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Probability distributions unimodality of finite sample extremes of independent Erlang random variables

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Abstract. Samples of independent identically distributed random non-negative values $\tilde{r}_1, ..., \tilde{r}_N$ with a finite size $N \geq 2$ are studied. It is posed the problem to find the sufficient conditions for their common probability distribution $Q(x) = \Pr\{\tilde{r}_j < x\}$, $j = 1 \div N$ which guarantee the unimodality of the probability distributions $F_N^{(+)}(x) = \Pr\{\tilde{r}_+ < x\}$ and $F_N^{(-)}(x) = \Pr\{\tilde{r}_- < x\}$ which correspond to the maximum $\tilde{r}_+ = \max\{\tilde{r}_j; j = 1 \div N\}$ and to the minimum $\tilde{r}_- = \max\{\tilde{r}_j; j = 1 \div N\}$ of the sample, respectively. It is proved that if the distribution Q is determined by a continuously differentiable Erlang probability density q of an arbitrary order $n \in \mathbf{N}$ then distributions $F_N^{(\pm)}$ are unimodal.

1. Introduction

Historically, the theory of probability distributions of extremes of random variables samples and the theory of the probability distributions of extreme values of trajectories of stationary random processes which is connected with it have arisen on the basis of the necessity to find some answers of quite practical questions. Now, this theory has extracted into a separate direction of research in the probability theory. In the framework of this scientific direction, the theory of probability distributions of extremes of independent random variables sequences occupies a special place [1]. One can be seen a detailed overview of the current state of this theory in [2].

Many authors became to interest the problems of mathematical modeling associated with the study of statistics of extremes of sequences of independent random variables at the beginning of the 20th century in connection with the request for this kind of research of the rapidly developing industry in European countries (see [3–7]). The statistical characteristics of extremes of independent random variables were studied in these works, where some probability distributions were chosen for them which were already widely used in probability theory. In contrary to such investigations, in the works of M. Frechet [8] and R. A. Fisher with L. Tippett [9], apparently, the problem of calculating of probability distribution of extreme values has been posed and has been solved for the first time on the basis of a priori reasonable assumptions from the point of view of applications.

In the first of these works, the distribution of the maximum of a sequence of independent identically distributed random variables was investigated, and in the second one, it was studied the problem of the probability distribution of the minimum of such a random positive variables set. Significantly, the latter work arose as a result of Leonard Tippett's statistical analysis of thread breakage in textile production. Thus, already at the initial stage of development of the theory, it has been appeared the fact that the theory of extreme values of independent random variables is closely related to the solution of problems about the reliability of certain technical elements (see [10, 11]).

The limit theorems for probability distributions of extreme values are the main object of study in the theory of probability distributions of extremes of sequences $\{\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_N\}$ of random variables. Here and later, we note random variables by means of the tilde sign.Limiting distributions can be observed statistically due to appropriate circumstances of the experiment organization. The works [12–14] are classical of this direction.In these works it has been given the final solution of the questions which have been put at the first stage of theory development. Their results are summarized in the monograph [15]. Now, the subject of research related to the limiting behavior of probability distributions of extremums of random variables sequences is currently far from exhausted, that one may see due to the emergence of some works in this direction (see, for example, [16–21]).

Note that the application of limit theorems is useful in practice in the case when there is a minimal a priori information about the probability distribution Q of a typical random variable of the sample. In this case, the appropriate processing of a large number N of statistical data allows to establish some, quite rough, characteristics of the distribution Q.

At the same time, in some problems of mathematical modeling, when constructing the corresponding stochastic models is done, it is possible a situation when the model formulated in terms of the extremes of the sample of random variables is such that one cannot consider it as large. Therefore, the use of limit theorems for the analysis of this model becomes inadequate. But, it is possible to choose the distribution Q in essentially their more wide class when the model constructing. In this case we must base only on some a priori considerations regarding to nature of the random variable. Such a situation takes place , for example, in problems of equilibrium statistical physics. It becomes possible to do the mathematical analysis of such a model and the obtained results can have predictive cense. The adequacy verification of such models is connected with the solution of inverse problem of the distribution theory of extremes of random variables sequences for a small number of N components. One the basis of such solutions it becomes possible to restore the distribution Q using the empirical distribution function F_N, found by processing a large number of series of similar experiments connected with samples of small size.

