Algorithmic Persuasion with Evidence

Martin Hoefer

Goethe Universität Frankfurt, Germany
mhoefer@cs.uni-frankfurt.de

Pasin Manurangsi

Google Research, Mountain View, CA, USA
pasin@google.com

Alexandros Psomas

Purdue University, West Lafayette, IN, USA
apsomas@cs.purdue.edu

Abstract

We consider a game of persuasion with evidence between a sender and a receiver. The sender has private information. By presenting evidence on the information, the sender wishes to persuade the receiver to take a single action (e.g., hire a job candidate, or convict a defendant). The sender's utility depends solely on whether or not the receiver takes the action. The receiver's utility depends on both the action as well as the sender's private information. We study three natural variations. First, we consider sequential equilibria of the game without commitment power. Second, we consider a persuasion variant, where the sender commits to a signaling scheme and then the receiver, after seeing the evidence, takes the action or not. Third, we study a delegation variant, where the receiver first commits to taking the action if being presented certain evidence, and then the sender presents evidence to maximize the probability the action is taken. We study these variants through the computational lens, and give hardness results, optimal approximation algorithms, as well as polynomial-time algorithms for special cases. Among our results is an approximation algorithm that rounds a semidefinite program that might be of independent interest, since, to the best of our knowledge, it is the first such approximation algorithm for a natural problem in algorithmic economics.

2012 ACM Subject Classification Theory of computation → Algorithmic game theory and mechanism design; Theory of computation → Approximation algorithms analysis

Keywords and phrases Bayesian Persuasion, Semidefinite Programming, Approximation Algorithms

Digital Object Identifier 10.4230/LIPIcs.ITCS.2021.3

Related Version A full version of the paper is available at `http://arxiv.org/abs/2008.12626`.

Funding *Martin Hoefer*: GIF grant I-1419-118.4/2017 and DFG grants Ho 3831/5-1, 6-1, and 7-1.

Acknowledgements This work was done in part while Martin Hoefer and Alexandros Psomas were visiting the Simons Institute for the Theory of Computing. The authors acknowledge financial support and the invitation by the organizers to join the stimulating work environment. This work was done in part while Alexandros Psomas was visiting Google Research, Mountain View, USA.

1 Introduction

Persuasion is a fundamental challenge arising in diverse areas such as recommendation problems in the Internet, consulting and lobbying, employee hiring, and many more. Persuasion problems occupy a central role in economics and received significant interest over the last two decades. A prominent approach is *persuasion with evidence* as introduced by Glazer and Rubinstein [13, 14], which has attracted a lot of subsequent work. In this problem, a sender wishes to persuade a receiver to take a single action by presenting evidence. The sender's utility depends solely on whether or not the action is taken, while the receiver's utility depends on both the action as well as the sender's private information. Consider, for example, a prosecutor trying to convince a judge that a defendant is guilty and should be convicted, or a job candidate trying to convince a company that she has the best qualifications and should be hired. How should these pairs of agents interact?

The literature on persuasion games in economics and game theory is vast; see Sobel [27] for a survey. In sharp contrast, very little is known about *computation* in this domain, especially for the persuasion problem with evidence. How does the restriction to evidence impact the computational complexity of the problem? Our main contribution of this paper is to initiate the systematic study of *persuasion with evidence though a computational lens*. We examine three natural model variants that arise from the power to commit to certain behavior.

If there is no commitment power, the scenario is an extensive-form game. We prove that finding a sequential equilibrium is always possible in polynomial time. However, the sender and the receiver can significantly improve their utility when they enjoy commitment power.

If the sender has commitment power, then she can commit in advance which evidence is presented in each possible instantiation of her private information, and the receiver seeing the evidence then takes the action or not. We refer to this situation as *constrained persuasion*, since the sender with commitment power wants to persuade the rational receiver to take the action. The sender is constrained to providing concrete evidence instead of just making a recommendation as is the case in the so called Bayesian persuasion paradigm [19]. Constrained persuasion is a natural model in the example of prosecutor and judge, where the prosecutor (sender) with private information would first present evidence before the judge (receiver) makes a decision. Although this scenario seems structurally rather simple, we show that the sender's task in constrained persuasion is computationally (highly) intractable. Unless $\mathsf{P} = \mathsf{NP}$, optimal persuasion can become hard to approximate within a polynomial factor of the input size.

If the receiver has commitment power, she commits to taking the action if and only if being faced with a specific set of evidence. We refer to this situation as *constrained delegation*, since we assume that the receiver with commitment power delegates inspection of the state of nature to a sender, whose incentive becomes to provide convincing evidence to support taking the action. Constrained delegation better fits the second example, where the company (receiver) can give the candidate (sender) a test to present evidence on the private information about qualifications, and commit to hiring her if she performs well. We show that the receiver's task in delegation is also intractable – unless $\mathsf{P} = \mathsf{NP}$, optimal delegation can become hard to approximate within a factor of $2 - \varepsilon$, for any constant $\varepsilon > 0$.

