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ORIGINAL ARTICLE

Construction of a new class of symmetric function of binary products of (*p*, *q*)-numbers with 2-orthogonal Chebyshev polynomials

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Abstract
In this paper, we give some new generating functions of the products of (p, q)-Fibonacci numbers, (p, q) -Lucas numbers, (p, q)-Pell numbers, (p, q)-Pell Lucas numbers, (p, q)-Jacobsthal numbers, and (p, q)-Jacobsthal Lucas numbers with 2-orthogonal Chebyshev polynomials and trivariate Fibonacci polynomials.

Keywords Symmetric functions · Generating functions · (p, q)-Fibonacci numbers · (p, q)-Lucas · 2-Orthogonal Chebyshev polynomials · Trivariate Fibonacci polynomials

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1 Introduction and preliminaries

The authors in [11] defined the trivariate Fibonacci and Lucas polynomials $H_n(x,y,t)$ and $K_n(x,y,t)$. They gave generating functions, Binet's formulas, and explicit formulas of these trivariate polynomials.

The trivariate Fibonacci polynomials are the terms of the sequence $\{0,1,x,x^2+y,x^3+2xy+t,\ldots\}$ that satisfy the recurrence relation:

$$H_n(x,y,t)=xH_{n-1}(x,y,t)+yH_{n-2}(x,y,t)+tH_{n-3}(x,y,t) \text{ for } n\geq 3,$$

beginning with the values $H_0(x,y,t)=0$, $H_1(x,y,t)=1$ and $H_2(x,y,t)=x$.

Special cases of trivariate Fibonacci polynomials are Tribonacci polynomials $H_n(x^2,x,1)=T_n(x)$, Tribonacci numbers $H_n(1,1,1)=T_n$, and bivariate Fibonacci polynomials $H_n(x,y,0)=F_n(x,y)$.

The 2-orthogonal monic Chebyshev polynomials MPS (2-classical) of the first kind $\{\widehat{T}_n(x)\}_{n\geq 0}$ studied in [15], and defined by the next relation:

$$\begin{cases}\widehat{T}_{n+3}(x)=x\widehat{T}_{n+2}(x)-\alpha\widehat{T}_{n+1}(x)-\gamma\widehat{T}_n(x), \ n\geq 0, \gamma\neq 0\\ \widehat{T}_0(x)=1, \widehat{T}_1(x)=x, \widehat{T}_2(x)=x^2-\alpha,\end{cases} \tag{1}$$

where α and γ are constants (see also [22]). If $\alpha=0$ in the relationship (1), we get the 2-orthogonal monic Chebyshev polynomial MPS of the second kind $\{\widehat{U}_n(x)\}_{n\geq 0}$.

In [13], the (p,q)-Pell numbers $\{P_{p,q,n}\}_{n\in\mathbb{N}}$ and (p,q)-Pell Lucas numbers $\{Q_{p,q,n}\}_{n\in\mathbb{N}}$ are defined, respectively, by:

$$\begin{aligned}P_{p,q,n}&=2pP_{p,q,n-1}+qP_{p,q,n-2}, \text{ for } n\geq 2,\\ Q_{p,q,n}&=2pQ_{p,q,n-1}+qQ_{p,q,n-2}, \text{ for } n\geq 2,\end{aligned}$$

with initial conditions $P_{p,q,0}=0$, $P_{p,q,1}=1$ and $Q_{p,q,0}=2$, $Q_{p,q,1}=2p$. The first few terms of $\{P_{p,q,n}\}_{n\in\mathbb{N}}$ are $0, 1, 2p, 4p^2+q, 8p^3+4pq$ and so on. Also, the first few terms of $\{Q_{p,q,n}\}_{n\in\mathbb{N}}$ are $2, 2p, 4p^2+2q, 8p^3+6pq$, and so on.

In [10], Suvarnamani and Tatong studied the (p,q)-Fibonacci numbers $\{F_{p,q,n}\}_{n\in\mathbb{N}}$, they gave the Binet's formula and some results of these numbers. Then, Suvarnamani in [9] found some properties of (p,q) -Lucas numbers.

The (p,q)-Fibonacci numbers $\{F_{p,q,n}\}_{n\in\mathbb{N}}$ are defined by:

$$F_{p,q,n}=pF_{p,q,n-1}+qF_{p,q,n-2} \text{ for } n\geq 2, \text{ with } F_{p,q,0}=0, F_{p,q,1}=1.$$

The (p,q)-Lucas numbers $\{L_{p,q,n}\}_{n\in\mathbb{N}}$ are defined by the following recurrence relation:

$$L_{p,q,n}=pL_{p,q,n-1}+qL_{p,q,n-2}, \text{ for } n\geq 2, \text{ with } L_{p,q,0}=2, L_{p,q,1}=p.$$

The well-known Binet's formulas for (p,q)-Fibonacci and (p,q)-Lucas numbers are given by:

$$F_{p,q,n} = \frac{x_1^n - x_2^n}{x_1 - x_2} \text{ and } L_{p,q,n} = x_1^n + x_2^n,$$

where $x_1 = \frac{p+\sqrt{p^2+4q}}{2}$ and $x_2 = \frac{p-\sqrt{p^2+4q}}{2}$ are roots of the characteristic equation $x^2 - px - q = 0$. We note that:

$$x_1 + x_2 = p, x_1x_2 = -q \text{ and } x_1 - x_2 = \sqrt{p^2 + 4q}.$$

In 2015, Uygun [21] introduced the (p, q)-Jacobsthal and (p, q)-Jacobsthal Lucas numbers $\{J_{p,q,n}\}_{n\in\mathbb{N}}$ and $\{j_{p,q,n}\}_{n\in\mathbb{N}}$, which are defined by the second-order linear recurrence sequences, for any integer $n \geq 2$:

$$J_{p,q,n} = pJ_{p,q,n-1} + 2qJ_{p,q,n-2},$$

and

$$j_{p,q,n} = pj_{p,q,n-1} + 2qj_{p,q,n-2},$$

where $J_{p,q,0} = 0$, $J_{p,q,1} = 1$, $j_{p,q,0} = 2$, and $j_{p,q,1} = p$, respectively. The first few terms of these sequences are listed in the Table 1.

Corollary 1.1 *Given an alphabet* $A = \{a_1, a_2, a_3\}$, *we have*:

$$\begin{aligned} \widehat{T}_n(x) =& h_n(a_1, a_2, a_3), \\ H_n(x, y, t) =& h_{n-1}(a_1, a_2, a_3), \ n \in \mathbb{N}. \end{aligned}$$

An integer partition [12] is a finite sequence $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ of positive integers. The λ_i are parts of the partition λ and the k is called the number of parts of λ, denoted by $l(\lambda)$. The multiplicity of the part i in λ, denoted by $t_i(\lambda)$, is the number of parts of λ equal to i .