On the basis of solutions of inverse problems it is possible to study the relationship of qualitative properties of probability distributions $Q(x) = \Pr\{\tilde{r}_j < x\}$, $j = 1 \div N$ with qualitative properties of probability distributions $F_N^{(\pm)}(x) = \Pr\{\tilde{r}_\pm < x\}$ where $\tilde{r}_+ = \max\{\tilde{r}_j; j = 1 \div N\}$, $\tilde{r}_- = \min\{\tilde{r}_j; j = 1 \div N\}$ which are revealed at the analysis of statistical data and which constitute a significant predictive value.

One of the most important qualitative properties of one-dimensional probability distributions, which is quite easily identified during statistical data processing, is the number of their modes. In particular, the distributions used in mathematical statistics (e.g. Pearson distributions [22]) are unimodal. The unimodality problem of one-dimensional probability distributions was given much attention in the scientific literature of probability theory (see, for example, [23]). From the physical point of view, the violation of the distribution unimodality means the presence of some specific physical mechanism that leads to the appearance of this effect. Its manifestation at statistical data processing can be quite an unexpected surprise and, therefore, it is subject to theoretical understanding. Therefore, the mathematical interest consists of determining such cases when, despite the unimodality of the "origin" distribution Q, the probability distributions of random variables $\tilde{r}_\pm$ are still unimodal.

This paper is devoted to proving the unimodality of distributions $F_N^{(\pm)}(x)$ in the case when $Q(x)$ is represented by the Erlang distribution of arbitrary order. In spite of the fact that this two-parameter class of distributions is rather special, the solution of such a problem nevertheless seems actual, since there are no proofs of statements of any general character in scientific literature which concern the unimodality of functions $F_N^{(\pm)}(x)$ at finite N. Moreover, there are no effective methods for studying the unimodality of such functions that do not use the explicit form of the distribution $Q(x)$. This is due to the fact that methods for analyzing the unimodality of one-dimensional distributions are usually focused on the distribution functions of sums of independent random variables, which are obtained by convolution of distributions. In this regard, specific effective methods related to the theory of characteristic functions are used to analyze the unimodality of such distributions. These methods are based on the representation of the characteristic function of each studied distribution as a product of suitable characteristic functions. For example, it is a method based on the concept of *strictly unimodal distribution* [28] (see also, [23, P. 127]) and also it is a method based on the Khinchin criterion (see [23, P. 118, 29]).

The problem solved in this paper was initiated by connection with identifying the physical reasons which that lead to the loss of unimodality of the distribution function $F_N^{(+)}(x)$ that arises in the study of the probabilistic model of the electrical strength of polymer enamel-lacquer coatings [30]. In this model, the random variables $\tilde{r}_j$, $j = 1 \div N$ represent some random sizes of air inclusions in the material. These inclusions arise as a result of the technological process of creating a coating. In this model, the local electrical strength is a decreasing function of the maximum $\tilde{r}_+ = \max\{\tilde{r}_j; j = 1 \div N\}$ of random radii of air bubbles.

Thus, it is necessary to study the probability distribution of a random variable $\tilde{r}_+$ when $\tilde{r}_j$, $j = 1 \div N$ are statistically independent and equivalently distributed random positive variables with a distribution function $Q(x)$. In this case, the sample size $N \geq 2$. It is equal to the number of air inclusions per 1cm^2 of the coverage area. Due to the relatively small value of the number N, it is not possible to use limit theorems for the probability distribution of sample maxima at the analysis of the model. In conditions of great uncertainty of choice of the function $Q(x)$ for the purpose of mathematical modeling, it is necessary to study the function $F_N(x)^{(+)}$ at very broad assumptions about the probability distribution of the random size of air pores.

