_0} > 0$, $V_{\lambda_0} > 0$, $W_{\lambda_0} > 0$ and $G_{\lambda_0} > 0$ in $B_{\lambda_0}\backslash\{0\}$, we deduce from (3.5) that

$$\begin{aligned}\liminf_{|x|\to 0} U_{\lambda_0}(x) &\geq \alpha_1 R_N \int_{B_{\lambda_0}} \left(\frac{1}{|y|^{N-2}} - \frac{1}{\lambda_0^{N-2}}\right) \times (u_{\lambda_0}(y)w_{\lambda_0}(y) - u(y)w(y))dy \\ &\quad + \beta R_N \int_{B_{\lambda_0}} \left(\frac{1}{|y|^{N-2}} - \frac{1}{\lambda_0^{N-2}}\right) \times (u_{\lambda_0}(y)g_{\lambda_0}(y) - u(y)g(y))dy \\ &> 0\end{aligned}$$

and, similarly, $\liminf_{|x|\to 0} V_{\lambda_0}(x) > 0$. Then (3.13) holds for small $\rho > 0$ and $\delta > 0$.

Let $C > 0$ be the constant given in Lemma 3.1 and fix a number $r_0 \in (0, \frac{\lambda_0}{2})$ such that

$$\begin{aligned}&\alpha_1|u|^2_{2^*,B_{\lambda_0+r_0}\backslash B_{\lambda_0-r_0}} + \alpha_1|w|_{\frac{N}{2},B_{\lambda_0+r_0}\backslash B_{\lambda_0-r_0}} + \beta|g|_{\frac{N}{2},B_{\lambda_0+r_0}\backslash B_{\lambda_0-r_0}} \\ &\qquad + \beta|u|_{2^*,B_{\lambda_0+r_0}\backslash B_{\lambda_0-r_0}}|v|_{2^*,B_{\lambda_0+r_0}\backslash B_{\lambda_0-r_0}} \leq \frac{1}{4C}\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\alpha_2|v|^2_{2^*,B_{\lambda_0+r_0}\setminus B_{\lambda_0-r_0}} &+ \alpha_2|g|_{\frac{N}{2},B_{\lambda_0+r_0}\setminus B_{\lambda_0-r_0}} + \beta|w|_{\frac{N}{2},B_{\lambda_0+r_0}\setminus B_{\lambda_0-r_0}} \\ &+ \beta|u|_{2^*,B_{\lambda_0+r_0}\setminus B_{\lambda_0-r_0}}|v|_{2^*,B_{\lambda_0+r_0}\setminus B_{\lambda_0-r_0}} \le \frac{1}{4C}.\end{aligned} \tag{3.15}$$

Since $U_{\lambda_0} > 0$ and $V_{\lambda_0} > 0$ in $B_{\lambda_0}\setminus\{0\}$, we see from (3.13) that $U_{\lambda_0} \ge \delta'$ and $V_{\lambda_0} \ge \delta'$ in $B_{\lambda_0-r_0}\setminus\{0\}$ for some $\delta' > 0$. By uniform continuity, there exists $\tau_0 \in (0, r_0)$ such that, for any $\lambda \in (\lambda_0, \lambda_0 + \tau_0)$, $U_\lambda \ge \delta'/2$ and $V_\lambda \ge \delta'/2$ in $B_{\lambda_0-r_0}\setminus\{0\}$. Then $B^u_\lambda \subset B_{\lambda_0+r_0}\setminus B_{\lambda_0-r_0}$ and $B^v_\lambda \subset B_{\lambda_0+r_0}\setminus B_{\lambda_0-r_0}$ for any $\lambda \in (\lambda_0, \lambda_0 + \tau_0)$. Using Lemma 3.1, (3.14) and (3.15) leads to $|U_\lambda|_{2^*,B^u_\lambda} = 0$ when $\lambda \in (\lambda_0, \lambda_0 + \tau_0)$, which means meas $(B^u_\lambda) = 0$. Therefore, we have $U_\lambda \ge 0$ in $B_\lambda\setminus\{0\}$ for any $\lambda \in (\lambda_0, \lambda_0 + \tau_0)$, which contradicts the definition of λ^u_0. The proof is complete. □

The following lemma is proved in [27,28].

Lemma 3.4. *Let $N \ge 1$, $\nu \in \mathbb{R}$ and $u \in C^1(\mathbb{R}^N, \mathbb{R})$. For every $x_0 \in \mathbb{R}^N$ and $\lambda > 0$, define*

$$u_{x_0,\lambda}(x) = \Big(\frac{\lambda}{|x - x_0|}\Big)^\nu u\Big(\frac{\lambda^2(x - x_0)}{|x - x_0|^2} + x_0\Big), \quad x \in \mathbb{R}^N\setminus\{x_0\}.$$

(i) If for every $x_0 \in \mathbb{R}^N$ there exists $\lambda_{x_0} < +\infty$ such that

$$u_{x_0,\lambda_{x_0}}(x) = u(x), \quad \text{for all } x \in \mathbb{R}^N\setminus\{x_0\},$$

then

$$u(x) = C\Big(\frac{\tau}{\tau^2 + |x - \overline{x}|^2}\Big)^{\frac{\nu}{2}}$$

for some $C \in \mathbb{R}$, $\tau > 0$ and $\overline{x} \in \mathbb{R}^N$. Moreover, we have $\lambda_{x_0} = \sqrt{\tau^2 + |x_0 - \overline{x}|^2}$.
(ii) If for every $x_0 \in \mathbb{R}^N$ there holds

$$u_{x_0,\lambda}(x) \ge u(x), \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \in B_\lambda(x_0)\setminus\{x_0\},$$

then $u \equiv C$ for some $C \in \mathbb{R}$.

We are in a position to prove the main result in this section.

Proof of Theorem 1.2. Let (u, v) be a positive solution of (3.1) and recall that

$$\lambda^u_{x_0} := \sup \Gamma^u_{x_0} > 0, \quad \lambda^v_{x_0} := \sup \Gamma^v_{x_0} > 0, \quad \lambda_{x_0} := \min\{\lambda^u_{x_0}, \lambda^v_{x_0}\} > 0,$$

where

$$\Gamma^u_{x_0} := \{\lambda > 0 : U_{x_0,\sigma} \ge 0 \text{ in } B_\sigma(x_0)\setminus\{x_0\} \text{ for all } \sigma \in (0, \lambda]\}$$

and

$$\Gamma^v_{x_0} := \{\lambda > 0 : V_{x_0,\sigma} \ge 0 \text{ in } B_\sigma(x_0)\setminus\{x_0\} \text{ for all } \sigma \in (0, \lambda]\}.$$

We first claim that $\lambda_{x_0} < +\infty$ for any $x_0 \in \mathbb{R}^N$. If not, then we have the following two cases.

Case 1. $\lambda_{x_0} = +\infty$ for any $x_0 \in \mathbb{R}^N$. In this case, we see from Lemma 3.4(ii) that

$$(u, v) \equiv (C_1, C_2)$$

for some constants C_1, $C_2 > 0$. Then (u, v) could not be a solution of (3.1), yielding a contradiction.

Case 2. There exist x_0, $y_0 \in \mathbb{R}^N$ such that $\lambda_{x_0} = +\infty$ and $\lambda_{y_0} < +\infty$. In this case, since $\lambda^u_{x_0} \ge \lambda_{x_0} = +\infty$, we have, for any $\lambda > 0$, $U_{x_0,\lambda} \ge 0$ for $x \in B_\lambda(x_0)\setminus\{x_0\}$ which implies that $u(x) \ge u_{x_0,\lambda}(x)$ for $x \in \mathbb{R}^N\setminus B_\lambda(x_0)$. Then we obtain $|x - x_0|^{N-2}u(x) \ge \lambda^{N-2}u(x_{x_0,\lambda})$ for $x \in \mathbb{R}^N\setminus B_\lambda(x_0)$ and so

$$\liminf_{|x|\to\infty} |x|^{N-2}u(x) \ge \lambda^{N-2}u(x_0).$$

Since $\lambda > 0$ is arbitrary and $u(x_0) > 0$, we obtain

$$\lim_{|x|\to\infty} |x|^{N-2}u(x) = +\infty. \tag{3.16}$$

On the other hand, since $\lambda_{y_0} < +\infty$, we see from Lemma 3.3 that

$$u_{y_0,\lambda_{y_0}}(x) = u(x), \quad \text{for } x \in \mathbb{R}^N\setminus\{y_0\}.$$

Then we have $\lim_{|x|\to\infty}|x|^{N-2}u(x)=\lambda_{y_0}^{N-2}u(y_0)<+\infty$, yielding a contradiction with (3.16).

Since $\lambda_{x_0}<+\infty$ for any $x_0\in\mathbb{R}^N$, we deduce from Lemma 3.3 that

$$u_{x_0,\lambda_{x_0}}(x)=u(x)\ \text{ and }\ v_{x_0,\lambda_{x_0}}(x)=v(x),\quad \text{for all } x\in\mathbb{R}^N\backslash\{x_0\}.$$

In view of Lemma 3.4(i), (u,v) must be of the form

$$u(x)=C_1\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^{\frac{N-2}{2}},\quad v(x)=C_2\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^{\frac{N-2}{2}} \tag{3.17}$$

for some C_1, C_2, $\tau>0$ and $\overline{x}\in\mathbb{R}^N$.

Using (3.17) and the identity (see [16, (37)] for example)

$$\int_{\mathbb{R}^N}\frac{1}{|x-y|^{2s}}\Big(\frac{1}{1+|y|^2}\Big)^{N-s}dy=I(s)\Big(\frac{1}{1+|x|^2}\Big)^{s},\quad 0<s<\frac{N}{2}, \tag{3.18}$$

we have

$$w(x)=\int_{\mathbb{R}^N}\frac{u^2(y)}{|x-y|^4}dy=C_1^2I(2)\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^2$$

and

$$g(x)=\int_{\mathbb{R}^N}\frac{v^2(y)}{|x-y|^4}dy=C_2^2I(2)\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^2.$$

Then we deduce from (3.2) and (3.18) that

$$\begin{aligned}u(x)&=\alpha_1R_N\int_{\mathbb{R}^N}\frac{u(y)w(y)}{|x-y|^{N-2}}dy+\beta R_N\int_{\mathbb{R}^N}\frac{u(y)g(y)}{|x-y|^{N-2}}dy\\&=\alpha_1R_NC_1^3I(2)I\Big(\frac{N-2}{2}\Big)\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^{\frac{N-2}{2}}+\beta R_NC_1C_2^2I(2)I\Big(\frac{N-2}{2}\Big)\Big(\frac{\tau}{\tau^2+|x-\overline{x}|^2}\Big)^{\frac{N-2}{2}},\end{aligned}$$

which combined with (3.17) leads to

$$\alpha_1R_NC_1^2I(2)I\Big(\frac{N-2}{2}\Big)+\beta R_NC_2^2I(2)I\Big(\frac{N-2}{2}\Big)=1.$$

Similarly, we also have

$$\alpha_2R_NC_2^2I(2)I\Big(\frac{N-2}{2}\Big)+\beta R_NC_1^2I(2)I\Big(\frac{N-2}{2}\Big)=1.$$

A simple calculation shows that

$$C_1=\frac{\sqrt{k_0}}{\sqrt{R_NI(2)I\big(\frac{N-2}{2}\big)}},\quad C_2=\frac{\sqrt{l_0}}{\sqrt{R_NI(2)I\big(\frac{N-2}{2}\big)}}.$$

The proof is completed. □

As a direct consequence of Theorem 1.2 and Lemma 2.3, we have the following corollary.

