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On a Class of Semicommutative Rings

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Abstract. In this paper, a generalization of the class of semicommutative rings is investigated. A ring R is called *central semicommutative* if for any $a, b \in R$, $ab = 0$ implies arb is a central element of R for each $r \in R$. We prove that some results on semicommutative rings can be extended to central semicommutative rings for this general settings.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Hence R is a semicommutative ring if and only if every right (or left) ideal annihilator in R is an ideal of R. A ring R is called *reduced* if it does not have any nonzero nilpotent elements. A ring R is called *weakly semicommutative* [7], if for any $a, b \in R$, $ab = 0$ implies arb is nilpotent for each $r \in R$. Semicommutative rings have also been studied under the names IFP rings and zero-insertive (ZI) rings in the literature. There are some generalization of semicommutative rings. Namely, a ring R is called *g-IFP* whenever $ab = 0$ for any a, $b \in R$ with $b \neq 0$, there exists a nonzero $c \in R$ such that $aRc = 0$ (see [5] in detail). In this paper we give another generalization of semicommutative rings. A ring R is called *central semicommutative* if for any $a, b \in R$, $ab = 0$ implies arb is a central element of R for each $r \in R$. It is clear that every semicommutative ring is central semicommutative. For any positive integer n and a ring R, $R^{n\times n}$ and $T_n(R)$ are the ring of $n \times n$ matrices and

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the $n \times n$ upper triangular matrix ring over the ring R respectively. Let R_n denote the subring $\{(a_{ij}) \in T_n(R) \mid$ all a_{ii} ' s are equal for $i = 1, 2, ..., n\}$ of $T_n(R)$. If R is a reduced ring, then R_n is not semicommutative for $n \geq 4$ from [6, Example 1.3]. But R_n is weakly semicommutative for all $n \geq 1$ by [7, Example 2.1]. We show that for some rings R, $R^{n\times n}$ for every $n \geq 5$ and $T_n(R)$ for every $n \geq 2$ are not central semicommutative rings. Moreover we prove that if R is a commutative reduced ring and k is a positive integer, then $T^k_{2k+2}(R)$ being a subring of $T_{2k+2}(R)$ is a central semicommutative ring, and R_4 is central semicommutative but not semicommutative. But in general we prove that R_n is not central semicommutative for $n \geq 5$. It is also proved that every central semicommutative ring is 2-primal.

Throughout this paper, the center of a ring R will be denoted by $C(R)$. For a positive integer n, Z_n denotes the ring of integers Z modulo n. We write $R[x]$ and $R[x, x^{-1}]$ for the polynomial ring and the Laurent polynomial ring over a ring R, respectively.

2. Central semicommutative rings

In this section we introduce a class of rings which is a generalization of semicommutative rings. We investigate some properties of this class of rings.

Lemma 2.1. *If R is a prime central semicommutative ring, then R does not have any nonzero divisors of zero.*

Proof. Let a,$b \in R$ with $ab = 0$. Then for any $r \in R$, arb is a central element and so a^2rb, arb^2 are central. For any $r \in R$, $b(arb)a = ba(arb) = b(a^2rb) = a^2rb^2 = a(arb)b = ab(arb) = 0$. Hence $baRba = 0$. By hypothesis $ba = 0$, and so $aRb = 0$. Hence $a = 0$ or $b = 0$. □

Proposition 2.2. *Let R be a semiprime central semicommutative ring. Then R is semicommutative.*

Proof. Let a,$b \in R$ with $ab = 0$. As in the proof of Lemma 2.1, $baRba = 0$ and so baR is a nilpotent right ideal. By hypothesis $ba = 0$ implies $arb = 0$ for all $r \in R$.□

A ring R is called *directly finite* whenever a, $b \in R$, $ab = 1$ implies $ba = 1$.