2. Problem statement

The probability distribution $G(x) = \Pr\{\tilde{\xi} < x\}$ of a random variable $\tilde{\xi}$ is named the unimodal one (see, for example, [23, P. 118]), if there is at least one such number $a \in \mathbf{R}$ that $G(x)$ is concave at $x > a$ and it is convex at $x < a$ where the value $x = a$ is called *the top* of the distribution $G(x)$.

If $G(x)$ is an absolutely continuous function, that is, it has almost everywhere on $\mathbf{R}$ the density $g(x) = dG(x)/dx$, then the unimodality of such a function consists of the existence such a number a that $g(x)$ does not increase at $x > a$ and it does not decrease at $x < a$.

Let $Q(x)$, $x > 0$ be the common probability distribution of random variables $\tilde{r}_1, ..., \tilde{r}_N$ and let the random variables $\tilde{r}_1, ..., \tilde{r}_N$ are statistically independent and nonnegative, then $F_N^{(+)}(x) = \prod_{j=1}^{N} \Pr\{\tilde{r}_j < x\} = [Q(x)]^N$. We will assume that $Q(x)$ has the density $q(x) = dQ(x)/dx > 0$ on $[0, \infty)$. In this case, the distribution function $F_N^{(+)}(x)$ also has the density $f_N^{(+)}(x)$ at $x > 0$, which is determined by the following formula

$$f_N^{(+)}(x) = Nq(x)Q^{N-1}(x) = Nq(x)\Big(\int_0^x q(y)dy\Big)^{N-1}, \qquad N \geq 2\,. \tag{1}$$

In general case, the inverse problem in the distribution theory of the sampling maximum of independent equally distributed random variables consists of the calculation of the distribution density q on the basis of the integral equation (1), considering the density $f_N^{(+)}$ as the given

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one. The latter is found as a result of processing of experimental data. The solution of this equation is easily found on the basis of the formula $F_N^{(+)}(x) = [Q(x)]^N$, since it follows by means of differentiation that $Q(x) = [F_N^{(+)}]^{1/N}$ and, therefore,

$$q(x) = \frac{1}{N} f_N^{(+)}(x)[F_N^{(+)}]^{1/N-1}(x) = \frac{1}{N} f_N^{(+)}(x)\Big(\int_0^x f_N^{(+)}(y)dy\Big)^{(1-N)/N}.$$

For statistically independent nonnegative equally distributed random variables $\langle r_j; j = 1 \div N\rangle$ the distribution function $F_N^{(-)}$ of the minimum $\tilde{r}_-$ of their values is given by the formula $F_N^{(-)}(x) = \prod_{j=1}^{N} \Pr\{\tilde{r}_j \geq x\} = 1 - [1 - Q(x)]^N$.

Just as above, in the case when the probability distribution Q has density q, the function $F_N^{(-)}$ also has density $f_N^{(-)}$ which is given by the formula

$$f_N^{(-)}(x) = Nq(x)[1 - Q(x)]^{N-1}(x) = Nq(x)\Big(\int_x^\infty q(y)dy\Big)^{N-1}, \qquad N \geq 2. \tag{2}$$

Then, the inverse problem in the distribution theory of the sampling minimum of independent equally distributed random variables, in general case, consists of the calculation of the distribution density q on the basis of the integral equation (2) with the given density $f_N^{(-)}$. The solution of this equation is written on the basis of the same argues as above. From the formula $F_N^{(-)}(x) = 1 - [1 - Q(x)]^N$ it follows that $Q(x) = 1 - [1 - F_N^{(-)}(x)]^{1/N}$, and so we have

$$q(x) = \frac{1}{N} f_N^{(-)}(x)\Big(\int_x^\infty f_N^{(-)}(y)dy\Big)^{(1-N)/N}.$$

Basing on integral equations (1), (2), the unimodality problem of distribution functions $F^{(\pm)}$ is to find effective sufficient conditions for the density q that would guarantee the presence of a unique top a of each densities $f_N^{(\pm)}$. In the special case, when solving such problems, we can assume that the density q corresponds to the unimodal probability distribution.

