On geometric applications of quaternions

Burcu BEKTAŞ DEMİRCİ[1,*] ⓘ**, Nazım AGHAYEV**[2] ⓘ
[1]Deparment of Civil Engineering, Faculty of Engineering, Fatih Sultan Mehmet Vakıf University, İstanbul, Turkey
[2]Department of Electrical and Electronics Engineering, Faculty of Engineering,
Fatih Sultan Mehmet Vakıf University, İstanbul, Turkey

Abstract: Quaternions have become a popular and powerful tool in various engineering fields, such as robotics, image and signal processing, and computer graphics. However, classical quaternions are mostly used as a representation of rotation of a vector in 3-dimensions, and connection between its geometric interpretation and algebraic structures is still not well-developed and needs more improvements. In this study, we develop an approach to understand quaternions multiplication defining subspaces of quaternion $\mathbb{H}$, called as $\mathrm{Plane}(N)$ and $\mathrm{Line}(N)$, and then, we observe the effects of sandwiching maps on the elements of these subspaces. Finally, we give representations of some transformations in geometry using quaternion.

Key words: Quaternions, reflection, orthogonal projection, sandwiching maps, involution and antiinvolution

1. Introduction

A quaternion, as an extension of the complex number, was first defined classically by Hamilton in 1866 [9]. Although the advantages of the quaternion appeared in the fundamental equations of some fields of science, Shoemake introduced formally using quaternions to specify rotations and orientation of coordinate system in 1985 [15]. In recent years, people are trying more and more to use algebraic properties of quaternions to make easy and efficient calculations. Some of these studies are given in [5, 13, 16, 17].

On the other hand, underlying geometric principles and characterizations of quaternions still not well-developed due to the difficulty of visualization in 4-dimensions. In this context, some authors give approaches to have a better intuitive geometric understanding of quaternion algebra [1, 7, 10, 11]. Especially, in [7], Goldman used three models to give an insight for quaternions, which are mass points in 3-dimensions, vectors in 4-dimensions and projections onto mutually orthogonal planes.

Affine and projective transformations in 3-dimensions and 4-dimensions, which are translations, rotations, reflections, shears, uniform and nonuniform scaling, orthogonal and perspective projection, play a key role in computer graphics [12]. These transformations can be expressed by linear transformations in 4-dimensions, which are represented by 4×4 matrices. For instance, [3, 8, 12] are some of the references about matrix representations of affine and projective transformations. Besides that, there are substantial results about how to show such transformations using quaternions in [2, 7].

In this paper, we continue to improve models based on projections onto mutually orthogonal planes and vectors in 3-dimensions given by Goldman in [7] to enrich geometric basics and interpretations of quaternions.

---

*Correspondence: bbektas@fsm.edu.tr

1289

First, we divide quaternion space $\mathbb{H}$ into two subspaces, called $\text{Plane}(N)$ and $\text{Line}(N)$, determined by a unit pure quaternion N. Then, we evaluate and examine the effects of sandwiching maps over these subspaces. In [14], the expressions for a rotation on a sphere from one point to another one were found. As a different perspective, we find how to interpret such transformations using L and R operators, which include quaternion multiplication. Furthermore, we provide an alternative geometric understanding of well-known transformations in computer graphics with respect to $\text{Plane}(N)$ and $\text{Line}(N)$, additionally paying attention to the reflection in 3-dimensions and 4-dimensions. In the last section, our main goal is to investigate expression of orthogonal projection in 3-dimensions by 4×4 matrix and sandwiching maps. Although the 4×4 matrix for orthogonal projection is given in [8] for a point in 3-dimensions, we investigate a solution for locations of different hidden points whose projections on a plane are the same.

2. Quaternion algebra and some properties

A quaternion given as the following set is an extension of complex numbers to 4-dimensions

$$\mathbb{H} = \{p = s_p + p_1 i + p_2 j + p_3 k \mid i^2 = j^2 = k^2 = ijk = -1 \text{ and } s_p, p_1, p_2, p_3 \in \mathbb{R}\}. \tag{2.1}$$

It is natural to split a quaternion p into the scalar part s_p and the vector part v_p as $p = s_p\mathbf{O} + v_p$ where $\mathbf{O}$ denotes $(1, 0, 0, 0)$ in 4-dimensions.

The quaternion multiplication for p and q is given by

$$pq = (s_p\mathbf{O} + v_p)(s_q\mathbf{O} + v_q) = (s_p s_q - v_p \cdot v_q)\mathbf{O} + s_p v_q + s_q v_p + v_p \times v_q \tag{2.2}$$

where $\cdot$ and $\times$ denote the scalar multiplication and the vector multiplication of v_p and v_q, respectively. It can be easily seen that $\mathbf{O} = \mathbf{O}^2$ is an identity of the quaternion multiplication. Quaternion multiplication is associative and distributes through addition, but it is not commutative. All scalars $r \in \mathbb{R}$ may be expressed as $r = r\mathbf{O}$.

The conjugate of a quaternion $p = s_p\mathbf{O} + v_p$, denoted by $\bar{p}$, is defined by $\bar{p} = s_p\mathbf{O} - v_p$ and the norm of p is $|p|^2 = (s_p)^2 + v_p \cdot v_p$. Thus, we have $p\bar{p} = (s_p^2 + v_p \cdot v_p)\mathbf{O} = |p|^2\mathbf{O}$. Also, $\overline{pq} = \bar{q}\bar{p}$ implies $|pq| = |p||q|$.

Every nonzero quaternion p has a multiplicative inverse, that is, $p^{-1} = \frac{\bar{p}}{|p|^2}$.

A quaternion p with $|p| = 1$ is called a unit quaternion. All unit quaternions form a unit hypersphere $\mathbb{S}^3$ in $\mathbb{R}^4$ given as

$$\mathbb{S}^3 = \{q \in \mathbb{R}^4 \mid |q|^2 = 1\}. \tag{2.3}$$

An angle θ between unit quaternions p and q satisfies the property

$$\cos\theta = \frac{1}{2}(p\bar{q} + q\bar{p}). \tag{2.4}$$

A quaternion p is said to be pure if $s_p = 0$. It is also identical with a vector in 3-dimensions. For any unit pure quaternion p, we have

$$\bar{p} = -p, \quad p^2 = -\mathbf{O}.$$

Also, for pure quaternions p and q, the quaternion multiplication becomes

$$pq = (-p \cdot q)\mathbf{O} + p \times q \tag{2.5}$$

Note that a set of all unit pure quaternions forms a unit sphere

$$\mathbb{S}^2 = \{p \in \mathbb{R}^3 \mid |p|^2 = 1 \text{ and } \bar{p} = -p\} \tag{2.6}$$

in $\mathbb{R}^3$ and since $\bar{p} = -p$ and $\bar{q} = -q$, we have $\cos\theta = -\frac{1}{2}(pq + qp)$.

