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ON RICKART MODULES

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Abstract. We investigate some properties of Rickart modules defined by Rizvi and Roman. Let R be an arbitrary ring with identity and M be a right R-module with $S = \text{End}_R(M)$. A module M is called to be *Rickart* if for any $f \in S$, $r_M(f) = Se$, for some $e^2 = e \in S$. We prove that some results of principally projective rings and Baer modules can be extended to Rickart modules for this general settings.

1. Introduction

Throughout this paper, R denotes an associative ring with identity, and modules will be unitary right R-modules. For a module M, $S = \text{End}_R(M)$ denotes the ring of right R-module endomorphisms of M. Then, M is a left S-module, right R-module and (S, R)-bimodule. In this work, for any rings S and R and any (S, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R and the left annihilator of a subset of R in M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ will be the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. A ring R is said to be *reduced* if it has no nonzero nilpotent elements. Recently, the reduced ring concept was extended to modules by Lee and Zhou, [12], that is, a module M is called *reduced* if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies

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$mR \cap Ma = 0$. According to Lambek [11], a ring R is called *symmetric* if $a, b, c \in R$ satisfy $abc = 0$, then we have $bac = 0$. This is generalized to modules in [11] and [14]. A module M is called *symmetric* if $a, b \in R$, $m \in M$ satisfy $mab = 0$, then we have $mba = 0$. Symmetric modules are also studied in [1] and [15]. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. A module M is called *semicommutative* [5] if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mRa = 0$. *Baer rings* [9] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring R is said to be *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. A ring R is called *right principally quasi-Baer* if the right annihilator of a principal right ideal of R is generated by an idempotent. Finally, a ring R is called *right (or left) principally projective* if every principal right (or left) ideal of R is a projective right (or left) R-module [4]. Baer property is considered in [18] by utilizing the endomorphism ring of a module. A module M is called *Baer* if for all R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$. A submodule N of M is said to be *fully invariant* if it is also left S-submodule of M. The module M is said to be *quasi-Baer* if for all fully invariant R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$, or equivalently, the right annihilator of a two-sided ideal is generated, as a right ideal, by an idempotent. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$, we mean, respectively, integers, rational numbers, real numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n.

2. **Rickart modules**

Let M be a right R-module with $S = \mathrm{End}_R(M)$. In [19], the module M is called *Rickart* if for any $f \in S$, $r_M(f) = r_M(Sf) = eM$, for some $e^2 = e \in S$. The ring R is called *right Rickart* if R_R is a Rickart module, that is, the right annihilator of any element is generated by an idempotent. Left Rickart rings are defined in a symmetric way. It is obvious that the module R_R is Rickart if and only if the ring R is right principally projective. This concept provides a generalization of a right principally projective ring to module theoretic setting. It is clear that every semisimple, Baer module is a Rickart module.

We now give an example for illustration.

Example 2.1. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Q}$. Then, endomorphism ring of M is $S = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$. It is easy to check that, for any $f \in S$, there exists an idempotent e in S such that $r_M(f) = eM$. Indeed, let namely $f = \begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix}$, where $0 \neq b$, $0 \neq c \in \mathbb{Q}$, and $m = \begin{bmatrix} x \\ y \end{bmatrix} \in r_M(f)$. Then, $bx + yc = 0$ and $e = \begin{bmatrix} 1 & 0 \\ -b/c & 0 \end{bmatrix}$ is an idempotent in S and $eM \leq r_M(f)$, since $feM = 0$. Let $m \in r_M(f)$. Then, $m = em$. Hence, $r_M(f) \leq eM$. Thus, $r_M(f) = eM$. The other possibilities for the picture of f give rise to an idempotent e such that $r_M(f) = eM$.*

Proposition 2.2. *Let M be an R-module with $S = End_R(M)$. If M is a Rickart module, then S is a right Rickart ring.*

Proof. Let $\varphi \in S$. By the hypothesis, we have $r_M(\varphi) = eM$, where $e^2 = e \in S$. We claim that $r_S(\varphi) = eS$. Since $0 = \varphi eM = \varphi eSM$, $eS \subseteq r_S(\varphi)$. For any $0 \neq f \in r_S(\varphi)$, we have $fM \subseteq r_M(\varphi)$, and so $f = ef$. Then, $f \in eS$. Therefore, $r_S(\varphi) = eS$. □

Proposition 2.3 is well known. We give a proof for the sake of completeness.