Further, we will assume that the density q is continuously differentiable on $(0, \infty)$. Then, according to (1), the density $f_N^{(+)}(x)$ is also continuously differentiable and its derivative is given by the formula

$$\dot{f}_N^{(+)}(x) = NQ^{N-2}(x)(\dot{q}(x)Q(x) + (N-1)q^2(x)), \quad N \geq 2.$$

In this case, the problem of determining the number of maxima at the density $f_N^{(+)}$ is reduced to the definition of the number of values $x > 0$, which are solutions of the equation

$$\dot{q}(x)\int_0^x q(y)dy + (N-1)q^2(x) = 0, \quad x > 0, \tag{3}$$

which defines all extremum points of the density $f_N^{(+)}$ excepting may be the point x = 0. From this equation, it immediately follows that the extremum points of the density $f_N^{(+)}$ are on the decreasing interval of the density q.

Similarly, under the assumption of continuous differentiability of the density q on $(0, \infty)$, the density $f_N^{(-)}(x)$ is also continuously differentiable according to (2). Its derivative is given by the formula

$$\dot{f}_N^{(-)}(x) = N[1 - Q(x)]^{N-2}(\dot{q}(x)(1 - Q(x)) - (N-1)q^2(x)), \quad N \geq 2.$$

Then, the problem of determining the number of maxima of the density $f_N^{(-)}$ reduces to determining the number of values $x > 0$ which satisfy the equation

$$\dot{q}(x) \int_x^\infty q(y)dy - (N-1)q^2(x) = 0\,, \quad x > 0\,, \quad x > 0\,. \tag{4}$$

Thus, in contrast with the density $f_N^{(+)}$, the extremum point of the density $f_N^{(-)}$ are on the interval where the density q increases.

At the end of the section, we prove the following simple statement.

Theorem 1. *Equations (3) and (4) are equivalent to the corresponding equations which are obtained to each of them by replacing the density $q(x)$ to the density $\mu q(\mu x)$ where $\mu > 0$ is an arbitrary positive value.*

Proof. The derivative $\dot{q}(x)$ is changed to $\mu^2\dot{q}(\mu x)$ when replacing $q(x)$ to $\mu q(\mu x)$ and hence the equations (3), (4) transform to the following ones

$$\mu^3\dot{q}(\mu x) \cdot \int_0^x q(\mu y)dy \pm (N-1)\mu^2 q^2(\mu x) = 0$$

or, after replacing the integration variable $\mu y \Rightarrow y$, we have

$$\dot{q}(\mu x) \cdot \int_0^{\mu x} q(y)dy + (N-1)q^2(\mu x) = 0\,.$$

The change $\mu x \Rightarrow x$ in these equations translates them into the original equations (3) and (4). *End of proof.*

In the following sections, we establish the unimodality of densities $f_N^{(\pm)}$ in the case when q belongs to the class of the Erlang distribution densities, that is, it is represented by the formula

$$q(x) = \frac{\lambda^n x^n}{n!}\exp(-\lambda x)\,,$$

where the number $n \in \mathbf{N}_+$ is the distribution order, and $\lambda > 0$ is a physically dimensional parameter that determines the statistical characteristics of random variables $\tilde{r}_j$, $j = 1 \div N$. The above-proved statement allows us to solve this problem only for the particular value $\lambda = 1$, replacing the density q by the density $\lambda^{-1}q(x/\lambda)$ in the equations (3), (6). Thus, we put further that

$$q(x) = \frac{x^n}{n!}\exp(-x)\,. \tag{5}$$

It is easily verified that the Erlang distribution is unimodal. Since, $\dot{q}(x) = (nx-1)q(x)/x$, then there is unique top with $a = 1/n$ at $n \in \mathbf{N}$, and the top is in $x = 0$ at $n = 0$.