These computational differences nicely reflect conceptual differences known from the economics literature. Namely, persuasion lacks a condition termed "credibility" that was shown for delegation. Formally, credibility implies that there is a deterministic optimal solution that does not require randomization, see Glazer and Rubinstein [14] for details. We proceed to study algorithms with matching approximation guarantees for constrained persuasion and delegation, as well as a number of exact and approximation algorithms for various special cases. This includes, in particular, an approximation algorithm for a class of delegation problems that solves and rounds a semidefinite program (SDP). This last result might be of independent interest and, to the best of our knowledge, it is the first natural problem in information structure design, as well as mechanism design, where the SDP toolbox is used.

2 Preliminaries

Following [13, 14, 25], we study the fundamental problem of persuasion with evidence. There are two players, a sender and a receiver. The receiver is tasked with either taking a specific action and "accept" (henceforth A), or sticking to the status quo and "reject" (henceforth R). The sender wants to convince the receiver to take action A. There is a state of nature θ drawn from a distribution $\mathcal{D}$ with support Θ of size n. We denote the probability that θ is drawn by q_θ. The set Θ is partitioned into the set of acceptable states Θ_A and the set of rejectable ones $\Theta_R = \Theta \setminus \Theta_A$. We denote the total probability on acceptable states by $q_A = \sum_{\theta \in \Theta_A} q_\theta$, and the total probability on rejectable states by $q_R = \sum_{\theta \in \Theta_R} q_\theta$.

Both players know $\mathcal{D}$. The sender knows the realization of the state of nature, the receiver does not. The sender has utility 1 whenever the receiver takes action A, and 0 otherwise. Formally, for the sender utility we have $u_s(A, \theta) = 1$ and $u_s(R, \theta) = 0$, for all $\theta \in \Theta$.

The utility of the receiver depends on the combination of the chosen action $a \in \{A, R\}$ and the state of nature θ. She has utility 1 if she makes the "right" decision – accept in an acceptable state of nature or reject in a rejectable state of nature – and 0 otherwise. Formally, $u_r(a, \theta) = 1$ when (1) $a = A$ and $\theta \in \Theta_A$, or (2) $a = R$ and $\theta \in \Theta_R$. Otherwise, $u_r(a, \theta) = 0$.

The sender strives to send a message to the receiver according to a public signaling strategy. This message should persuade the receiver to accept. On the other hand, upon receiving the message, the receiver strives to infer the state of nature and make the right accept/reject decision. We focus on games with evidence, where the messages that can be sent are not arbitrary. Every state of nature has intrinsic characteristics (e.g., a candidate for a position has grades, degrees, or test scores) that *can* be (but don't *have* to be) revealed to the receiver and cannot be forged.

More formally, there is a set Σ of m possible messages or *signals* that the sender can report to the receiver. We are given as input a bipartite graph $H = (\Theta \cup \Sigma, E)$, where an edge $e = (\theta, \sigma) \in E$ implies that signal σ is allowed to be sent in state θ. We use $N(\theta) \subseteq \Sigma$ to denote the neighborhood of θ, i.e., the set of allowed signals for state θ. Similarly, $N(\sigma) \subseteq \Theta$ is the set of states in which signal σ can be sent. To avoid trivialities, we assume that none of the neighborhoods $N(\cdot)$ are empty, i.e., there are no isolated nodes in H.

We study the computational complexity of games with evidence for different forms of interaction between the sender and the receiver. In particular, in the case of *constrained persuasion*, the game starts with the sender committing to a *signaling scheme*. A signaling scheme φ is a mapping $\varphi : E \to [0, 1]$, where $\varphi(\theta, \sigma)$ is the joint probability that state θ is realized and signal σ is sent in state θ. Clearly, for any signaling scheme we have $\sum_{\sigma \in N(\theta)} \varphi(\theta, \sigma) = q_\theta$ for every state $\theta \in \Theta$. After the sender has committed to a scheme φ, nature draws $\theta \in \Theta$ with probability q_θ, and θ is revealed to the sender. Then, the sender sends signal σ with probability $\varphi(\theta, \sigma)/q_\theta$. The receiver then decides on an action A or R. Finally, depending on the (state of nature, action)-pair, the sender and receiver get payoffs as described by the utilities above.

▶ **Problem 1** (Constrained Persuasion). *Find a signaling scheme φ^* for commitment of the sender such that, upon a best response of the receiver, the sender utility is as high as possible.*

In the case of *constrained delegation*, the game starts with the receiver committing to an action for every possible signal $\sigma \in \Sigma$, according to a *decision scheme*. A decision scheme ψ is a mapping $\psi : \Sigma \to [0, 1]$, where $\psi(\sigma)$ is the probability to choose action A. After the receiver has committed to a scheme ψ, nature draws $\theta \in \Theta$ with probability q_θ, and θ is revealed to the sender. Then, the sender decides which signal σ she will report (under the constraint that $\sigma \in N(\theta)$). The receiver then takes action A with probability $\psi(\sigma)$, and R otherwise. Finally, depending on the (state of nature, action)-pair, the sender and receiver get payoffs as described by the utilities above.