Corollary 3.5. *Let $\beta>\max\{\alpha_1,\alpha_2\}$. If $(u,v)\in H$ is a nontrivial classical positive solution of (2.3), then we have*

$$(u,v)=(\sqrt{k_0}U_{\delta,z},\sqrt{l_0}U_{\delta,z})$$

for some $\delta>0$ and $z\in\mathbb{R}^N$. Moreover, each nontrivial classical positive solution $(u,v)\in H$ of (2.3) is a ground state solution.

4. A nonlocal global compactness lemma

In this section, we will prove a nonlocal global compactness result for (2.1), i.e., we will give a complete description for the Palais-Smale sequences of the functional J. We start with a Brézis-Lieb type lemma about the nonlocal term which is inspired by the Brézis-Lieb convergence lemma (see [8]). The proof is analogous to that of [23, Lemma 2.2] and [39, Lemma 2.4], but we exhibit it here for completeness.

Lemma 4.1. *Let $N \geq 5$ and assume $\{(u_n, v_n)\}$ to be a bounded sequence in $L^{\frac{2N}{N-2}}(\mathbb{R}^N) \times L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ such that $(u_n, v_n) \to (u, v)$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$. Then we have*

$$\int_{\mathbb{R}^N} (|x|^{-4} * |u_n^+|^2)|u_n^+|^2 dx - \int_{\mathbb{R}^N} (|x|^{-4} * |(u_n - u)^+|^2)|(u_n - u)^+|^2 dx \to \int_{\mathbb{R}^N} (|x|^{-4} * |u^+|^2)|u^+|^2 dx$$

and

$$\int_{\mathbb{R}^N} (|x|^{-4} * |u_n^+|^2)|v_n^+|^2 dx - \int_{\mathbb{R}^N} (|x|^{-4} * |(u_n - u)^+|^2)|(v_n - v)^+|^2 dx \to \int_{\mathbb{R}^N} (|x|^{-4} * |u^+|^2)|v^+|^2 dx$$

as $n \to \infty$.

Proof. Similar to the proof of the Brézis-Lieb Lemma in [8], we have

$$|u_n^+|^2 - |(u_n - u)^+|^2 \to |u^+|^2 \quad \text{in } L^{\frac{N}{N-2}}(\mathbb{R}^N) \tag{4.1}$$

and

$$|v_n^+|^2 - |(v_n - v)^+|^2 \to |v^+|^2 \quad \text{in } L^{\frac{N}{N-2}}(\mathbb{R}^N). \tag{4.2}$$

Using Proposition 1.1 yields

$$|x|^{-4} * (|u_n^+|^2 - |(u_n - u)^+|^2) \to |x|^{-4} * |u^+|^2 \quad \text{in } L^{\frac{N}{2}}(\mathbb{R}^N) \tag{4.3}$$

and

$$|x|^{-4} * (|v_n^+|^2 - |(v_n - v)^+|^2) \to |x|^{-4} * |v^+|^2 \quad \text{in } L^{\frac{N}{2}}(\mathbb{R}^N). \tag{4.4}$$

Note that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|x|^{-4} * |u_n^+|^2\big)|u_n^+|^2 dx - \int_{\mathbb{R}^N} \big(|x|^{-4} * |(u_n - u)^+|^2\big)|(u_n - u)^+|^2 dx \\ =& \int_{\mathbb{R}^N} \big(|x|^{-4} * (|u_n^+|^2 - |(u_n - u)^+|^2)\big)(|u_n^+|^2 - |(u_n - u)^+|^2) dx \\ &+ 2\int_{\mathbb{R}^N} \big(|x|^{-4} * (|u_n^+|^2 - |(u_n - u)^+|^2)\big)|(u_n - u)^+|^2 dx \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|x|^{-4} * |u_n^+|^2\big)|v_n^+|^2 dx - \int_{\mathbb{R}^N} \big(|x|^{-4} * |(u_n - u)^+|^2\big)|(v_n - v)^+|^2 dx \\ =& \int_{\mathbb{R}^N} \big(|x|^{-4} * (|u_n^+|^2 - |(u_n - u)^+|^2)\big)(|v_n^+|^2 - |(v_n - v)^+|^2) dx \\ &+ \int_{\mathbb{R}^N} \big(|x|^{-4} * (|u_n^+|^2 - |(u_n - u)^+|^2)\big)|(v_n - v)^+|^2 dx \\ &+ \int_{\mathbb{R}^N} \big(|x|^{-4} * (|v_n^+|^2 - |(v_n - v)^+|^2)\big)|(u_n - u)^+|^2 dx. \end{aligned} \tag{4.6}$$

Combining (4.1)−(4.6) with the fact that

$$|(u_n - u)^+|^2 \rightharpoonup 0 \quad \text{and} \quad |(v_n - v)^+|^2 \rightharpoonup 0 \quad \text{in } L^{\frac{N}{N-2}}(\mathbb{R}^N)$$

leads to the desired result. □

The global compactness lemma plays an important role in the study of critical problems, see [47,53] for a single elliptic equation with a local interaction, [34,43] for local system and [22] for a nonlocal Choquard equation. For $r \in \mathbb{R}^+$ and $z \in \mathbb{R}^N$, we denote the rescaling

$$(u,v)_{r,z} = r^{\frac{N-2}{2}}(u(rx+z), v(rx+z)).$$

Inspired by the above results, we can establish the global compactness lemma for nonlocal type systems.

Lemma 4.2. *Suppose that* $V_1, V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^{\infty}_{\text{loc}}(\mathbb{R}^N)$ *and* $\{(u_n, v_n)\} \subset H$ *is a* $(PS)_d$ *sequence for the functional* J*. Then there exist a number* $k \in \mathbb{N}$*, a solution* (u^0, v^0) *of* (2.1)*, nonzero solutions* $(u^1, v^1), \cdots, (u^k, v^k)$ *of* (2.3)*, sequences of points* $\{z_n^1\}, \cdots, \{z_n^k\}$ *in* $\mathbb{R}^N$ *and radii* $\{r_n^1\}, \cdots, \{r_n^k\}$ *such that, up to a subsequence,*

$$(u_n^0, v_n^0) := (u_n, v_n) \rightharpoonup (u^0, v^0) \quad \text{in } H$$

and

$$(u_n^j, v_n^j) := (u_n^{j-1} - u^{j-1}, v_n^{j-1} - v^{j-1})_{r_n^j, z_n^j} \rightharpoonup (u^j, v^j) \quad \text{in } H, \quad j = 1, ..., k.$$

Moreover, we have

$$\lim_{n\to\infty} \|(u_n, v_n)\|^2 = \sum_{j=0}^{k} \|(u^j, v^j)\|^2$$

and

$$\lim_{n\to\infty} J(u_n, v_n) = J(u^0, v^0) + \sum_{j=1}^{k} J_\infty(u^j, v^j).$$

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_d$ sequence for J, then it is bounded in H. We assume up to a subsequence that $(u_n, v_n) \rightharpoonup (u^0, v^0)$ in H, $(u_n, v_n) \to (u^0, v^0)$ almost everywhere in $\mathbb{R}^N$ and (u^0, v^0) is a weak solution of (2.1). Setting $(\overline{u}_n, \overline{v}_n) := (u_n - u^0, v_n - v^0)$, we have $(\overline{u}_n, \overline{v}_n) \rightharpoonup (0,0)$ in H. Using this together with the Brézis-Lieb Lemma [8] and Lemma 4.1, we deduce that

$$\|(\overline{u}_n, \overline{v}_n)\|^2 = \|(u_n, v_n)\|^2 - \|(u^0, v^0)\|^2 + o_n(1), \tag{4.7}$$

$$J(\overline{u}_n, \overline{v}_n) = J(u_n, v_n) - J(u^0, v^0) + o_n(1)$$

and

$$J'(\overline{u}_n, \overline{v}_n) = J'(u_n, v_n) - J'(u^0, v^0) + o_n(1) = o_n(1).$$

Since $V_1 \in L^{\frac{N}{2}}(\mathbb{R}^N)$, for any $\varepsilon > 0$ there exists a number $r = r(\varepsilon) > 0$ such that

$$\Big(\int_{\mathbb{R}^N \setminus B_r(0)} |V_1|^{\frac{N}{2}} dx\Big)^{\frac{2}{N}} < \varepsilon.$$

For such an r, we can find $n_0 \in \mathbb{N}$ such that

$$\int_{B_r(0)} \overline{u}_n^2 dx < \varepsilon, \quad \text{for } n \geq n_0.$$

Then, using $V_1 \in L^{\infty}_{\text{loc}}(\mathbb{R}^N)$ and the Hölder inequality, we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N} V_1(x)\overline{u}_n^2 dx\Big| &= \Big|\int_{B_r(0)} V_1(x)\overline{u}_n^2 dx + \int_{\mathbb{R}^N \setminus B_r(0)} V_1(x)\overline{u}_n^2 dx\Big| \\
&\leq |V_1|_{\infty, B_r(0)} \int_{B_r(0)} \overline{u}_n^2 dx + \Big(\int_{\mathbb{R}^N \setminus B_r(0)} |V_1|^{\frac{N}{2}} dx\Big)^{\frac{2}{N}} \Big(\int_{\mathbb{R}^N} |\overline{u}_n|^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} \\
&< C\varepsilon
\end{aligned}$$

for $n \geq n_0$, which means that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1(x)\overline{u}_n^2 dx = 0.$$