3. The unimodality of the function $F_N^{(+)}$

Let us note that it was made the division by the expression $(\int_0^x q(y)dy)^{N-2}$ at the obtaining of the equation (3). This expression is zero at $x = 0$ if $N > 2$. For this reason, the equation may not contain a zero solution that corresponds to minimum of the density $f_N^{(+)}(x)$ minimum. Consider first the following case which is a special one according to the above.

Example 1. Let $n = 0$, that is $q(x) = e^{-x}$, $\dot{q}(x) = -e^{-x}$. It leads to the equality

$$(1 - e^{-x}) = (N-1)e^{-x}\,,$$

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after the substitution the explicit form of the density $q(x)$ into the equation (2). Such an equality is only possible when $1 = Ne^{-x}$, that is, there is a unique solution $a = \ln N$ which is the maximum point. The point of the minimum $x = 0$ of the density $f_0^{(+)}$ dropped out of our consideration. In the general case, it is fair

Theorem 2. *Probability distribution* $F_N^{(+)}(x) = \Pr\{\tilde{r}_+ < x\} = Q^N(x)$ *of the sample maximum is unimodal at any* $N > 2$ *in the case when the sample consists of* N *independent equally distributed and nonnegative variables* $\tilde{r}_1, ..., \tilde{r}_N$ *and their common probability distribution* $Q(x)$, $x > 0$ *is determined by the Erlang density with arbitrary* $\lambda > 0$ *and* $n \in \mathbf{N}$.

Proof. It is sufficient to give a proof for $\lambda = 1$. We assume that $N \geq 2$ and $n \geq 2$, since its proof for $n = 0$ is given by the example 1 considered above.

Note that $Q(0) = 0$ for $n > 1$, and therefore it should be the equality $q(0) = 0$, i.e. the equation (3) always has a solution $x = 0$. At this x- point, the minimum of the nonnegative density $f_N(x)$ is realized. Thus, it is necessary to find positive solutions of the equation (3) which must exist in view of nonnegativity $f_N(x)$. The smallest of these solutions obviously corresponds to the maximum of the density $f_N(x)$.

For the density (5) it is fulfilled

$$\int_0^x q(y)dy = \frac{1}{(n-1)!}\int_0^x y^n e^{-y} dy = 1 - e^{-x}\sum_{l=0}^{n}\frac{x^l}{l!}\,.$$

Substitution this expression into the equation (3) gives

$$x^{n-1}(n-x)e^{-x}n!\Big(1 - e^{-x}\sum_{l=0}^{n}\frac{x^l}{l!}\Big) + (N-1)x^{2n}e^{-2x} = 0$$

or, after obvious algebraic transformations without taking into account the root $x = 0$ of the equation,

$$(n-x)\Big(1 - e^{-x}\sum_{l=0}^{n}\frac{x^l}{l!}\Big) + (N-1)\frac{x^{n+1}}{n!}\,e^{-x} = 0\,.$$

Separating the terms which are proportional e^{-x}, we write down this equation in the following form

$$(n-x) = e^{-x}\Big(n + \sum_{l=1}^{n-1}\frac{x^l}{(l-1)!}\Big[\frac{n}{l} - 1\Big] - \frac{Nx^{n+1}}{n!}\Big)\,. \tag{6}$$

Let us introduce the function $g(x) = e^{-x}P(x)$ where the polynomial $P(x)$ is represented by the formula

$$P(x) = n + \sum_{l=1}^{n-1}\frac{x^l}{(l-1)!}\Big[\frac{n}{l} - 1\Big] - \frac{Nx^{n+1}}{n!} \tag{7}$$

at $n \geq 2$. Thus, the solutions of the equation (6) are determined by the intersection points of schedules of the functions $n - x$ and $g(x)$. The graphs obviously are intersected at $x = 0$, since $g(0) = n$. According to the above, it is necessary to find positive solutions. Further,

$$\dot{g}(x) = e^{-x}(\dot{P}(x) - P(x))$$

where

$$\dot{P}(x) = \sum_{l=1}^{n-1}\frac{lx^{l-1}}{(l-1)!}\Big[\frac{n}{l} - 1\Big] - \frac{(n+1)Nx^n}{n!}\,.$$