If $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, then we say that λ is a partition of n and we use the notation $\lambda \mapsto n$. The fastest algorithms for enumerating all the partitions of an integer have recently been presented by Merca [17, 18].

We recall some basic facts about monomial symmetric functions. Proofs and details can be found in Macdonald's book [14]. Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ be a partition with $k \leq n$. Being given a set of variables $\{a_1, a_2, \ldots, a_n\}$, the monomial symmetric function

$$m_\lambda = m_{[\lambda_1, \lambda_2, \ldots, \lambda_k]}(a_1, a_2, \ldots, a_n),$$

on these variables is the sum of monomial $a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_k^{\lambda_k}$ and all distinct monomials obtained from it by a permutation of variables.

For instance, with $\lambda = [2, 1, 1]$ and $n = 4$, we have:

Table 1 $(p,\ q)$-Jacobsthal and (p,q) -Jacobsthal Lucas numbers for $0 \leq n \leq 6$

n	0	1	2	3	4	5	6
$J_{p,q,n}$	0	1	p	p^2+2q	p^3+4pq	$p^4+6p^2q+4q^2$	$p^5+8p^3q+12pq^2$
$j_{p,q,n}$	2	p	p^2+4q	p^3+6pq	$p^4+8p^2q+8q^2$	$p^5+10p^3q+20pq^2$	$p^6+12p^4q+36p^2q^2+16q^3$

$$\begin{aligned}m_{[2,1,1]} =& a_1^2a_2a_3 + a_1a_2^2a_3 + a_1a_2a_3^2 + a_1^2a_2a_4 \\ &+ a_1a_2^2a_4 + a_1a_2a_4^2 + a_1^2a_3a_4 + a_1a_3^2a_4 \\ &+ a_1a_3a_4^2 + a_2^2a_3a_4 + a_2a_3^2a_4 + a_2a_3a_4^2.\end{aligned}$$

We remark that the kth complete homogeneous symmetric function h_k is the sum of all monomials of total degree k in these variables, that is:

Definition 1.2 [16] Let k and n be two positive integers, and $\{a_1, a_2, \ldots, a_n\}$ are set of given variables the kth complete homogeneous symmetric function $h_k(a_1, a_2, \ldots, a_n)$ is defined by:

$$h_k^{(n)} = h_k(a_1, a_2, \ldots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1}a_2^{i_2}...a_n^{i_n} \qquad (0 \le k),$$

with $i_1, i_2, \ldots, i_n \geq 0$.

Definition 1.3 [16] Let k and n be two positive integers, and $\{a_1, a_2, \ldots, a_n\}$ are set of given variables the kth elementary symmetric function $e_k(a_1, a_2, \ldots, a_n)$ is defined by:

$$e_k^{(n)} = e_k(a_1, a_2, \ldots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1}a_2^{i_2}...a_n^{i_n} \qquad (0 \le k \le n),$$

with $i_1, i_2, \ldots, i_n = 0$ or 1.

Remark 1.4 Set $e_0(a_1, a_2, \ldots, a_n) = 1$ and $h_0(a_1, a_2, \ldots, a_n) = 1$, by usual convention. For $k < 0$, we set $e_k(a_1, a_2, \ldots, a_n) = 0$ and $h_k(a_1, a_2, \ldots, a_n) = 0$.

Definition 1.5 Given an alphabet $A = \{a_1, a_2, \ldots, a_n\}$, we have:

$$\sum_{k=0}^{+\infty} h_k(a_1, a_2, \ldots, a_n)z^k = \frac{1}{\prod_{a\in A}(1 - az)}.$$

In particular, $m_{[k]} = p_k$ is kth power sum symmetric function, that is:

$$p_k = p_k(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} a_i^k.$$

For $k = 0$, we consider $h_0(a_1, a_2, \ldots, a_n) = 1$, $e_0(a_1, a_2, \ldots, a_n) = 1$, and $p_0(a_1, a_2, \ldots, a_n) = n$.

There is a fundamental relation between the elementary symmetric polynomials and the complete homogeneous ones:

$$\sum_{j=0}^{k}(-1)^j e_j(a_1, a_2, \ldots, a_n)h_{k-j}(a_1, a_2, \ldots, a_n) = 0,$$

which is valid for all $k > 0$.

Definition 1.6 [2] Given an alphabet $B = \{b_1, b_2\}$, the symmetrizing operator $\delta^k_{b_1 b_2}$ is defined by:

$$\delta^k_{b_1 b_2}(f) = \frac{b_1^k f(b_1) - b_2^k f(b_2)}{b_1 - b_2} \text{ for all } k \in \mathbb{N}_0.$$

2 Main results

In this part, we are now in a position to provide new theorem.

Theorem 2.1 *Let A and B be two alphabets, respectively,* $\{a_1, a_2, \ldots, a_k\}$ *and* $\{b_1, b_2\}$, *and then, we have*:

$$\begin{aligned} &\sum_{n=0}^{+\infty} h_n(a_1, a_2, \ldots, a_k) h_{n+1-l}(b_1, b_2) z^n \\ &= \frac{h_{1-l}(b_1, b_2) - e_1(a_1, a_2, \ldots, a_k) b_1 b_2 h_{-l}(b_1, b_2) z}{\left(\sum_{n=0}^{+\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_2^n z^n\right)} \\ &- \frac{b_1^{2-l} b_2^{2-l} z^{3-l} \sum_{n=0}^{+\infty} (-1)^{n-l+3} e_{n-l+3}(a_1, a_2, \ldots, a_k) h_n(b_1, b_2) z^n}{\left(\sum_{n=0}^{+\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_2^n z^n\right)} \end{aligned} \tag{2}$$

for all $n,\ l,\ k \in \mathbb{N}_0$.

Proof By applying the operator $\delta^{2-l}_{b_1 b_2}$ to the series $f(b_1 z) = \sum_{n=0}^{+\infty} h_n(a_1, a_2, \ldots, a_k) b_1^n z^n$, we have:

$$\begin{aligned} \delta^{2-l}_{b_1 b_2} f(b_1 z) &= \frac{b_1^{2-l} \sum_{n=0}^{+\infty} h_n(a_1, a_2, \ldots, a_k) b_1^n z^n - b_2^{2-l} \sum_{n=0}^{+\infty} h_n(a_1, a_2, \ldots, a_k) b_2^n z^n}{b_1 - b_2} \\ &= \sum_{n=0}^{+\infty} h_n(a_1, a_2, \ldots, a_k) \left(\frac{b_1^{n-l+2} - b_2^{n-l+2}}{b_1 - b_2}\right) z^n \\ &= \sum_{n=0}^{+\infty} h_n(a_1, a_2, \ldots, a_k) h_{n+1-l}(b_1, b_2) z^n. \end{aligned}$$