Similarly, we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_2(x)\overline{v}_n^2 dx = 0.$$

Then

$$J_\infty(\overline{u}_n,\overline{v}_n)=J(\overline{u}_n,\overline{v}_n)+o_n(1)=J(u_n,v_n)-J(u^0,v^0)+o_n(1) \tag{4.8}$$

and

$$J'_\infty(\overline{u}_n,\overline{v}_n)=J'(\overline{u}_n,\overline{v}_n)+o_n(1)=o_n(1). \tag{4.9}$$

If $(\overline{u}_n,\overline{v}_n)\to(0,0)$ in H then we are done: k is just 0 and $(u_n^0,v_n^0):=(u_n,v_n)$. Now we consider the case where $(\overline{u}_n,\overline{v}_n)\not\to(0,0)$ in H. Assume up to a subsequence that $\lim_{n\to\infty}\|(\overline{u}_n,\overline{v}_n)\|^2=b>0$ and, by (4.9), there also holds

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\alpha_1|\overline{u}_n^+(x)|^2|\overline{u}_n^+(y)|^2+\alpha_2|\overline{v}_n^+(x)|^2|\overline{v}_n^+(y)|^2+2\beta|\overline{u}_n^+(x)|^2|\overline{v}_n^+(y)|^2}{|x-y|^4}dxdy=b.$$

For $t_n>0$ defined by

$$t_n^2=\frac{\|(\overline{u}_n,\overline{v}_n)\|^2}{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\alpha_1|\overline{u}_n^+(x)|^2|\overline{u}_n^+(y)|^2+\alpha_2|\overline{v}_n^+(x)|^2|\overline{v}_n^+(y)|^2+2\beta|\overline{u}_n^+(x)|^2|\overline{v}_n^+(y)|^2}{|x-y|^4}dxdy},$$

we have $(t_n\overline{u}_n,t_n\overline{v}_n)\in\mathcal{N}_\infty$ and $t_n^2=1+o_n(1)$ as $n\to\infty$. Then

$$c_\infty\le J_\infty(t_n\overline{u}_n,t_n\overline{v}_n)=\frac14 t_n^2\|(\overline{u}_n,\overline{v}_n)\|^2=\frac14 b+o_n(1),$$

which implies that $b\ge 4c_\infty$.

Claim: There exist sequences $\{r_n\}\subset\mathbb{R}^+$ and $\{z_n\}\subset\mathbb{R}^N$ such that

$$(\widetilde{u}_n,\widetilde{v}_n)=(\overline{u}_n,\overline{v}_n)_{r_n,z_n}\rightharpoonup(u,v)\ \text{ in } H,$$

where (u,v) is a nonzero solution of (2.3).

We see from (4.9) that

$$J_\infty(\overline{u}_n,\overline{v}_n)=\frac14\|(\overline{u}_n,\overline{v}_n)\|^2+o_n(1).$$

Define the Levy concentration function of $(\overline{u}_n,\overline{v}_n)$ by

$$Q_n(r):=\sup_{z\in\mathbb{R}^N}\int_{B_r(z)}(|\nabla\overline{u}_n|^2+|\nabla\overline{v}_n|^2)dx.$$

Let L be the least number of balls with radius 1 covering a ball of radius 2. We see from

$$\lim_{n\to\infty}\|(\overline{u}_n,\overline{v}_n)\|^2=b\ge 4c_\infty$$

that, for large n, there exist $r_n\in\mathbb{R}^+$ and $z_n\in\mathbb{R}^N$ such that

$$\sup_{z\in\mathbb{R}^N}\int_{B_r(z)}(|\nabla\overline{u}_n|^2+|\nabla\overline{v}_n|^2)dx=\int_{B_{r_n}(z_n)}(|\nabla\overline{u}_n|^2+|\nabla\overline{v}_n|^2)dx=\frac{2c_\infty}{L}.$$

Setting $(\widetilde{u}_n,\widetilde{v}_n):=(\overline{u}_n,\overline{v}_n)_{r_n,z_n}$, we have

$$\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx=\int_{B_1(0)}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx=\frac{2c_\infty}{L} \tag{4.10}$$

and $\{(\widetilde{u}_n,\widetilde{v}_n)\}$ is bounded in H. Assume by extracting a subsequence that $(\widetilde{u}_n,\widetilde{v}_n)\rightharpoonup(u,v)$ in H and $(\widetilde{u}_n,\widetilde{v}_n)\to(u,v)$ almost everywhere in $\mathbb{R}^N$. The scale invariance under translation and dilation implies that

$$\|(\overline{u}_n,\overline{v}_n)\|=\|(\widetilde{u}_n,\widetilde{v}_n)\|,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{u}_n^+(x)|^2|\overline{u}_n^+(y)|^2}{|x-y|^4}dxdy=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2|\widetilde{u}_n^+(y)|^2}{|x-y|^4}dxdy,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{v}_n^+(x)|^2|\overline{v}_n^+(y)|^2}{|x-y|^4}dxdy=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{v}_n^+(x)|^2|\widetilde{v}_n^+(y)|^2}{|x-y|^4}dxdy,$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\overline{u}_n^+(x)|^2|\overline{v}_n^+(y)|^2}{|x-y|^4}dxdy=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2|\widetilde{v}_n^+(y)|^2}{|x-y|^4}dxdy.$$

Then we have

$$J_\infty(\widetilde{u}_n,\widetilde{v}_n)=J_\infty(\overline{u}_n,\overline{v}_n)+o_n(1)$$

and

$$\|J'_\infty(\widetilde{u}_n,\widetilde{v}_n)\|=\|J'_\infty(\overline{u}_n,\overline{v}_n)\|=o_n(1).$$

Therefore, (u,v) is a solution of (2.3).

Next we show that $(u,v)\neq(0,0)$. In fact, using the arguments in [47], we can find $\rho\in[1,2]$ such that the solution $\widehat{\varphi}_n$ of the boundary value problem

$$\begin{cases} -\Delta\varphi=0 \ \text{ in } B_3(0)\backslash B_\rho(0),\\ \varphi|_{\partial B_\rho(0)}=\widetilde{u}_n-u,\ \varphi|_{\partial B_3(0)}=0\end{cases}$$

satisfies $\widehat{\varphi}_n\to 0$ in $H^1(B_3(0)\backslash B_\rho(0))$ and the solution $\widehat{\psi}_n$ of the problem in which the boundary condition $\varphi|_{\partial B_\rho(0)}=\widetilde{u}_n-u$ is replaced with $\varphi|_{\partial B_\rho(0)}=\widetilde{v}_n-v$ also satisfies $\widehat{\psi}_n\to 0$ in $H^1(B_3(0)\backslash B_\rho(0))$. Define

$$\widetilde{\varphi}_n(x)=\begin{cases}\widetilde{u}_n(x)-u(x), & x\in B_\rho(0),\\ \widehat{\varphi}_n, & x\in B_3(0)\backslash B_\rho(0),\\ 0, & x\in\mathbb{R}^N\backslash B_3(0).\end{cases}$$

Replace $\widetilde{u}_n$ and $\widehat{\varphi}_n$ with $\widetilde{v}_n$ and $\widehat{\psi}_n$ respectively in the definition of $\widetilde{\varphi}_n$, and denote this resulted new function by $\widetilde{\psi}_n$. Setting

$$\varphi_n=r_n^{-\frac{N-2}{2}}\widetilde{\varphi}_n\Big(\frac{\cdot-z_n}{r_n}\Big)\ \text{ and }\ \psi_n=r_n^{-\frac{N-2}{2}}\widetilde{\psi}_n\Big(\frac{\cdot-z_n}{r_n}\Big),$$

we have

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla\varphi_n|^2+|\nabla\psi_n|^2)dx&=\int_{\mathbb{R}^N}(|\nabla\widetilde{\varphi}_n|^2+|\nabla\widetilde{\psi}_n|^2)dx\\ &=\int_{B_\rho(0)}(|\nabla(\widetilde{u}_n-u)|^2+|\nabla(\widetilde{v}_n-v)|^2)dx+o_n(1)\\ &=\int_{B_\rho(0)}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx-\int_{B_\rho(0)}(|\nabla u|^2+|\nabla v|^2)dx+o_n(1)\\ &\leq\int_{B_\rho(0)}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx+o_n(1).\end{aligned}\tag{4.11}$$

Since $\widehat{\varphi}_n\to 0$ and $\widehat{\psi}_n\to 0$ in $H^1(B_3(0)\backslash B_\rho(0))$, the scale invariance implies that

$$\begin{aligned}o_n(1)&=\langle J'_\infty(\overline{u}_n,\overline{v}_n),(\varphi_n,\psi_n)\rangle=\langle J'_\infty(\widetilde{u}_n,\widetilde{v}_n),(\widetilde{\varphi}_n,\widetilde{\psi}_n)\rangle\\ &=\int_{B_\rho(0)}(\nabla\widetilde{u}_n\nabla(\widetilde{u}_n-u)+\nabla\widetilde{v}_n\nabla(\widetilde{v}_n-v))dx\\ &\quad-\alpha_1\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2\widetilde{u}_n^+(y)(\widetilde{u}_n-u)(y)}{|x-y|^4}dxdy-\beta\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{v}_n^+(x)|^2\widetilde{u}_n^+(y)(\widetilde{u}_n-u)(y)}{|x-y|^4}dxdy\\ &\quad-\alpha_2\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{v}_n^+(x)|^2\widetilde{v}_n^+(y)(\widetilde{v}_n-v)(y)}{|x-y|^4}dxdy-\beta\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2\widetilde{v}_n^+(y)(\widetilde{v}_n-v)(y)}{|x-y|^4}dxdy\\ &\quad+o_n(1).\end{aligned}\tag{4.12}$$

Since $\{\widetilde{u}_n^2\}$ is bounded in $L^{\frac{N}{N-2}}(\mathbb{R}^N)$ and $\widetilde{u}_n\to u$ almost everywhere in $\mathbb{R}^N$, we have $|\widetilde{u}_n^+|^2\rightharpoonup|u^+|^2$ in $L^{\frac{N}{N-2}}(\mathbb{R}^N)$. By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{N}{N-2}}(\mathbb{R}^N)$ to $L^{\frac{N}{2}}(\mathbb{R}^N)$ and then

$$\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2}{|x-y|^4}dx\rightharpoonup\int_{\mathbb{R}^N}\frac{|u^+(x)|^2}{|x-y|^4}dx\ \text{ in } L^{\frac{N}{2}}(\mathbb{R}^N).$$