▶ **Problem 2** (Constrained Delegation). *Find a decision scheme ψ^* for commitment of the receiver such that, upon a best response of the sender, the receiver utility is as high as possible.*

Finally, in the game without commitment power, we look for a pair (φ, ψ) of signaling and decision schemes that constitute a sequential equilibrium in the extensive-form game, where nature first determines the state of nature, the sender then picks φ to provide evidence, and then the receiver uses ψ to accept or reject based on the evidence provided. Given that the sender picks φ, the receiver shall pick ψ as a best response for every given evidence. Similarly, given that the receiver responds to evidence with ψ, the signaling scheme φ shall be a best response for the sender.

▶ **Problem 3** (Constrained Equilibrium). *Find a pair of signaling scheme φ and decision scheme ψ that represents a sequential equilibrium in the persuasion game with evidence and without commitment power.*

2.1 Structural Properties

While the persuasion problem with evidence appears rather elementary, it turns out that both persuasion and delegation variants are NP-hard, and even NP-hard to approximate in polynomial time. Hence, even in this seemingly simple domain, it is necessary to identify additional structure to obtain positive results. We mostly consider structural properties of the neighborhoods of the states of nature.

Unique Accepts and Rejects. In an instance with *unique accepts*, there is a single acceptable state, i.e., $|\Theta_A| = 1$. Similarly, for *unique rejects* we have $|\Theta_R| = 1$. This is equivalent to assuming that every acceptable (rejectable, resp.) state θ has the same neighborhood $N(\theta)$.

Degree-bounded States. In an instance with *degree-k states*, every state $\theta \in \Theta$ has $|N(\theta)| \leq k$. Similarly, for *degree-k accepts*, every acceptable state $\theta \in \Theta_A$ has $|N(\theta)| \leq k$, and for *degree-k rejects* every rejectable state $\theta \in \Theta_R$ has $|N(\theta)| \leq k$.

Foresight. Sher [25] considers instances with *foresight* defined as follows. For an acceptable state $\theta \in \Theta_A$, a signal $\sigma \in N(\theta)$ is called *minimally forgeable for θ* if $\sigma \in N(\theta')$ implies $\sigma' \in N(\theta')$ for every other signal $\sigma' \in N(\theta)$ and every rejectable state $\theta' \in \Theta_R$. In an instance *with foresight* every acceptable state has a minimally forgeable signal. Intuitively, in such a problem every acceptable state θ has a (not necessarily unique) signal that is maximally informative about θ with respect to the set of rejectable states. Foresight strictly generalizes other properties studied in previous work, e.g. normality [4]. Normality requires a signal for every state (not only the acceptable ones) that satisfies the condition of minimally forgeable, and it satisfies the condition w.r.t. all states (not only w.r.t. rejectable ones). In addition, foresight is a generalization of instances with unique rejects, as well as a generalization the class of degree-1 accepts.

Table 1 Approximation results shown in this paper, as well as results shown or implied by [25].

Scenario	Constrained Delegation		Constrained Persuasion	
	Upper	Lower	Upper	Lower
General	2	$2-\varepsilon$ (P $\neq$ NP)	$O(n)$	$n^{1-\varepsilon}$ (P $\neq$ NP)
Degree-2 States	1.1	APX-hard [25]	$O(n)$	$n^{1-\varepsilon}$ (P $\neq$ NP)
Degree-d States	$2-1/d^2$	APX-hard [25]	$O(n)$	$n^{1-\varepsilon}$ (P $\neq$ NP)
Degree-1 Rejects	2	APX-hard [25]	$O(n)$	$n^{1-\varepsilon}$ (P $\neq$ NP)
Degree-1 Accepts	1 [25]		$O(n)$	$n^{1-\varepsilon}$ (P $\neq$ NP)
Foresight	1 [25]		$O(n)$	$n^{1-\varepsilon}$ (P $\neq$ NP)
Unique Rejects	1 [25]		1	
Unique Accepts	1		PTAS	Strongly NP-hard

2.2 Results and Contribution

We provide polynomial-time exact and approximation algorithms as well as hardness results for the general problems and the domains with more structure described above.

We first consider the case of the constrained equilibrium problem. The existence of a sequential equilibrium is implied by [14]; we show that it can always be computed in polynomial time by repeatedly solving a maximum flow problem. We compare the utility obtained in an equilibrium with the one achievable with commitment power, for the sender and the receiver, respectively. Formally, we define and bound the ratio of the utilities for best and worst-case equilibria, in the spirit of prices of anarchy and stability. For the receiver, it is known that the price of stability is 1 [14]; we show that the price of anarchy is 2. For the sender we show that both ratios are unbounded. This substantial utility gain provides further motivation to study problems with commitment power.