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Due to this formula and the formula (7)

$$\dot{P}(x) - P(x) =$$

$$= -1 + \sum_{l=1}^{n-2} \frac{x^l}{(l-1)!}\Big(\frac{l+1}{l}\Big[\frac{n}{l+1} - 1\Big] - \Big[\frac{n}{l} - 1\Big]\Big) - \frac{x^{n-1}}{(n-1)!} + \frac{Nx^n}{n!}(x-n-1) =$$

$$= -1 - \sum_{l=1}^{n-1} \frac{x^l}{l!} + \frac{Nx^n}{n!}(x-n-1)\,. \tag{8}$$

Therefore, $\dot{g}(0) = \dot{P}(0) - P(0) = -1$, that is the schedule $g(x)$ touches the graph $n - x$ at the point $x = 0$. The second derivative of the function $g(x)$ is equal

$$\ddot{g}(x) = e^{-x}(P(x) - 2\dot{P}(x) + \ddot{P}(x))\,. \tag{9}$$

Since, on the basis of (8),

$$\ddot{P}(x) - \dot{P}(x) = -1 - \sum_{l=1}^{n-2} \frac{x^l}{l!} + \frac{Nx^{n-1}}{n!}(n+1)(x-n)\,,$$

then

$$P(x) - 2\dot{P}(x) + \ddot{P}(x) = \Big\{ -1 - \sum_{l=1}^{n-2} \frac{x^l}{l!} + \frac{Nx^{n-1}}{n!}(n+1)(x-n)\Big\} -$$

$$-\Big\{ -1 - \sum_{l=1}^{n-1} \frac{x^l}{l!} + \frac{Nx^n}{n!}(x-n-1)\Big\} =$$

$$= \frac{x^{n-1}}{(n-1)!} - \frac{Nx^{n-1}}{n!}\Big[x^2 - 2x(n+1) + n(n+1)\Big]\,. \tag{10}$$

It follows that $\ddot{g}(0) = 0$ at the point $x = 0$. However, $\ddot{g}(x) < 0$ in any sufficiently small neighborhood of this point, that is the function $g(x)$ is concave in this neighborhood. In addition, $g(x)$ has one inflection point $x_* > 0$ where $\ddot{g}(x_*) = 0$, since it satisfies the quadratic equation

$$x^2 - 2x(n+1) + n(n+1) - \frac{n}{N} = 0\,,$$

which has one positive solution $x_* = n + 1 + \sqrt{n + 1 + n/N}$.

According to (9), (10), the function g is concave at $x \in (0, x_*)$. By virtue of its due to concavity and its touching with a straight line $n-x$ at the point $x = 0$, the inequality $g(x) < n-x$ should take place everywhere on the semi-interval $(0, x*]$, particularly it is fulfilled $g(x_*) < n-x_*$. Therefore, the equation $g(x) = n-x$ has no roots on the semi-interval $(0, x*]$. We prove that the equation has exactly one root x_+ on the semi-interval $[x_*\infty)$ where the function $g(x)$ is convex.

Firstly, because of the convexity $g(x)$, the schedule of the function g has no more than two intersections with the line $n - x$, that is, the equation $g(x) = n - x$ has no more than two roots. Secondly, in the case of no roots or the presence of exactly two roots, in view of the inequality $g(x_*) < n - x_*$, such an inequality $g(x) < n - x$ must be satisfied for all sufficiently large values of x. However, the function $g(x)$ tends to zero and $n - x$ tends to $-\infty$ when $x \to \infty$. Hence, there is the single root x_+ of the equation $g(x) = n - x$.