The set given in (2.1) is a Cartesian form of quaternion p and $\{1, i, j, k\}$ is used as a specific basis. However, it is possible to define different bases for quaternions. Let N and v be any two unit pure quaternions which are orthogonal to each other. Since $N \perp v$ and $Nv = N \times v$ is exactly perpendicular to N and v, $\{\mathbf{O}, N, v, N \times v\}$ gives another basis of $\mathbb{H}$. Then, a quaternion p can be expressed according to the last basis as follows:

$$p = s_p\mathbf{O} + a_pN + b_pv + c_pN \times v$$

for $s_p, a_p, b_p, c_p \in \mathbb{R}$. Since there are infinitely many pairs of orthogonal unit pure quaternions, we have infinitely many choices for bases of $\mathbb{H}$.

Multiplication by quaternion is a linear transformation that preserves lengths, so multiplication by unit quaternions represents a rotation in 4-dimensions. To visualize geometric effects of quaternion multiplication, we will use the same notations used in [7]:

$$L_q(p) = qp \text{ and } R_q(p) = pq \tag{2.7}$$

and for the composition of these transformations, called sandwiching operator, we will use:

$$S_q(p) = L_q(R_{\bar{q}}(p)) = qp\bar{q} \text{ and } T_q(p) = L_q(R_q(p)) = qpq \tag{2.8}$$

for a unit quaternion $q(N, \theta) = \cos\theta\mathbf{O} + \sin\theta N$. It can be easily seen that the sandwiching operators S_q and T_q are linear transformations over quaternions.

As a convenience, we will use the lower case letters p, q, r to represent arbitrary nonpure quaternions and the lower case letters u, v, w to represent pure quaternions. Upper case letters P, Q will be used to denote points in 3-dimensional space.

3. The geometry of quaternion multiplication

As mentioned before in preliminaries, for any unit pure quaternion N, $\{\mathbf{O}, N, v, N \times v\}$ is a basis for $\mathbb{H}$. Then, every quaternion in the plane spanned by $\mathbf{O}, N$ is orthogonal to every pure quaternion in the plane spanned by $v, N \times v$. Hence, span$\{\mathbf{O}, N\}$ and span$\{v, N \times v\}$ determines orthogonal planes whose union gives the set of all quaternions and intersection is an origin of 4-dimensional coordinate system.

For a unit pure quaternion N, let us define the following sets:

$$\text{Plane}(N) = \{p \in \mathbb{H} \mid Np = -pN\}, \tag{3.1}$$

$$\text{Line}(N) = \{p \in \mathbb{H} \mid Np = pN\}. \tag{3.2}$$

It can be easily shown that Plane(N) and Line(N) are the subspaces of $\mathbb{H}$. In case $N = i$, Line$(i) = \mathbb{C}$ and Plane$(i) = \mathbb{C}j$.

Now, we will give a relation between these two sets with span$\{\mathbf{O}, N\}$ and span$\{v, N \times v\}$. This new presentations of subspaces will give us new relations over projections of quaternions.

Theorem 3.1 *For a unit pure quaternion N and an arbitrary quaternion p, the following statements are equivalent:*

(i) $p \in Plane(N)$.

(ii) $p \in span\{v, N \times v\}$.

(iii) $T_N(p) = -S_N(p) = p$.

Proof (i) $\Rightarrow$ (ii) Suppose $p = s_p\mathbf{O} + v_p$ and $p \in \text{Plane}(N)$. Using $N\mathbf{O} = \mathbf{O}N = N$ and the equation (2.5), we calculate Np as follows:

$$Np = N(s_p\mathbf{O} + v_p) = s_pN\mathbf{O} + Nv_p = (-N \cdot v_p)\mathbf{O} + s_pN + N \times v_p. \tag{3.3}$$

Similarly, we find that

$$pN = (s_p\mathbf{O} + v_p)N = s_p\mathbf{O}N + v_pN = (-v_p \cdot N)\mathbf{O} + s_pN + v_p \times N. \tag{3.4}$$

Since $p \in \text{Plane}(N)$, that is, $Np = -pN$, equations (3.3) and (3.4) imply:

$$(-2v_p \cdot N)\mathbf{O} + 2s_pN = 0.$$

Hence, $s_p = 0$ and $N \cdot v_p = 0$, which means that $p = v_p$ is a pure quaternion and $p \in span\{v, N \times v\}$, respectively.

(ii) $\Rightarrow$ (iii) Assume $p \in span\{v, N \times v\}$. Then, there exist constants b_p and c_p such that $p = b_pv + c_pN \times v$. Using the linearity of S, $S_N(v) = -v$ and $S_N(N \times v) = -N \times v$, we find

$$S_N(p) = S_N(b_pv + c_pN \times v) = b_pS_N(v) + c_pS_N(N \times v) = -b_pv - c_pN \times v = -p. \tag{3.5}$$

Since $T_N(p) = -S_N(p)$, $T_N(p) = p$ for $p \in span\{v, N \times v\}$.

(iii) $\Rightarrow$ (i) Assume $T_N(p) = -S_N(p) = p$. Then, we have $NpN = p$. Since $N\bar{N} = \bar{N}N = \mathbf{O}$ and $\bar{N} = -N$, $Np = -pN$ which means $p \in \text{Plane}(N)$. $\qquad\square$

From (ii) of Theorem 3.1, we get the following corollary:

Corollary 3.2 *All elements of $Plane(N)$ are pure quaternions.*

Note that from equation (2.4), we have $Np = -pN$ for unit pure quaternions N and p when N is perpendicular to p. Thus, all pure quaternions orthogonal to N are elements of $\text{Plane}(N)$.

Theorem 3.3 *For a unit pure quaternion N and an arbitrary quaternion p, the following statements are equivalent:*

(i) $p \in Line(N)$.

(ii) $p \in span\{O, N\}$.