Our results for constrained delegation and persuasion are summarized in Table 1. We discuss a selected subset of our most interesting contributions in the main part of the paper. All missing proofs are deferred to the full version of this paper. In addition, in the full version, we prove additional results that omitted from this version due to spatial constraints.

For the constrained delegation problem, we show two interesting non-trivial approximation results. For degree-2 states, we propose a semidefinite-programming algorithm to compute a 1.1-approximation. To the best our knowledge, this is the first application of advanced results from the SDP toolbox in the context of information design, as well as mechanism design. For instances with degree-d states we give a $(2-\frac{1}{d^2})$-approximation algorithm via LP rounding.

For constrained persuasion, the strong hardness arises from deciding which action should be preferred by the receiver for each signal. It holds even in several seemingly special cases with degree-1 accepts, degree-1 rejects and degree-2 states. As a consequence, good approximation algorithms can be obtained only in significantly more limited scenarios than for delegation. For unique accepts, we prove strong NP-hardness (i.e. there is no FPTAS unless P= NP) and provide a polynomial-time approximation scheme (PTAS).

2.3 Related Work

There is a large literature on strategic communication, see Sobel [27] for an extensive review. The works most closely related to ours are [14, 25]. Glazer and Rubinstein [14] introduce the problem of constrained delegation. They show, among other things, that the optimal decision scheme in constrained delegation is deterministic. Furthermore, they prove that there is always a sequential equilibrium where the receiver plays the optimal decision scheme from constrained delegation, i.e., the price of stability for the receiver is 1. This condition is termed "credibility". It is easy to see that this is not true when sender moves first. This conceptual difference between persuasion and delegation is reflected as a difference in the problems' computational complexity. Deterministic optimal strategies and "credibility" hold also beyond the simple model with 2 actions – when receiver utility is a concave transformation of sender utility, see [24]. Sher [25] builds on the model of [14] and characterizes optimal rules for static as well as dynamic persuasion. Furthermore, and more relevant to our interest here, he proves an NP-hardness result for constrained delegation, as well as provides a polynomial-time algorithm for optimal delegation in instances with foresight. Here we strengthen this hardness result to a hardness of approximation within a factor of $2 - \varepsilon$ (and provide a matching, alas trivial, approximation algorithm). While this subsumes NP-hardness in general, we observe that his hardness proof applies in case of degree-2 states and degree-1 rejects, and that it even implies APX-hardness for such instances.

Glazer and Rubinstein [13] study a related setting, where the state of nature is multi-dimensional, and the receiver can verify at most one dimension. The authors characterize the optimal mechanism as a solution to a particular linear programming problem, show that it takes a fairly simple form, and show that random mechanisms may be necessary to achieve the optimum. Carroll and Egorov [5] study the problem of fully revealing the sender's information in a setting with multidimensional states, where the receiver can verify a single dimension. Importantly, the dimension the receiver chooses to reveal depends on the sender's message.

A number of works in the algorithmic economics literature investigate the computational complexity of persuasion and information design. Computational aspects of the Bayesian persuasion model [19] are studied in, e.g., [10, 6, 9, 8, 11, 18, 17], but in these works there are no limits on the senders' signals, i.e., H is the complete bipartite graph. More closely related to our work are [7, 16] who study computational problems in Bayesian persuasion with limited signals, where the number of signals is smaller than the number of actions.

3 Sequential Equilibria

We first study the scenario without commitment power. Our interest here is to obtain a signaling scheme $\varphi : E \to [0, 1]$ and a decision scheme $\psi : \Sigma \to [0, 1]$, such that the pair (φ, ψ) forms a sequential equilibrium.

▶ **Theorem 4.** *A sequential equilibrium can be computed in polynomial time.*

Our algorithm repeatedly sets up a flow network based on the graph H. In each iteration, it computes a maximum s-t flow and identifies suitable regions of the graph where it fixes the equilibrium schemes of sender and receiver. Then it removes the fixed regions and repeats the construction on the graph with the remaining states and signals. After at most $\min\{n, m\}$ iterations, the algorithm finishes the construction of the equilibrium.

How desirable is an equilibrium for the sender and the receiver? By how much can each player benefit when he or she enjoys commitment power? Towards this end, we bound the ratios of the optimal utility achievable with commitment power over the utilities in the worst and best equilibrium. Intuitively, commitment power might be interpreted as a form of control over the game, so we use the term *price of anarchy* and *price of stability* to refer to the ratios, respectively.

More formally, for the price of anarchy we bound the ratio of the optimal utility achievable with commitment over the *worst* utility in any sequential equilibrium. For the price of stability we bound the ratio of the optimal utility achievable with commitment over the *best* utility in any sequential equilibrium.

For the receiver, the optimal scheme with commitment leads to an equilibrium [14], so the price of stability is 1. The price of anarchy is 2 (c.f. Proposition 7 below). For the sender, both prices of anarchy and stability are unbounded.

