The Complexity of Finding Fair Independent Sets in Cycles

Ishay Haviv
School of Computer Science, The Academic College of Tel Aviv-Yaffo, Tel Aviv, Israel

Abstract

Let G be a cycle graph and let $V_1, \ldots, V_m$ be a partition of its vertex set into m sets. An independent set S of G is said to *fairly represent* the partition if $|S \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$. It is known that for every cycle and every partition of its vertex set, there exists an independent set that fairly represents the partition (Aharoni et al., A Journey through Discrete Math., 2017). We prove that the problem of finding such an independent set is PPA-complete. As an application, we show that the problem of finding a monochromatic edge in a Schrijver graph, given a succinct representation of a coloring that uses fewer colors than its chromatic number, is PPA-complete as well. The work is motivated by the computational aspects of the "cycle plus triangles" problem and of its extensions.

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1 Introduction

In 1986, Du, Hsu, and Hwang [19] conjectured that if a graph on $3n$ vertices is the disjoint union of a Hamilton cycle of length $3n$ and n pairwise vertex-disjoint triangles then its independence number is n. The conjecture has become known as the "cycle plus triangles" problem and has been strengthened by Erdös [20], who conjectured that such a graph is 3-colorable. Fleischner and Stiebitz [26] confirmed these conjectures in a strong form and proved, using an algebraic approach of Alon and Tarsi [6], that such a graph is in fact 3-choosable. Their proof can also be viewed as an application of Alon's Combinatorial Nullstellensatz technique [4]. Slightly later, an alternative elementary proof of the 3-coloring result was given by Sachs [39]. However, none of these proofs supplies an efficient algorithm that given a graph on $3n$ vertices whose set of edges is the disjoint union of a Hamilton cycle and n pairwise vertex-disjoint triangles finds a 3-coloring of the graph or an independent set of size n. Questions on the computational aspects of the problem were posed in several works over the years (see, e.g., [27, 5, 9, 1]).

A natural extension of the problem of Du et al. [19] is the following. Let G be a cycle and let $V_1, \ldots, V_m$ be a partition of its vertex set into m sets. We are interested in an independent set of G that (almost) *fairly represents* the given partition, that is, an independent set S of G satisfying $|S \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m] = \{1, \ldots, m\}$. The existence of such an independent set was proved in a recent work of Aharoni, Alon, Berger, Chudnovsky, Kotlar, Loebl, and Ziv [1]. For the special case where all the sets V_i are of size 3, the proof technique of Aharoni et al. [1] allowed them to show that there are *two* disjoint independent

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sets that fairly represent the partition, providing a new proof of a stronger form of the original conjecture of Du et al. [19]. The results of [1] were extended in a work of Alishahi and Meunier [3] who proved the following.

▶ **Theorem 1** ([3]). *Let G be a cycle on n vertices and let $V_1, \ldots, V_m$ be a partition of its vertex set into m sets. Suppose that n and m have the same parity. Then, there exist two disjoint independent sets S_1 and S_2 of G covering all vertices but one from each V_i such that for each $j \in \{1,2\}$, it holds that $|S_j \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$.*

As shown by Black et al. [10], analogues of Theorem 1 for paths and for partitions into sets of odd sizes can also be proved using the approach of Aharoni et al. [1].

It is interesting to mention that although the statements of Theorem 1 and of its aforementioned variants are purely combinatorial, all of their known proofs are based on tools from topology. The use of topological methods in combinatorics was initiated by Lovász [32] who applied the Borsuk-Ulam theorem [11] from algebraic topology to prove a conjecture of Kneser [31] on the chromatic number of Kneser graphs. For integers $n \geq 2k$, the *Kneser graph* $K(n,k)$ is the graph whose vertices are all the k-subsets of $[n]$ where two sets are adjacent if they are disjoint. It was proved in [32] that the chromatic number of $K(n,k)$ is $n-2k+2$, a result that was strengthened and generalized by several researchers (see, e.g., [34, Chapter 3]). One such strengthening was obtained by Schrijver [40], who studied the subgraph of $K(n,k)$ induced by the collection of all k-subsets of $[n]$ with no two consecutive elements modulo n. This graph is denoted by $S(n,k)$ and is commonly referred to as the *Schrijver graph*. It was proved in [40], again by a topological argument, that the chromatic number of $S(n,k)$ is equal to that of $K(n,k)$. As for Theorem 1, the proof of Alishahi and Meunier [3] employs the Octahdral Tucker lemma that was applied by Matoušek [33] in an alternative proof of Kneser's conjecture and can be viewed as a combinatorial formulation of the Borsuk-Ulam theorem (see also [42]). The approach of Aharoni et al. [1] and of Black et al. [10], however, is based on a direct application of the chromatic number of the Schrijver graph. As before, these proofs are not constructive, in the sense that they do not suggest efficient algorithms for the corresponding search problems. Understanding the computational complexity of these problems is the main motivation for the current work.

In 1994, Papadimitriou [38] has initiated the study of the complexity of total search problems in view of the mathematical argument that lies at the existence proof of their solutions. Let TFNP be the complexity class, defined in [35], of total search problems in NP, that is, the class of search problems in which a solution is guaranteed to exist and can be verified in polynomial running-time. Papadimitriou has introduced in [38] several subclasses of TFNP, each of which consists of the total search problems that can be reduced to a problem that represents some mathematical argument. For example, the class PPA (Polynomial Parity Argument) corresponds to the simple fact that every graph with maximum degree 2 that has a vertex of degree 1 must have another degree 1 vertex. Hence, PPA is defined as the class of all problems in TFNP that can be efficiently reduced to the LEAF problem, in which given a succinct representation of a graph with maximum degree 2 and given a vertex of degree 1 in the graph, the goal is to find another such vertex. The class PPAD (Polynomial Parity Argument in Directed graphs) is defined similarly with respect to directed graphs. Another complexity class defined in [38] is PPP (Polynomial Pigeonhole Principle) whose underlying mathematical argument is the pigeonhole principle. Additional examples of complexity classes defined in this way are PLS (Polynomial Local Search) [30], CLS (Continuous Local Search) [15], and EOPL (End of Potential Line) [21].