(iii) $S_N(p) = -T_N(p) = p$.

Proof (i) $\Rightarrow$ (ii) Suppose $p = s_p\mathbf{O} + v_p$ and $p \in \text{Line}(N)$. From equations (3.3) and (3.4), $Np = pN$ gives $N \times v_p = 0$, that is, v_p is parallel to N. Hence, $p \in \text{span}\{\mathbf{O}, N\}$.

(ii) $\Rightarrow$ (iii) Assume $p \in \text{span}\{\mathbf{O}, N\}$. Then, there exist constants s_p and a_p such that $p = s_p\mathbf{O} + a_pN$. Using the linearity of S, $S_N(\mathbf{O}) = \mathbf{O}$ and $S_N(N) = N$, we find

$$S_N(p) = S_N(s_p\mathbf{O} + a_pN) = s_pS_N(\mathbf{O}) + a_pS_N(N) = s_p\mathbf{O} + a_pN = p. \tag{3.6}$$

Since $T_N(p) = -S_N(p)$, $T_N(p) = -p$ for $p \in \text{span}\{\mathbf{O}, N\}$.

(iii) $\Rightarrow$ (i) Assume $S_N(p) = -T_N(p) = p$, that is, $NpN = -p$. Since $N\bar{N} = \bar{N}N = \mathbf{O}$ and $\bar{N} = -N$, $Np = pN$. That is $p \in \text{Line}(N)$. $\qquad\square$

In [14], Perwin and Webb obtained some results for a rotation of points on a sphere $\mathbb{S}^2$. In this part of the section, we will try to express such a rotation using the transformation L_q and R_q defined in equation (2.7).

Proposition 3.4 *For an arbitrary unit pure quaternions v and w in $\text{Plane}(N)$, we have the following statements:*

(i) $L_{(-vw)}$ rotates w by the angle θ in $\text{Plane}(N)$,

(ii) $R_{(-wv)}$ rotates v by the angle $-\theta$ in $\text{Plane}(N)$

where θ is an angle between w and v.

Proof (i) Using $w^2 = -\mathbf{O}$, we have $L_{(-vw)}(w) = (-vw)w = v$. Also, from equation (2.5) we know that

$$-vw = \cos\theta\mathbf{O} + \sin\theta N.$$

Thus, from Proposition 7.3 in [7], it can be said that $L_{(-vw)}$ rotates w by the angle θ in $\text{Plane}(N)$.

(ii) Similarly, using Proposition 7.3 in [7], $R_{(-wv)}(v) = (-wv)v = w$ and $R_{(-wv)}$ rotates v by the angle $-\theta$ in $\text{Plane}(N)$. $\qquad\square$

Let g be a great circle of a sphere $\mathbb{S}^2$ defined by the intersection of $\text{Plane}(N)$ and $\mathbb{S}^2$. From [5], we know that two unit pure quaternions v, w in $\text{Plane}(N)$ define an arc of a great circle g, whose arclength is an angle between v and w. Using Proposition 3.4, we give the following statement:

Corollary 3.5 *Let v and w be two unit pure quaternions of $\text{Plane}(N)$. Then, $L_{(-vw)}$ gives a rotation of w along g until it reaches v. Moreover, $R_{(-wv)}$ also gives same rotation in the reverse direction (Figure 1).*
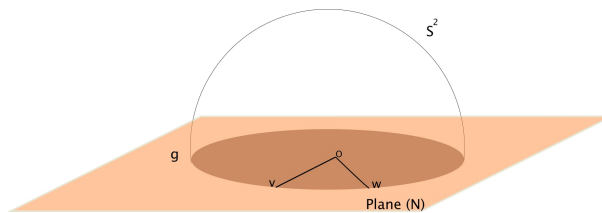

Figure 1. Rotation in $\mathbb{S}^2 \cap \text{Plane}(N)$.

For any pure quaternion v, it can be easily seen that $v \times N$ is an element of $\text{Plane}(N)$. Thus, the following statement is a straightforward consequence of Corollary 3.5.

Corollary 3.6 *Let v and w be two unit pure quaternions which are not in $Plane(N)$. Then, $L_{(-v \times N)(w \times N)}$ gives a rotation of w about a line parallel to N until it reaches v. Moreover, $R_{(-w \times N)(v \times N)}$ also gives the same rotation in the reverse direction (Figure 2).*
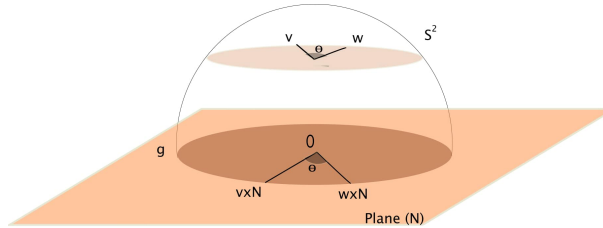

Figure 2. Rotation between two points of $\mathbb{S}^2$.

If we associate the point V on a sphere $\mathbb{S}^2$ with a pure quaternion v and the point W on a sphere $\mathbb{S}^2$ with a pure quaternion w, then the angle of the arc from V to W is in fact the angle between v and w. Thus, Corollaries 3.5 and 3.6 exactly give a motion between from one point to another one on a sphere $\mathbb{S}^2$.

Theorem 3.7 *Let v and w be unit pure quaternions in $Plane(N)$. Then, v bisects an angle between w and $S_{vw}(w)$.*

Proof Suppose v and w are unit pure quaternions in $Plane(N)$, that is, $Nv = -vN$ and $Nw = -wN$. Let us call an angle between pure quaternions w and v by θ and $\theta \in (0, \pi]$.