▶ **Proposition 5.** *The price of anarchy for the receiver is 2 and this is tight. The prices of anarchy and stability for the sender are unbounded.*

4 Constrained Delegation

In constrained delegation, the game starts with the receiver committing to a decision scheme $\psi : \Sigma \to [0, 1]$, where $\psi(\sigma)$ is the probability to choose action A if the sender reports signal σ. The first insight is due to [14, Proposition 1].

▶ **Lemma 6** (Glazer and Rubinstein [14]). *In constrained delegation, there is an optimal decision scheme ψ^* that is deterministic, i.e., $\psi^*(\sigma) \in \{0, 1\}$ for all $\sigma \in \Sigma$.*

Given a deterministic decision scheme ψ, the sender's problem is trivial: after learning θ, report an arbitrary signal $\sigma \in N(\theta)$ such that $\psi(\sigma) = 1$ if one exists. Otherwise, report an arbitrary signal $\sigma \in N(\theta)$. In the following, we focus on the computational complexity of the receiver's problem: How hard is it to compute the optimal ψ? What about a good approximation algorithm?

This problem turns out to be much easier than the sender's problem in constrained persuasion studied below. It readily admits a trivial 2-approximation algorithm. Let ψ_A be the scheme that accepts all signals, i.e., $\psi_A(\sigma) = 1$ for all σ, and ψ_R the scheme that rejects all signals. The better of ψ_A and ψ_R results in utility $\max\{q_A, q_R\}$ for the receiver, which is at least $1/2$. Clearly, the receiver can obtain at most a utility of 1.

▶ **Proposition 7.** *For constrained delegation, the better of ψ_A and ψ_R is a 2-approximation to the optimal decision scheme ψ^*.*

In Section 4.1 we show that the factor 2 is essentially optimal in the worst case, unless P = NP. In Section 4.2 we present our results on approximation algorithms.

4.1 Hardness

Sher [25, Theorem 7] shows NP-hardness of constrained delegation, even in the special case with degree-1 rejects and degree-2 states. His construction can be extended easily to show APX-hardness (we provide the details in the full version of this paper). Our main result in this section is a stronger hardness result that matches the guarantee of the trivial algorithm in Proposition 7.

▶ **Theorem 8.** *For any constant $\varepsilon \in (0, 1)$, it is NP-hard to approximate constrained delegation within a factor of $(2 - \varepsilon)$.*

For simplicity, we sketch below an outline for a reduction that does *not* give the NP-hardness, but nonetheless encapsulates the main ideas of the proof. After the outline, we roughly explain the changes needed to achieve the NP-hardness; the full proof is deferred to the full version of this paper.

We reduce from the Bipartite Vertex Expansion problem. In this problem, we are given a bipartite graph (U, V, E) and positive real number β. The goal is to select (at least) $\beta|U|$ vertices from U such that their neighborhood (in V) is as small as possible. Khot and Saket [20] show the following strong inapproximability result:

▶ **Theorem 9** ([20]). *Assuming* $\mathsf{NP} \not\subseteq \bigcap_{\delta>0} \mathsf{DTIME}(2^{n^\delta})$*, for any positive constants* $\tau, \gamma > 0$*, there exists* $\beta \in (0, 1)$ *such that no polynomial-time algorithm can, given a bipartite graph* (U, V, E)*, distinguish between the following two cases:*
- *(YES) There exists* $S^* \subseteq U$ *of size at least* $\beta|U|$ *where* $|N(S^*)| \leq \gamma|V|$*.*
- *(NO) For any* $S \subseteq U$ *of size at least* $\tau\beta|U|$*,* $|N(S)| > (1 - \gamma)|V|$*.*

The main idea of our reduction is as follows. Roughly speaking, given a bipartite graph (U, V, E), we set $\Sigma = U$, $\Theta_R = V$ and the edge set between them is exactly E. To get a high utility on Θ_R, we must pick a signal set $T \subseteq \Sigma$ such that $|N(T)|$ is small, and set $\psi(\sigma) = 1$ for all $\sigma \in T$; this does not mean much so far, since we could just pick $T = \emptyset$. This is where the set of acceptable states comes in: we let Θ_A be equal to $U^\ell = \{(u_1, \ldots, u_\ell) | u_i \in U\}$ for some appropriate $\ell \in \mathbb{N}$, and there is an edge between $\theta = (u_1, \ldots, u_\ell)$ and $\sigma = u$ if $u_i = u$ for some $i \in [\ell]$. Intuitively, this forces us to pick T that is not too small as otherwise Θ_A won't contribute to the total utility. Finally, we need to pick a distribution $\mathcal{D}$ over Θ such that $q_A = q_R$, as otherwise the trivial algorithm already gets better than a 2-approximation.