On the other hand, by applying the operator $\delta^{2-l}_{b_1 b_2}$ to the series:

$$f(b_1 z) = \frac{1}{\sum_{n=0}^{+\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n},$$

we obtain:

$$\begin{aligned}
\delta_{b_1 b_2}^{2-l} f(b_1 z) =& \delta_{b_1 b_2}^{2-l}\left(\frac{1}{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n}\right)\\
=& \frac{\frac{b_1^{2-l}}{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n} - \frac{b_2^{2-l}}{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n}}{b_1 - b_2}\\
=& \frac{b_1^{2-l}\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n - b_2^{2-l}\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n}{(b_1-b_2)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
=& \frac{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n b_2^n \frac{b_1^{2-l-n}-b_2^{2-l-n}}{b_1-b_2} z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
=& \frac{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n b_2^n h_{1-n-l}(b_1,b_2) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
=& \frac{\sum_{n=0}^{1-l}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n b_2^n h_{1-n-l}(b_1,b_2) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
&+ \frac{\sum_{n=3-l}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n b_2^n h_{1-n-l}(b_1,b_2) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
=& \frac{\sum_{n=0}^{1-l}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n b_2^n h_{1-n-l}(b_1,b_2) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
&- \frac{\sum_{n=3-l}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^{2-l} b_2^{2-l}\left(\frac{b_1^{n+l-2}-b_2^{n+l-2}}{b_1-b_2}\right) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)},
\end{aligned}$$

accordingly:

$$\begin{aligned}
\delta_{b_1 b_2}^{2-l} f(b_1 z) =& \frac{h_{1-l}(b_1,b_2) - e_1(a_1,a_2,a_3,\ldots,a_k) b_1 b_2 h_{-l}(b_1,b_2) z}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
&- \frac{b_1^{2-l} b_2^{2-l}\sum_{n=3-l}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) h_{n+l-3}(b_1,b_2) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
=& \frac{h_{1-l}(b_1,b_2) - e_1(a_1,a_2,\ldots,a_k) b_1 b_2 h_{-l}(b_1,b_2) z}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
&- \frac{b_1^{2-l} b_2^{2-l} z^{3-l}\sum_{n=0}^{+\infty}(-1)^{n-l+3} e_{n-l+3}(a_1,a_2,\ldots,a_k) h_n(b_1,b_2) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\sum_{n=0}^{+\infty} h_n(a_1,a_2,\ldots,a_k) h_{n+1-l}(b_1,b_2) z^n\\
&= \frac{h_{1-l}(b_1,b_2) - e_1(a_1,a_2,\ldots,a_k) b_1 b_2 h_{-l}(b_1,b_2) z}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}\\
&\quad - \frac{b_1^{2-l} b_2^{2-l} z^{3-l}\sum_{n=0}^{+\infty}(-1)^{n-l+3} e_{n-l+3}(a_1,a_2,\ldots,a_k) h_n(b_1,b_2) z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_1^n z^n\right)\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k) b_2^n z^n\right)}.
\end{aligned}$$

Thus, this completes the proof. □

- For $A=\{a_1,a_2,a_3\}$, $B=\{b_1,b_2\}$, $l=0$, $l=1$ and $l=2$ in Theorem 2.1, we deduce the following lemmas.

Lemma 2.2 *Given two alphabets $A=\{a_1,a_2,a_3\}$ and $B=\{b_1,b_2\}$, we have:*

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_{n+1}(b_1,b_2)z^n = \frac{b_1+b_2-b_1b_2(a_1+a_2+a_3)z+b_1^2b_2^2a_1a_2a_3z^3}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1-a_ib_2z)}. \tag{3}$$

Based on the relationship (3), we get:

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_n(b_1,b_2)z^n = \frac{(b_1+b_2)z-b_1b_2(a_1+a_2+a_3)z^2+b_1^2b_2^2a_1a_2a_3z^4}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1-a_ib_2z)}. \tag{4}$$

Lemma 2.3 *Given two alphabets $A=\{a_1,a_2,a_3\}$ and $B=\{b_1,b_2\}$, we have:*

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_n(b_1,b_2)z^n = \frac{1-b_1b_2(a_1a_2+a_1a_3+a_2a_3)z^2+b_1b_2(b_1+b_2)a_1a_2a_3z^3}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1-a_ib_2z)}. \tag{5}$$

Based on the relationship (5), we get:

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_{n-1}(b_1,b_2)z^n = \frac{z-b_1b_2(a_1a_2+a_1a_3+a_2a_3)z^3+b_1b_2(b_1+b_2)a_1a_2a_3z^4}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1-a_ib_2z)}. \tag{6}$$

Lemma 2.4 *Given two alphabets $A=\{a_1,a_2,a_3\}$ and $B=\{b_1,b_2\}$, we have:*

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_{n-1}(b_1,b_2)z^n = \frac{(a_1+a_2+a_3)z-(b_1+b_2)(a_1a_2+a_1a_3+a_2a_3)z^2}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1-a_ib_2z)} + \frac{a_1a_2a_3((b_1+b_2)^2-b_1b_2)z^3}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1-a_ib_2z)}. \tag{7}$$

3 Generating functions of the products of 2-orthogonal Chebyshev polynomials with (p,q)-numbers

In this part, we now derive the new generating functions of the products of 2-orthogonal Chebyshev polynomials with (p,q)-Fibonacci numbers, (p, q)-Lucas numbers, (p, q)-Pell numbers, (p, q)-Pell Lucas numbers, (p, q) -Jacobsthal numbers, and (p,q)-Jacobsthal Lucas numbers.

- Replacing b_2 by $(-b_2)$ in (5) and (7), we obtain:

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n = \frac{1+b_1b_2(a_1a_2+a_1a_3+a_2a_3)z^2 - b_1b_2(b_1-b_2)a_1a_2a_3z^3}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1+a_ib_2z)}. \tag{8}$$

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n = \frac{(a_1+a_2+a_3)z-(b_1-b_2)(a_1a_2+a_1a_3+a_2a_3)z^2}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1+a_ib_2z)} + \frac{a_1a_2a_3((b_1-b_2)^2+b_1b_2)z^3}{\prod_{i=1}^{3}(1-a_ib_1z)\prod_{i=1}^{3}(1+a_ib_2z)}. \tag{9}$$

This case consists of three related parts. First, the substitutions:

$$\begin{cases} a_1+a_2+a_3 = x \\ a_1a_2+a_1a_3+a_2a_3 = \alpha \\ a_1a_2a_3 = -\gamma \end{cases} \text{and} \begin{cases} b_1-b_2 = p \\ b_1b_2 = q \end{cases},$$

in (8) and (9), we obtain:

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n = \frac{1+\alpha qz^2+\gamma pqz^3}{f_{\alpha,\gamma}(z)}, \tag{10}$$

$$\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n = \frac{xz-\alpha pz^2-\gamma(p^2+q)z^3}{f_{\alpha,\gamma}(z)}, \tag{11}$$

with

$$f_{\alpha,\gamma}(z) = 1 - pxz + \left(\alpha\left(p^2+2q\right)-qx^2\right)z^2 + p\left(\gamma\left(p^2+3q\right)+\alpha qx\right)z^3 + q\left(\gamma x\left(p^2+2q\right)+\alpha^2q\right)z^4 + \alpha\gamma pq^2z^5 - \gamma^2q^3z^6,$$

thus, we get the following both corollary and theorem.