First, we need to show $S_{vw}(w) \in Plane(N)$. From the fact that $\overline{vw} = \bar{w}\bar{v}$ and $w\bar{w} = \mathbf{O}$, we have $S_{vw}(w) = vw\bar{v}$. Also, it can be easily said that $S_{vw}(w)$ is a unit quaternion. Then, using $v, w \in Plane(N)$ and $\bar{v} = -v$, we get

$$NS_{vw}(w) = N(vw\bar{v}) = -(vN)(w\bar{v}) = v(wN)\bar{v} = vw(vN) = -vw\bar{v}N = -S_{vw}(w)N \qquad (3.7)$$

which implies that $S_{vw}(w)$ is in $Plane(N)$. Thus, from Corollary 3.2, $S_{vw}(w)$ is a unit pure quaternion. Assume that α is an angle between w and $S_{vw}(w)$. From equation (2.4), we have

$$\cos\theta = -\frac{1}{2}(vw + wv) \quad \text{and} \quad \cos\alpha = -\frac{1}{2}(wS_{vw}(w) + S_{vw}(w)w). \qquad (3.8)$$

Since $S_{vw}(w) = vw\bar{v}$ and $\bar{v} = -v$, we get $\cos\alpha = \frac{1}{2}(w(vwv) + (vwv)w)$. On the other hand,

$$vw = (-v \cdot w)\mathbf{O} + v \times w = -\cos\theta\mathbf{O} - \sin\theta N. \qquad (3.9)$$

Similarly, we have $wv = -\cos\theta\mathbf{O} + \sin\theta N$. Thus, we find

$$(vw)(vw) = \cos 2\theta\mathbf{O} + \sin 2\theta N \quad \text{and} \quad (wv)(wv) = \cos 2\theta\mathbf{O} - \sin 2\theta N. \qquad (3.10)$$

From equations (3.8) and (3.10), we get $\cos\alpha = (\cos 2\theta)\mathbf{O}$, that is, $\cos\alpha = \cos 2\theta$. Thus, $\alpha = 2\theta$ which implies that v is the bisector of the angle between w and $S_{vw}(w)$. □

From Theorem 3.7, we give the following statement:

Corollary 3.8 *For unit pure orthogonal quaternions v and w in $Plane(N)$, $S_{vw}(w)$ gives antipodal point corresponding to w on a great circle g.*

In [4], Ell and Sangwine studied quaternion involutions and antiinvolutions to express reflections and projections of arbitrary quaternions. Now, we will observe if the sandwiching operators S and T are involutions or antiinvolutions.

Definition 3.9 *[5] A transformation $f : \mathbb{H} \longrightarrow \mathbb{H}$ is an involution such that for $p, q \in \mathbb{H}$:*

(i) An involution is its own inverse, i.e. $f(f(p)) = p$.

(ii) An involution is linear, i.e. $f(p + q) = f(p) + f(q)$ and $f(\lambda p) = \lambda f(p)$ for a real constant λ.

(iii) The involution of a product is the product of the involution, i.e. $f(pq) = f(p)f(q)$.

An antiinvolution is a self inverse transformation similar to an involution which satisfies $f(pq) = f(q)f(p)$ instead of (iii). A trivial example for antiinvolution is a quaternion conjugation.

The following remark is found in [6] and [7].

Remark 3.10 *For a unit quaternion $q(N, \theta) = \cos\theta \mathbf{O} + \sin\theta N$, S_q and T_q have the following properties over basis vectors of $\mathbb{H}$:*

(i) $S_q(\mathbf{O}) = \mathbf{O}$, $S_q(N) = N$,
$S_q(v) = \cos 2\theta v + \sin 2\theta(N \times v)$, $S_q(N \times v) = \sin(-2\theta)v + \cos(-2\theta)(N \times v)$.

(ii) $T_q(\mathbf{O}) = \cos 2\theta \mathbf{O} + \sin 2\theta N$, $T_q(N) = \sin(-2\theta)\mathbf{O} + cos(-2\theta)N$,
$T_q(v) = v$, $T_q(N \times v) = N \times v$.

Theorem 3.11 *The following statements are equivalent:*

(i) S_q^2 is an identity tranformation.

(ii) $\theta = \frac{k\pi}{2}$ $k \in \mathbb{Z}$, that is, $q = \pm\mathbf{O}$ or $q = \pm N$.

(iii) T_q^2 is an identity transformation.

Proof Let S_q and T_q be operators given by (2.8). Assume $p = s_p\mathbf{O} + v_p$. Then, there exist constants a_p, b_p, c_p such that $p = s_p\mathbf{O} + a_pN + b_pv + c_pN \times v$.

(i) $\Leftrightarrow$ (ii) Since S_q is a linear transformation, we have

$$S_q(p) = s_pS_q(\mathbf{O}) + a_pS_q(N) + b_pS_q(v) + c_pS_q(N \times v). \tag{3.11}$$

Using Remark 3.10, we obtain

$$S_q(p) = s_p\mathbf{O} + a_pN + (b_p\cos 2\theta - c_p\sin 2\theta)v + (b_p\sin 2\theta + c_p\cos 2\theta)N \times v. \tag{3.12}$$

Applying again we get

$$S_q^2(p) = s_p\mathbf{O} + a_pN + (b_p\cos 4\theta - c_p\sin 4\theta)v + (b_p\sin 4\theta + c_p\cos 4\theta)N \times v. \tag{3.13}$$

Hence, $S_q^2(p) = p$ if and only if the following system of equations satisfy:

$$b_p(\cos 4\theta - 1) - c_p\sin 4\theta = 0, \text{ and } b_p\sin 4\theta + c_p(\cos 4\theta - 1) = 0 \tag{3.14}$$

If $b_p = c_p = 0$, then $p = s_p \mathbf{O} + a_p N$. From Corollary 7.2 in [7], we know that S_q^2 is an identity transformation. For nonzero constants b_p, c_p, (3.14) has a solution if and only if $\theta = \frac{k\pi}{2}$ for $k \in \mathbb{Z}$.

(ii) $\Leftrightarrow$ (iii) Similarly S_q^2, by using Remark 3.10, we obtain the desired $T_q^2(p) = p$ if and only if b_p and c_p satisfy the equation system (3.14). Thus, we obtain the desired result. $\qquad\square$

Using Definition 3.9 and Theorem 3.11, we give the following corollary:

Corollary 3.12 $S_\mathbf{O}, T_\mathbf{O}$, and S_N are involutions, but T_N is neither involution nor antiinvolution.

Proof From Theorem 3.11, $S_\mathbf{O}^2 = T_\mathbf{O}^2 = S_N^2(p) = T_N^2(p) = p$ for any quaternion p. It is known that they are linear transformations. Due to the fact that $N\bar{N} = \mathbf{O}$, we have

$$S_N(p_1 p_2) = N(p_1 p_2)\bar{N} = N p_1 (\bar{N}N) p_2 \bar{N} = S_N(p_1) S_N(p_2) \qquad (3.15)$$

for quaternions p_1, p_2. Thus, S_N is an involution. For $S_\mathbf{O}, T_\mathbf{O}$, it is trivial that the condition (iii) of Definition 3.9 is satisfied.