Combining this with $\widetilde{u}_n^+ \rightharpoonup u^+$ in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ leads to

$$\widetilde{u}_n^+(y)\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2}{|x-y|^4}dx \rightharpoonup u^+(y)\int_{\mathbb{R}^N}\frac{|u^+(x)|^2}{|x-y|^4}dx \quad \text{in } L^{\frac{2N}{N+2}}(\mathbb{R}^N),$$

which implies that

$$\begin{aligned}\lim_{n\to\infty}\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2\widetilde{u}_n^+(y)u(y)}{|x-y|^4}dxdy &= \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|u^+(x)|^2u^+(y)u(y)}{|x-y|^4}dxdy\\ &= \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|u^+(x)|^2|u^+(y)|^2}{|x-y|^4}dxdy.\end{aligned}$$

Then

$$\begin{aligned}&\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2\widetilde{u}_n^+(y)(\widetilde{u}_n-u)(y)}{|x-y|^4}dxdy\\ &= \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2|\widetilde{u}_n^+(y)|^2}{|x-y|^4}dxdy - \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|u^+(x)|^2|u^+(y)|^2}{|x-y|^4}dxdy + o_n(1)\\ &= \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{u}_n-u)^+(x)|^2|(\widetilde{u}_n-u)^+(y)|^2}{|x-y|^4}dxdy + o_n(1).\end{aligned} \tag{4.13}$$

Similarly, we also have

$$\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{v}_n^+(x)|^2\widetilde{u}_n^+(y)(\widetilde{u}_n-u)(y)}{|x-y|^4}dxdy = \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{v}_n-v)^+(x)|^2|(\widetilde{u}_n-u)^+(y)|^2}{|x-y|^4}dxdy + o_n(1), \tag{4.14}$$

$$\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{v}_n^+(x)|^2\widetilde{v}_n^+(y)(\widetilde{v}_n-v)(y)}{|x-y|^4}dxdy = \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{v}_n-v)^+(x)|^2|(\widetilde{v}_n-v)^+(y)|^2}{|x-y|^4}dxdy + o_n(1) \tag{4.15}$$

and

$$\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_n^+(x)|^2\widetilde{v}_n^+(y)(\widetilde{v}_n-v)(y)}{|x-y|^4}dxdy = \int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{u}_n-u)^+(x)|^2|(\widetilde{v}_n-v)^+(y)|^2}{|x-y|^4}dxdy + o_n(1). \tag{4.16}$$

Substituting (4.13)–(4.16) into (4.12) and using $(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (u, v)$ in H, we obtain

$$\begin{aligned}o_n(1) = &\int_{B_\rho(0)}(|\nabla(\widetilde{u}_n-u)|^2 + |\nabla(\widetilde{v}_n-v)|^2)dx\\ &- \alpha_1\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{u}_n-u)^+(x)|^2|(\widetilde{u}_n-u)^+(y)|^2}{|x-y|^4}dxdy - \beta\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{v}_n-v)^+(x)|^2|(\widetilde{u}_n-u)^+(y)|^2}{|x-y|^4}dxdy\\ &- \alpha_2\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{v}_n-v)^+(x)|^2|(\widetilde{v}_n-v)^+(y)|^2}{|x-y|^4}dxdy - \beta\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{|(\widetilde{u}_n-u)^+(x)|^2|(\widetilde{v}_n-v)^+(y)|^2}{|x-y|^4}dxdy.\end{aligned}$$

Using $\widehat{\varphi}_n \to 0$ and $\widehat{\psi}_n \to 0$ in $H^1(B_3(0)\backslash B_\rho(0))$ and the scale invariance again

$$\begin{aligned}o_n(1) &= \int_{\mathbb{R}^N}(|\nabla\widetilde{\varphi}_n|^2 + |\nabla\widetilde{\psi}_n|^2)dx - \alpha_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{\varphi}_n^+(x)|^2|\widetilde{\varphi}_n^+(y)|^2}{|x-y|^4}dxdy\\ &\quad - \alpha_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{\psi}_n^+(x)|^2|\widetilde{\psi}_n^+(y)|^2}{|x-y|^4}dxdy - 2\beta\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{\varphi}_n^+(x)|^2|\widetilde{\psi}_n^+(y)|^2}{|x-y|^4}dxdy\\ &= \int_{\mathbb{R}^N}(|\nabla\varphi_n|^2 + |\nabla\psi_n|^2)dx - \alpha_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_n^+(x)|^2|\varphi_n^+(y)|^2}{|x-y|^4}dxdy\\ &\quad - \alpha_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi_n^+(x)|^2|\psi_n^+(y)|^2}{|x-y|^4}dxdy - 2\beta\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_n^+(x)|^2|\psi_n^+(y)|^2}{|x-y|^4}dxdy.\end{aligned} \tag{4.17}$$

If $(\varphi_n^+, \psi_n^+) \neq (0,0)$, we define $t_n > 0$ by

$$t_n^2 = \frac{\|(\varphi_n, \psi_n)\|^2}{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\alpha_1|\varphi_n^+(x)|^2|\varphi_n^+(y)|^2 + \alpha_2|\psi_n^+(x)|^2|\psi_n^+(y)|^2 + 2\beta|\varphi_n^+(x)|^2|\psi_n^+(y)|^2}{|x-y|^4}dxdy}.$$

Then $(t_n\varphi_n, t_n\psi_n) \in \mathcal{N}_\infty$ and

$$c_\infty \leq J_\infty(t_n\varphi_n, t\psi_n) = \frac{1}{4}\frac{\|(\varphi_n, \psi_n)\|^4}{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\alpha_1|\varphi_n^+(x)|^2|\varphi_n^+(y)|^2 + \alpha_2|\psi_n^+(x)|^2|\psi_n^+(y)|^2 + 2\beta|\varphi_n^+(x)|^2|\psi_n^+(y)|^2}{|x-y|^4}dxdy},$$

which indicates that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\alpha_1|\varphi_n^+(x)|^2|\varphi_n^+(y)|^2 + \alpha_2|\psi_n^+(x)|^2|\psi_n^+(y)|^2 + 2\beta|\varphi_n^+(x)|^2|\psi_n^+(y)|^2}{|x-y|^4}dxdy \leq \frac{1}{4c_\infty}\|(\varphi_n, \psi_n)\|^4.$$

Note that the above inequality also holds if $(\varphi_n^+, \psi_n^+) = (0,0)$. Combining this with (4.17) and (4.11) yields

$$\begin{aligned} o_n(1) &\geq \Big(1 - \frac{1}{4c_\infty}\int_{\mathbb{R}^N}(|\nabla\varphi_n|^2 + |\nabla\psi_n|^2)dx\Big)\int_{\mathbb{R}^N}(|\nabla\varphi_n|^2 + |\nabla\psi_n|^2)dx \\ &\geq \Big(1 - \frac{1}{4c_\infty}\int_{B_\rho(0)}(|\nabla\widetilde{u}_n|^2 + |\nabla\widetilde{v}_n|^2)dx\Big)\int_{\mathbb{R}^N}(|\nabla\varphi_n|^2 + |\nabla\psi_n|^2)dx + o_n(1). \end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty}\int_{B_\rho(0)}(|\nabla(\widetilde{u}_n - u)|^2 + |\nabla(\widetilde{v}_n - v)|^2)dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla\varphi_n|^2 + |\nabla\psi_n|^2)dx = 0,$$

since the definition of L implies

$$\int_{B_\rho(0)}(|\nabla\widetilde{u}_n|^2 + |\nabla\widetilde{v}_n|^2)dx \leq L\int_{B_1(0)}(|\nabla\widetilde{u}_n|^2 + |\nabla\widetilde{v}_n|^2)dx = 2c_\infty.$$

Then we see from (4.10) that $(u,v) \neq (0,0)$ and conclude the proof of the claim.

Set $(u_n^1, v_n^1) := (\overline{u}_n, \overline{v}_n)$, $(u^1, v^1) := (u,v)$, $r_n^1 := r_n$ and $z_n^1 := z_n$. Doing iteration, we obtain sequences $\{r_n^j\}$ and $\{z_n^j\}$ such that $(u_n^j, v_n^j) := (u_n^{j-1} - u^{j-1}, v_n^{j-1} - v^{j-1})_{r_n^j, z_n^j} \rightharpoonup (u^j, v^j)$ in H, where (u^j, v^j) are nonzero solutions of (2.3). Moreover, we see from (4.7) and (4.8) that, by induction,

$$\|(u_n^j, v_n^j)\|^2 = \|(u_n, v_n)\|^2 - \sum_{i=0}^{j-1}\|(u^i, v^i)\|^2 + o_n(1)$$

and

$$J_\infty(u_n^j, v_n^j) = J(u_n, v_n) - J(u^0, v^0) - \sum_{i=1}^{j-1}J_\infty(u^i, v^i) + o_n(1).$$

The iterating process must terminate in finite steps, because, for any nonzero solution (u,v) of (2.3), there holds $J_\infty(u,v) \geq c_\infty > 0$. Moreover, the last Palais-Smale sequence for J_∞ must converge to $(0,0)$ strongly in H. This finishes the proof. □

Corollary 4.3. *Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a $(PS)_d$ sequence for the constrained functional $J|_{\mathcal{N}}$ at the level $d \in (c_\infty, \min\{S_{H,L}^2/4\alpha_1, S_{H,L}^2/4\alpha_2, 2c_\infty\})$, then $\{(u_n, v_n)\}$ is relatively compact in H.*

Proof. It is easy to see that $\{(u_n, v_n)\} \subset \mathcal{N}$ is a $(PS)_d$ sequence for the functional J. According to Lemma 4.2, there exist a number $k \in \mathbb{N}$, a solution (u^0, v^0) of (2.1) and nonzero solutions $(u^1, v^1), \cdots, (u^k, v^k)$ of (2.3) such that, up to a subsequence,

$$\lim_{n\to\infty}\|(u_n, v_n)\|^2 = \sum_{j=0}^{k}\|(u^j, v^j)\|^2$$

and

$$\lim_{n\to\infty} J(u_n, v_n) = J(u^0, v^0) + \sum_{j=1}^{k} J_\infty(u^j, v^j).$$