Proposition 2.3. *Let R be a right Rickart ring and $e^2 = e \in R$. Then, eRe is a right Rickart ring.*

Proof. Let $a \in eRe$ and $r_R(a) = fR$, for some $f^2 = f \in R$. Then, $1-e \in fR$ and $r_{eRe}(a) = (eRe) \cap r_R(a)$. Multiplying $1-e$ from the left by f, we obtain $f - fe = 1-e$, and so $ef = efe$ by multiplying $f - fe$ from the left by e. Set $g = ef$. Then, $g \in eRe$, and $g^2 = efef = ef^2 = ef = g$. We prove $(eRe) \cap r_R(a) = g(eRe)$. Let $t \in (eRe) \cap r_R(a)$. Since $t = ete$ and $t \in fR$, $t = fr$, for some $r \in R$. Multiplying $t = fr$ from the left by f, we have $t = ft = fete$. Again, multiplying $t = ft = fete$ from the left by e, we obtain $t = et = efete = gete \in g(eRe)$. So, $(eRe) \cap r_R(a) \leq g(eRe)$. For the converse inclusion, let $gete \in g(eRe)$. Then, $gete = efete \in eRe$. On the other hand, $agete = aefete = afete = 0$ implies $gete \in r_R(a)$. Hence, $g(eRe) \leq (eRe) \cap r_R(a)$. Therefore, $g(eRe) = (eRe) \cap r_R(a)$. □

Proposition 2.4. *Let M be a Rickart module. Then, every direct summand N of M is a Rickart module.*

Proof. Let $M = N \oplus P$. Let $S^{'}$ =End$_R(N)$. Then, for any $\varphi' \in S'$, there exists $\varphi \in S$, defined by $\varphi = \varphi' \oplus 0_{|_P}$. By the hypothesis, $r_M(\varphi)$ is a direct summand of M. Let $M = r_M(\varphi) \oplus Q$. Since $P \subseteq r_M(\varphi)$, there exists $L \leq r_M(\varphi)$ such that $r_M(\varphi) = P \oplus L$. So, we have $M = r_M(\varphi) \oplus Q = P \oplus L \oplus Q$. Let $\pi_N : M \to N$ be the projection of M onto N. Then, $\pi_N \mid_{Q \oplus L}: Q \oplus L \to N$ is an isomorphism. Hence, $N = \pi_N(Q) \oplus \pi_N(L)$. We will show that $r_N(\varphi') = \pi_N(L)$. Since $\varphi(P \oplus L) = 0$, we get $\varphi(L) = 0$. But, for all $l \in L$, $l = \pi_N(l) + \pi_P(l)$. Since $\varphi\pi_P(l) = 0$, we have $\varphi^{'}(\pi_N(L)) = 0$. So, $\pi_N(L) \subseteq r_N(\varphi')$.

Let $n \in N \backslash \pi_N(L)$. Then, $n = n_1 + n_2$, for some $n_1 \in \pi_N(L)$ and some $0 \neq n_2 \in \pi_N(Q)$. Since $\pi_N \mid_{Q \oplus L}$ is an isomorphism, there exists a $\overline{n_2} \in Q$ such that $\pi_N(\overline{n_2}) = n_2$. Since $Q \cap r_M(\varphi) = 0$, we have $\varphi(\overline{n_2}) = \varphi' \oplus 0_{|_P}(\overline{n_2}) \neq 0$. Since $\overline{n_2} = \pi_N(\overline{n_2}) + \pi_P(\overline{n_2})$, we get $\varphi'\pi_N(\overline{n_2}) \neq 0$. So, $\varphi'(\overline{n_2}) \neq 0$. This implies $n \notin r_N(\varphi')$. Therefore, $r_N(\varphi') = \pi_N(L)$. □

Corollary 2.5. *Let R be a right Rickart ring and let e be any idempotent in R. Then, $M = eR$ is a Rickart module.*

Proposition 2.6. *Let M be an R-module with $S = End_R(M)$. If S is a von Neumann regular ring, then M is a Rickart module.*

Proof. For any $\alpha \in S$, there exists $\beta \in S$ such that $\alpha = \alpha\beta\alpha$. Define $e = \beta\alpha$. Then, $e^2 = e$ and $\alpha = \alpha e$. Hence, $r_M(\alpha) = r_M(e) = (1 - e)M$. This completes the proof. □

Recall that M is called *a duo module* if every submodule N of M is fully invariant, i.e., $f(N) \leq N$, for all $f \in S$, while M is said to be *a weak duo module*, if every direct summand of M is fully invariant. Every duo module is weak duo (see [13] for details).