For T_N, if $T_N \neq 0$ were an involution, then it could satisfy the property $T_N(p_1 p_2) = T_N(p_1) T_N(p_2)$ for quaternions p_1, p_2. From this equality, we get $N p_1 p_2 N = (N p_1 N)(N p_2 N) = -N p_1 p_2 N$ which implies $N p_1 p_2 N = 0$, so $T_N = 0$, which is a contradiction. Similary, it is proven that T_N cannot be antiinvolution. $\square$

For a quaternion p, $S_\mathbf{O}(p)$ and $T_\mathbf{O}(p)$ leave the quaternion p unchanged, that is, they are identity transformations. On the other hand, $S_{m\mathbf{O}}(p)$ and $T_{m\mathbf{O}}(p)$ scale the quaternion p by a factor m^2 for a nonzero constant scalar m.

4. Reflection and projection

In this section, we will represent a reflection of a quaternion across $\mathrm{Plane}(N)$ and $\mathrm{Line}(N)$, orthogonal projection of a quaternion onto $\mathrm{Plane}(N)$ and $\mathrm{Line}(N)$ by using sandwiching operators.

4.1. Reflection

In [14], Pervin and Webb expressed reflections and projections of a vector onto a line or a plane. Also, Goldman in [6, 7] stated theorems for reflections of a vector across a plane perpendicular to a unit pure quaternion N by using sandwiching operator T_N.

Theorem 4.1 *[7] Let N be a unit vector and v be a vector in 3-dimensions. Then, $T_N(v) = NvN = -S_N(v)$ is the mirror image of v in the plane perpendicular to N.*

Therefore, $T_N(v)$ gives the reflection of a pure quaternion v across $\mathrm{Plane}(N)$.

Theorem 4.2 *For an arbitrary nonpure quaternion p, $S_N(\bar{p})$ and $-T_N(\bar{p})$ leaves the scalar part of p invariant and reflects the vector part of p across $\mathrm{Plane}(N)$.*

Proof Suppose that $p = s_p \mathbf{O} + v_p$ is a nonpure quaternion. Since S_N is a linear transformation, we obtain

$$S_N(\bar{p}) = s_p S_N(\mathbf{O}) - S_N(v_p). \qquad (4.1)$$

From Remark 3.10, we have $S_N(\mathbf{O}) = \mathbf{O}$. Thus, $S_N(p) = s_p\mathbf{O} - S_N(v_p)$ is a nonpure quaternion. From Theorem 4.1, it can be said that $-S_N(v_p)$ gives the reflection of pure quaternion v_p across $\mathrm{Plane}(N)$. Similarly, since $T_N(\mathbf{O}) = -\mathbf{O}$, we have

$$T_N(\bar{p}) = s_p T_N(\mathbf{O}) - T_N(v_p) = -s_p\mathbf{O} - T_N(v_p). \tag{4.2}$$

Thus, $-T_N(\bar{p})$ is also a nonpure quaternion and its pure part gives the reflection of v_p across $\mathrm{Plane}(N)$. $\square$

Theorem 4.3 *A reflection of a pure quaternion v across $\mathrm{Line}(N)$ is given by $S_N(v)$ or $-T_N(v)$.*

Proof For any pure quaternion v, it can be written as $v = v_1 + v_2$ where $v_1 \in \mathrm{span}\{\mathbf{O}, N\}$ and $v_2 \in \mathrm{span}\{v, N \times v\}$. Using Theorems 3.1 and 3.3, it can be easily seen that $S_N(v_1) = v_1$ and $S_N(v_2) = -v_2$. By using linearity of the transformation S_N, we have

$$S_N(v) = S_N(v_1) + S_N(v_2) = v_1 - v_2. \tag{4.3}$$

Similarly, using Theorems 3.1 and 3.3 we have $T_N(v_1) = -v_1$ and $T_N(v_2) = v_2$ and

$$T_N(v) = T_N(v_1) + T_N(v_2) = -v_1 + v_2. \tag{4.4}$$

Thus, $S_N(v) = -T_N(v)$. Also, S_N and $-T_N$ have the same effect on a pure quaternion v as reflecting v respect to the $\mathrm{Line}(N)$. $\square$

Theorem 4.4 *For a nonpure quaternion p, $S_N(p)$ or $-T_N(p)$ is a nonpure quaternion whose pure part is the reflection of a pure part of p across $\mathrm{Line}(N)$.*

Proof Suppose that $p = s_p\mathbf{O} + v_p$ is a nonpure quaternion. Since S_N is a linear transformation, we obtain

$$S_N(p) = s_p S_N(\mathbf{O}) + S_N(v_p). \tag{4.5}$$

From Remark 3.10, we have $S_N(\mathbf{O}) = \mathbf{O}$. Thus, $S_N(p) = s_p\mathbf{O} + S_N(v_p)$ is a nonpure quaternion. From Theorem 4.3, it can be said that $S_N(v_p)$ gives the reflection of pure quaternion v_p across $\mathrm{Line}(N)$. Similarly, since $T_N(\mathbf{O}) = -\mathbf{O}$, we have

$$T_N(p) = s_p T_N(\mathbf{O}) + T_N(v_p) = -s_p\mathbf{O} + T_N(v_p). \tag{4.6}$$

Thus, $-T_N(p)$ is also a nonpure quaternion and its pure part gives the reflection of v_p across $\mathrm{Line}(N)$. $\square$

Theorem 4.5 *The compositions $S_N \circ T_N$ and $T_N \circ S_N$ are symmetries of a pure quaternion with respect to the origin.*

Proof For a pure quaternion v, using Theorem 4.1 we have

$$S_N \circ T_N(v) = S_N(T_N(v)) = -S_N(S_N(v)) = -S_N^2(v). \tag{4.7}$$

From Theorem 3.11, $S_N^2(v) = v$. Thus, we get $S_N \circ T_N(v) = -v$. Hence, $S_N \circ T_N$ has the effect on a pure quaternion v as reflecting it with respect to the origin.