Proposition 2.3. *Every central semicommutative ring is directly finite.*

Proof. Let R be a central semicommutative ring and a, $b \in R$ with $ab = 1$. Then $a(ba - 1) = 0$. For any $r \in R$, $ar(ba - 1)$ is central in R. By commuting with b, we have $bar(ba - 1) = 0$. Multiplying the latter by a from the left we obtain $ar(ba - 1) = 0$. Replacing r by b we have $ba = 1$. □

Let R be a ring, $P(R)$ the prime radical and $N(R)$ the set of all nilpotent elements of the ring R. Since $P(R)$ is the intersection of all prime ideals of R, it is a nil ideal, therefore $P(R) \subseteq N(R)$. The ring R is called 2-*primal* if $P(R) = N(R)$ (see [3] and [5]). In [8, Theorem 1.5] it is proved that every semicommutative ring is 2-primal. In this direction we prove the following theorem.

Theorem 2.4. *Every central semicommutative ring is 2-primal.*

Proof. Let $a \in N(R)$. Assume $a^2 = 0$. Then ara is central and so $asara = ara^2s = 0$ for r, $s \in R$. Hence for any prime ideal P, since r and s are arbitrary elements in R, $asara \in P$ implies $a \in P$. Then $a \in P(R)$. Now assume $a^3 = 0$. Then for any $r \in R$, ara^2 is central. We commute the latter by a we obtain $a^2ra^2 = 0$. By hypothesis, for any $s \in R$, $asara^2$ is central. Again for any $t \in R$ $atasara^2 = 0$. Similarly for any $u \in R$, $atasaraua$ is central. By commuting with av for any $v \in R$ we have $avatasaraua = 0$. Then for any prime ideal P, $avatasaraua \in P$. Hence $a \in P$, and so is $a \in P(R)$. By an induction on the index of nilpotency of a, we may conclude that $N(R) \subseteq P(R)$. □

Lemma 2.5. *Every subring of a central semicommutative ring is central semicommutative.*

Proof. Let S be a subring of central semicommutative ring R, and a, $b \in R$ with $ab = 0$. Then arb is central for all $r \in R$. Hence arb commutes with every element of R, in particular it commutes with every element of S. □

Lemma 2.6. *Let R be a central semicommutative ring. Then every idempotent is central.*

Proof. Let $e^2 = e \in R$. By hypothesis $e(1-e) = 0$ implies $er(1-e)$ is central for all $r \in R$. Commuting e by $er(1-e)$ we obtain $er(1-e) = 0$. Similarly we have $(1-e)re = 0$. Hence $er = ere = re$. □

The following example shows that, the converse of the Lemma 2.6 may not be true in general.

Example 2.7. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z, a - d \equiv b \equiv c \equiv 0(\bmod 2) \right\}$$

Then only idempotents of R are zero and identity matrices, and

$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

is not central.

Lemma 2.8. *Let R be a commutative or reduced ring. Then R_2 and R_3 are central semicommutative.*

Proof. If R is a reduced ring, then R_2 and R_3 are semicommutative by [6], therefore they are central semicommutative. Assume that R is commutative. We prove R_3 is central semicommutative. Let $A = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}$,

$A_2 = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix}$ and $AA_2 = 0$. Then $aa_2 = 0$, $ab_2+ba_2 = 0$, $ac_2+bd_2+ca_2 = 0$ and $ad_2 + da_2 = 0$. We use these to obtain, for any elements a_1, b_1, c_1 and d_1 in R, $aa_1a_2 = 0$, $aa_1b_2 + (ab_1 + ba_1)a_2 = 0$, $aa_1d_2 + (ad_1 + da_1)a_2 = 0$. Then for any $A_1 = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \in R_3$, $AA_1A_2 = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for some $u \in R$. It is clear that $AA_1A_2 \in C(R_3)$. The rest is clear since the commutativity of R implies that of R_2. □