Proposition 2.7. *Let M be a quasi-Baer and weak duo module with $S = End_R(M)$. Then, M is Rickart.*

Proof. Let $f \in S$. By the hypothesis, there exists $e^2 = e \in S$ such that $eM = r_M(SfS)$. Since $f \in Sf \leq SfS$, $eM = r_M(SfS) \leq r_M(Sf) = r_M(f)$. There exists $K \leq M$ such that $r_M(f) = eM \oplus K$. Assume that $K \neq 0$ to reach a contradiction. Since K is fully invariant and $K \leq r_M(f)$, we have $SK \leq K \leq r_M(f)$. So, $fSK = 0$ and $SfSK = 0$. Therefore, $K \leq r_M(SfS) = eM$. This is the required contradiction. Thus, M is a Rickart module. □

Let M be an R-module with $S = \mathrm{End}_R(M)$. Some properties of R-modules do not characterize the ring R, namely there are reduced

R-modules but R need not be reduced and there are abelian R-modules but R is not an abelian ring. Because of this, we are currently investigating the reduced, rigid, symmetric, semicommutative, Armendariz and abelian modules in terms of endomorphism ring S. In the sequel, we continue studying relations between reduced, rigid, symmetric, semicommutative, Armendariz and abelian modules by using Rickart modules.

Definition 2.8. Let M be an R-module with $S = \mathrm{End}_R(M)$. A module M is called *reduced* if $fm = 0$ implies $Imf \cap Sm = 0$, for each $f \in S$, and $m \in M$.

Following the definition of reduced module in [12] and [15], M is a reduced module if and only if $f^2m = 0$, implies $fSm = 0$ for each $f \in S$, and $m \in M$. The ring R is called *reduced* if the right R-module R is reduced by considering $\mathrm{End}_R(R) \cong R$, that is, for any a, $b \in R$, $ab = 0$ implies $aR \cap Rb = 0$, or equivalently R does not have any nonzero nilpotent elements.

Example 2.9. *Let p be any prime integer and M denote the $\mathbb{Z}$-module $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$. Then, $S = End_R(M)$ is isomorphic to the matrix ring $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\}$ and M is a reduced module.*

In [10], Krempa introduced the notion of rigid ring. An endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$. According to Hong et al. [8], R is said to be an *α-rigid ring* if there exits a rigid endomorphism α of R. This "rigid ring" notion depends heavily on the endomorphism of the ring R. In the following, we redefine rigidness so that it will be independent of endomorphism and also will be extended to modules.

Proof of the Proposition 2.10 is obvious.

Proposition 2.10. *Let M be an R-module with $S = End_R(M)$. For any $f \in S$, the followings are equivalent.*
(1) $Kerf \cap Imf = 0$.
(2) *For $m \in M$, $f^2m = 0$ if and only if $fm = 0$.*

A module M is called *rigid* if it satisfies Proposition 2.10 for every $f \in S$. The ring R is said to be *rigid* if the right R-module R is rigid by considering $\mathrm{End}_R(R) \cong R$, that is, for any a, $b \in R$, $a^2b = 0$ implies $ab = 0$.

Lemma 2.11. *Let M be an R-module with $S = End_R(M)$. If M is a rigid module, then S is a reduced ring, and therefore idempotents in S are central.*

Proof. Let $f, g \in S$ with $fg = 0$ and $fg' = f'g$, for some f', $g' \in S$. For any $m \in M$, $(gf)^2 m = 0$. By the hypothesis, $(gf)m = 0$. Hence, $gf = 0$. So, $gfg' = gf'g = 0$. From what we have proved, we obtain $f'g = 0$. The rest is clear. □

Recall that the module M is called *extending* if every submodule of M is essential in a direct summand of M. We have the following result.