The complexity class PPAD is known to perfectly capture the complexity of many important search problems. Notable examples of PPAD-complete problems are those associated with Sperner's lemma [38, 12], the Nash Equilibrium theorem [13, 14], the Envy-Free Cake

Cutting theorem [18], and the Hairy Ball theorem [29]. For PPA, the undirected analogue of PPAD, until recently there were not known complete problems that are "natural", i.e., do not involve circuits and Turing machines in their definitions. In the last few years, the situation was changed following a breakthrough result of Filos-Ratsikas and Goldberg [23, 24], who proved that the Consensus Halving problem with inverse-polynomial precision parameter is PPA-complete (see also [25]) and used it to derive the PPA-completeness of the classical Splitting Necklace with two thieves and Discrete Sandwich problems. This was obtained building on the PPA-hardness, proved by Aisenberg, Bonet, and Buss [2], of the search problem associated with Tucker's lemma. The variant of the problem that corresponds to the Octahedral Tucker lemma was suggested for study by Pálvölgyi [37] and proved to be PPA-complete by Deng, Feng, and Kulkarni [17]. Additional PPA-complete problems, related to the Combinatorial Nullstellensatz and the Chevalley-Warning theorem, were provided by Belovs et al. [8].

1.1 Our Contribution

The present work initiates the study of the complexity of finding independent sets that fairly represent a given partition of the vertex set of a cycle. It is motivated by the computational aspects of combinatorial existence statements, such as the "cycle plus triangles" conjecture of Du et al. [19] proved by Fleischner and Stiebitz [26] and its extensions by Aharoni et al. [1], Alishahi and Meunier [3], and Black et al. [10]. As mentioned before, the challenge of understanding the complexity of the corresponding search problems was explicitly raised by several authors, including Fleischner and Stiebitz [27], Alon [5], and Aharoni et al. [1]. In this work we demonstrate that this research avenue may illuminate interesting connections between this family of problems and the complexity class PPA.

We start by introducing the Fair Independent Set in Cycle Problem, which we denote by FAIR-IS-CYCLE and define as follows.

▶ **Definition 2** (Fair Independent Set in Cycle Problem). *In the FAIR-IS-CYCLE problem, the input consists of a cycle G and a partition $V_1, \ldots, V_m$ of its vertex set into m sets. The goal is to find an independent set S of G satisfying $|S \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$.*

The existence of a solution to every input of FAIR-IS-CYCLE is guaranteed by a result of Aharoni et al. [1, Theorem 1.8]. Since such a solution can be verified in polynomial running-time, the total search problem FAIR-IS-CYCLE lies in the complexity class TFNP. We prove that the class PPA captures the complexity of the problem.

▶ **Theorem 3.** *The FAIR-IS-CYCLE problem is PPA-complete.*

In view of the "cycle plus triangles" problem of Du et al. [19], it would be interesting to understand the complexity of the FAIR-IS-CYCLE problem restricted to partitions into sets of size 3. While Theorem 3 immediately implies that this restricted problem lies in PPA, the question of determining its precise complexity remains open.

We proceed by considering the search problem associated with Theorem 1. In the Fair Splitting of Cycle Problem, denoted FAIR-SPLIT-CYCLE, we are given a cycle and a partition of its vertex set and the goal is to find *two* disjoint independent sets that fairly represent the partition and cover all vertices but one from every part of the partition. We define below an approximation version of this problem, in which the fairness requirement is replaced with the relaxed notion of ε-fairness, where the independent sets should include at least $\frac{1}{2} - \varepsilon$ fraction of the vertices of every part.

▶ **Definition 4** (Approximate Fair Splitting of Cycle Problem). *In the ε-FAIR-SPLIT-CYCLE problem with parameter $\varepsilon \geq 0$, the input consists of a cycle G on n vertices and a partition $V_1, \ldots, V_m$ of its vertex set into m sets, such that n and m have the same parity. The goal is to find two disjoint independent sets S_1 and S_2 of G covering all vertices but one from each V_i such that for each $j \in \{1,2\}$, it holds that $|S_j \cap V_i| \geq (\frac{1}{2} - \varepsilon) \cdot |V_i| - 1$ for all $i \in [m]$. For $\varepsilon = 0$, the problem is denoted by FAIR-SPLIT-CYCLE.*

The existence of a solution to every input of ε-FAIR-SPLIT-CYCLE, already for $\varepsilon = 0$, is guaranteed by Theorem 1 proved in [3]. For $\varepsilon = 0$, it can be seen that FAIR-SPLIT-CYCLE is at least as hard as FAIR-IS-CYCLE. Yet, it turns out that FAIR-SPLIT-CYCLE lies in PPA and is thus also PPA-complete.

▶ **Theorem 5.** *The FAIR-SPLIT-CYCLE problem is* PPA*-complete.*

For the approximation version of the problem, we provide the following hardness result.