Corollary 3.1 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with (p, q)-Fibonacci numbers is given by*:

$$\sum_{n=0}^{+\infty} \widehat{T}_n(x) F_{p,q,n} z^n = \frac{xz - \alpha p z^2 - \gamma(p^2 + q)z^3}{f_{\alpha,\gamma}(z)}, \tag{12}$$

with $\widehat{T}_n(x) F_{p,q,n} = h_n(a_1, a_2, a_3) h_{n-1}(b_1, [-b_2])$.

Theorem 3.2 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with (p, q)-Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} \widehat{T}_n(x) L_{p,q,n} z^n = \frac{2 - pxz + \alpha(p^2 + 2q)z^2 + \gamma p(p^2 + 3q)z^3}{f_{\alpha,\gamma}(z)}. \tag{13}$$

Proof We know that:

$$L_{p,q,n} = 2h_n(b_1, [-b_2]) - p h_{n-1}(b_1, [-b_2]), \text{ (see [20]).}$$

We see that:

$$\begin{aligned}
\sum_{n=0}^{+\infty} \widehat{T}_n(x) L_{p,q,n} z^n &= \sum_{n=0}^{+\infty} h_n(a_1, a_2, a_3)(2h_n(b_1, [-b_2]) - p h_{n-1}(b_1, [-b_2]))z^n \\
&= 2 \sum_{n=0}^{+\infty} h_n(a_1, a_2, a_3) h_n(b_1, [-b_2]) z^n \\
&\quad - p \sum_{n=0}^{+\infty} h_n(a_1, a_2, a_3) h_{n-1}(b_1, [-b_2]) z^n.
\end{aligned}$$

Then, according to the relationships (10) and (11), we obtain:

$$\begin{aligned}
\sum_{n=0}^{+\infty} \widehat{T}_n(x) L_{p,q,n} z^n &= \frac{2(1 + \alpha q z^2 + \gamma p q z^3)}{f_{\alpha,\gamma}(z)} - \frac{p(xz - \alpha p z^2 - \gamma(p^2 + q)z^3)}{f_{\alpha,\gamma}(z)} \\
&= \frac{2 - pxz + \alpha(p^2 + 2q)z^2 + \gamma p(p^2 + 3q)z^3}{f_{\alpha,\gamma}(z)}.
\end{aligned}$$

This completes the proof. □

- By the relationships (12) and (13), we have two cases.

Case 1. With $\alpha = 0$, we obtain:

$$\begin{aligned} f_{0,\gamma}(z) =& 1 - pxz - qx^2z^2 + \gamma p\left(p^2+3q\right)z^3 \\ & + \gamma qx\left(p^2+2q\right)z^4 - \gamma^2 q^3 z^6, \end{aligned}$$

and we have the following corollaries.

Corollary 3.3 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with (p, q)-Fibonacci numbers is given by*:

$$\sum_{n=0}^{+\infty} \widehat{U}_n(x) F_{p,q,n} z^n = \frac{xz - \gamma(p^2+q)z^3}{f_{0,\gamma}(z)}.$$

Corollary 3.4 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with (p, q)-Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} \widehat{U}_n(x) L_{p,q,n} z^n = \frac{2 - pxz + \gamma p(p^2+3q)z^3}{f_{0,\gamma}(z)}.$$

Case 2. With $\alpha = 3$ and $\gamma = -1$, we obtain:

$$\begin{aligned} f_{3,-1}(z) =& 1 - pxz + \left(3p^2 + 6q - qx^2\right)z^2 + p\left(3qx - p^2 - 3q\right)z^3 \\ & + q\left(9q - x\left(p^2+2q\right)\right)z^4 - 3pq^2z^5 - q^3z^6, \end{aligned}$$

and we have the following corollaries.

Corollary 3.5 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with (p, q)-Fibonacci numbers is given by*:

$$\sum_{n=0}^{+\infty} \widetilde{T}_n(x) F_{p,q,n} z^n = \frac{xz - 3pz^2 + (p^2+q)z^3}{f_{3,-1}(z)}.$$

Corollary 3.6 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with (p, q)-Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} \widetilde{T}_n(x) L_{p,q,n} z^n = \frac{2 - pxz + 3(p^2+2q)z^2 - p(p^2+3q)z^3}{f_{3,-1}(z)}.$$

Second, the substitutions:

$$\begin{cases} a_1 + a_2 + a_3 = x \\ a_1a_2 + a_1a_3 + a_2a_3 = \alpha \\ a_1a_2a_3 = -\gamma \end{cases} \text{ and } \begin{cases} b_1 - b_2 = 2p \\ b_1b_2 = q \end{cases},$$

in (8) and (9), we obtain:

$$\sum_{n=0}^{+\infty} h_n(a_1, a_2, a_3)h_n(b_1, [-b_2])z^n = \frac{1 + \alpha qz^2 + 2\gamma pqz^3}{g_{\alpha,\gamma}(z)}, \tag{14}$$

$$\sum_{n=0}^{+\infty} h_n(a_1, a_2, a_3)h_{n-1}(b_1, [-b_2])z^n = \frac{xz - 2\alpha pz^2 - \gamma(4p^2 + q)z^3}{g_{\alpha,\gamma}(z)}, \tag{15}$$

with

$$\begin{aligned} g_{\alpha,\gamma}(z) =& 1 - 2pxz + \left(2\alpha\left(2p^2 + q\right) - qx^2\right)z^2 + 2p\left(\gamma\left(4p^2 + 3q\right) + \alpha qx\right)z^3 \\ &+ q\left(\alpha^2 q + 2\gamma x\left(2p^2 + q\right)\right)z^4 \\ &+ 2\alpha\gamma pq^2z^5 - \gamma^2q^3z^6, \end{aligned}$$

and we deduce the following both corollary and theorem.

Corollary 3.7 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with (p, q)-Pell numbers is given by:*

$$\sum_{n=0}^{+\infty} \widehat{T}_n(x)P_{p,q,n}z^n = \frac{xz - 2\alpha pz^2 - \gamma(4p^2 + q)z^3}{g_{\alpha,\gamma}(z)}, \tag{16}$$

with $\widehat{T}_n(x)P_{p,q,n} = h_n(a_1, a_2, a_3)h_{n-1}(b_1, [-b_2])$.