Theorem 2.12. *If M is a rigid and extending module, then M is a Rickart module.*

Proof. Let $f \in S$ and $m \in Kerf$. If mR is essential in M, then $Kerf$ is essential in M. Since M is rigid, i.e., $Ker(f) \cap Im(f) = 0$, $f = 0$. Assume that mR is not essential in M. There exists a direct summand K of M such that mR is essential in K and $M = K \oplus K'$. Let π_K denote the canonical projection from M onto K. Then, the composition map $f\pi_K$ has kernel $mR + K'$, that is an essential submodule of M. By assumption, $f\pi_K = 0$. Hence, $f(K) = 0$, and $Kerf = K \oplus (Kerf) \cap K'$. Similarly, there exists a direct summand U of K' containing $(Kerf) \cap K'$ essentially so that $K' = U \oplus U'$. Let π_U denote the canonical projection from M onto U. Then, $Ker(f\pi_U)$ is essential in M. Hence, $Ker(f\pi_U) = 0$. So, $f(U) = 0$. Thus, $Kerf = K \oplus U$. This is a direct summand of M. □

Proposition 2.13. *Let R be a ring. Then, the followings are equivalent.*
(1) *R is a reduced ring.*
(2) *R_R is a reduced module.*
(3) *R_R is a rigid module.*

Proof. Clear by definitions. □

In the module case, Proposition 2.13 does not hold in general.

Proposition 2.14. *If M is a reduced module, then M is a rigid module. The converse holds if M is a Rickart module.*

Proof. For any $f \in S$, $(SKerf) \cap Imf = 0$, by the hypothesis. Since $Kerf \cap Imf \subset (Skerf) \cap Imf$, $Kerf \cap Imf = 0$. Then, M is a rigid module. Conversely, let M be a Rickart and rigid module. Assume that $fm = 0$, for $f \in S$ and $m \in M$. Then, there exists $e^2 = e \in S$ such that $r_M(f) = eM$. By Lemma 2.11, e is central in S. Then, $fe = ef = 0$,

$m = em$. Let $fm' = gm \in fM \cap Sm$. We multiply $fm' = gm$ from the left by e to obtain $efm' = fem' = egm = gem = gm = 0$. Therefore, M is a reduced module. □

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$, for any $e^2 = e$, $a \in R$. Abelian modules are introduced in the context by Roos [20] and studied by Goodearl and Boyle [7], Rizvi and Roman [17]. A module M is called *abelian* if for any $f \in S$, $e^2 = e \in S$, $m \in M$, we have $fem = efm$. Note that M is an abelian module if and only if S is an abelian ring.

We mention some classes of abelian modules.

Examples 2.15. (*1*) *Every weak duo module is abelian. In fact, let* $e^2 = e \in S$, $f \in S$. *For any* $m \in M$, *write* $m = em + (1-e)m$. M *Being weak duo, we have* $fem \in eM$ *and* $f(1-e)m \in (1-e)M$. *Multiplying* $fm = fem + f(1-e)m$ *by* e *from the left, we have* $efm = fem$.
(*2*) *Let* M *be a torsion* $\mathbb{Z}$*-module. Then,* M *is abelian if and only if* $M = \bigoplus_{i=1}^{t} \mathbb{Z}_{p_i^{n_i}}$ *where the* p_i *are distinct prime integers and the* $n_i \geq 1$ *are integers.*
(*3*) *Cyclic* $\mathbb{Z}$*-modules are always abelian, but non-cyclic finitely generated torsion-free* $\mathbb{Z}$*-modules are not abelian.*

Lemma 2.16. *If* M *is a reduced module, then it is abelian. The converse is true if* M *is a Rickart module.*

Proof. One way is clear. For the converse, assume that M is a Rickart and abelian module. Let $f \in S, m \in M$ with $fm = 0$. We want to show that $fM \cap Sm = 0$. There exists $e^2 = e \in S$ such that $m \in r_M(f) = eM$. Then, $em = m$ and $fe = 0$. Let $fm_1 = gm \in fM \cap Sm$, where $m_1 \in M$, $g \in S$. Multiplying $fm_1 = gm$ by e from the left. Then, we have $0 = fem_1 = efm_1 = egm = gem = gm$. This completes the proof. □

Recall that a ring R is *symmetric* if $abc = 0$, implies $acb = 0$, for any $a, b, c \in R$. For the module case, we have the following definition.

Definition 2.17. Let M be an R-module with $S = \mathrm{End}_R(M)$. A module M is called *symmetric* if for any $m \in M$ and f, $g \in S$, $fgm = 0$ implies $gfm = 0$.