▶ **Theorem 6.** *There exists a constant $\varepsilon > 0$ for which the ε-FAIR-SPLIT-CYCLE problem is* PPAD*-hard.*

We finally consider the complexity of the SCHRIJVER problem. In this problem we are given a succinct representation of a coloring of the Schrijver graph $S(n,k)$ with $n-2k+1$ colors, which is one less than its chromatic number [40], and the goal is to find a monochromatic edge (see Definition 16). The study of the SCHRIJVER problem is motivated by a question raised by Deng et al. [17] regarding the complexity of the analogue problem for Kneser graphs. Note that the latter is not harder than the SCHRIJVER problem, because $S(n,k)$ is a subgraph of $K(n,k)$ with the same chromatic number. As an application of our Theorem 3, we prove the following.

▶ **Theorem 7.** *The SCHRIJVER problem is* PPA*-complete.*

1.2 Overview of Proofs

To obtain our results we present a chain of reductions, as described in Figure 1. Our starting point is the Consensus Halving problem with precision parameter ε, in which given a collection of m probability measures on the interval $[0,1]$ the goal is to partition the interval into two pieces using relatively few cuts, so that each of the measures has the same mass on the two pieces up to an error of ε (see Definition 8). It is known that every input to this problem has a solution with at most m cuts even for $\varepsilon = 0$ [41] (see also [28, 7]). The problem of finding a solution is PPA-hard when ε is inverse-polynomial in m [23, 24, 25] and PPAD-hard when ε is some positive constant [22].

In Section 2, we reduce the Consensus Halving problem to an intermediate variant of the ε-FAIR-SPLIT-CYCLE problem, which we call ε-FAIR-SPLIT-PATH$'$ (see Definition 12). Then, we use this reduction to obtain our hardness results for the FAIR-IS-CYCLE and FAIR-SPLIT-CYCLE problems. The reduction borrows a discretization argument that was used in [23] to reduce the Consensus Halving problem to the Splitting Necklace problem with two thieves. This argument enables us to transform a Consensus Halving instance into a path and a partition of its vertex set, for which the goal is to partition the path using relatively few cuts into two parts, each of which contains roughly half of the vertices of every set in the partition. In order to relate this problem to independent sets that fairly represent the partition, we need an additional simple trick. Between every two consecutive vertices of the path we add a new vertex and put all the new vertices in a new set added to the partition

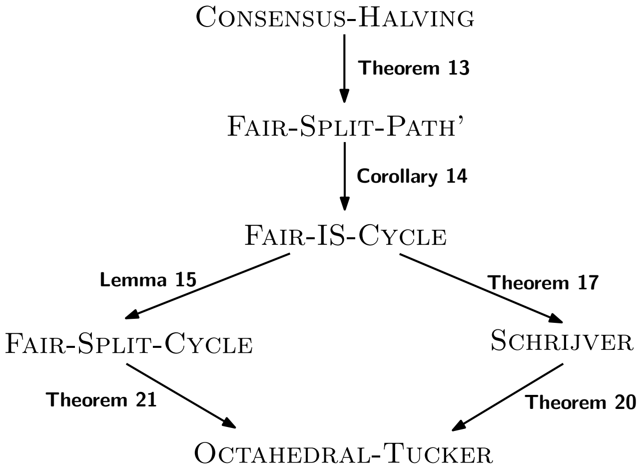

Figure 1 Chain of reductions.

of the vertex set. We then argue, roughly speaking, that two disjoint independent sets in the obtained path, which fairly represent the partition and cover almost all of the vertices, can be used to obtain a solution to the original instance. The high-level idea is that those few vertices that are uncovered by the two independent sets can be viewed as cuts, and every path between two such vertices alternates between the two given independent sets. By construction, it means that only one of the two independent sets contains in such a path original vertices (that is, vertices that were not added in the last phase of the reduction), hence every such path can be naturally assigned to one of the two pieces required by the Consensus Halving problem. Combining our reduction with the known hardness results of Consensus Halving, we derive the PPA-hardness of Fair-IS-Cycle and Fair-Split-Cycle and the PPAD-hardness of ε-Fair-Split-Cycle for a constant $\varepsilon > 0$, as needed for Theorems 3, 5, and 6.

In Section 3, we introduce and study the Schrijver problem. We reduce the Fair-IS-Cycle problem to the Schrijver problem, implying that the latter is PPA-hard. The reduction follows an argument of Aharoni et al. [1] who used the chromatic number of the Schrijver graph [40] to prove the existence of the independent set required in Fair-IS-Cycle. Finally, employing arguments of Meunier [36] and Alishahi and Meunier [3], we reduce the Schrijver and Fair-Split-Cycle problems to the search problem associated with the Octahedral Tucker lemma (see Definition 18). Since it is known, already from [38], that this problem lies in PPA, we get that Fair-IS-Cycle, Fair-Split-Cycle, and Schrijver are all members of PPA, completing the proofs of Theorems 3, 5, and 7.

We remark that one could consider analogues of the Fair-IS-Cycle and Fair-Split-Cycle problems for paths rather than for cycles and obtain similar results. We have chosen to focus here on the cycle setting, motivated by the computational aspects of the "cycle plus triangles" problem [19, 20, 26].

2 Fair Independent Sets in Cycles

In this section we prove our hardness results for the Fair-IS-Cycle and Fair-Split-Cycle problems. We first recall the definition of the Consensus Halving problem and gather some of the hardness results known for it. Then, we present an efficient reduction from this problem to an intermediate problem, which is used to obtain the hardness results of Theorems 3, 5, and 6.

2.1 Consensus Halving

Consider the following variant of the Consensus Halving problem, denoted Con-Halving.