Theorem 3.8 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the first kind with (p, q)-Pell Lucas numbers is given by:*

$$\sum_{n=0}^{+\infty} \widehat{T}_n(x)Q_{p,q,n}z^n = \frac{2 - 2pxz + 2\alpha(2p^2 + q)z^2 + 2\gamma p(4p^2 + 3q)z^3}{g_{\alpha,\gamma}(z)}. \tag{17}$$

Proof Recall that, we have $Q_{p,q,n} = 2h_n(b_1, [-b_2]) - 2ph_{n-1}(b_1, [-b_2])$ (see [20]). We see that:

$$\begin{aligned}\sum_{n=0}^{+\infty}\widehat{T}_n(x)Q_{p,q,n}z^n =&\sum_{n=0}^{+\infty}h_n(a_1,a_2,a_3)(2h_n(b_1,[-b_2])-2ph_{n-1}(b_1,[-b_2]))z^n\\ =&2\sum_{n=0}^{+\infty}h_n(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n\\ &-2p\sum_{n=0}^{+\infty}h_n(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n.\end{aligned}$$

Using the relationships (14) and (15), we obtain:

$$\begin{aligned}\sum_{n=0}^{+\infty}\widehat{T}_n(x)Q_{p,q,n}z^n =&\frac{2(1+\alpha qz^2+2\gamma pqz^3)}{g_{\alpha,\gamma}(z)}-\frac{2p(xz-2\alpha pz^2-\gamma(4p^2+q)z^3)}{g_{\alpha,\gamma}(z)}\\ =&\frac{2-2pxz+2\alpha(2p^2+q)z^2+2\gamma p(4p^2+3q)z^3}{g_{\alpha,\gamma}(z)}.\end{aligned}$$

This completes the proof. □

- By the relationships (16) and (17), we have two cases.

Case 1. With $\alpha=0$, we obtain:

$$\begin{aligned}g_{0,\gamma}(z) =&1-2pxz-qx^2z^2+2\gamma p\left(4p^2+3q\right)z^3\\ &+2\gamma qx\left(2p^2+q\right)z^4-\gamma^2q^3z^6,\end{aligned}$$

and we have the following corollaries.

Corollary 3.9 *For $n\in\mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with (p,q)-Pell numbers is given by*:

$$\sum_{n=0}^{+\infty}\widehat{U}_n(x)P_{p,q,n}z^n=\frac{xz-\gamma(4p^2+q)z^3}{g_{0,\gamma}(z)}.$$

Corollary 3.10 *For $n\in\mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with (p,q)-Pell Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty}\widehat{U}_n(x)Q_{p,q,n}z^n=\frac{2-2pxz+2\gamma p(4p^2+3q)z^3}{g_{0,\gamma}(z)}.$$

Case 2. With $\alpha=3$ and $\gamma=-1$, we obtain:

$$g_{3,-1}(z) = 1 - 2pxz + \left(12p^2 + 6q - qx^2\right)z^2 + 2p\left(3qx - 4p^2 - 3q\right)z^3 + q\left(9q - 2x\left(2p^2 + q\right)\right)z^4 - 6pq^2z^5 - q^3z^6,$$

and we have the following corollaries.

Corollary 3.11 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with (p, q)-Pell numbers is given by*:

$$\sum_{n=0}^{+\infty} \widetilde{T}_n(x) P_{p,q,n} z^n = \frac{xz - 6pz^2 + (4p^2 + q)z^3}{g_{3,-1}(z)}.$$

Corollary 3.12 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with (p, q)-Pell Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} \widetilde{T}_n(x) Q_{p,q,n} z^n = \frac{2 - 2pxz + 6(2p^2 + q)z^2 - 2p(4p^2 + 3q)z^3}{g_{3,-1}(z)}.$$

Third, the substitutions:

$$\begin{cases} a_1 + a_2 + a_3 = x \\ a_1a_2 + a_1a_3 + a_2a_3 = \alpha \\ a_1a_2a_3 = -\gamma \end{cases} \text{ and } \begin{cases} b_1 - b_2 = p \\ b_1b_2 = 2q \end{cases},$$

in (8) and (9), we obtain:

$$\sum_{n=0}^{+\infty} h_n(a_1, a_2, a_3) h_n(b_1, [-b_2]) z^n = \frac{1 + 2\alpha qz^2 + 2\gamma pqz^3}{K_{\alpha,\gamma}(z)}, \tag{18}$$

$$\sum_{n=0}^{+\infty} h_n(a_1, a_2, a_3) h_{n-1}(b_1, [-b_2]) z^n = \frac{xz - \alpha pz^2 - \gamma(p^2 + 2q)z^3}{K_{\alpha,\gamma}(z)}, \tag{19}$$

with

$$K_{\alpha,\gamma}(z) = 1 - pxz + \left(\alpha\left(p^2 + 4q\right) - 2qx^2\right)z^2 + p\left(\gamma\left(p^2 + 6q\right) + 2\alpha qx\right)z^3 + 2q\left(\gamma x\left(p^2 + 4q\right) + 2\alpha^2 q\right)z^4 + 4\alpha\gamma pq^2z^5 - 8\gamma^2q^3z^6,$$

thus, we get the following both corollary and theorem.

Corollary 3.13 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials MPS of the first kind with (p, q)-Jacobsthal numbers is given by*:

$$\sum_{n=0}^{+\infty} \widehat{T}_n(x)J_{p,q,n}z^n = \frac{xz - \alpha pz^2 - \gamma(p^2+2q)z^3}{K_{\alpha,\gamma}(z)}, \tag{20}$$

with $\widehat{T}_n(x)J_{p,q,n} = h_n(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])$.

Theorem 3.14 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials MPS of the first kind with (p, q)-Jacobsthal Lucas numbers is given by:*

$$\sum_{n=0}^{+\infty} \widehat{T}_n(x)j_{p,q,n}z^n = \frac{2 - pxz + \alpha(p^2+4q)z^2 + \gamma p(p^2+6q)z^3}{K_{\alpha,\gamma}(z)}. \tag{21}$$

Proof By [20], we have $j_{p,q,n} = 2h_n(b_1,[-b_2]) - ph_{n-1}(b_1,[-b_2])$. Then, we can see that:

$$\begin{aligned}\sum_{n=0}^{+\infty} \widehat{T}_n(x)j_{p,q,n}z^n &= \sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)(2h_n(b_1,[-b_2]) - ph_{n-1}(b_1,[-b_2]))z^n \\ &= 2\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n \\ &\quad - p\sum_{n=0}^{+\infty} h_n(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n.\end{aligned}$$

Using the relationships (18) and (19) , we obtain:

$$\begin{aligned}\sum_{n=0}^{+\infty} \widehat{T}_n(x)j_{p,q,n}z^n &= \frac{2(1+2\alpha qz^2+2\gamma pqz^3)}{K_{\alpha,\gamma}(z)} - \frac{p(xz - \alpha pz^2 - \gamma(p^2+2q)z^3)}{K_{\alpha,\gamma}(z)} \\ &= \frac{2 - pxz + \alpha(p^2+4q)z^2 + \gamma p(p^2+6q)z^3}{K_{\alpha,\gamma}(z)}.\end{aligned}$$

This completes the proof. □

- By the relationships (20) and (21) , we have two cases.