Lemma 2.18. *If* M *is a reduced module, then it is symmetric. The converse holds if* M *is a Rickart module.*

Proof. Let $fgm = 0$, $f, g \in S$. Then, $(fg)^2(m) = 0$. By the hypothesis, $fgSm \leq (fgM) \cap Sm = 0$. So, $fgfm = 0$ and $(gf)^2 m = 0$. Similarly, $gfSm = 0$, and so $gfm = 0$. Therefore, M is symmetric. For inverse implication, let $f \in S$ and $m \in M$ with $fm = 0$. We prove that $fM \cap Sm = 0$. Let $fm_1 = gm \in fM \cap Sm$, where $m_1 \in M$, $g \in S$. There exists a central idempotent $e \in S$ such that $r_M(f) = eM$. Then, $feM = efM = 0$ and $em = m$. Multiplying $fm_1 = gm$ from the left by e, we have $0 = efm_1 = egm = gem = gm$. This completes the proof. □

The next example shows that the reverse implication of the first statement in Lemma 2.18 is not true, in general, i.e., there exists a symmetric module which is neither reduced nor Rickart.

Example 2.19. *Let $\mathbb{Z}$ denote the ring of integers. Consider the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{Z} \right\}$ and R-module $M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} : a, b \in \mathbb{Z} \right\}$. Let $f \in S$ and $f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$. Multiplying the latter by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ from the right, we have $f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix}$. Similarly, let $g \in S$ and $g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix}$. Then, $g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c' \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $g \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix}$. Then, it is easy to check that for any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$,*

$$fg \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = f \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix} = \begin{bmatrix} 0 & ac'c \\ ac'c & ad'c + adc' + bc'c \end{bmatrix},$$

and

$$gf \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = g \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix} = \begin{bmatrix} 0 & acc' \\ acc' & acd' + ac'd + bcc' \end{bmatrix}.$$

Hence, $fg = gf$, for all $f, g \in S$. Therefore, S is commutative, and so M is symmetric.

Let $f \in S$ *be defined by* $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$, *where* $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$. *Then,*
$f \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $f^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$. *Hence,* M *is not rigid, and so* M *is not reduced. Also, since* $r_M(f) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} : b \in \mathbb{Z} \right\}$ *and* M *is indecomposable as a right* R*-module,* $r_M(f)$ *can not be generated by an idempotent as a direct summand of* M. *Hence,* M *is not Rickart.*

For an R-module M with $S = \mathrm{End}_R(M)$, M is called *semicommutative* if for any $f \in S$ and $m \in M$, $fm = 0$ implies $fSm = 0$; see [3] for details.

Proposition 2.20. *Let* M *be an* R*-module with* $S = End_R(M)$. *If* M *is a semicommutative module, then* S *is semicommutative, and hence an abelian ring.*

Proof. Let $f, g \in S$ and assume $fg = 0$. Then, $fgm = 0$ for all $m \in M$. By the hypothesis, $fhgm = 0$, for all $m \in M$ and $h \in S$. Hence, $fhg = 0$, for all $h \in S$ and so $fSg = 0$. Let $e, f \in S$ with $e^2 = e$. Then, $e(1-e)M = 0$. By the hypothesis, $ef(1-e)M = 0$. Hence, $ef(1-e) = 0$, for all $f \in S$. Similarly, $(1-e)fe = 0$, for all $f \in S$. Thus, $ef = fe$, for all $f \in S$. □

Proposition 2.21. *Let* M *be a semicommutative module. Consider the followings.*
(1) M *is a Baer module.*
(2) M *is a quasi-Baer module.*
(3) M *is a Rickart module.*
Then, (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1) Let N be any submodule of M and $n \in N$. By the hypothesis, $l_S(n) = l_S(SnR)$. Hence $l_S(N) = l_S(SN)$. Since SN is a fully invariant submodule of M, by (2), $l_S(SN) = Se$, for some $e^2 = e \in S$. Then, M is a Baer module.
(2) $\Rightarrow$ (3) Let φ be in S. Since $S\varphi S$ is a two sided ideal of S, there exists an idempotent $e \in S$ such that $r_M(S\varphi S) = eM$. Also, since M is semicommutative, $r_M(\varphi) = r_M(\varphi S) = r_M(S\varphi S)$, and so $r_M(\varphi) = eM$. This completes the proof. □

Lemma 2.22. *If M is semicommutative, then it is abelian. The converse holds if M is Rickart.*

Proof. Let M be a semicommutative module and $g \in S$, $e^2 = e \in S$. Then, $e(1-e)m = 0$, for all $m \in M$. Since M is semicommutative, $eg(1-e)m = 0$. So, we have $egm = egem$. Similarly, $(1-e)em = 0$. Then, $gem = egem$. Therefore, $egm = gem$. Suppose now that M is abelian and Rickart module. Let $f \in S$, $m \in M$ with $fm = 0$. Then, $m \in r_M(f)$. Since M is a Rickart module, there exists an idempotent e in S such that $r_M(f) = eM$. Then, $m = em$, $fe = 0$. For any $h \in S$, since M is abelian, $fhm = fhem = fehm = 0$. Therefore, $fSm = 0$. □