▶ **Definition 8** (Consensus Halving Problem). *In the ε-Con-Halving(m,ℓ) problem with precision parameter $\varepsilon = \varepsilon(m)$, the input consists of m probability measures $\mu_1, \ldots, \mu_m$ on the interval $I = [0,1]$, given by their density functions. The goal is to partition the interval I using at most ℓ cuts into two (not necessarily connected) pieces I^+ and I^-, so that for every $i \in [m]$ it holds that $|\mu_i(I^+) - \mu_i(I^-)| \leq \varepsilon$.*

For $\ell \geq m$, every input of ε-Con-Halving(m,ℓ) has a solution even for $\varepsilon = 0$ [41]. We state below two hardness results known for Con-Halving. Here, a function on an interval is said to be *piecewise constant* if its domain can be partitioned into a finite set of intervals such that the function is constant on each of them. We refer to the intervals of the partition on which the function is nonzero as the *blocks* of the function.

▶ **Theorem 9** ([23, 24, 25]). *There exists an inverse-polynomial $\varepsilon = \varepsilon(m)$ such that for every constant $c \geq 0$, the ε-Con-Halving$(m, m+c)$ problem, restricted to inputs with piecewise constant density functions with at most 2 blocks, is* PPA*-hard.*

▶ **Theorem 10** ([22]). *There exist an absolute constant $\varepsilon > 0$ and a polynomial p such that for every constant $c \geq 0$, the ε-Con-Halving$(m, m+c)$ problem, restricted to inputs with piecewise constant density functions with at most $p(m)$ blocks, is* PPAD*-hard.*

▶ Remark 11. We note that, as explained in [25], the constant c given in Theorems 9 and 10 can be replaced by $m^{1-\alpha}$ for any constant $\alpha > 0$. This stronger hardness, however, is not required to obtain our results. We also note that our results do not rely on the fact that the hardness given in Theorem 9 holds for instances with density functions with at most 2 blocks, as proved in [25], rather than polynomially many blocks as was previously proved in [24].

2.2 The Main Reduction

To obtain our hardness results for the Fair-IS-Cycle and Fair-Split-Cycle problems, we consider the following intermediate problem.

▶ **Definition 12.** *In the ε-Fair-Split-Path$'$ problem with parameter $\varepsilon \geq 0$, the input consists of a path G and a partition $V_1, \ldots, V_m$ of its vertex set into m sets such that $|V_i|$ is odd for all $i \in [m]$. The goal is to find two disjoint independent sets S_1 and S_2 of G covering all but at most m of the vertices of G such that*

$$|S_1 \cap V_i| \in \Big[(\tfrac{1}{2} - \varepsilon) \cdot |V_i| - 1, (\tfrac{1}{2} + \varepsilon) \cdot |V_i|\Big]$$

for all $i \in [m]$. When $\varepsilon = 0$, the problem is denoted by Fair-Split-Path$'$.

Note that the ε-Fair-Split-Path$'$ problem differs from the ε-Fair-Split-Cycle problem (see Definition 4) in the following respects: (a) The input graph is a path rather than a cycle, (b) an ε-fairness property is required only for the independent set S_1 rather than for both S_1 and S_2, (c) there is no requirement regarding the sets V_i to which the vertices that are uncovered by S_1 and S_2 belong, and (d) the sets V_i are required to be of odd sizes. Yet, it is easy to check that Theorem 1 implies that every instance of the ε-Fair-Split-Path$'$ problem has a solution already for $\varepsilon = 0$.

We turn to prove the following.

▶ **Theorem 13.** *Let p be a polynomial and suppose that $\varepsilon = \varepsilon(m)$ is bounded from below by some inverse-polynomial in m. Then, the ε-*Con-Halving*$(m, m+1)$ problem, restricted to inputs with piecewise constant density functions with at most $p(m)$ blocks, is polynomial-time reducible to the $\frac{\varepsilon}{4}$-*Fair-Split-Path*$'$ problem.*

Proof. Consider an instance of ε-Con-Halving$(m, m+1)$ consisting of m probability measures $\mu_1, \dots, \mu_m$ on the interval $I = [0,1]$, given by their piecewise constant density functions $g_1, \dots, g_m$, each of which has at most $p(m)$ blocks. The reduction constructs an instance of $\frac{\varepsilon}{4}$-Fair-Split-Path$'$, namely, a path G and a partition $V_1, \dots, V_{m+1}$ of its vertex set into $m+1$ sets of odd sizes.

We start with a high-level description of the reduction. The reduction associates with every density function g_i a collection V_i of vertices located in the (at most $p(m)$) intervals on which g_i is nonzero. To do so, we partition every block of g_i into sub-intervals such that the measure of μ_i on each of them is δ, where $\delta > 0$ is some small parameter (assuming, for now, that the measure of μ_i on every block is an integer multiple of δ). At the middle of every such sub-interval we locate a vertex and put it in V_i. Then, we construct a path G that alternates between the vertices of $V_1 \cup \dots \cup V_m$ ordered according to their locations in I and additional vertices which we put in another set V_{m+1}. We also take care of the requirement that each $|V_i|$ is odd.