Similarly, using Theorems 3.11 and 4.3 we compute $T_N \circ S_N(v)$ as follows:

$$T_N \circ S_N(v) = T_N(S_N(v)) = -T_N(T_N(v)) = -T_N^2(v) = -v \tag{4.8}$$

Hence, $T_N \circ S_N(v)$ also gives symmetry of pure quaternion v with respect to the origin. □

Theorem 4.6 *The compositions $S_N \circ T_N$ and $T_N \circ S_N$ are symmetries of a nonpure quaternion with respect to the origin in 4-dimensions.*

Proof Assume that $p = s_p \mathbf{O} + v_p$ is a nonpure quaternion. From Remark 3.10, we have $S_N(\mathbf{O}) = \mathbf{O}$ and $T_N(\mathbf{O}) = -\mathbf{O}$. Then, we get $S_N \circ T_N(p)$ as follows:

$$S_N \circ T_N(p) = s_p S_N(T_N(\mathbf{O})) + S_N(T_N(v_p)) = -s_p \mathbf{O} + S_N(T_N(v_p)) \tag{4.9}$$

From Theorem 4.5, it can be said that $S_N \circ T_N(v_p)$ give symmetry of v_p with respect to the origin, that is, $S_N \circ T_N(v_p) = -v_p$. Thus, $S_N \circ T_N(p) = -p$, which is symmetry of p with respect to the origin.

Similarly, it can be easily shown that $T_N \circ S_N(p) = -p$ which gives symmetry of a nonpure quaternion p with respect to the origin in 4-dimensions. □

4.2. Projection

To display three dimensional geometry on a two dimensional screen, the projection from 3-dimensions to 2-dimensions is needed. The two most important types of projections are orthogonal projection and perspective projection. In this part, we will find representations for orthogonal projection in 3-dimensions and 4-dimensions by matrices and sandwiching maps.

4.3. Representation of orthogonal projection onto a plane in 3-dimensions by 4×4 matrices

Suppose we want to project a point P onto a plane S passing through a point Q and perpendicular to a unit vector N in 3-dimensions (see Figure 3). It can be easily seen that the projection point P_{proj} of P is obtained by adding a vector $P_{\text{proj}} - P$ to a point P, that is, $P_{\text{proj}} = P + (P_{\text{proj}} - P)$. We know that the vector $P_{\text{proj}} - P$ is parallel to the vector N and the length of it is $(Q - P) \cdot N$. Thus, P_{proj} is given by

$$P_{\text{proj}} = P + ((Q - P) \cdot N)N \tag{4.10}$$

where $|(Q - P) \cdot N|$ is a distance from the point P to the plane S.

In [8], such an orthogonal projection in 3-dimensions is represented by 4×4 matrix given as

$$\text{Orth}(Q, N) = \begin{pmatrix} I - (N^T \star N) & 0 \\ (Q \cdot N)N & 1 \end{pmatrix} \tag{4.11}$$

where I denotes a 3×3 identity matrix, N^T is the tranpose of the unit vector N and $\star$ denotes matrix multiplication. On the other hand, we know that points on the line through the point P and perpendicular to the plane S project to the same point P_{proj} on the plane S. To detect such points, called hidden points, we give the following method.
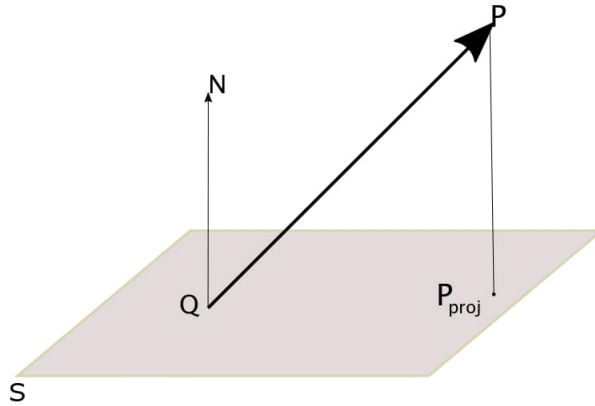
Figure 3. Orthogonal projection of a point P onto the plane S through the point Q with unit normal N.

Let us define $\tilde{P}_{\text{proj}}$ as

$$\tilde{P}_{\text{proj}} = ((Q - P) \cdot N)\mathbf{O} + P + ((Q - P) \cdot N)N. \tag{4.12}$$

Then, $\tilde{P}_{\text{proj}}$ is a nonpure quaternion whose absolute value of scalar part is the distance of a point P from a plane S and vector part gives the location of P_{proj}. If two points project to the same point, then the smaller absolute value of their scalar part gives us the closer point to the projection plane S. Thus, we can use $\tilde{P}_{\text{proj}}$ to detect hidden points.

On the other side, we can represent this transformation by a 4×4 matrix. Let I be 3×3 identity matrix, N^T be the tranpose of a unit vector N and $k = (Q - P) \cdot N$. Then, we define $\text{Orth}(P, Q, N)$ as

$$\text{Orth}(P, Q, N) = \begin{pmatrix} I & -N^T \\ kN & Q \cdot N \end{pmatrix}. \tag{4.13}$$

It can be easily shown that $\text{Orth}(P, Q, N)$ is a nonsingular matrix.

Theorem 4.7 *Let P be a point and S be a plane through a point Q perpendicular to a unit vector N in 3-dimensions. Then,*

$$\tilde{P}_{proj} = (P, 1) \star Orth(P, Q, N) \tag{4.14}$$

where $Orth(P, Q, N)$ is a 4×4 nonsingular matrix given by (4.13). The first three components of $\tilde{P}_{proj}$ gives the location of orthogonal projection of P onto the plane S and the last component of $\tilde{P}_{proj}$ is equal to k whose absolute value is the distance of P from the projection plane S.

Proof To show that $\tilde{P}_{\text{proj}}$ is located at the orthogonal projection of the point P onto the projection plane S, we need to show that $\tilde{P}_{\text{proj}} - Q$ is perpendicular to N. From (4.13), we get

$$\begin{aligned} \tilde{P}_{\text{proj}} &= (P, 1) \star \begin{pmatrix} I & -N^T \\ kN & Q \cdot N \end{pmatrix} \\ &= (P + kN, k) = (P + ((Q - P) \cdot N)N, ((Q - P) \cdot N)). \end{aligned} \tag{4.15}$$

Moreover,

$$(\tilde{P}_{\text{proj}} - Q) \cdot N = ((P - Q) + ((Q - P) \cdot N)N) \cdot N = 0. \tag{4.16}$$

Thus, $\tilde{P}_{\text{proj}} - Q$ is perpendicular to the unit vector N. □

Remark that for $k = 1$, the matrix given by (4.13) gives the same result for the location of projection point which is obtained from (4.11).