We first claim that $(u^0, v^0) \neq (0,0)$. If not, we see from $d < 2c_\infty$ that $k = 1$. By Corollary 3.5 and the uniqueness of positive solutions for the Choquard equation $-\Delta u = \alpha_i(|x|^{-4} * u^2)u$ in $\mathbb{R}^N$, (u^1, v^1) must be, up to translation and dilation, one of the three solutions $(\sqrt{k_0}U_{1,0}, \sqrt{l_0}U_{1,0})$, $(\frac{1}{\sqrt{\alpha_1}}U_{1,0}, 0)$, and $(0, \frac{1}{\sqrt{\alpha_2}}U_{1,0})$. Then either $d = c_\infty$ or $d = S_{H,L}^2/4\alpha_1$ or $d = S_{H,L}^2/4\alpha_2$, which contradicts the assumption. since $(u^0, v^0) \neq (0,0)$, using $d < 2c_\infty$ again and Lemma 2.4 leads to $k = 0$. Therefore, we conclude that $\{(u_n, v_n)\}$ is relatively compact in H. □

5. Existence of a positive solution

For $(u, v) \in H$, set

$$\|(u,v)\|_{NL} := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\alpha_1|u^+(x)|^2|u^+(y)|^2 + \alpha_2|v^+(x)|^2|v^+(y)|^2 + 2\beta|u^+(x)|^2|v^+(y)|^2}{|x-y|^4}dxdy\Big)^{\frac{1}{4}}.$$

Following the idea in [10], we introduce a barycenter map $\beta : H \to \mathbb{R}^N$ defined as

$$\beta(u,v) = \frac{1}{\|(u,v)\|_{NL}^4}\int_{\mathbb{R}^N}\frac{x}{1+|x|}\int_{\mathbb{R}^N}\frac{\alpha_1|u^+(x)|^2|u^+(y)|^2 + \alpha_2|v^+(x)|^2|v^+(y)|^2 + 2\beta|u^+(x)|^2|v^+(y)|^2}{|x-y|^4}dy\,dx.$$

We also define a functional

$$\gamma(u,v) = \frac{1}{\|(u,v)\|_{NL}^4}\int_{\mathbb{R}^N}\Big|\frac{x}{1+|x|} - \beta(u,v)\Big|\int_{\mathbb{R}^N}\frac{\alpha_1|u^+(x)|^2|u^+(y)|^2 + \alpha_2|v^+(x)|^2|v^+(y)|^2 + 2\beta|u^+(x)|^2|v^+(y)|^2}{|x-y|^4}dy\,dx$$

to estimate the concentration of (u, v) around its barycenter. Denote

$$\mathcal{M} := \Big\{(u,v) \in \mathcal{N} : \beta(u,v) = 0, \gamma(u,v) = \frac{1}{2}\Big\}$$

and consider the infimum

$$c^\star = \inf_{(u,v)\in\mathcal{M}} J(u,v).$$

Lemma 5.1. *If $\beta > \max\{\alpha_1, \alpha_2\}$ and V_1, $V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N)$ are nonnegative functions such that*

$$|V_1|_{\frac{N}{2}} + |V_2|_{\frac{N}{2}} > 0,$$

then $c^\star > c_\infty$.

Proof. We first see from $\mathcal{M} \subset \mathcal{N}$ and Lemma 2.4 that $c^\star \geq c = c_\infty$. Assume to the contrary that $c^\star = c_\infty$. Then, by Ekeland's variational principle, there exists a sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ such that, as $n \to \infty$,

$$\beta(u_n, v_n) \to 0, \quad \gamma(u_n, v_n) \to \frac{1}{2} \tag{5.1}$$

and

$$J(u_n, v_n) \to c_\infty, \quad (J|_{\mathcal{N}})'(u_n, v_n) \to 0.$$

By Lemmas 2.4 and 4.2, there exist $\delta_n > 0$, $z_n \in \mathbb{R}^N$ and $(\varphi_n, \psi_n) \in H$ such that

$$(u_n, v_n) = (\sqrt{k_0}U_{\delta_n,z_n}, \sqrt{l_0}U_{\delta_n,z_n}) + (\varphi_n, \psi_n),$$

where $(\varphi_n, \psi_n) \to (0,0)$ in H. Then we have

(5.2)

$$
\begin{aligned}
\frac{1}{2} &= \lim_{n\to\infty} \gamma(u_n, v_n) \\
&= \lim_{n\to\infty} \frac{1}{\|(u_n,v_n)\|_{NL}^4} \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} \int_{\mathbb{R}^N} \frac{\alpha_1 |u_n^+(x)|^2|u_n^+(y)|^2 + \alpha_2 |v_n^+(x)|^2|v_n^+(y)|^2 + 2\beta |u_n^+(x)|^2|v_n^+(y)|^2}{|x-y|^4} dy\, dx \\
&= \lim_{n\to\infty} \frac{1}{S_{H,L}^2} \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} \int_{\mathbb{R}^N} \frac{U^2_{\delta_n,z_n}(x) U^2_{\delta_n,z_n}(y)}{|x-y|^4} dy\, dx
\end{aligned}
$$

and

$$
0 = \lim_{n\to\infty} \beta(u_n, v_n) = \lim_{n\to\infty} \frac{1}{S_{H,L}^2} \int_{\mathbb{R}^N} \frac{x}{1+|x|} \int_{\mathbb{R}^N} \frac{U^2_{\delta_n,z_n}(x) U^2_{\delta_n,z_n}(y)}{|x-y|^4} dy\, dx. \tag{5.3}
$$

We divide into the following four cases. In each case, we will come to a contradiction and finish the proof.

Case 1. Up to a subsequence, there holds $\lim_{n\to\infty} \delta_n = +\infty$.

In this case, we have

$$
\lim_{n\to\infty} \int_{B_r(0)} \int_{\mathbb{R}^N} \frac{U^2_{\delta_n,z_n}(x) U^2_{\delta_n,z_n}(y)}{|x-y|^4} dy\, dx = 0
$$

for any $r > 0$. Combining this with (5.2) leads to

$$
\frac{1}{2} = \lim_{n\to\infty} \frac{1}{S_{H,L}^2} \int_{\mathbb{R}^N \setminus B_r(0)} \frac{|x|}{1+|x|} \int_{\mathbb{R}^N} \frac{U^2_{\delta_n,z_n}(x) U^2_{\delta_n,z_n}(y)}{|x-y|^4} dy\, dx \geq \frac{r}{1+r},
$$

which is impossible when $r > 1$.

Case 2. Up to a subsequence, there hold $\lim_{n\to\infty} \delta_n = \delta > 0$ and $\lim_{n\to\infty} z_n = z$.

In this case, one can prove that

$$
(\sqrt{k_0} U_{\delta_n,z_n}, \sqrt{l_0} U_{\delta_n,z_n}) \to (\sqrt{k_0} U_{\delta,z}, \sqrt{l_0} U_{\delta,z}) \text{ in } H
$$

and then $(u_n, v_n) \to (\sqrt{k_0} U_{\delta,z}, \sqrt{l_0} U_{\delta,z})$ in H. We come to a contradiction as shown by

$$
\begin{aligned}
c_\infty &= \lim_{n\to\infty} J(u_n, v_n) \\
&= \frac{1}{4} \lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + V_1(x) u_n^2 + V_2(x) v_n^2) dx \\
&= \frac{k_0 + l_0}{4} \int_{\mathbb{R}^N} |\nabla U_{\delta,z}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^4} (k_0 V_1(x) U^2_{\delta,z} + l_0 V_2(x) U^2_{\delta,z}) \\
&> \frac{k_0 + l_0}{4} S_{H,L}^2 \\
&= c_\infty.
\end{aligned}
$$

Case 3. Up to a subsequence, there holds $\lim_{n\to\infty} \delta_n = \delta > 0$ and $\lim_{n\to\infty} |z_n| = +\infty$.

In this case, we have

$$
\begin{aligned}
&\frac{1}{S_{H,L}^2} \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} \int_{\mathbb{R}^N} \frac{U^2_{\delta_n,z_n}(x) U^2_{\delta_n,z_n}(y)}{|x-y|^4} dy\, dx \\
&= 1 - \frac{1}{S_{H,L}^2} \int_{\mathbb{R}^N} \frac{1}{1+|x|} \int_{\mathbb{R}^N} \frac{U^2_{\delta_n,z_n}(x) U^2_{\delta_n,z_n}(y)}{|x-y|^4} dy\, dx \\
&= 1 - o_n(1),
\end{aligned}
$$

which contradicts with (5.2).

Case 4. Up to a subsequence, there holds $\lim_{n\to\infty} \delta_n = 0$.

In this case, we have

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N \setminus B_r(0)} \int_{\mathbb{R}^N} \frac{U^2_{\delta_n,0}(x) U^2_{\delta_n,0}(y)}{|x-y|^4} dy\, dx = 0
$$

for any $r > 0$. Combining this with (5.1), (5.3) and the inequality $\left|\frac{z_n}{1+|z_n|} - \frac{x}{1+|x|}\right| \le r$ for $x \in B_r(z_n)$ yields

$$\begin{aligned}\frac{|z_n|}{1+|z_n|} &= \Big|\frac{z_n}{1+|z_n|} - \beta(u_n, v_n)\Big| + o_n(1)\\ &\le \frac{1}{S_{H,L}^2}\int_{\mathbb{R}^N}\Big|\frac{z_n}{1+|z_n|} - \frac{x}{1+|x|}\Big|\int_{\mathbb{R}^N}\frac{U^2_{\delta_n,z_n}(x)U^2_{\delta_n,z_n}(y)}{|x-y|^4}dy\,dx + o_n(1)\\ &= \frac{1}{S_{H,L}^2}\int_{B_r(z_n)}\Big|\frac{z_n}{1+|z_n|} - \frac{x}{1+|x|}\Big|\int_{\mathbb{R}^N}\frac{U^2_{\delta_n,z_n}(x)U^2_{\delta_n,z_n}(y)}{|x-y|^4}dy\,dx + o_n(1)\\ &\le r + o_n(1),\end{aligned}$$

which implies that $|z_n| \to 0$ as $n \to \infty$, since $r > 0$ is arbitrary. Then we have

$$\begin{aligned}&\frac{1}{S_{H,L}^2}\int_{\mathbb{R}^N}\frac{|x|}{1+|x|}\int_{\mathbb{R}^N}\frac{U^2_{\delta_n,z_n}(x)U^2_{\delta_n,z_n}(y)}{|x-y|^4}dy\,dx\\ &= \frac{1}{S_{H,L}^2}\int_{B_r(z_n)}\frac{|x|}{1+|x|}\int_{\mathbb{R}^N}\frac{U^2_{\delta_n,z_n}(x)U^2_{\delta_n,z_n}(y)}{|x-y|^4}dy\,dx + o_n(1)\\ &\le r + o_n(1)\end{aligned}$$

for any $r > 0$. This contradicts with (5.2) again when $r < \frac{1}{2}$. □

Let $\eta \in \mathcal{C}_0^\infty(B_1(0))$ be a radially decreasing function such that $\eta \equiv 1$ on $B_\rho(0)$ for some $0 < \rho < 1$ and define $u_\varepsilon(x) = \eta(x)U_{\varepsilon,0}(x)$ for $\varepsilon > 0$. By [23, Section 3], we have

$$\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2dx = S_{H,L}^2 + O(\varepsilon^{N-2}), \tag{5.4}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_\varepsilon^2(x)u_\varepsilon^2(y)}{|x-y|^4}dxdy \ge S_{H,L}^2 - O(\varepsilon^{N-2}) \tag{5.5}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_\varepsilon^2(x)u_\varepsilon^2(y)}{|x-y|^4}dxdy \le S_{H,L}^2 + O(\varepsilon^{2N-4}). \tag{5.6}$$