We now introduce a notation for some subrings of $T_n(R)$. Let k be a natural number smaller than n. Say

$$T_n^k(R) = \{\sum_{i=j}^{n}\sum_{j=1}^{k} x_j e_{(i-j+1)i} + \sum_{i=j}^{n-k}\sum_{j=1}^{n-k} a_{ij}e_{j(k+i)} : x_j, a_{ij} \in R\}$$

where e_{ij}' s are matrix units. Elements of $T_n^k(R)$ are in the form

$$\begin{bmatrix} x_1 & x_2 & \dots & x_k & a_{1(k+1)} & a_{1(k+2)} & \dots & a_{1n} \\ 0 & x_1 & \dots & x_{k-1} & x_k & a_{2(k+2)} & \dots & a_{2n} \\ 0 & 0 & x_1 & \dots & & & & a_{3n} \\ & & & \dots & & & & \\ & & & & & & & x_1 \end{bmatrix}$$

where $x_i \in R$, $a_{js} \in R$, $1 \le i \le k$, $1 \le j \le n-k$ and $k+1 \le s \le n$.

Lemma 2.9. *Let R be any ring. Then*

1. *R_n is not central semicommutative for all $n \ge 5$.*
2. *$T_n(R)$ is not central semicommutative for all $n \ge 2$.*
3. *$R^{n\times n}$ is not central semicommutative for all $n \ge 2$.*
4. *If R is reduced, then for $n \ge 4$ and $k = [\frac{n}{2}]$, the subring $T_n^k(R)$ is central semicommutative.*

Proof. (1) Let e_{ij} denote the $n\times n$ matrix units. Then $e_{12}e_{34} = 0$. But $e_{12}e_{23}e_{34} = e_{14}$ and $e_{15} = e_{14}e_{45} \ne e_{45}e_{14} = 0$. Hence $e_{12}e_{23}e_{34}$ is not central and so R_5 is not central semicommutative. Since R_5 may be embedded, as a subring, in R_n for any $n \ge 5$, by Lemma 2.5 R_n for any $n \ge 5$ is not a central semicommutative ring.
(2) Assume that $T_n(R)$ is central semicommutative for some $n \ge 2$. Let $e^2 = e \in T_n(R)$. By Lemma 2.6 e is a central element of $T_n(R)$. Hence $e = 0$ or e is the identity. So it cannot be central semicommutative.
(3) Assume that $R^{n\times n}$ is central semicommutative for all $n \ge 2$. By Lemma 2.5, $T_n(R)$ will be central semicommutative. This is not the case.

(4) By [1, Theorem 2.5] $T_n^k(R)$ is semicommutative for $n \geq 4$ and $k = [\frac{n}{2}]$ and so it is central semicommutative. □

In [7] it is proved that R_5 is a weakly semicommutative ring. But in Lemma 2.9(1) we prove that R_5 is not central semicommutative. So, weakly semicommutative rings are not central semicommutative. But we have the following lemma.

Proposition 2.10. *Every central semicommutative ring is weakly semicommutative.*

Proof. Let $a, b \in R$ and $ab = 0$. We will prove $(arb)^2 = 0$ for any $r \in R$. Since R is a central semicommutative ring, for any $r \in R$, arb is in $C(R)$. Then $(arb)^2 = (arba)rb = (a^2rb)rb = (a^2rbr)b = (ra^2rb)b = r(a^2rb^2) = (ra)(arb)b = rab(arb) = 0$. □

Theorem 2.11. (i) *For any ring R, $T_n^k(R)$ is not a central semicommutative ring, where $n \in \mathbb{N}$, $n \geq 4$ and $0 \leq k \leq [\frac{n-3}{2}]$.*

(ii) *If $T_n^k(R)$ is a central semicommutative ring, where $n \in \mathbb{N}$, $n \geq 4$ and $2k+2 \leq n$, then R is commutative and $n = 2k+2$.*

Proof. (i) $e_{1(k+1)}e_{(k+2)(2k+2)} = 0$ and $e_{1(k+1)}(e_{12} + e_{23} + ... + e_{(k+1)(k+2)} + ... + e_{(n-1)n})e_{(k+2)(2k+2)} = e_{1(2k+2)} \in C(T_n^k(R))$. But $e_{1(2k+2)}(e_{12} + e_{23} + ... + e_{(2k+2)(2k+3)} + ... + e_{(n-1)n}) \neq 0 = (e_{12} + e_{23} + ... + e_{(2k+2)(2k+3)} + ... + e_{(n-1)n})e_{1(2k+2)}$. Therefore, if $2k+3 \leq n$, then $T_n^k(R)$ cannot be central semicommutative.