As stated earlier, the above reduction does not yet give NP-hardness, because Theorem 9 relies on a stronger assumption[1] that $\mathsf{NP} \not\subseteq \bigcap_{\delta>0} \mathsf{DTIME}(2^{n^\delta})$. To overcome this, we instead use a "colored version" of the problem, where every vertex in U is colored and the subset $S \subseteq U$ must only contain vertices of different colors (i.e., be "colorful"). It turns out that the above reduction can be adapted to work with such a variant as well, by changing the acceptable states Θ_A to "test" this condition instead of the condition that $|S|$ is small. Furthermore, we show, via a reduction from the Label Cover problem, that this colored version of Bipartite Vertex Expansion is NP-hard to approximate. Together, these imply Theorem 8. Our proof formalizes this outline; see the full version for details.

4.2 Approximation Algorithms for Constrained Delegation

By Theorem 8 there is no hope for a $(2 - \epsilon)$-approximation algorithm for the constrained delegation problem. Proposition 7 provides a matching guarantee.

As a consequence, we examine in which way instance parameters influence the existence of polynomial-time approximation algorithms. In particular, the maximum degree d is a main force that drives the hardness result. For the case of degree at most d, we give a $2 - \frac{1}{d^2}$ approximation algorithm via LP rounding. When $d = 2$, we improve upon this by giving a 1.1-approximation algorithm via SDP rounding.

4.2.1 Better than 2 via LP Rounding

For instances with degree-d-states we take the better of (1) rounding the natural linear program for constrained delegation and (2) the trivial scheme of Proposition 7.

▶ **Theorem 10.** *For constrained delegation with degree-d states there is a polynomial-time* $\left(2 - \frac{1}{d^2}\right)$*-approximation algorithm.*

[1] We remark that it is entirely possible that Theorem 9 holds under NP-hardness (instead of under the assumption $\mathsf{NP} \not\subseteq \bigcap_{\delta>0} \mathsf{DTIME}(2^{n^\delta})$) but this is not yet known.

Proof. Consider the following integer program for constrained delegation (c.f. [14, 25]).

$$\max \quad \sum_{\theta \in \Theta} c_\theta q_\theta \tag{1a}$$

$$\text{s.t.} \quad \sum_{\sigma \in N(\theta)} \psi_\sigma \geq c_\theta, \qquad \text{for all } \theta \in \Theta_A \tag{1b}$$

$$\sum_{\sigma \in N(\theta)} \psi_\sigma \leq |N(\theta)|(1 - c_\theta) \qquad \text{for all } \theta \in \Theta_R \tag{1c}$$

$$\psi_\sigma \in \{0,1\}, \text{for all } \sigma \in \Sigma \quad \text{and} \quad c_\theta \in \{0,1\}, \text{for all } \theta \in \Theta \tag{1d}$$

The variable ψ_σ encodes whether the action is accept or reject for signal σ. The variable c_θ encodes whether the receiver makes the correct choice when the state of nature is θ. Constraint (1b) states that, if $\theta \in \Theta_A$, she can't make the correct choice when she rejects all signals available from θ. Constraint (1c) states that, if $\theta \in \Theta_R$, making the correct choice means rejecting all signals available from θ; the $|N(\theta)|$ term ensures that the constraint can still be satisfied even when $c_\theta = 0$.

Our algorithm first solves the linear relaxation of this integer program; let $\hat{\psi}_\sigma$ and $\hat{c}_\theta$ be the fractional optimum. We round this solution by setting $\psi_\sigma = 1$ with probability $\hat{\psi}_\sigma$, and 0 otherwise. We can optimally pick c_θ given the ψ_σ's. The rounded solution is feasible by definition; we show that it is a good approximation to the optimal LP value, i.e., $\sum_{\theta \in \Theta} \hat{c}_\theta q_\theta$.

Let $G = \frac{1}{|\Theta_A|} \sum_{\theta \in \Theta_A} \hat{c}_\theta q_\theta$ and $B = \frac{1}{|\Theta_R|} \sum_{\theta \in \Theta_R} \hat{c}_\theta q_\theta$ be the average contribution to the LP objective from the acceptable and rejectable states, respectively. The LP value is $G|\Theta_A| + B|\Theta_R|$. We start by showing the following lower bound on the expected value of the rounded solution.

▶ **Lemma 11.** $\mathbb{E}[\sum_{\theta \in \Theta} c_\theta q_\theta] \geq \frac{G|\Theta_A|}{d} + q_R(1-d) + dB|\Theta_R|$.