Case 1. With $\alpha = 0$, we obtain:

$$K_{0,\gamma}(z) = 1 - pxz - 2qx^2z^2 + \gamma p\left(p^2+6q\right)z^3 + 2\gamma qx\left(p^2+4q\right)z^4 - 8\gamma^2q^3z^6,$$

and we have the following corollaries.

Corollary 3.15 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with (p, q)-Jacobsthal numbers is given by:*

$$\sum_{n=0}^{+\infty} \widehat{U}_n(x) J_{p,q,n} z^n = \frac{xz - \gamma(p^2+2q)z^3}{K_{0,\gamma}(z)}.$$

Corollary 3.16 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev MPS of the second kind with (p, q)-Jacobsthal Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} \widehat{U}_n(x) j_{p,q,n} z^n = \frac{2 - pxz + \gamma p(p^2+6q)z^3}{K_{0,\gamma}(z)}.$$

Case 2. With $\alpha = 3$ and $\gamma = -1$, we obtain:

$$\begin{aligned} K_{3,-1}(z) =& 1 - pxz + \left(3p^2 + 12q - 2qx^2\right)z^2 + p\left(6qx - p^2 - 6q\right)z^3 \\ &+ 2q\left(18q - x\left(p^2+4q\right)\right)z^4 - 12pq^2z^5 - 8q^3z^6, \end{aligned}$$

and we have the following corollaries.

Corollary 3.17 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with (p, q)-Jacobsthal numbers is given by*:

$$\sum_{n=0}^{+\infty} \widetilde{T}_n(x) J_{p,q,n} z^n = \frac{xz - 3pz^2 + (p^2+2q)z^3}{K_{3,-1}(z)}.$$

Corollary 3.18 *For $n \in \mathbb{N}$, the new generating function of the product of 2-orthogonal Chebyshev polynomials of the first kind with (p, q)-Jacobsthal Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} \widetilde{T}_n(x) j_{p,q,n} z^n = \frac{2 - pxz + 3(p^2+4q)z^2 - p(p^2+6q)z^3}{K_{3,-1}(z)}.$$

4 Generating functions of the products of trivariate Fibonacci polynomials with (*p*,*q*)-numbers

In this part, we now derive the new generating functions of the products of trivariate Fibonacci polynomials with (p, q)-Fibonacci numbers, (p, q)-Lucas numbers, (p, q)-Pell numbers, (p, q)-Pell Lucas numbers, (p, q) -Jacobsthal numbers, and (p, q)-Jacobsthal Lucas numbers.

- Replacing b_2 by $(-b_2)$ in (4) and (6), we obtain:

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n = \frac{(b_1-b_2)z + b_1b_2(a_1+a_2+a_3)z^2 + b_1^2b_2^2a_1a_2a_3z^4}{\prod_{i=1}^3(1-a_ib_1z)\prod_{i=1}^3(1+a_ib_2z)}. \tag{22}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n = \frac{z + b_1b_2(a_1a_2+a_1a_3+a_2a_3)z^3 - b_1b_2(b_1-b_2)a_1a_2a_3z^4}{\prod\limits_{i=1}^{3}(1-a_ib_1z)\prod\limits_{i=1}^{3}(1+a_ib_2z)}. \tag{23}$$

This case consists of three related parts. First, the substitutions:

$$\begin{cases} a_1+a_2+a_3 = x \\ a_1a_2+a_1a_3+a_2a_3 = -y \\ a_1a_2a_3 = t \end{cases} \text{ and } \begin{cases} b_1-b_2 = p \\ b_1b_2 = q \end{cases},$$

in (22) and (23), we obtain:

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n = \frac{pz+qxz^2+q^2tz^4}{f_{x,y,t}(z)}, \tag{24}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n = \frac{z-qyz^3-pqtz^4}{f_{x,y,t}(z)}, \tag{25}$$

with

$$\begin{aligned} f_{x,y,t}(z) =& 1 - pxz - \big(y\big(p^2+2q\big)+qx^2\big)z^2 - p\big(t\big(p^2+3q\big)+qxy\big)z^3 \\ &+ q\big(qy^2 - xt\big(p^2+2q\big)\big)z^4 + pq^2ytz^5 - q^3t^2z^6, \end{aligned}$$

and we deduce the following both corollary and theorem.

Corollary 4.1 *For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with (p,q)-Fibonacci numbers is given by:*

$$\sum_{n=0}^{+\infty} H_n(x,y,t)F_{p,q,n}z^n = \frac{z-qyz^3-pqtz^4}{f_{x,y,t}(z)}, \tag{26}$$

with $H_n(x,y,t)F_{p,q,n} = h_{n-1}(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])$.

Theorem 4.2 *For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with (p,q)-Lucas numbers is given by:*

$$\sum_{n=0}^{+\infty} H_n(x,y,t)L_{p,q,n}z^n = \frac{pz + 2qxz^2 + pqyz^3 + qt(p^2+2q)z^4}{f_{x,y,t}(z)}. \qquad (27)$$

Proof By referred to [20], we have:

$$L_{p,q,n} = 2h_n(b_1,[-b_2]) - ph_{n-1}(b_1,[-b_2]).$$

We see that:

$$\begin{aligned}\sum_{n=0}^{+\infty} H_n(x,y,t)L_{p,q,n}z^n =& \sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)(2h_n(b_1,[-b_2]) - ph_{n-1}(b_1,[-b_2]))z^n \\ =& 2\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n \\ & - p\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n.\end{aligned}$$

According to relationships (24) and (25), this gives the following equality:

$$\begin{aligned}\sum_{n=0}^{+\infty} H_n(x,y,t)L_{p,q,n}z^n =& \frac{2(pz+qxz^2+q^2tz^4)}{f_{x,y,t}(z)} - \frac{p(z-qyz^3-pqtz^4)}{f_{x,y,t}(z)} \\ =& \frac{pz + 2qxz^2 + pqyz^3 + qt(p^2+2q)z^4}{f_{x,y,t}(z)}.\end{aligned}$$

This completes the proof. □

- By the relationships (26) and (27) , we have three cases.

Case 1. Writing x^2 instead of x, x instead of y and taking $t = 1$, we obtain:

$$\begin{aligned}f_{x^2,x,1}(z) =& 1 - px^2z - x\left(p^2+2q+qx^3\right)z^2 - p\left(p^2+3q+qx^3\right)z^3 \\ & - qx^2\left(p^2+q\right)z^4 + pq^2xz^5 - q^3z^6,\end{aligned}$$

and we have the following corollaries.