(ii) Assume that R is not commutative. Then there are elements $a, b \in R$ such that $ab \neq ba$. Since $e_{1(k+1)}e_{(k+2)(2k+2)} = 0$ where $2k+2 \leq n$ and $T_n^k(R)$ is central semicommutative, we can write that $e_{1(k+1)}a(e_{12} + e_{23} + ... + e_{(k+1)(k+2)} + ... + e_{(n-1)n})e_{(k+2)(2k+2)} = ae_{1(2k+2)} \in C(T_n^k(R))$ and so $ae_{1(2k+2)}b = bae_{1(2k+2)}$, that is $ab = ba$. But this is a contradiction. By (i) $2k+3 > n$ and $2k+2 \leq n$, $n = 2k+2$. □

Example 2.12 shows that the converse of Theorem 2.11 (ii) may not be true in general.

Example 2.12. Let $R = Z_4$ be the ring of integers modulo 4 and $A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ in R_4. Then $AB = 0$. For $C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in R_4, it is easy to check that $ACB = D$ and D is not central in R_4. Hence R_4 is not central semicommutative.

Theorem 2.13. *Let R be a commutative reduced ring and k a positive integer. Then $T_{2k+2}^k(R)$ is a central semicommutative ring.*

Proof. Note that $T^k_{2k+2}(R)$ is equal to the following set
$\left\{\begin{pmatrix} A & B \\ 0 & a_{11}\end{pmatrix} : A \in T^k_{2k+1}(R), B = \begin{pmatrix} b_1 & ... & b_{k+2} & a_{1k} & ... & a_{12}\end{pmatrix}^T\right\}$. Let $X = \begin{pmatrix} A & B \\ 0 & a_{11}\end{pmatrix}$ and $Y = \begin{pmatrix} A_1 & B_1 \\ 0 & a'_{11}\end{pmatrix} \in T^k_{2k+2}(R)$ and $XY = 0$. Then $AA_1 = 0$, $AB_1 + Ba'_{11} = 0$ and $a_{11}a'_{11} = 0$. Since $AA_1 = 0$ and R is a reduced ring, we have the following equalities:

$$\begin{array}{l} a_{11}a'_{ij} = a_{12}a'_{ij} = ... = a_{1k}a'_{ij} = 0 \\ a_{ij}a'_{11} = a_{ij}a'_{12} = ... = a_{ij}a'_{1k} = 0 \qquad ... \qquad (*) \end{array}$$

Since $AB_1 + Ba'_{11} = 0$, $AB_1a'_{11} + B(a'_{11})^2 = 0$. From being R commutative reduced and the equalities (*) we have $AB_1 = Ba'_{11} = 0$. Now we investigate that $AB_1 = 0$. We can write that $A = \begin{pmatrix} C & D \\ 0 & E\end{pmatrix}$ and $B_1 = \begin{pmatrix} x_1 & ... & x_{k+2} & a'_{1k} & ... & a'_{12}\end{pmatrix}^T$ where $C \in T^k_{k+2}(R)$, $E \in T^{k-2}_{k-1}(R)$ and D is a $(k+2)\times(k-1)$ matrix and $x_1, ..., x_{k+2} \in R$. Therefore, by $\begin{pmatrix} C & D \\ 0 & E\end{pmatrix}\begin{pmatrix} x_1 \\ ... \\ x_{k+2} \\ a'_{1k} \\ ... \\ a'_{12}\end{pmatrix} = \begin{pmatrix} C\begin{pmatrix} x_1 \\ ... \\ x_{k+2}\end{pmatrix} \\ 0\end{pmatrix} = 0$ we can obtain that $C\begin{pmatrix} x_1 \\ ... \\ x_{k+2}\end{pmatrix} = 0$. This implies the following equalities:

$$\begin{array}{ll} a_{11}x_2 = ... = a_{11}x_{k+2} & = 0 \\ a_{12}x_3 = ... = a_{12}x_{k+2} & = 0 \\ ... & \\ a_{1k}x_{k+1} = a_{1k}x_{k+2} & = 0 \\ a_{2(k+2)}x_{k+2} & = 0 \qquad .... \qquad (**) \end{array}$$