The intuitive idea behind this reduction is the following. Suppose that we are given a solution to the constructed instance, i.e., two disjoint independent sets S_1 and S_2 of the path G covering all but $m+1$ of the vertices such that S_1 contains roughly half of the vertices of V_i for each $i \in [m+1]$. Observe that by removing from G the $m+1$ vertices that *do not* belong to $S_1 \cup S_2$, we essentially get a partition of the vertices of $S_1 \cup S_2$ into $m+2$ paths. Since S_1 and S_2 are independent sets in G, it follows that each such path alternates between S_1 and S_2. However, recalling that G alternates between $V_1 \cup \dots \cup V_m$ and V_{m+1}, it follows that ignoring the vertices of V_{m+1}, each such path contains either only vertices of S_1 or only vertices of S_2. Now, one can view the $m+1$ locations of the vertices that do not belong to $S_1 \cup S_2$ as cuts in the interval I which partition it into $m+2$ sub-intervals, each of which includes vertices from either S_1 or S_2 (again, ignoring the vertices of V_{m+1}). Let I^+ and I^- be the pieces of I obtained from the sub-intervals that correspond to S_1 and S_2 respectively. Since the number of vertices from V_i in every path is approximately proportional to the measure of μ_i in the corresponding sub-interval, it can be shown that the probability measure of μ_i on I^+ is approximately $\frac{1}{2}$. This yields that the probability measure μ_i is approximately equal on the pieces I^+ and I^-, as needed for the Con-Halving$(m, m+1)$ problem.

We turn to the formal description of the reduction. Define $\delta = \frac{\varepsilon}{4 \cdot (2p(m)+m+3)}$. The reduction acts as follows.

1. For every $i \in [m]$, do the following:
   - We are given a partition of the interval I into intervals such that on at most $p(m)$ of them the function g_i is equal to a nonzero value and is zero everywhere else. For every such interval, let γ denote the volume of g_i on it, and divide it into $\lceil \gamma/\delta \rceil$ sub-intervals of volume δ each, possibly besides the last one whose volume might be smaller. We refer to a sub-interval of volume smaller than δ as an *imperfect* sub-interval. The number of imperfect sub-intervals associated with g_i is clearly at most $p(m)$. At the middle point of every sub-interval of g_i, locate a vertex and put it in the set V_i.
   - If the number of vertices in V_i is even, then add another vertex to V_i and locate it arbitrarily in I.
   - Note that, by $\mu_i(I) = 1$, we have
     $$|V_i| \cdot \delta \in [1, 1 + (p(m)+1) \cdot \delta]. \quad (1)$$

2. Consider the path on the vertices of $V_1 \cup \dots \cup V_m$ ordered according to their locations in the interval I, breaking ties arbitrarily.
3. Add a new vertex before every vertex in this path, locate it at the middle of the sub-interval between its two adjacent vertices (where the first new vertex is located at 0), and put these new vertices in the set V_{m+1}. If the number of vertices in V_{m+1} is even then add one more vertex to the end of the path, locate it at 1, and put it in V_{m+1} as well. Denote by G the obtained path, and note that G alternates between $V_1 \cup \dots \cup V_m$ and V_{m+1}.
4. The output of the reduction is the path G and the partition $V_1, \dots, V_{m+1}$ of its vertex set V into $m+1$ sets. By construction, $|V_i|$ is odd for every $i \in [m+1]$.

It is easy to verify that the reduction can be implemented in polynomial running-time. Indeed, every density function g_i is piecewise constant with at most $p(m)$ blocks, hence for every $i \in [m]$ the number of vertices that the reduction defines for V_i is at most $1/\delta + p(m) + 1$, and the latter is polynomial in the input size because of the definition of δ and the fact that ε is at least inverse-polynomial in m. The additional set V_{m+1} doubles the number of vertices, possibly with one extra vertex, preserving the construction polynomial in the input size.

We turn to prove the correctness of the reduction, that is, that a solution to the constructed instance of $\frac{\varepsilon}{4}$-Fair-Split-Path′ can be used to efficiently compute a solution to the original instance of ε-Con-Halving$(m, m+1)$. Suppose we are given a solution to $\frac{\varepsilon}{4}$-Fair-Split-Path′ for the path G and the partition $V_1, \dots, V_{m+1}$ of its vertex set V. Such a solution consists of two disjoint independent sets S_1 and S_2 of G covering all but at most $m+1$ of the vertices of G such that

$$|S_1 \cap V_i| \in \Big[(\tfrac{1}{2} - \tfrac{\varepsilon}{4}) \cdot |V_i| - 1, (\tfrac{1}{2} + \tfrac{\varepsilon}{4}) \cdot |V_i| \Big] \tag{2}$$

for all $i \in [m+1]$. Put $S_3 = V \setminus (S_1 \cup S_2)$. It can be assumed that $|S_3| = m+1$ (otherwise, remove some arbitrary vertices from S_2). Denote the vertices of S_3 by $u_1, \dots, u_{m+1}$ ordered according to their order in G. Let $P_1, \dots, P_{m+2}$ be the $m+2$ paths obtained from G by removing the vertices of S_3 (where some of the paths might be empty). Since S_1 and S_2 are independent sets, every path P_j alternates between S_1 and S_2. By our construction, this implies that in every path P_j either the vertices of S_1 are from $V \setminus V_{m+1}$ and those of S_2 are from V_{m+1}, or the vertices of S_2 are from $V \setminus V_{m+1}$ and those of S_1 are from V_{m+1}. We define $b_j = 1$ in the former case and $b_j = 2$ in the latter. Thus, for every $i \in [m]$, the number of vertices of V_i that appear in the paths P_j with $b_j = 1$ is precisely $|S_1 \cap V_i|$.

Now, let $\beta_1, \dots, \beta_{m+1} \in I$ be the locations of the vertices $u_1, \dots, u_{m+1}$ in the interval I as defined by the reduction. We interpret these locations as $m+1$ cuts of the interval I. Set $\beta_0 = 0$ and $\beta_{m+2} = 1$, and for every $j \in [m+2]$, let I_j denote the interval $[\beta_{j-1}, \beta_j]$. Consider the partition of I into two pieces I^+ and I^-, where I^+ includes all the parts I_j with $b_j = 1$ and I^- includes all the parts I_j with $b_j = 2$. We claim that this partition, which is obtained using $m+1$ cuts in I, forms a valid solution to the original instance of ε-Con-Halving$(m, m+1)$. To this end, we show that for every $i \in [m]$ it holds that $|\mu_i(I^+) - \frac{1}{2}| \leq \frac{\varepsilon}{2}$, which is equivalent to $|\mu_i(I^+) - \mu_i(I^-)| \leq \varepsilon$.