The unique positive root x_+ of the equation $n - x = g(x)$ corresponds to the position of the unique maximum of the density q. *End of proof.*

4. Unimodality of the function $F_N^{(-)}$

First of all, we note that the density $f_N^{(-)}$ has the view $f_N^{(-)}(x) = \lambda N e^{-\lambda N}$ for exponential density $q(x) = \lambda e^{-\lambda x}$ (the case $n = 0$) according to (2). Hence, in this case, there is only one top $a = 0$.

In general case, i.e. for an arbitrary $n \in \mathbf{N}$, the unimodality of the distribution $F_N^{(-)}$ is established by Theorem 3, formulated below. To prove it, we need the following lemmas.

Lemma 1. *Let the functions $f(x)$ and $g(x)$ satisfy the conditions $f(a) \leq g(a)$ and $f'(a) \leq g'(a)$ on the segment $[a, b]$ and the equation $f'(x) = g'(x)$ has no more than one solution on $[a, b]$. Then the equation $f(x) = g(x)$ has no more than one solution on $[a, b]$.*

Proof. Let there be at least two solutions x_1 and x_2, $x_1 < x_2$ where $x_1 \in [a, b]$ is the minimum among all possible solutions on the segment $[a, b]$, and x_2 is the solution following after it in ascending order. Then, the equalities $f(x_1) = g(x_1)$ and $f'(x_1) \geq g'(x_1)$ take place. In view of the fact that $f(a) \leq g(a)$, the unique solution of the equation $f'(x) = g'(x)$ is in the segment $[a, x_1]$. (This does not exclude the case of $x_1 = a$.) Therefore, due to following equalities

$$f(x_2) = f(x_1) + \int\limits_{x_1}^{x_2} f'(y)dy\,; \quad g(x_1) + \int\limits_{x_1}^{x_2} g'(y)dy = g(x_2)\,,$$

$f(x_1) = g(x_1)$ and due to the inequality $f'(y) > g'(y)$ when $y \in (x_1, x_2)$, the inequality $f(x_2) > g(x_2)$ holds. The resulting contradiction proves the statement of the theorem. *End of proof.*

From the proved lemma it follows

Lemma 2. *If the polynomial takes the form*

$$P_{n+1}(x) = x^{n+1} - \sum_{l=0}^{n} a_l x^l\,, \quad a_l \geq 0$$

and $a_0 > 0$, then it has exactly one positive root.

Proof. We differentiate m times the polynomial. In a result, we have

$$P_{n+1}^{(m)}(x) = \frac{(n+1)!}{(n+1-m)!}\, x^{n+1-m} - m! a_m\,,$$

where $m = \max\{l \in \{0, 1, ..., n\}\,; a_l > 0\}$. Then $P_{n+1}^{(m)}(0) = -m! a_m < 0$, since $n + 1 - m > 0$. We put $g'(x) = m! a_m$ with $g(x) = m! a_m x + (m-1)! a_{m-1}$ and

$$f'(x) = \frac{(n+1)!}{(n+1-m)!}\, x^{n+1-m}\,, \quad f(x) = \frac{(n+1)!}{(n+2-m)!}\, x^{n+2-m}$$

so that $f(0) = 0 \leq a_{m-1} = g(0)$. In this case, the equation $f'(x) = g'(x)$ which has the form

$$\frac{(n+1)!}{(n+1-m)!}\, x^{n+1-m} = a_m m!\,.$$

It is obviously has no more one root in the segment $[0, b]$ at any value b. Then the equation $f(x) = g(x)$ has no more than one root in $[0, b]$.

Next, putting

$$f_k(x) = \frac{d}{dx^{m-k}} x^{n+1} = \frac{(n+1)!}{(n+1+k-m)!}\, x^{n+1+k-m}\,,$$

$$g_k(x) = \frac{d}{dx^{m-k}} \sum_{l=0} a_l x^l = \sum_{l=m-k}^{m} \frac{l!}{(l-m+k)!} a_l x^{l-m+k}, \quad k = 1 \div m,$$

we argue by induction on $k = 1, ..., m$.