Proof. First, consider a state $\theta \in \Theta_A$. The probability that $c_\theta = 1$ is at least the probability that we rounded one of the ψ_σ variables to 1, for $\sigma \in N(\theta)$, i.e.,

$$\mathbf{Pr}\left[c_\theta = 1\right] \geq \max_{\sigma \in N(\theta)} \hat{\psi}_\sigma \geq \frac{\hat{c}_\theta}{|N(\theta)|} \geq \frac{\hat{c}_\theta}{d} \quad , \tag{2}$$

where we used the fact that $\hat{c}_\theta$ satisfies Constraint (1b). For a state $\theta \in \Theta_R$, the probability that $c_\theta = 1$ is exactly the probability that none of its signals were selected, which is $\prod_{\sigma \in N(\theta)} (1 - \hat{\psi}_\sigma) \geq 1 - \sum_{\sigma \in N(\theta)} \hat{\psi}_\sigma$. Thus

$$\mathbf{Pr}\left[c_\theta = 1\right] \geq 1 - \sum_{\sigma \in N(\theta)} \hat{\psi}_\sigma \geq 1 - |N(\theta)|(1 - \hat{c}_\theta) \geq 1 - d + d\hat{c}_\theta \quad , \tag{3}$$

where we used the fact that $\hat{c}_\theta$ satisfies Constraint (1c). Adding up (2) and (3), the expected value of our rounded solution is

$$\mathbb{E}\left[\sum_{\theta \in \Theta} c_\theta q_\theta\right] \geq \sum_{\theta \in \Theta_A} \frac{q_\theta \hat{c}_\theta}{d} + \sum_{\theta \in \Theta_R} q_\theta(1 - d + d\hat{c}_\theta) \geq \frac{G|\Theta_A|}{d} + q_R(1-d) + dB|\Theta_R|. \quad ◀$$

Our final algorithm, i.e., the better of the trivial scheme and the rounded LP solution, has expected value at least $\max\{q_A, q_R, \mathbb{E}[\sum_{\theta\in\Theta} c_\theta q_\theta]\}$. We have that

$$\begin{aligned}\left(2d-\frac{1}{d}\right)\max\left\{q_A, q_R, \mathbb{E}\left[\sum_{\theta\in\Theta} c_\theta q_\theta\right]\right\} &\geq \left(d-\frac{1}{d}\right)q_A + (d-1)q_R + \mathbb{E}\left[\sum_{\theta\in\Theta} c_\theta q_\theta\right] \\ &\overset{\text{Lemma 11}}{\geq} \left(d-\frac{1}{d}\right)q_A + (d-1)q_R + \frac{G|\Theta_A|}{d} + q_R(1-d) + dB|\Theta_R| \\ &\overset{(G|\Theta_A|\leq q_A)}{\geq} dG|\Theta_A| + dB|\Theta_R| \ ,\end{aligned}$$

which is d times the value of the optimum fractional value of the LP. The theorem follows. ◀

4.2.2 Better than 2 via Semidefinite Programming

In this subsection we give a 1.1-approximation algorithm for constrained delegation with degree-2 states, where every state of nature θ has at most two allowed signals, σ_u and σ_v. The approach stems from an observation that the problem belongs to the class of *constraint satisfaction problems (CSPs)*; we make use of the toolbox for semidefinite program (SDP) rounding in approximating CSPs (e.g. [15, 12, 21]).

Consider the integer program (4a) for our problem below. We assume w.l.o.g. that every state has *exactly* two adjacent signals; if there is a state θ with a single neighbor σ, we can add a parallel edge (θ, σ) in H and the analysis remains valid. Note that the integer program here is *not* the same as the one used in the previous subsection. An intuitive reason for the change is that the variables c_θ there are redundant: given $\{\psi_\sigma\}_{\sigma\in\Sigma}$, the values of $\{c_\theta\}_{\theta\in\Theta}$ are already fixed. In particular, each c_θ can be expressed as a degree-d polynomial[2] in $\{\psi_\sigma\}_{\sigma\in N(\theta)}$, which is exactly how the integer program below is written.

$$\max_{x\in\{-1,1\}^m} \quad \frac{1}{4}\sum_{\substack{\theta\in\Theta_A\\ \theta=(\sigma_i,\sigma_j)}} (3-x_i-x_j-x_ix_j)q_\theta + \frac{1}{4}\sum_{\substack{\theta\in\Theta_R\\ \theta=(\sigma_i,\sigma_j)}} (1+x_i+x_j+x_ix_j)q_\theta \tag{4a}$$

In the program above $x_i = -1$ is interpreted as accepting when the signal is σ_i. One can check that $\frac{1}{4}(3-x_i-x_j-x_ix_j)$ is equal to 1 iff at least one of x_i, x_j is -1 (and zero otherwise), i.e., a state of nature $\theta\in\Theta_A$ contributes to the objective only when at least one of its allowed signals is accepted. Similarly, $\frac{1}{4}(1+x_i+x_j+x_ix_j)$ is equal to 1 if and only if both x_i and x_j are equal to 1.

We will solve the semidefinite relaxation of this program, and give a rounding algorithm. The SDP is the following, where we replaced x_i by w_i, to distinguish these vector variables from the variables of our integer program above.