Set

$$t_\varepsilon = \left(\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2dx}{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_\varepsilon^2(x)u_\varepsilon^2(y)}{|x-y|^4}dxdy}\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}\|t_\varepsilon u_\varepsilon\|^2 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|t_\varepsilon u_\varepsilon(x)|^2|t_\varepsilon u_\varepsilon(y)|^2}{|x-y|^4}dxdy\\ &= \frac{\|u_\varepsilon\|^4}{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_\varepsilon^2(x)u_\varepsilon^2(y)}{|x-y|^4}dxdy}\\ &\ge \frac{\left[S_{H,L}^2 + O(\varepsilon^{N-2})\right]^2}{S_{H,L}^2 + O(\varepsilon^{2N-4})}\\ &= S_{H,L}^2 + O(\varepsilon^{N-2})\\ &> S_{H,L}^2\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. By (5.4)–(5.6), it is easy to see that

$$\lim_{\varepsilon\to 0}\|t_\varepsilon u_\varepsilon\|^2 = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|t_\varepsilon u_\varepsilon(x)|^2|t_\varepsilon u_\varepsilon(y)|^2}{|x-y|^4}dxdy = S_{H,L}^2.$$

Take $w = t_\varepsilon u_\varepsilon$ with $\varepsilon > 0$ small enough, we can see that the nonnegative radial function $w \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ satisfies the following properties: $\operatorname{supp} w \subset B_1(0)$, w is non-increasing with respect to $r = |x|$,

$$\|w\|^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{w^2(x)w^2(y)}{|x-y|^4}dxdy > S_{H,L}^2,$$

$$\frac{k_0 + l_0}{4}\|w\|^2 < c^\star \tag{5.7}$$

and

$$\begin{aligned}\frac{k_0+l_0}{4}\|w\|^2 S_{H,L}^{-2}\Big(S_{H,L} + \frac{\beta-\alpha_2}{2\beta-\alpha_1-\alpha_2}C(N,4)^{-\frac{1}{2}}|V_1|_{\frac{N}{2}} + \frac{\beta-\alpha_1}{2\beta-\alpha_1-\alpha_2}C(N,4)^{-\frac{1}{2}}|V_2|_{\frac{N}{2}}\Big)^2 \\ < \min\Big\{\frac{S_{H,L}^2}{4\alpha_1}, \frac{S_{H,L}^2}{4\alpha_2}, 2c_\infty\Big\}\end{aligned} \tag{5.8}$$

which is equivalent to the second inequality in (1.12).

The proof of the following Lemma is similar to Lemma 3.6 in [10], Lemma 4.2 in [33], we only state the main results here.

Lemma 5.2. *Denote* $w_{\delta,z} = \delta^{-\frac{N-2}{2}} w(\frac{x-z}{\delta})$ *for* $\delta > 0$ *and* $z \in \mathbb{R}^N$. *If* $a \in L^{\frac{N}{2}}(\mathbb{R}^N)$, *then*

$$\lim_{\delta\to 0^+}\int_{\mathbb{R}^N} a(x)w_{\delta,z}^2 dx = \lim_{\delta\to+\infty}\int_{\mathbb{R}^N} a(x)w_{\delta,z}^2 dx = 0$$

uniformly for $z \in \mathbb{R}^N$ *and*

$$\lim_{|z|\to+\infty}\int_{\mathbb{R}^N} a(x)w_{\delta,z}^2 dx = 0$$

uniformly for $\delta > 0$.

Lemma 5.3. *Denote the inner product in* $\mathbb{R}^N$ *by* $\langle\cdot,\cdot\rangle_{\mathbb{R}^N}$ *and let* $r > 0$ *be a fixed number. Then*
(1) $\langle \beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}), z\rangle_{\mathbb{R}^N} > 0$ *for any* $\delta > 0$ *and* $z \in \mathbb{R}^N\backslash\{0\}$;
(2) $\lim_{\delta\to0^+}\gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) = 0$ *uniformly for* $z \in \mathbb{R}^N$;
(3) $\lim_{\delta\to+\infty}\gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) = 1$ *uniformly for* $z \in B_r(0)$.

Proof. (1) Let $\delta > 0$ and $z \in \mathbb{R}^N\backslash\{0\}$. For any $x \in \mathbb{R}^N$ with $\langle x, z\rangle_{\mathbb{R}^N} > 0$, there holds $|-x-z| > |x-z|$. Then from the properties of w we see that $w_{\delta,z}(x) \geq w_{\delta,z}(-x)$ for any $x \in \mathbb{R}^N$ with $\langle x, z\rangle_{\mathbb{R}^N} > 0$ and $\operatorname{meas}\{x \in \mathbb{R}^N | \langle x, z\rangle_{\mathbb{R}^N} > 0, w_{\delta,z}(x) > w_{\delta,z}(-x)\} > 0$. Thus we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{\langle x,z\rangle_{\mathbb{R}^N}}{1+|x|}\int_{\mathbb{R}^N}\frac{w_{\delta,z}^2(x)w_{\delta,z}^2(y)}{|x-y|^4}dy\,dx \\ =& \int_{\{x\in\mathbb{R}^N|\langle x,z\rangle_{\mathbb{R}^N}>0\}}\frac{\langle x,z\rangle_{\mathbb{R}^N}}{1+|x|}\int_{\mathbb{R}^N}\frac{w_{\delta,z}^2(x)w_{\delta,z}^2(y)}{|x-y|^4}dy\,dx \\ &+ \int_{\{x\in\mathbb{R}^N|\langle x,z\rangle_{\mathbb{R}^N}<0\}}\frac{\langle x,z\rangle_{\mathbb{R}^N}}{1+|x|}\int_{\mathbb{R}^N}\frac{w_{\delta,z}^2(x)w_{\delta,z}^2(y)}{|x-y|^4}dy\,dx \\ =& \int_{\{x\in\mathbb{R}^N|\langle x,z\rangle_{\mathbb{R}^N}>0\}}\frac{\langle x,z\rangle_{\mathbb{R}^N}}{1+|x|}\int_{\mathbb{R}^N}\frac{(w_{\delta,z}^2(x)-w_{\delta,z}^2(-x))w_{\delta,z}^2(y)}{|x-y|^4}dy\,dx \\ >& \, 0,\end{aligned}$$

which indicates that

$$\langle\beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}), z\rangle_{\mathbb{R}^N} = \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\frac{\langle x,z\rangle_{\mathbb{R}^N}}{1+|x|}\int_{\mathbb{R}^N}\frac{w_{\delta,z}^2(x)w_{\delta,z}^2(y)}{|x-y|^4}dy\,dx > 0$$

for any $\delta > 0$ and $z \in \mathbb{R}^N\backslash\{0\}$.

(2) For any $\delta > 0$ and $z \in \mathbb{R}^N$, we have

$$\Big|\frac{z}{1+|z|} - \beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})\Big| \le \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\Big|\frac{z}{1+|z|} - \frac{x}{1+|x|}\Big|\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx \le \delta,$$

where we have used the fact that $|\frac{z}{1+|z|} - \frac{x}{1+|x|}| < \delta$ for any $x \in B_\delta(z)$. Then

$$\begin{aligned}
0 &\le \gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) \\
&= \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\Big|\frac{x}{1+|x|} - \beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})\Big|\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx \\
&\le \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\Big|\frac{x}{1+|x|} - \frac{z}{1+|z|}\Big|\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx \\
&\quad + \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\Big|\frac{z}{1+|z|} - \beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})\Big|\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx \\
&\le 2\delta,
\end{aligned}$$

which implies that

$$\lim_{\delta\to 0^+}\gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) = 0$$

uniformly for $z \in \mathbb{R}^N$.

(3) We first claim that

$$\lim_{\delta\to+\infty}\beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) = 0 \tag{5.9}$$

uniformly for $z \in B_r(0)$. Indeed, since $w_{\delta,0}$ is radially symmetric, we have

$$\int_{\mathbb{R}^N}\frac{x}{1+|x|}\int_{\mathbb{R}^N}\frac{w^2_{\delta,0}(x)w^2_{\delta,0}(y)}{|x-y|^4}dy\,dx = 0$$

and so

$$\begin{aligned}
\Big|\beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})\Big| &= \frac{1}{\|w\|^2}\Big|\int_{\mathbb{R}^N}\frac{x}{1+|x|}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,)dx\Big| \\
&= \frac{1}{\|w\|^2}\Big|\int_{\mathbb{R}^N}\frac{x}{1+|x|}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y) - w^2_{\delta,0}(x)w^2_{\delta,0}(y)}{|x-y|^4}dy\,dx\Big| \\
&\le \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w^2_{\delta,z}(x)w^2_{\delta,z}(y) - w^2_{\delta,0}(x)w^2_{\delta,0}(y)|}{|x-y|^4}dy\,dx \\
&= \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w^2_{1,z/\delta}(x)w^2_{1,z/\delta}(y) - w^2_{1,0}(x)w^2_{1,0}(y)|}{|x-y|^4}dy\,dx \\
&\to 0
\end{aligned}$$

as $\delta \to +\infty$, uniformly for $z \in B_r(0)$.