Example 4.8 *Let S be a plane in 3-dimensions passing through a point $Q(1,0,0)$ with the unit normal vector $N = \frac{1}{\sqrt{3}}(1,1,1)$. Now we are going to find orthogonal projection of an arbitrary point P in 3-dimensions onto the plane S. For any point $P(x,y,z)$, we find $k = (Q-P) \cdot N = \frac{1}{\sqrt{3}}(1 - x - y - z)$ and then using the equation (4.10), we obtain the orthogonal projection of P onto the plane S as*

$$P_{proj} = \left(x + \frac{k}{\sqrt{3}}, y + \frac{k}{\sqrt{3}}, z + \frac{k}{\sqrt{3}} \right).$$

Also, from equation (4.12), we write

$$\tilde{P}_{proj} = k\mathbf{O} + \left(x + \frac{k}{\sqrt{3}}, y + \frac{k}{\sqrt{3}}, z + \frac{k}{\sqrt{3}} \right). \tag{4.17}$$

Moreover, from equation (4.13), we have

$$Orth(P,Q,N) = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 1 & -\frac{1}{\sqrt{3}} \\ \frac{k}{\sqrt{3}} & \frac{k}{\sqrt{3}} & \frac{k}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \tag{4.18}$$

Using Theorem 4.7, we obtain

$$\tilde{P}_{proj} = (P,1) \star Orth(P,Q,N) = \left(x + \frac{k}{\sqrt{3}}, y + \frac{k}{\sqrt{3}}, z + \frac{k}{\sqrt{3}}, k \right). \tag{4.19}$$

Let us take two points $P_1(0,0,2)$ and $P_2(-1,-1,1)$. Then, we have $k_1 = (Q - P_1) \cdot N = -\frac{1}{\sqrt{3}}$ and $k_2 = (Q - P_2) \cdot N = \frac{2}{\sqrt{3}}$. From equation (4.17), we get

$$(\tilde{P}_1)_{proj} = -\frac{1}{\sqrt{3}}\mathbf{O} + \frac{1}{3}(-1,-1,5)$$

and

$$(\tilde{P}_2)_{proj} = \frac{2}{\sqrt{3}}\mathbf{O} + \frac{1}{3}(-1,-1,5).$$

The vector part of $(\tilde{P}_1)_{proj}, (\tilde{P}_2)_{proj}$ are equal to each other but P_1 is closer to the projection plane S than P_2 because of $|k_1| < |k_2|$.

Theorem 4.9 *Orthogonal projection and translation commute, that is, projecting a point P in 3-dimensions into a plane S passing through Q and then translating the projection point by the vector v is equivalent to the translating of point P by the vector v and then projecting the resulting point into the translated plane of S by v.*

Proof Let P be a point in 3-dimensions and S be a plane passing through Q with the unit normal vector N. Then, we know that the projection point P_{proj} of P onto the plane S is given by equation (4.10). If we translate the point P and the projection plane S by the vector v, we obtain $P + v$ and the translated plane S' passing through $Q + v$ with the unit normal vector N. From equation (4.10), the projection point $\tilde{P}_{\text{proj}}$ of $P + v$ onto the translated plane S' is given by

$$\tilde{P}_{\text{proj}} = (P + v) + (((Q + v) - (P + v)) \cdot N)N. \tag{4.20}$$

Thus, $\tilde{P}_{\text{proj}} = P_{\text{proj}} + v$ which implies that the orthogonal projection and translation commutes. $\square$

4.3.1. Representation of orthogonal projection onto $\mathbf{Plane}(N)$ and $\mathbf{Line}(N)$ by sandwiching maps

In this section, we will study how to express orthogonal projection in 3-dimensions and 4-dimensions using sandwiching maps.

Theorem 4.10 *For a pure quaternion w, $T_{q\left(N, \frac{\pi}{4}\right)}(w)$ is a nonpure quaternion whose vector part gives the location of projection of a pure quaternion w onto $\text{Plane}(N)$ and an absolute value of scalar part is equal to the distance of a point corresponding to the pure quaternion w from $\text{Plane}(N)$.*

Proof Suppose $w = a_w N + v_w$ for $v_w \in \text{Plane}(N)$. Since T_q is a linear transformation, we obtain

$$T_{q\left(N, \frac{\pi}{4}\right)}(p) = a_w T_{q\left(N, \frac{\pi}{4}\right)}(N) + T_{q\left(N, \frac{\pi}{4}\right)}(v_w). \tag{4.21}$$

From Theorem 3.1, we know that $v_w \in \text{span}\{v, N \times v\}$. Thus, using Remark 3.10, $T_{q\left(N, \frac{\pi}{4}\right)}(v_w) = v_w$ and $T_{q\left(N, \frac{\pi}{4}\right)}(N) = -\mathbf{O}$. Hence, $T_{q\left(N, \frac{\pi}{4}\right)}(p) = -a_w \mathbf{O} + v_w$ for $a_w = w \cdot N = -k$. $\square$

Since quaternion can be thought as a vector in 4-dimensions, we give the following statements for orthogonal projection from 4-dimensions to 2-dimensions.

Theorem 4.11 *Let $\widetilde{T}_N : \mathbb{H} \longrightarrow \mathbb{H}$ be a linear transformation defined by $\widetilde{T}_N(p) = \dfrac{p + T_N(p)}{2}$ for any $p \in \mathbb{H}$. Then, $\widetilde{T}_N(p)$ gives an orthogonal projection of a nonpure quaternion p onto $\text{Plane}(N)$.*

Proof Suppose $p = s_p \mathbf{O} + a_p N + w_p$ for $w_p \in \text{Plane(N)}$. Since T_N is a linear transformation, we have

$$T_N(p) = s_p T_N(O) + a_p T_N(N) + T_N(w_p) = -s_p \mathbf{O} - a_p N + T_N(w_p). \tag{4.22}$$

From Theorem 3.1, we know that $T_N(w_p) = w_p$ where $w_p \in \text{Plane(N)}$ and then, $T_N(p) = -s_p \mathbf{O} - a_p N + w_p$. Thus, $\widetilde{T}_N(p) = w_p$ which is a projection of p onto $\text{Plane}(N)$. $\square$