For each of the values $k = 1, ..., m-1$ it is fulfilled $f_k(0) = 0$, $g_k(0) = a_{m-k} \geq 0$, that is $f_k(0) \leq g_k(0)$ and the equalities

$$\frac{df_{k+1}}{dx} = f_k, \quad \frac{dg_{k+1}}{dx} = g_k$$

take place.

When $k = 1$, the equation $f_1 = g_1$ has no more than one solution in any segment $[0, b]$, since $f_1 = f$, $g_1 = g$. Then, assuming that the equation $f_k = g_k$ at the k-th induction step has no more than one solution in $[0, b]$, we obtain on the basis of Lemma 1, that the equation $f_{k+1} = g_{k+1}$ has no more than one solution on this segment. Therefore, at the m-th induction step, we conclude that the equation $f_m = g_m$ has at most one solution in $[0, b]$.

Since the equations $f_k = g_k$ present the equalities $P_{n+1}^{(m-k)}(x) = 0$, then we obtain at $k = m$ that the equation $f_m = g_m$ we may rewrite in the form $P_{n+1}(x) = 0$. It has no more than one solution on the segment $[0, b]$ at any $b > 0$. So as under this sircumstance, it is fulfilled

$$\lim_{x \to \infty} \frac{f_m(x)}{g_m(x)} = \infty,$$

then last equation necessarily has a root in $[0, b]$ at all sufficiently large values $b > 0$. *End of proof.*

Now we are able to prove the following statement.

Theorem 3. *Probability distribution* $F_N^{(-)}(x) = \Pr\{\tilde{r}_- < x\} = 1 - (1 - Q(x))^N$ *of the sample minimum of* N *independent identically distributed nonnegative variables* $\tilde{r}_1, ..., \tilde{r}_N$ *with their common probability distribution* $Q(x)$, $x > 0$ *is unimodal at any* $N \in \mathbf{N}$ *in the case when* $Q(x)$ *is defined by the Erlang density with any* $\lambda > 0$ *and any* $n \in \mathbf{N}_+$.

Proof. By virtue of theorem 1, it is sufficient to give a proof in the case of $\lambda = 1$. We assume that $N \geq 2$ and $n \in \mathbf{N}$, since in the cases $N = 1$ and $n = 0$ the statement is trivial. At $n \in \mathbf{N}$ we have $\dot{q}(x) = (n - x)q(x)/x$,

$$\frac{1}{n!} \int_x^\infty q(y)dy = e^{-x} \sum_{l=0}^{n} \frac{x^l}{l!}.$$

The use of these expressions results when the equation (6) is converted at $x > 0$ into the equation

$$(n - x)e^{-x} \sum_{l=0}^{n} \frac{x^l}{l!} = (N - 1)xq(x).$$

As a result of obvious algebraic transformations, we obtain

$$n + \sum_{l=1}^{n} n \frac{x^l}{l!} = \sum_{l=1}^{n} \frac{x^l}{(l-1)!} + N \frac{x^{n+1}}{n!},$$

$$g(x) \equiv n + \sum_{l=1}^{n-1} \frac{x^l}{(l-1)!} \Big(\frac{n}{l} - 1 \Big) = N \frac{x^{n+1}}{n!} \equiv f(x),$$

where functions f and g are defined when $x > 0$. We put $P_n(x) = n![f(x) - g(x)]/N$. This polynomial, has the form specified in the condition of Lemma 2. Hence, it has one positive root. *End of proof.*

5. Conclusions

In conclusion, we note that despite the fact that Erlang's distributions constitute a two-parameter class of distributions of mathematical statistics and they have applications in various applications of probability theory, it is desirable to extend the obtained result to a much wider set of distributions. Its characteristic must be qualitative, rather than purely analytical, and they could be justified as natural from the point of view of those applications where sample extremums of independent positive random variables $\tilde{r}_1, ..., \tilde{r}_N$ are used and where the the fact of unimodality of probability distributions $\tilde{r}_\pm$ is essential.

6. References

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