In [16], the ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$, for all i and j. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module M is called *Armendariz* if the following condition (1) is satisfied, and M is called *Armendariz of power series type* if the following condition (2) is satisfied:

(1) For any $m(x) = \sum\limits_{i=0}^{n} m_i x^i \in M[x]$ and $f(x) = \sum\limits_{j=0}^{s} a_j x^j \in S[x]$, $f(x)m(x) = 0$ implies $a_j m_i = 0$, for all i and j.

(2) For any $m(x) = \sum\limits_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $f(x) = \sum\limits_{j=0}^{\infty} a_j x^j \in S[[x]]$, $f(x)m(x) = 0$ implies $a_j m_i = 0$, for all i and j.

Lemma 2.23. *If the module M is Armendariz, then M is abelian. The converse holds if M is a Rickart module.*

Proof. Let $m \in M$, $f^2 = f \in S$ and $g \in S$. Consider

$$m_1(x) = (1-f)m + fg(1-f)mx,\ m_2(x) = fm + (1-f)gfmx \in M[x],$$
$$h_1(x) = f - fg(1-f)x,\ h_2(x) = (1-f) - (1-f)gfx \in S[x].$$

Then, $h_i(x)m_i(x) = 0$, for $i = 1, 2$. Since M is Armendariz, $fg(1-f)m = 0$ and $(1-f)gfm = 0$. Therefore, $fgm = gfm$.

Suppose that M is an abelian and Rickart module. Let $m(t) = \sum\limits_{i=0}^{s} m_i t^i \in M[t]$ and $f(t) = \sum\limits_{j=0}^{t} f_j t^j \in S[t]$. If $f(t)m(t) = 0$, then

(1) $f_0 m_0 = 0$
(2) $f_0 m_1 + f_1 m_0 = 0$
(3) $f_0 m_2 + f_1 m_1 + f_2 m_0 = 0$
$\cdots$

By the hypothesis, there exists an idempotent $e_0 \in S$ such that $r_M(f_0) = e_0M$. Then, (1) implies $f_0e_0 = 0$ and $m_0 = e_0m_0$. Multiplying (2) by e_0 from the left, we have $0 = e_0f_0m_1 + e_0f_1m_0 = f_1e_0m_0 = f_1m_0$. By (2), $f_0m_1 = 0$. Let $r_M(f_1) = e_1M$. So, $f_1e_1 = 0$ and $m_0 = e_1m_0$. Multiplying (3) by e_0e_1 from the left and using abelianness of S and $e_0e_1f_2m_0 = f_2m_0$, we have $f_2m_0 = 0$. Then, (3) becomes $f_0m_2 + f_1m_1 = 0$. Multiplying this equation by e_0 from left and using $e_0f_0m_2 = 0$ and $e_0f_1m_1 = f_1m_1$, we have $f_1m_1 = 0$. From (3), $f_2m_0 = 0$. Continuing in this way, we may conclude that $f_jm_i = 0$, for all $1 \leq i \leq s$ and $1 \leq j \leq t$. Hence, M is Armendariz. This completes the proof. □

Corollary 2.24. *If M is Armendariz of power series type, then M is abelian. The converse holds if M is a Rickart module.*

Proof. Similar to the proof of Lemma 2.23. □

We end with some observations concerning relationships between reduced, rigid, symmetric, semicommutative, Armendariz and abelian modules by using Rickart modules.

Theorem 2.25. *If M is a Rickart module, then the followings are equivalent.*

(1) *M is a rigid module.*
(2) *M is a reduced module.*
(3) *M is a symmetric module.*
(4) *M is a semicommutative module.*
(5) *M is an abelian module.*
(6) *M is an Armendariz module.*
(7) *M is an Armendariz of power series type module.*

Proof. (1) ⇔ (2) Use Proposition 2.14. (2) ⇔ (3) Use Lemma 2.18. (2) ⇔ (5) Use Lemma 2.16. (4) ⇔ (5) Use Lemma 2.22. (5) ⇔ (6) Use Lemma 2.23. (5) ⇔ (7) Use Corollary 2.24. □

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