Let $T \in T^k_{2k+2}(R)$ where $T = \begin{pmatrix} A_2 & B_2 \\ 0 & a''_{11}\end{pmatrix}$. Since $T^k_{2k+1}(R)$ is semicommutative, when R is reduced, by [1] we can obtain that $AA_2A_1 = 0$ and $a_{11}a''_{11}a'_{11} = 0$. Hence $XTY = \begin{pmatrix} AA_2A_1 & AA_2B_1 + (AB_2 + Ba''_{11})a'_{11} \\ 0 & a_{11}a''_{11}a'_{11}\end{pmatrix} = \begin{pmatrix} 0 & AA_2B_1 \\ 0 & 0\end{pmatrix}$. Also since R is a commutative reduced ring and by the equalities in (*) we get that $AA_2B_1 + (AB_2 + Ba''_{11})a'_{11} = AA_2B_1$. Let $A_2 = \begin{pmatrix} C_2 & D_2 \\ 0 & E_2\end{pmatrix}$ where $C_2 \in T^k_{k+2}(R)$, $E_2 \in T^{k-2}_{k-1}(R)$ and D_2 is a $(k+2)\times(k-1)$ matrix. Since R is a commutative reduced ring, by using (*) we can write the following:

$$AA_2B_1 = \begin{pmatrix} CC_2\begin{pmatrix} x_1 \\ \dots \\ x_{k+2} \end{pmatrix} + (CD_2+DE_2)\begin{pmatrix} a'_{1k} \\ \dots \\ a'_{12} \end{pmatrix} \\ EE_2\begin{pmatrix} a'_{1k} \\ \dots \\ a'_{12} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} CC_2\begin{pmatrix} x_1 \\ \dots \\ x_{k+2} \end{pmatrix} \\ 0 \end{pmatrix}.$$

By (**) there is $y \in R$ such that $CC_2\begin{pmatrix} x_1 \\ \dots \\ x_{k+2} \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ \dots \\ 0 \end{pmatrix}$. Take any $T_1 \in T^k_{2k+2}(R)$ then $T_1 = \begin{pmatrix} A_3 & B_3 \\ 0 & a'''_{11} \end{pmatrix}$ for suitable A_3, B_3 and a'''_{11}. Therefore

$$XTYT_1 = \begin{pmatrix} 0 & \begin{pmatrix} y \\ 0 \\ \dots \\ 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_3 & B_3 \\ 0 & a'''_{11} \end{pmatrix} = \begin{pmatrix} 0 & \begin{pmatrix} ya'''_{11} \\ 0 \\ \dots \\ 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix}$$ and $T_1XTY =$

$$\begin{pmatrix} A_3 & B_3 \\ 0 & a'''_{11} \end{pmatrix}\begin{pmatrix} 0 & \begin{pmatrix} y \\ 0 \\ \dots \\ 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{pmatrix} a'''_{11}y \\ 0 \\ \dots \\ 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix},$$ that is, $XTYT_1 = T_1XTY$

and then we get that $XTY \in C(T^k_{2k+2}(R))$. Thus $T^k_{2k+2}(R)$ is a central semicommutative ring. □

Corollary 2.14. *Let R be a commutative reduced ring. Then R_4 is a central semicommutative ring which is not semicommutative.*

Let S denote a multiplicatively closed subset of R consisting of central regular elements. Let $S^{-1}R$ be the localization of R at S. Then we have the following.