Fix some $i \in [m]$. We turn to estimate the quantity $\mu_i(I^+)$, i.e., the total measure of μ_i on the intervals I_j with $b_j = 1$. By our construction, every vertex of V_i corresponds to a sub-interval whose measure by μ_i is δ (except for at most $p(m) + 1$ of them). Since the intervals of I^+ correspond to the paths P_j whose vertices in $V \setminus V_{m+1}$ are precisely the vertices of $S_1 \setminus V_{m+1}$, one would expect $\mu_i(I^+)$ to measure the number of vertices in $S_1 \cap V_i$, with a contribution of δ per every such vertex. This suggests an estimation of $|S_1 \cap V_i| \cdot \delta$ for $\mu_i(I^+)$. However, several issues might prevent from this estimation to be accurate:

- The set V_i might include vertices that correspond to imperfect sub-intervals whose measure by μ_i is smaller than δ. Since there are at most $p(m)$ such vertices in V_i, they can cause an error of at most $p(m) \cdot \delta$ in the above estimation.
- To make sure that $|V_i|$ is odd, the reduction might add one extra vertex to V_i. This might cause an error of at most δ in the above estimation.
- The precise locations β_j of the cuts of I might fall inside sub-intervals that correspond to vertices of V_i. Since the sub-intervals that correspond to vertices of V_i are disjoint, every such cut can cause an error of at most δ in the above estimation, and since there are $m+1$ cuts the error here is bounded by $(m+1) \cdot \delta$.

We conclude that $\mu_i(I^+)$ differs from the aforementioned estimation $|S_1 \cap V_i| \cdot \delta$ by not more than $(p(m)+m+2) \cdot \delta$. Combining (1) and (2), it can be verified that

$$\Big| |S_1 \cap V_i| \cdot \delta - \tfrac{1}{2} \Big| \leq \tfrac{\varepsilon}{4} + (p(m)+1) \cdot \delta,$$

hence

$$\begin{aligned} \Big| \mu_i(I^+) - \tfrac{1}{2} \Big| &\leq \Big| \mu_i(I^+) - |S_1 \cap V_i| \cdot \delta \Big| + \Big| |S_1 \cap V_i| \cdot \delta - \tfrac{1}{2} \Big| \\ &\leq (p(m)+m+2) \cdot \delta + \tfrac{\varepsilon}{4} + (p(m)+1) \cdot \delta \\ &= \tfrac{\varepsilon}{4} + (2p(m)+m+3) \cdot \delta = \tfrac{\varepsilon}{2}, \end{aligned}$$

where the last equality holds by the definition of δ. This completes the proof. ◀

2.3 Hardness of Fair-IS-Cycle and Fair-Split-Cycle

Equipped with Theorem 13, we are ready to derive the hardness of the FAIR-IS-CYCLE and FAIR-SPLIT-CYCLE problems (see Definitions 2 and 4).

▶ **Corollary 14.** *The FAIR-IS-CYCLE problem is* PPA*-hard.*

Proof. By Theorem 9, the ε-CON-HALVING$(m, m+1)$ problem is PPA-hard for input density functions that are piecewise constant with at most 2 blocks, where $\varepsilon = \varepsilon(m)$ is inverse-polynomial. By Theorem 13, this problem is polynomial-time reducible to the $\frac{\varepsilon}{4}$-FAIR-SPLIT-PATH$'$ problem, implying that FAIR-SPLIT-PATH$'$, with $\varepsilon = 0$, is PPA-hard. Hence, to prove the corollary, it suffices to show that FAIR-SPLIT-PATH$'$ is polynomial-time reducible to FAIR-IS-CYCLE.

Consider an instance of FAIR-SPLIT-PATH$'$, that is, a path G on n vertices and a partition $V_1, \ldots, V_m$ of its vertex set into m sets such that $|V_i|$ is odd for all $i \in [m]$. The reduction simply returns the cycle G', obtained from the path G by connecting its endpoints by an edge, and the same partition $V_1, \ldots, V_m$ of its vertex set. For correctness, suppose that we are given a solution to this instance of FAIR-IS-CYCLE, i.e., an independent set S_1 of G' satisfying $|S_1 \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$. Since each $|V_i|$ is odd, it can be assumed that $|S_1 \cap V_i| = \frac{1}{2} \cdot (|V_i| - 1)$ for all $i \in [m]$ (by removing some vertices from S_1 if needed), implying that

$$|S_1| = \sum_{i=1}^{m} |S_1 \cap V_i| = \tfrac{1}{2} \cdot \sum_{i=1}^{m} (|V_i| - 1) = \frac{n-m}{2}.$$

For every vertex of S_1 consider the vertex that follows it in the cycle G' (say, oriented clockwise), and let S_2 be the set of vertices that follow those of S_1. Since S_1 is an independent set in G', we get that S_2 is another independent set in G' which is disjoint from S_1 and has the same size. We obtain that

$$|S_1 \cup S_2| = |S_1| + |S_2| = 2 \cdot \frac{n-m}{2} = n - m,$$

hence S_1 and S_2 are two disjoint independent sets of G' covering all but m of its vertices. In particular, S_1 and S_2 are independent sets in the path G, and as such, they form a valid solution to the Fair-Split-Path$'$ instance. This solution can clearly be constructed in polynomial running-time given S_1, completing the proof. ◀

The following simple lemma allows us to derive the PPA-hardness of Fair-Split-Cycle.