For $\varepsilon > 0$, we fix a constant $\rho = \rho(\varepsilon) > 0$ such that $\frac{1}{1+\rho} < \frac{\varepsilon}{3}$. For such a ρ, we see from (5.9) that

$$\lim_{\delta\to+\infty}\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx = 0$$

uniformly for $z \in B_r(0)$ and that there exists $\delta_0 > 0$ such that

$$\big|\beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})\big| < \frac{\varepsilon}{3}$$

and

$$\frac{1}{\|w\|^2}\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx < \frac{\varepsilon}{3}$$

for all $\delta \in (\delta_0, +\infty)$ and $z \in B_r(0)$. Observe that

$$\gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) = \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\Big|\frac{x}{1+|x|} - \beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})\Big|\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx < 1 + \frac{\varepsilon}{3}$$

for all $\delta \in (\delta_0, +\infty)$ and $z \in B_r(0)$. On the other hand, for all $\delta \in (\delta_0, +\infty)$ and $z \in B_r(0)$ we have

$$\begin{aligned}
\gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) &= \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\Big|\frac{x}{1+|x|} - \beta(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})\Big|\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx\\
&\geq \frac{1}{\|w\|^2}\int_{\mathbb{R}^N}\frac{|x|}{1+|x|}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx - \frac{\varepsilon}{3}\\
&\geq \frac{1}{\|w\|^2}\int_{\mathbb{R}^N\setminus B_\rho(0)}\frac{|x|}{1+|x|}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx - \frac{\varepsilon}{3}\\
&\geq \frac{\rho}{1+\rho} - \frac{1}{\|w\|^2}\int_{B_\rho(0)}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dy\,dx - \frac{\varepsilon}{3}\\
&\geq 1 - \frac{1}{1+\rho} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3}\\
&> 1 - \varepsilon.
\end{aligned}$$

Therefore, we have

$$\lim_{\delta\to+\infty}\gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) = 1$$

uniformly for $z \in B_r(0)$. □

For simplicity, we define $T : \mathbb{R}^+ \times \mathbb{R}^N \to H$ by

$$T(\delta, z) = (\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z})$$

and $\Theta : H\setminus\{(0,0)\} \to \mathcal{N}$ by

$$\Theta(u,v) = (t_{(u,v)}|u|, t_{(u,v)}|v|),$$

where $t_{(u,v)} > 0$ is given by

$$t^2_{(u,v)} = \frac{\|(u,v)\|^2 + \int_{\mathbb{R}^N}(V_1(x)u^2 + V_2(x)v^2)dx}{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\alpha_1 u^2(x)u^2(y) + \alpha_2 v^2(x)v^2(y) + 2\beta u^2(x)v^2(y)}{|x-y|^4}dxdy}.$$

We have, for $\delta > 0$ and $z \in \mathbb{R}^N$,

$$\begin{aligned}
J(\Theta\circ T(\delta,z)) &= \frac{1}{4}\frac{[(k_0+l_0)\|w_{\delta,z}\|^2 + \int_{\mathbb{R}^N}(k_0V_1(x)w^2_{\delta,z} + l_0V_2(x)w^2_{\delta,z})dx]^2}{(k_0+l_0)\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w^2_{\delta,z}(x)w^2_{\delta,z}(y)}{|x-y|^4}dxdy}\\
&= \frac{1}{4}\frac{[(k_0+l_0)\|w\|^2 + \int_{\mathbb{R}^N}(k_0V_1(x)w^2_{\delta,z} + l_0V_2(x)w^2_{\delta,z})dx]^2}{(k_0+l_0)\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w^2(x)w^2(y)}{|x-y|^4}dxdy}.
\end{aligned}$$

Then, by Lemma 5.2 and by (5.7), we can fix $R_0 > 0$ such that $J(\Theta\circ T(\delta,z)) < c^\star$ for all $\delta > 0$ and $|z| \geq R_0$. Moreover, as a consequence of Lemmas 5.2 and 5.3, we have

Lemma 5.4. (1) *There exists* $\delta_1 \in (0, \frac{1}{2})$ *such that*

$$J(\Theta\circ T(\delta,z)) < c^\star$$

and

$$\gamma(\sqrt{k_0}w_{\delta,z}, \sqrt{l_0}w_{\delta,z}) < \frac{1}{2}$$

for all $\delta \in (0, \delta_1]$ *and* $z \in \mathbb{R}^N$.

(2) *There exists* $\delta_2 \in (\frac{1}{2}, +\infty)$ *such that*

$$J(\Theta \circ T(\delta, z)) < c^\star$$

and

$$\gamma(\sqrt{k_0} w_{\delta,z}, \sqrt{l_0} w_{\delta,z}) > \frac{1}{2}$$

for all $\delta \in [\delta_2, +\infty)$ *and* $z \in B_{R_0}(0)$.

Lemma 5.5. *Denote* $D = [\delta_1, \delta_2] \times B_{R_0}(0)$ *and define a map* $g : D \to \mathbb{R}^+ \times \mathbb{R}^N$ *by*

$$g(\delta, z) = (\gamma \circ \Theta \circ T(\delta, z), \beta \circ \Theta \circ T(\delta, z)).$$

Then we have

$$\deg\Big(g, D, \Big(\frac{1}{2}, 0\Big)\Big) = 1.$$

Proof. Consider the homotopy map $G : [0, 1] \times D \to \mathbb{R}^+ \times \mathbb{R}^N$ defined by

$$G(s, \delta, z) = (1 - s)(\delta, z) + sg(\delta, z).$$

We claim

$$\Big(\frac{1}{2}, 0\Big) \notin G([0, 1] \times \partial D). \tag{5.10}$$

If this is true, then the conclusion follows easily from the homotopy invariance and normalization of degree.

Now we verify (5.10). If $\delta = \delta_1$ and $z \in B_{R_0}(0)$, then we see from Lemma 5.4(1) that

$$(1 - s)\delta_1 + s\gamma \circ \Theta \circ T(\delta_1, z) = (1 - s)\delta_1 + s\gamma(\sqrt{k_0} w_{\delta_1,z}, \sqrt{l_0} w_{\delta_1,z}) < \frac{1}{2}.$$

If $\delta = \delta_2$ and $z \in B_{R_0}(0)$, then it follows from Lemma 5.4(2) that

$$(1 - s)\delta_1 + s\gamma \circ \Theta \circ T(\delta_2, z) = (1 - s)\delta_2 + s\gamma(\sqrt{k_0} w_{\delta_2,z}, \sqrt{l_0} w_{\delta_2,z}) > \frac{1}{2}.$$

If $\delta \in [\delta_1, \delta_2]$ and $|z| = R_0$, then using Lemma 5.3(1) yields

$$\langle (1 - s)z + s\beta \circ \Theta \circ T(\delta, z), z\rangle = \langle (1 - s)z + s\beta(\sqrt{k_0} w_{\delta,z}, \sqrt{l_0} w_{\delta,z}), z\rangle_{\mathbb{R}^N} > 0,$$

which implies that $(1 - s)z + s\beta \circ \Theta \circ T(\delta, z) \neq 0$. Therefore, (5.10) holds. □

Setting $\mathbb{A} = \Theta \circ T(D)$ and $\Gamma = \{h \in C(\mathbb{A}, \mathcal{N}) : h|_{\partial \mathbb{A}} = id\}$, we have

Lemma 5.6. $\mathcal{M}$ *and* $\partial \mathbb{A}$ *link with respect to* Γ.

Proof. Assume that $(u, v) \in \partial \mathbb{A} = \Theta \circ T(\partial D)$. From the choice of R_0 and Lemma 5.4, we see that $J(u, v) < c^\star$ which implies $(u, v) \notin \mathcal{M}$. Therefore, we have $\mathcal{M} \cap \partial \mathbb{A} = \emptyset$.

For any $h \in \Gamma$, we define a continuous map $\overline{g} : D \to \mathbb{R}^+ \times \mathbb{R}^N$ by

$$\overline{g}(\delta, z) = (\gamma \circ h \circ \Theta \circ T(\delta, z), \beta \circ h \circ \Theta \circ T(\delta, z)).$$

Since $h|_{\partial \mathbb{A}} = id$, we have $\overline{g}|_{\partial D} = g|_{\partial D}$. Then it follows from Lemma 5.5 that

$$\deg\Big(\overline{g}, D, \Big(\frac{1}{2}, 0\Big)\Big) = \deg\Big(g, D, \Big(\frac{1}{2}, 0\Big)\Big) = 1.$$

By the Kronecker existence theorem, there is $(\overline{\delta}, \overline{z}) \in D$ such that $h \circ \Theta \circ T(\overline{\delta}, \overline{z}) \in \mathcal{M}$. Then we conclude that $\mathcal{M} \cap h(\mathbb{A}) \neq \emptyset$. □

Now we are in a position to prove the main result.

Proof of Theorem 1.3. Define the minimax value

$$d = \inf_{h\in\Gamma} \max_{(u,v)\in\mathbb{A}} J(h(u,v)).$$

Lemma 5.6 indicates that $\mathcal{M}\cap h(\mathbb{A}) \neq \emptyset$ for any $h \in \Gamma$. Then $d \geq c^\star > c_\infty$. Since $id \in \Gamma$, we have

$$\begin{aligned} d &\leq \max_{(u,v)\in\mathbb{A}} J(u,v) \\ &\leq \max_{(\delta,z)\in\mathbb{R}^+\times\mathbb{R}^N} J(\Theta\circ T(\delta,z)) \\ &\leq \max_{(\delta,z)\in\mathbb{R}^+\times\mathbb{R}^N} \frac{1}{4}\frac{\Big[(k_0+l_0)\|w_{\delta,z}\|^2 + \displaystyle\int_{\mathbb{R}^N}(k_0V_1(x)w_{\delta,z}^2 + l_0V_2(x)w_{\delta,z}^2)dx\Big]^2}{(k_0+l_0)\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w_{\delta,z}^2(x)w_{\delta,z}^2(y)}{|x-y|^4}dxdy}. \end{aligned}$$