Corollary 4.3 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with (p,q)-Fibonacci numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n(x)F_{p,q,n}z^n = \frac{z - qxz^3 - pqz^4}{f_{x^2,x,1}(z)}.$$

Corollary 4.4 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with $(p,\ q)$-Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n(x) L_{p,q,n} z^n = \frac{pz + 2qx^2z^2 + pqxz^3 + q(p^2 + 2q)z^4}{f_{x^2,x,1}(z)}.$$

Case 2. With $x = y = t = 1$, we obtain:

$$f_{1,1,1}(z) = 1 - pz - \left(p^2 + 3q\right)z^2 - p\left(p^2 + 4q\right)z^3 - q\left(p^2 + q\right)z^4 + pq^2z^5 - q^3z^6,$$

and we have the following corollaries.

Corollary 4.5 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with (p, q)-Fibonacci numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n F_{p,q,n} z^n = \frac{z - qz^3 - pqz^4}{f_{1,1,1}(z)}.$$

Corollary 4.6 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with (p, q)-Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n L_{p,q,n} z^n = \frac{pz + 2qz^2 + pqz^3 + q(p^2 + 2q)z^4}{f_{1,1,1}(z)}.$$

Case 3. With $t = 0$, we obtain:

$$f_{x,y,0}(z) = 1 - pxz - \left(y\left(p^2 + 2q\right) + qx^2\right)z^2 - pqxyz^3 + q^2y^2z^4,$$

and we have the following corollaries.

Corollary 4.7 *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with (p, q)-Fibonacci numbers is given by*:

$$\sum_{n=0}^{+\infty} F_n(x, y) F_{p,q,n} z^n = \frac{z - qyz^3}{f_{x,y,0}(z)}.$$

Corollary 4.8 *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with (p, q)-Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} F_n(x, y) L_{p,q,n} z^n = \frac{pz + 2qxz^2 + pqyz^3}{f_{x,y,0}(z)}.$$

Second, the substitutions:

$$\begin{cases} a_1 + a_2 + a_3 = x \\ a_1a_2 + a_1a_3 + a_2a_3 = -y \\ a_1a_2a_3 = t \end{cases} \text{ and } \begin{cases} b_1 - b_2 = 2p \\ b_1b_2 = q \end{cases},$$

in (22) and (23), we obtain:

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1, a_2, a_3)h_n(b_1, [-b_2])z^n = \frac{2pz + qxz^2 + q^2tz^4}{g_{x,y,t}(z)}, \tag{28}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1, a_2, a_3)h_{n-1}(b_1, [-b_2])z^n = \frac{z - qyz^3 - 2pqtz^4}{g_{x,y,t}(z)}, \tag{29}$$

with

$$\begin{aligned} g_{x,y,t}(z) =& 1 - 2pxz - \left(2y\left(2p^2 + q\right) + qx^2\right)z^2 - 2p\left(t\left(4p^2 + 3q\right) + qxy\right)z^3 \\ &+ q\left(qy^2 - 2xt\left(2p^2 + q\right)\right)z^4 + 2pq^2ytz^5 - q^3t^2z^6; \end{aligned}$$

thus, we get the following both corollary and theorem.

Corollary 4.9 *For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with (p, q)-Pell numbers is given by:*

$$\sum_{n=0}^{+\infty} H_n(x, y, t)P_{p,q,n}z^n = \frac{z - qyz^3 - 2pqtz^4}{g_{x,y,t}(z)}, \tag{30}$$

with $H_n(x, y, t)P_{p,q,n} = h_{n-1}(a_1, a_2, a_3)h_{n-1}(b_1, [-b_2])$.

Theorem 4.10 *For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with (p, q)-Pell Lucas numbers is given by:*

$$\sum_{n=0}^{+\infty} H_n(x, y, t)Q_{p,q,n}z^n = \frac{2pz + 2qxz^2 + 2pqyz^3 + 2qt(2p^2 + q)z^4}{g_{x,y,t}(z)}. \tag{31}$$

Proof By [20], we have $Q_{p,q,n} = 2h_n(b_1, [-b_2]) - 2ph_{n-1}(b_1, [-b_2])$. Then, We see that:

$$\begin{aligned} \sum_{n=0}^{+\infty} H_n(x, y, t)Q_{p,q,n}z^n =& \sum_{n=0}^{+\infty} h_{n-1}(a_1, a_2, a_3)(2h_n(b_1, [-b_2]) - 2ph_{n-1}(b_1, [-b_2]))z^n \\ =& 2\sum_{n=0}^{+\infty} h_{n-1}(a_1, a_2, a_3)h_n(b_1, [-b_2])z^n \\ & - 2p\sum_{n=0}^{+\infty} h_{n-1}(a_1, a_2, a_3)h_{n-1}(b_1, [-b_2])z^n, \end{aligned}$$

using the relationships (28) and (29), we obtain:

$$\sum_{n=0}^{+\infty} H_n(x,y,t)Q_{p,q,n}z^n = \frac{2(2pz+qxz^2+q^2tz^4)}{g_{x,y,t}(z)} - \frac{2p(z-qyz^3-2pqtz^4)}{g_{x,y,t}(z)}$$
$$= \frac{2pz+2qxz^2+2pqyz^3+2qt(2p^2+q)z^4}{g_{x,y,t}(z)}.$$

This completes the proof. □

- By the relationships (30) and (31), we have three cases.

Case 1. Writing x^2 instead of x, x instead of y and taking $t=1$, we obtain:

$$g_{x^2,x,1}(z) = 1 - 2px^2z - x\left(4p^2+2q+qx^3\right)z^2 - 2p\left(4p^2+3q+qx^3\right)z^3$$
$$- qx^2\left(4p^2+q\right)z^4 + 2pq^2xz^5 - q^3z^6,$$

and we have the following corollaries.

Corollary 4.11 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with (p, q)-Pell numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n(x)P_{p,q,n}z^n = \frac{z-qxz^3-2pqz^4}{g_{x^2,x,1}(z)}.$$

Corollary 4.12 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with (p, q)-Pell Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n(x)Q_{p,q,n}z^n = \frac{2pz+2qx^2z^2+2pqxz^3+2q(2p^2+q)z^4}{g_{x^2,x,1}(z)}.$$

Case 2. With $x=y=t=1$, we obtain:

$$g_{1,1,1}(z) = 1 - 2pz - \left(4p^2+3q\right)z^2 - 8p\left(p^2+q\right)z^3 - q\left(4p^2+q\right)z^4 + 2pq^2z^5 - q^3z^6,$$

and we have the following corollaries.

Corollary 4.13 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with (p,q)-Pell numbers is given by*:

$$\sum_{n=0}^{+\infty} T_nP_{p,q,n}z^n = \frac{z-qz^3-2pqz^4}{g_{1,1,1}(z)}.$$

Corollary 4.14 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with (p,q)-Pell Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n Q_{p,q,n} z^n = \frac{2pz + 2qz^2 + 2pqz^3 + 2q(2p^2+q)z^4}{g_{1,1,1}(z)}.$$

Case 3. With $t = 0$, we obtain:

$$g_{x,y,0}(z) = 1 - 2pxz - \left(2y\left(2p^2+q\right) + qx^2\right)z^2 - 2pqxyz^3 + q^2y^2z^4,$$

and we have the following corollaries.