▶ **Lemma 15.** *The Fair-IS-Cycle problem is polynomial-time reducible to Fair-Split-Cycle.*

Proof. Consider an instance of Fair-IS-Cycle, that is, a cycle G on n vertices and a partition $V_1, \ldots, V_m$ of its vertex set into m sets. If n and m have the same parity then the reduction returns the input as is. Otherwise, there exists some $i \in [m]$ for which the size of V_i is even. In this case, the reduction adds to the cycle G a new vertex located between two arbitrary consecutive vertices and puts it in V_i. Now, the number of vertices and the number of sets in the partition have the same parity, so the reduction can output the obtained cycle and partition.

A solution to the constructed instance of Fair-Split-Cycle involves two disjoint independent sets that fairly represent the partition. Clearly, at least one of the sets does not include the two neighbors of the vertex that was possibly added to G. Letting S denote the set of vertices of G in this set, we get that S is independent in G, and it is easy to check that $|S \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$, so we are done. ◀

We end this section with a proof of Theorem 6.

Proof of Theorem 6. By Theorem 10, the ε-Con-Halving$(m, m+1)$ problem is PPAD-hard for input density functions that are piecewise constant with at most $p(m)$ blocks, where p is a polynomial and ε is a positive constant. Applying Theorem 13, we get that the $\frac{\varepsilon}{4}$-Fair-Split-Path$'$ problem is PPAD-hard. To complete the proof, we show that for every $\varepsilon \geq 0$ the ε-Fair-Split-Path$'$ problem is polynomial-time reducible to the ε-Fair-Split-Cycle problem.

Consider again the reduction that given a path G and a partition $V_1, \ldots, V_m$ of its vertex set into sets of odd sizes returns the cycle G', obtained from the path G by connecting its endpoints by an edge, and the same partition $V_1, \ldots, V_m$. Since the sets of the partition have odd sizes, it follows that the number of vertices and the number of sets in the partition have the same parity, hence the reduction provides an appropriate instance of the ε-Fair-Split-Cycle problem.

For correctness, consider a solution to the constructed instance, i.e., two disjoint independent sets S_1 and S_2 of G' covering all vertices but one from each part V_i such that for each $j \in \{1, 2\}$, it holds that $|S_j \cap V_i| \geq (\frac{1}{2} - \varepsilon) \cdot |V_i| - 1$ for all $i \in [m]$. We claim that S_1 and S_2 form a valid solution to the original ε-Fair-Split-Path$'$ instance. Indeed, an independent set in G' is also an independent set in G. In addition, the set S_1 satisfies $|S_1 \cap V_i| \in \big[(\frac{1}{2} - \varepsilon) \cdot |V_i| - 1, (\frac{1}{2} + \varepsilon) \cdot |V_i|\big]$ for all $i \in [m]$, where the upper bound holds because

$$|S_1 \cap V_i| = |V_i| - |S_2 \cap V_i| - 1 \leq |V_i| - \Big((\tfrac{1}{2} - \varepsilon) \cdot |V_i| - 1\Big) - 1 = (\tfrac{1}{2} + \varepsilon) \cdot |V_i|.$$

This completes the proof. ◀

3 The Schrijver Problem

In this section we introduce and study the SCHRIJVER problem, a natural analogue of the Kneser problem defined by Deng et al. [17].

We start with some definitions. A set $A \subseteq [n]$ is said to be *stable* if it does not contain two consecutive elements modulo n (that is, if $i \in A$ then $i+1 \notin A$, and if $n \in A$ then $1 \notin A$). In other words, a stable subset of $[n]$ is an independent set in the cycle on n vertices with the numbering from 1 to n along the cycle. For integers $n \geq 2k$, let $\binom{[n]}{k}_{\text{stab}}$ denote the collection of all stable k-subsets of $[n]$. Recall that the Schrijver graph $S(n,k)$ is the graph on the vertex set $\binom{[n]}{k}_{\text{stab}}$, where two sets are adjacent if they are disjoint. We define the search problem SCHRIJVER as follows.

▶ **Definition 16** (Schrijver Graph Problem). *In the* SCHRIJVER *problem, the input consists of a Boolean circuit that represents a coloring*

$$c : \binom{[n]}{k}_{\text{stab}} \to [n-2k+1]$$

of the Schrijver graph $S(n,k)$ using $n-2k+1$ colors, where n and k are integers satisfying $n \geq 2k$. The goal is to find a monochromatic edge, i.e., two disjoint sets $S_1, S_2 \in \binom{[n]}{k}_{\text{stab}}$ such that $c(S_1) = c(S_2)$.

As mentioned earlier, it was proved by Schrijver [40] that the chromatic number of $S(n,k)$ is precisely $n-2k+2$. Therefore, every input to the SCHRIJVER problem has a solution.

3.1 From Fair-IS-Cycle to Schrijver

The following theorem is used to obtain the hardness result for the SCHRIJVER problem. The proof applies an argument of [1] (see also [10]).