Proposition 2.15. *Let R be a ring. Then R is central semicommutative if and only if $S^{-1}R$ is central semicommutative.*

Proof. Assume that R is a central semicommutative ring and let $a_1 = s^{-1}a$, $b_1 = t^{-1}b \in S^{-1}R$, where $t,s \in S$, and $a_1b_1 = 0$. Since s and t are central, $a_1b_1 = s^{-1}t^{-1}ab = 0$, and so $ab = 0$. By assumption $arb \in C(R)$ for all $r \in R$. Let $r \in R$ and $u \in S$. Then $s^{-1}t^{-1}u^{-1}$ and arb are central, and so $s^{-1}t^{-1}u^{-1}arb = (s^{-1}a)(u^{-1}r)(t^{-1}b)$ is central for every $u^{-1}r \in S^{-1}R$. Converse is clear since R may be embedded in $S^{-1}R$ as a subring and central semicommutativity is preserved under subrings. □

Corollary 2.16. *Let R be a ring. Then $R[x]$ is central semicommutative if and only if $R[x, x^{-1}]$ is central semicommutative.*

Proof. Let $S = \{1, x, x^2, x^3, x^4, \dots\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. It follows from Proposition 2.15. □

If R is a central semicommutative ring, then R/I may not be a central semicommutative ring in general, as the following example shows.

Example 2.17. Let D be a division ring, $R = D[x, y]$ and $I =< x^2 >$ with $xy \neq yx$. Then R is a semicommutative ring and so central semicommutative. Since $(x + I)^2 = I$ and $(x + I)(y + I)(x + I) = xyx + I \notin C(R/I)$, R/I is not a central semicommutative ring.

The next example shows that for a ring R and an ideal I, if both R/I and I are central semicommutative, then R need not be central semicommutative.

Example 2.18. Let F be a field. By Lemma 2.9(2), $R = T_2(F)$ is not a central semicommutative ring. Let $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then I is an ideal of R and $R/I \cong F$. Hence R/I and I are central semicommutative, but R is not.

Lemma 2.19. *Let R be a ring and I an ideal of R. If R/I is a central semicommutative ring and I is reduced, then R is a central semicommutative ring.*

Proof. Let $ab = 0$. Since $bIa \subseteq I$ and $(bIa)^2 = 0$, $bIa = 0$. Therefore $((aRb)I)^2 = 0$ and so $(aRb)I = 0$. Since R/I is central semicommutative and $(a + I)(b + I) = I$, $aRb + I \in C(R/I)$, that is, $arbr_1 - r_1arb \in I$ for all $r, r_1 \in R$. So $(arbr_1 - r_1arb)^2 \in (arbr_1 - r_1arb)I = 0$ by $(aRb)I = 0$. Then for all $r, r_1 \in R$ $arbr_1 = r_1arb$ and so $aRb \in C(R)$. □

For a commutative or reduced ring R, it is shown that R_2 is semicommutative, and so central semicommutative. One may suspect that if R is semicommutative or central semicommutative, then R_2 is central semicommutative. But the following example erases the possibility. This example appeared also in [4, Example 11].

Example 2.20. Let F be a field, $K = F[y]$ and $\alpha : K \longrightarrow K$, $\alpha(f(y) = f(y^2)$ be a ring homomorphism. Let $S = K[x; \alpha] = F[y][x; \alpha]$ be an Ore extension of K. Then S satisfies following condition: $xf(y) = \alpha(f(y))x = f(y^2)x$. Also from the fact that S is a noncommutative integral domain, S is a reduced ring. By Lemma 2.8, $U = S_2$ is semicommutative and so a central semicommutative ring. But $R = U_2$ is not a central semicommutative ring. For if

$$a = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \text{ and } b = \begin{pmatrix} \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -y^2x & 0 \\ 0 & -y^2x \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

then $ab = 0$ since $xy = y^2x \in S$. Let

$$r = \begin{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \end{pmatrix}. \text{ Since } y^2xy = y^4x \in S,$$

we have $arb = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & (-y^3+y^4)x \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$

which is not in $C(R)$. So R is not a central semicommutative ring.

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