Corollary 4.15 *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with (p, q)-Pell numbers is given by*:

$$\sum_{n=0}^{+\infty} F_n(x,y) P_{p,q,n} z^n = \frac{z - qyz^3}{g_{x,y,0}(z)}.$$

Corollary 4.16 *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with (p, q)-Pell Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} F_n(x,y) Q_{p,q,n} z^n = \frac{2pz + 2qxz^2 + 2pqyz^3}{g_{x,y,0}(z)}.$$

Third, the substitutions:

$$\begin{cases} a_1 + a_2 + a_3 = x \\ a_1a_2 + a_1a_3 + a_2a_3 = -y \\ a_1a_2a_3 = t \end{cases} \text{ and } \begin{cases} b_1 - b_2 = p \\ b_1b_2 = 2q \end{cases},$$

in (22) and (23), we obtain:

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3) h_n(b_1,[-b_2]) z^n = \frac{pz + 2qxz^2 + 4q^2tz^4}{k_{x,y,t}(z)}, \tag{32}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3) h_{n-1}(b_1,[-b_2]) z^n = \frac{z - 2qyz^3 - 2pqtz^4}{k_{x,y,t}(z)}, \tag{33}$$

with

$$\begin{aligned} k_{x,y,t}(z) = & 1 - pxz - \left(y\left(p^2+4q\right) + 2qx^2\right)z^2 - p\left(t\left(p^2+6q\right) + 2qxy\right)z^3 \\ & + 2q\left(2qy^2 - xt\left(p^2+4q\right)\right)z^4 + 4pq^2ytz^5 - 8q^3t^2z^6, \end{aligned}$$

and we deduce the following both corollary and theorem.

Corollary 4.17 *For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with (p, q)-Jacobsthal numbers is given by*:

$$\sum_{n=0}^{+\infty} H_n(x,y,t)J_{p,q,n}z^n = \frac{z-2qyz^3-2pqtz^4}{k_{x,y,t}(z)}, \tag{34}$$

with $H_n(x,y,t)J_{p,q,n} = h_{n-1}(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])$.

Theorem 4.18 *For $n \in \mathbb{N}$, the new generating function of the product of trivariate Fibonacci polynomials with (p,q)-Jacobsthal Lucas numbers is given by:*

$$\sum_{n=0}^{+\infty} H_n(x,y,t)j_{p,q,n}z^n = \frac{pz+4qxz^2+2pqyz^3+2qt(p^2+4q)z^4}{k_{x,y,t}(z)}. \tag{35}$$

Proof We know that:

$$j_{p,q,n} = 2h_n(b_1,[-b_2]) - ph_{n-1}(b_1,[-b_2]), \text{ (see [20])}.$$

We see that:

$$\begin{aligned}\sum_{n=0}^{+\infty} H_n(x,y,t)j_{p,q,n}z^n =& \sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)(2h_n(b_1,[-b_2]) - ph_{n-1}(b_1,[-b_2]))z^n \\ =& 2\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_n(b_1,[-b_2])z^n \\ & - p\sum_{n=0}^{+\infty} h_{n-1}(a_1,a_2,a_3)h_{n-1}(b_1,[-b_2])z^n.\end{aligned}$$

Then, according to the relationships (32) and (33), we obtain:

$$\begin{aligned}\sum_{n=0}^{+\infty} H_n(x,y,t)j_{p,q,n}z^n =& \frac{2(pz+2qxz^2+4q^2tz^4)}{k_{x,y,t}(z)} - \frac{p(z-2qyz^3-2pqtz^4)}{k_{x,y,t}(z)} \\ =& \frac{pz+4qxz^2+2pqyz^3+2qt(p^2+4q)z^4}{k_{x,y,t}(z)}.\end{aligned}$$

This completes the proof. □

- By the relationships (34) and (35), we have three cases.

Case 1. Writing x^2 instead of x, x instead of y and taking $t=1$, we obtain:

$$\begin{aligned}k_{x^2,x,1}(z) =& 1 - px^2z - x\left(p^2+4q+2qx^3\right)z^2 - p\left(p^2+6q+2qx^3\right)z^3 \\ & - 2qx^2\left(p^2+2q\right)z^4 + 4pq^2xz^5 - 8q^3z^6,\end{aligned}$$

and we have the following corollaries.

Corollary 4.19 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with (p,q)-Jacobsthal numbers is given by:*

$$\sum_{n=0}^{+\infty} T_n(x) J_{p,q,n} z^n = \frac{z - 2qxz^3 - 2pqz^4}{k_{x^2,x,1}(z)}.$$

Corollary 4.20 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials with (p, q)-Jacobsthal Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n(x) j_{p,q,n} z^n = \frac{pz + 4qx^2z^2 + 2pqxz^3 + 2q(p^2 + 4q)z^4}{k_{x^2,x,1}(z)}.$$

Case 2. With $x = y = t = 1$, we obtain:

$$k_{1,1,1}(z) = 1 - pz - \left(p^2 + 6q\right)z^2 - p\left(p^2 + 8q\right)z^3 - 2q\left(p^2 + 2q\right)z^4 + 4pq^2z^5 - 8q^3z^6,$$

and we have the following corollaries.

Corollary 4.21 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with (p, q)-Jacobsthal numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n J_{p,q,n} z^n = \frac{z - 2qz^3 - 2pqz^4}{k_{1,1,1}(z)}.$$

Corollary 4.22 *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci numbers with (p, q)-Jacobsthal Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} T_n j_{p,q,n} z^n = \frac{pz + 4qz^2 + 2pqz^3 + 2q(p^2 + 4q)z^4}{k_{1,1,1}(z)}.$$

Case 3. With $t = 0$, we obtain:

$$k_{x,y,0}(z) = 1 - pxz - \left(y\left(p^2 + 4q\right) + 2qx^2\right)z^2 - 2pqxyz^3 + 4q^2y^2z^4,$$

and we have the following corollaries.

Corollary 4.23 *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with (p, q)-Jacobsthal numbers is given by*:

$$\sum_{n=0}^{+\infty} F_n(x, y) J_{p,q,n} z^n = \frac{z - 2qyz^3}{k_{x,y,0}(z)}.$$

Corollary 4.24 *For $n \in \mathbb{N}$, the new generating function of the product of bivariate Fibonacci polynomials with (p, q)-Jacobsthal Lucas numbers is given by*:

$$\sum_{n=0}^{+\infty} F_n(x,y) j_{p,q,n} z^n = \frac{pz + 4qxz^2 + 2pqyz^3}{k_{x,y,0}(z)}.$$

5 Conclusion

In this paper, by making use of Theorem 2.1, we have derived some new generating functions of the products of trivariate Fibonacci polynomials and 2-orthogonal Chebyshev polynomials with (p, q)-numbers. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

Compliance with ethical standards

Conflict of interest The authors declare that they do not have conflict of interests.

Ethical standard This research complies with ethical standards.

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