Using the Hölder inequality, the definition of S, (1.9) and (5.8) leads to

$$\begin{aligned} d &\leq \frac{1}{4}\frac{\big[(k_0+l_0)\|w\|^2 + k_0|V_1|_{\frac{N}{2}}|w|_{2^*}^2 + l_0|V_2|_{\frac{N}{2}}|w|_{2^*}^2\big]^2}{(k_0+l_0)\|w\|^2} \\ &\leq \frac{1}{4}\frac{\big[(k_0+l_0)\|w\|^2 + k_0S^{-1}|V_1|_{\frac{N}{2}}\|w\|^2 + l_0S^{-1}|V_2|_{\frac{N}{2}}\|w\|^2\big]^2}{(k_0+l_0)\|w\|^2} \\ &= \frac{k_0+l_0}{4}\|w\|^2S_{H,L}^{-2}\Big(S_{H,L} + \frac{\beta-\alpha_2}{2\beta-\alpha_1-\alpha_2}C(N,4)^{-\frac{1}{2}}|V_1|_{\frac{N}{2}} + \frac{\beta-\alpha_1}{2\beta-\alpha_1-\alpha_2}C(N,4)^{-\frac{1}{2}}|V_2|_{\frac{N}{2}}\Big)^2 \\ &< \min\Big\{\frac{S_{H,L}^2}{4\alpha_1}, \frac{S_{H,L}^2}{4\alpha_2}, 2c_\infty\Big\}, \end{aligned} \tag{5.11}$$

where $2^* = \frac{2N}{N-2}$. According to the deformation lemma (see [48, Theorem 8.4 in Chapter II]), the constrained functional $J|_{\mathcal{N}}$ has a Palais-Smale sequence $\{(u_n,v_n)\}\subset\mathcal{N}$ at the level d. By Corollary 4.3, we conclude that $\{(u_n,v_n)\}$ contains a convergent subsequence in H, and there is a critical point (u,v) of the constrained functional $J|_{\mathcal{N}}$ with $J(u,v) = d$. Then, by Lemma 2.1 and $d < \min\{S_{H,L}^2/4\alpha_1, S_{H,L}^2/4\alpha_2\}$, it is easy to see that $u \neq 0$, $v\neq 0$, and (u,v) is a critical point of the functional J. By the maximum principle, we see that (u,v) is a positive solution of (1.7). □

References

[1] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z., **248** (2004), 423–443. 3

[2] C. O. Alves, G. M. Figueiredo & R. Molle, *Multiple positive bound state solutions of a critical Choquard equation*, arXiv:1812.04875. 3

[3] C.O. Alves, F. Gao, M. Squassina & M. Yang, *Singularly perturbed critical Choquard equations*, J. Differential Equations., **263** (2017), 3943–3988. 3

[4] C.O. Alves, A.B. Nóbrega & M. Yang, *Multi-bump solutions for Choquard equation with deepening potential well*, Calc. Var. Partial Differential Equations, **55** (2016), 48. 3

[5] A. Ambrosetti & E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations*, J. Lond. Math. Soc., **75** (2007), 67–82. 2

[6] T. Bartsch, Z.-Q. Wang & J. Wei, *Bound states for a coupled Schrödinger system*, J. Fixed Point Theory Appl., **2** (2007), 353–367. 2

[7] V. Benci & G. Cerami, *Existence of positive solutions of the equation* $-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}$ *in* $\mathbb{R}^N$, J. Funct. Anal., **88** (1990), 90–117. 2, 3

[8] H. Brézis & E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., **88** (1983), 486–490. 16, 17

[9] H. Brézis & L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), 437–477. 4

[10] G. Cerami & R. Molle, *Multiple positive bound states for critical Schrödinger-Poisson systems*, ESAIM Control Optim. Calc. Var., **25** (2019), 29 pp. 3, 22, 25

[11] W. Chen, C. Li & B. Ou, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math., **59** (2006), 330–343. 4, 9, 10

[12] Z. Chen, C.-S. Lin & W. Zou, *Sign-changing solutions and phase separation for an elliptic system with critical exponent*, Comm. Partial Differential Equations., **39** (2014), 1827–1859. 2

[13] Z. Chen & W. Zou, *Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent*, Arch. Ration. Mech. Anal., **205** (2012), 515–551. 2

[14] Z. Chen & W. Zou, *Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent: higher dimensional case*, Calc. Var. Partial Differential Equations, **52** (2015), 423–467. 2

[15] M. Clapp, A. Pistoia, Existence and phase separation of entire solutions to a pure critical competitive elliptic system, Calc. Var. Partial Differential Equations, **57** (2018), Paper No. 23, 20 pp. 2

[16] W. Dai, J. Huang, Y. Qin, B. Wang & Y. Fang, *Regularity and classification of solutions to static Hartree equations involving fractional Laplacians*, Discrete Contin. Dyn. Syst., **39** (2019), 1389–1403. 15

[17] F. Dalfovo, S. Giorgini, L.P. Pitaevskii & S. Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys., **71** (1999), 463–512. 1

[18] E.N. Dancer & J. Wei, *Spike Solutions in coupled nonlinear Schrödinger equations with attractive interaction*, Trans. Amer. Math. Soc., **361** (2009), 1189-1208. 2

[19] L. Du & M. Yang, *Uniqueness and nondegeneracy of solutions for a critical nonlocal equation*, Discrete Contin. Dyn. Syst., **39** (2019), 5847–5866. 3, 4, 6, 9

[20] A. Elgart & B. Schlein, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math., **60** (2007), 500–545. 3

[21] B. Esry, C. Greene, J. Burke Jr. & J. Bohn, *Hartree-Fock theory for double condensates*, Phys. Rev. Lett., **78** (1997), 3594–3597. 1

[22] F. Gao, E. Silva, M. Yang & J. Zhou, *Existence of solutions for critical Choquard equations via the concentration compactness method*, Proc. Roy. Soc. Edinburgh Sect. A, **150** (2020), 921–954. 3, 5, 17

[23] F. Gao & M. Yang, *The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation*, Sci China Math, **61** (2018), 1219–1242. 4, 16, 24

[24] Y. Lei, *Liouville theorems and classification results for a nonlocal Schrödinger equation*, Discrete Contin. Dyn. Syst., **38** (2018), 5351–5377. 3, 4, 9

[25] M. Lewin, P. T. Nam & N. Rougerie, *Derivation of Hartree's theory for generic mean-field Bose systems*, Adv. Math., **254** (2014), 570–621. 1

[26] M. Lewin, P. T. Nam & N. Rougerie, *The mean-field approximation and the non-linear Schrdinger functional for trapped Bose gases*, Trans. Amer. Math. Soc., **368** (2016), 6131–6157. 1

[27] Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc., **6** (2004), 153–180. 14

[28] Y. Li & L. Zhang, *Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations*, J. Anal. Math., **90** (2003), 27–87. 14

[29] E. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math., **57** (1976/77), 93–105. 3

[30] E. Lieb & M. Loss, Analysis, Gradute Studies in Mathematics, AMS, Providence, RI, 2001. 3

[31] T.-C. Lin & J. Wei, *Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$*, Comm. Math. Phys., **255** (2005), 629–653. 2

[32] P.-L. Lions, *The Choquard equation and related questions*, Nonlinear Anal., **4** (1980), 1063–1072. 3

[33] H. Liu & Z. Liu, *Positive solutions of a nonlinear Schrödinger system with nonconstant potentials*, Discrete Contin. Dyn. Syst., **36** (2016), 1431–1464. 2, 3, 25

[34] H. Liu & Z. Liu, *A coupled Schrödinger system with critical exponent*, Calc. Var. Partial Differential Equations, **59** (2020), 145. 2, 7, 17

[35] Z. Liu & Z.-Q. Wang, *Multiple bound states of nonlinear Schrödinger systems*, Comm. Math. Phys., **282** (2008), 721–731. 2

[36] L. Ma & L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal., **195** (2010), 455–467. 3

[37] L.A. Maia, E. Montefusco & B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system*, J. Differential Equations, **229** (2006), 743–767. 2

[38] M. Mitchell & M. Segev, *Self-trapping of incoherent white light*, Nature, **387** (1997), 880–883. 1

[39] V. Moroz & J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics*, J. Funct. Anal., **265** (2013), 153–184. 3, 16

[40] V. Moroz & J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc., **367** (2015), 6557–6579. 3

[41] V. Moroz & J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl., **19** (2017), 773–813. 3

[42] S. Pekar, Untersuchung über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954. 3

[43] S. Peng, Y. Peng & Z.-Q. Wang *On elliptic systems with Sobolev critical growth*, Calc. Var. Partial Differential Equations, **55** (2016), 142. 2, 17

[44] A. Pistoia, N.Soave, H. Tavares, A fountain of positive bubbles on a Coron's problem for a competitive weakly coupled gradient system. J. Math. Pures Appl. **135**(2020), 159–198. 2

[45] Y. Sato & Z.-Q. Wang, *On the multiple existence of semi-positive solutions for a nonlinear Schrödinger system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **30** (2013), 1–22. 2

[46] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in* $\mathbb{R}^N$, Comm. Math. Phys., **271** (2007), 199–221. 2

[47] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z., **187** (1984), 511–517. 5, 17, 19

[48] M. Struwe, Variational Methods, Springer, Berlin, 1996. 29

[49] S. Terracini & G. Verzini, *Multipulse phases in k-mixtures of Bose-Einstein condensates*, Arch. Ration. Mech. Anal., **194** (2009), 717–741. 2

[50] J. Wang & J. Shi, *Standing waves for a coupled nonlinear Hartree equations with nonlocal interaction*, Calc. Var. Partial Differential Equations, **56** (2017), Paper No. 168, 36 pp. 2

[51] J. Wei & M. Winter, *Strongly interacting bumps for the Schrödinger-Newton equations*, J. Math. Phys., **50** (2009), 012905. 3

[52] J. Wei & W. Yao, *Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations*, Comm. Pure Appl. Anal., **11** (2012), 1003–1011. 2

[53] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996. 17

[54] M. Yang, Y. Wei & Y. Ding, *Existence of semiclassical states for a coupled Schrödinger system with potentials and nonlocal nonlinearities*, Z. Angew. Math. Phys., **65** (2014), 41–68. 2

[55] Y. Zheng, C.A. Santos, Z. Shen & M. Yang, *Least energy solutions for coupled Hartree system with Hardy-Littlewood-Sobolev critical exponents*, Comm. Pure Appl. Anal., **19** (2020), 329–369. 2

Fashun Gao,
Department of Mathematics and Physics, Henan University of Urban Construction,
Pingdingshan, Henan, 467044, People's Republic of China
E-mail address: fsgao@zjnu.edu.cn

Haidong Liu,
Institute of Mathematics, Jiaxing University,
Jiaxing, Zhejiang, 314000, People's Republic of China
E-mail address: liuhaidong@mail.zjxu.edu.cn

Vitaly Moroz,
Mathematics Department, Swansea University,
Bay Campus, Fabian Way, Swansea SA1 8EN, Wales, United Kingdom
E-mail address: v.moroz@swansea.ac.uk

Minbo Yang,
Department of Mathematics, Zhejiang Normal University,
Jinhua, Zhejiang, 321004, People's Republic of China
E-mail address: mbyang@zjnu.edu.cn