Theorem 4.12 *Let $\hat{T}_N : \mathbb{H} \longrightarrow \mathbb{H}$ be a linear transformation defined by $\hat{T}_N(p) = \dfrac{p - T_N(p)}{2}$ for any $p \in \mathbb{H}$. Then, $\hat{T}_N(p)$ gives an orthogonal projection of a nonpure quaternion p onto $Line(N)$.*

Proof Suppose $p = s_p \mathbf{O} + a_p N + w_p$ for $w_p \in \text{Plane(N)}$. Since T_N is a linear transformation, we have

$$T_N(p) = s_p T_N(O) + a_p T_N(N) + T_N(w_p) = -s_p \mathbf{O} - a_p N + T_N(w_p). \tag{4.23}$$

From Theorem 3.1, we know $T_N(w_p) = w_p$ for $w_p \in \text{Plane(N)}$ and then, $T_N(p) = -s_p \mathbf{O} - a_p N + w_p$. Hence, $\hat{T}_N(p) = s_p \mathbf{O} + a_p N$ which is a projection of p onto Line(N). $\qquad\square$

For particular cases, using the result in [8] we give the following consequences:

Corollary 4.13 *Let $\widetilde{T}_N : \mathbb{R}^3 \longrightarrow \mathbb{H}$ be a linear transformation defined by $\widetilde{T}_N(w) = \dfrac{w + T_N(w)}{2}$ for any $w \in \mathbb{R}^3$. Then, $\widetilde{T}_N(w)$ gives an orthogonal projection of a pure quaternion w onto $Plane(N)$.*

Corollary 4.14 *Let $\hat{T}_N : \mathbb{R}^3 \longrightarrow \mathbb{H}$ be a linear transformation defined by $\hat{T}_N(w) = \dfrac{w - T_N(w)}{2}$ for any $w \in \mathbb{R}^3$. Then, $\hat{T}_N(w)$ gives an orthogonal projection of a pure quaternion w onto $Line(N)$.*

Example 4.15 *$Plane(N)$ is a plane passing through origin perpendicular to the unit vector N. The plane S given in Example 4.8 is a translated version of a $Plane(N)$ by the vector $v = (1, 0, 0)$. Let us find the orthogonal projection of $P(-1, 0, 2)$ onto $Plane(N)$. Denote the corresponding pure quaternion of P by w. Then, $w = \frac{1}{\sqrt{3}} N + w_p$ for the unit vector $N = \frac{1}{\sqrt{3}}(1, 1, 1)$ $w_p = \frac{1}{3}(-4, -1, 5) \in Plane(N)$ and then, $T_N(w) = -\frac{1}{\sqrt{3}} N + w_p$. From Corollary 4.13, the orthogonal projection of w onto $Plane(N)$ is*

$$\frac{1}{2}(w + T_N(w)) = \frac{1}{2}\left(\frac{1}{\sqrt{3}}N + w_p - \frac{1}{\sqrt{3}}N + w_p\right) = w_p = \frac{1}{3}(-4, -1, 5). \tag{4.24}$$

On the other side, $P_1(0, 0, 2)$ is a translated point of P by the vector v. From Theorem 4.9, $\frac{1}{2}(w + T_N(w)) + v = \frac{1}{3}(-1, -1, 5)$ gives the projection of point P_1 onto the translated plane S, which is the same as in Example 4.8.

Notice that the orthogonal projection of any quaternion onto $\text{Plane}(N)$ and $\text{Line}(N)$ can be also expressed using sandwiching operator S_N due to the fact that $T_N = -S_N$.

References

[1] Baek J, Jean H, Kim G, Han S. Visualizing quaternion multiplication. IEEE Access 2017; 5: 8948-8955. doi: 10.1109/ACCESS.2017.2705196

[2] Coxeter HSM. Quaternions and reflections. The American Mathematical Monthly 1946; 53 (3): 136-146.

[3] Du J, Goldman R, Mana S. Modeling Affine and Projective Transformations in 3-Dimensions by Linear Transformations in 4-Dimensions. University of Waterloo Technical Report CS-2018-05, 2018.

[4] Ell TA, Sangwine SJ. Quaternions involutions and antiinvolutions. Computers and Mathematics with Applications 2007; 53: 137-143.

[5] Ell TA, Bihan NL, Sangwine SJ. Quaternion Fourier Transforms for Signal and Image Processing. New York, NY, USA: Wiley, 2014.

[6] Goldman R. Rethinking Quaternions: Theory and Computation. San Rafael, CA, USA: Morgan and Claypool Publishers, 2010.

[7] Goldman R. Understanding quaternions. Graphical Models 2011; 73 (2): 21-49. doi: 10.1016/j.gmod.2010.10.004

[8] Goldman R. An Integrated Introduction to Computer Graphics and Geometric Modeling. Boca Raton, FL, USA: CRC Press, 2009.

[9] Hamilton WR. Elements of Quaternions. London, Longmans, Greens,$ Company, 1866.

[10] Hart JC, Francis GK, Kaufman LH. Visualizing quaternion rotation. ACM Transactions on Graphics 1994; 13 (3): 256-276.

[11] Hanson AJ. Visualizing Quaternions. San Francisco, CA, United States: Morgan Kaufmann Publishers Inc., 2006.

[12] Hughes JF, Dam AV, Mcguire M, Sklar DF, Foley JD et al. Computer Graphics Principles and Practice. Boston, MA, USA: Addison-Wesley, 2014.

[13] Kuipers JB. Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality. Princeton, NJ, USA: Princeton University Press, 1999.

[14] Pervin E, Webb JA. Quaternions in Computer Vision and Robotics. Carnegie-Mellon University CMU-CS-82-150, 1982.

[15] Shoemake K. Animating rotation with quaternion curves. In SIGGRAPH'85: Proceedings of the 12th Annual Conference on Computer Graphics and Interactive Techniques; ACM Press; 1985. pp. 245-254.

[16] Xing Y, Tan J, Hang P. Perspective projection based on quaternion interpolation. Applied Mechanics and Materials 2013; 303, 2130-2133. doi: 10.4028/www.scientific.net/amm.303-306.2130

[17] Yang Y. Spacecraft Modeling, Attitude Determination, and Control: Quaternion Based Approach. Boca Raton, FL, USA: CRC Press, 2019.