▶ **Theorem 17.** *The* FAIR-IS-CYCLE *problem is polynomial-time reducible to the* SCHRIJVER *problem.*

Proof. Consider an instance of the FAIR-IS-CYCLE problem, namely, a cycle G and a partition $V_1, \ldots, V_m$ of its vertex set into m sets. For every $i \in [m]$, let V_i' be the set obtained from V_i by removing one arbitrary vertex if $|V_i|$ is even, and let $V_i' = V_i$ otherwise. Since the size of every set V_i' is odd, we can write $|V_i'| = 2r_i + 1$ for an integer $r_i \geq 0$. Let G' be the cycle obtained from G by removing the vertices that do not belong to the sets V_i' and connecting the remaining vertices according to their order in G. Letting n denote the number of vertices in G', it can be assumed that its vertex set is $[n]$ with the numbering from 1 to n along the cycle. Put $k = \sum_{i=1}^{m} r_i$, and notice that $n = 2k + m$. Define a coloring c of the Schrijver graph $S(n,k)$ as follows. The color $c(S)$ of a vertex $S \in \binom{[n]}{k}_{\text{stab}}$ is defined as the smallest integer $i \in [m]$ for which $|S \cap V_i'| > r_i$ in case that such an i exists, and $m+1$ otherwise. This gives us a coloring of $S(n,k)$ with $n-2k+1$ colors, and thus an instance of the SCHRIJVER problem. It can be seen that a Boolean circuit that computes the coloring c can be constructed in polynomial running-time.

To prove the correctness of the reduction, consider a solution to the constructed SCHRIJVER instance, i.e., two disjoint sets $S_1, S_2 \in \binom{[n]}{k}_{\text{stab}}$ with $c(S_1) = c(S_2)$. It is impossible that for some $i \in [m]$ it holds that $|S_1 \cap V_i'| > r_i$ and $|S_2 \cap V_i'| > r_i$, because S_1 and S_2 are disjoint and $|V_i'| = 2r_i + 1$. It follows that $c(S_1) = c(S_2) = m+1$, meaning that $|S_1 \cap V_i'| \leq r_i$ and $|S_2 \cap V_i'| \leq r_i$ for all $i \in [m]$. Since $|S_1| = |S_2| = k$, it follows that

$|S_1 \cap V_i'| = r_i$ and $|S_2 \cap V_i'| = r_i$ for all $i \in [m]$, hence S_1 and S_2 are two disjoint independent sets of G' covering all vertices but one from each V_i' and for each $j \in \{1, 2\}$, we have $|S_j \cap V_i'| = \frac{1}{2} \cdot (|V_i'| - 1) \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$. Since S_1 and S_2 are also independent sets of the original cycle G, each of them forms a valid solution to the FAIR-IS-CYCLE instance, completing the proof. ◀

3.2 Membership in PPA

We now show that the SCHRIJVER and FAIR-SPLIT-CYCLE problems lie in PPA by reductions to the search problem associated with the Octahderal Tucker lemma. The reductions follow the proofs of the corresponding mathematical statements by Meunier [36] and by Alishahi and Meunier [3]. The proofs can be found in the full version of the paper.

We start with some notation (following [16, Section 2]). The partial order $\preceq$ on the set $\{+, -, 0\}$ is defined by $0 \preceq +$ and by $0 \preceq -$, where $+$ and $-$ are incomparable. The definition is extended to vectors, so that for two vectors x, y in $\{+, -, 0\}^n$, we have $x \preceq y$ if for all $i \in [n]$ it holds that $x_i \preceq y_i$ (equivalently, $x_i = y_i$ whenever $x_i \neq 0$). The Octahedral Tucker lemma, given implicitly in [33] and explicitly in [42], asserts that for every function $\lambda : \{+, -, 0\}^n \setminus \{0\} \rightarrow \{\pm 1, \dots, \pm(n-1)\}$ satisfying $\lambda(-x) = -\lambda(x)$ for all x, there exist vectors x, y such that $x \preceq y$ and $\lambda(x) = -\lambda(y)$. This guarantees the existence of a solution to every input of the following search problem, denoted OCTAHEDRAL-TUCKER.

▶ **Definition 18** (Octahedral Tucker Problem). *In the OCTAHEDRAL-TUCKER problem, the input consists of a Boolean circuit that represents a function* $\lambda : \{+, -, 0\}^n \setminus \{0\} \rightarrow \{\pm 1, \pm 2, \dots, \pm(n-1)\}$ *satisfying* $\lambda(-x) = -\lambda(x)$ *for all* x*. The goal is to find vectors* x, y *such that* $x \preceq y$ *and* $\lambda(x) = -\lambda(y)$*.*

The OCTAHEDRAL-TUCKER problem is known to be PPA-complete [17], where its membership in PPA follows already from [38] (see also [17, Appendix A]).

▶ **Proposition 19** ([38]). *The OCTAHEDRAL-TUCKER problem lies in* PPA*.*

The SCHRIJVER problem is reduced to OCTAHEDRAL-TUCKER, applying an argument of [36]. The proof is omitted.

▶ **Theorem 20.** *SCHRIJVER is polynomial-time reducible to OCTAHEDRAL-TUCKER.*

The FAIR-SPLIT-CYCLE problem (see Definition 4) is reduced to OCTAHEDRAL-TUCKER, applying an argument of [3]. The proof is omitted.

▶ **Theorem 21.** *FAIR-SPLIT-CYCLE is polynomial-time reducible to OCTAHEDRAL-TUCKER.*

3.3 Putting All Together

The presented reductions complete the proofs of our results. Indeed, the FAIR-IS-CYCLE problem is PPA-hard by Corollary 14, and is polynomial-time reducible to the FAIR-SPLIT-CYCLE and SCHRIJVER problems by Lemma 15 and Theorem 17 respectively. By Theorems 20 and 21, each of the two is efficiently reducible to the OCTAHEDRAL-TUCKER problem, which by Proposition 19 lies in PPA. It thus follows that all of these problems are PPA-complete (see Figure 1), confirming Theorems 3, 5, and 7.

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