[2] Note that linear functions do not suffice to express c_θ. In particular, if we rewrite (1c) for $\theta = (\sigma_i, \sigma_j)$ as $c_\theta \leq 1 - \frac{\psi_{\sigma_i}+\psi_{\sigma_j}}{2}$, then it is still possible to have $c_\theta = 1/2$ when $\psi_{\sigma_i} = 1, \psi_{\sigma_j} = 0$.

$$\begin{aligned}
\max \quad & \frac{1}{4} \sum_{\theta=(\sigma_i,\sigma_j)\in\Theta_A} (3 - w_i \cdot w_0 - w_j \cdot w_0 - w_i \cdot w_j) q_\theta \\
& \quad + \frac{1}{4} \sum_{\theta=(\sigma_i,\sigma_j)\in\Theta_R} (1 + w_i \cdot w_0 + w_j \cdot w_0 + w_i \cdot w_j) q_\theta && \text{(5a)} \\
\text{s.t.} \quad & w_i \cdot w_i = 1 \quad \text{for all } i \in [m] \cup \{0\} && \text{(5b)} \\
& w_i \cdot w_0 + w_j \cdot w_0 + w_i \cdot w_j \geq -1 \quad \text{for all } i, j \in [m] && \text{(5c)} \\
& - w_i \cdot w_0 + w_j \cdot w_0 - w_i \cdot w_j \geq -1 \quad \text{for all } i, j \in [m] && \text{(5d)} \\
& - w_i \cdot w_0 - w_j \cdot w_0 + w_i \cdot w_j \geq -1 \quad \text{for all } i, j \in [m] && \text{(5e)} \\
& w_i \in \mathbb{R}^{m+1} \quad \text{for all } i \in [m] \cup \{0\}
\end{aligned}$$

Constraint (5b) is standard. Constraints (5c)-(5e) encode the triangle inequalities, which are satisfied by every valid solution to the original program; these strengthen the relaxation a bit (see [12, 21]). Let $\mathcal{V}_{SDP}$ denote the optimal value of this semidefinite program (SDP). We generally cannot find the exact solution to an SDP, but it is possible to find a feasible solution with value at least $\mathcal{V}_{SDP} - \epsilon$ in time polynomial in $1/\epsilon$ (see [1]). In our analysis we will (as is typically the case) ignore the ϵ factor as it can be made arbitrarily small given sufficient time.

It is known that the SDP written above provides the optimal approximation achievable in polynomial time for any 2-CSPs [22, 23] including our problem, assuming the Unique Games Conjecture (UGC). However, a generic rounding algorithm from this line of work (see e.g. [23]) does not give a concrete approximation ratio. Below, we describe a specific family of rounding algorithms for which we can provide the concrete approximation ratio of 1.1.

Rounding Algorithm

Given solution vectors $\{w_0, w_1, \dots w_m\}$, $w_i \in \mathbb{R}^{m+1}$, for this SDP we produce a feasible solution $x_i \in \{-1, 1\}$ (for $i \in [m]$) to the original integer program as follows. Let $\xi_i = w_0 \cdot w_i$, and $\tilde{w}_i = \frac{w_i - \xi_i w_0}{\sqrt{1-\xi_i^2}}$ be the part of w_i orthogonal to w_0, normalized to a unit vector. Our rounding algorithm mostly follows the rounding procedure of [21], which they call $\mathcal{THRESH}^-$. First, pick a $(m+1)$-dimensional vector[3] $r \sim \mathcal{N}(0, 1)$ $r \in \mathbb{R}^{m+1}$. Then, set $x_i = -1$ (which corresponds to accepting signal σ_i) if and only if $\tilde{w}_i \cdot r \leq T(\xi_i)$, where $T(.)$ is a threshold function, and set $x_i = 1$ otherwise. Specifically, $T(x) = \Phi^{-1}(\frac{1-\nu(x)}{2})$, where $\Phi^{-1}(.)$ is the inverse of the normal distribution function, and $\nu : [-1, 1] \to [-1, 1]$ is a function. Later in the analysis – and this is essentially the point in which various SDP rounding methods diverge from each other, e.g. see [26] for the different choices for MAX-2-SAT and MAX-2-AND – we will optimize over a family of $\nu(.)$, exploiting structure in our problem, in order to improve our approximation ratio.

Generic Analysis

We now derive a generic analysis for $\mathcal{THRESH}^-$ algorithms; note that these are similar arguments as in [21, 2]. However, in the end, we will pick a different function ν than previous works, which results in better approximation ratios for our problem.

[3] In other words, the i-th dimension r_i is sampled independently from a Gaussian with zero mean and variance one.

First, notice that $\tilde{w}_i \cdot r$ is a standard $\mathcal{N}(0,1)$ variable, and therefore by the choice of $T(.)$ we have that $\mathbf{Pr}\left[x_i = -1\right] = \frac{1-\nu(\xi_i)}{2}$, which implies that

$$\mathbb{E}\left[x_i\right] = \nu(\xi_i) \ . \tag{6}$$

Now, we need to also analyze the quadratic terms. Let $\Gamma_c(\mu_1,\mu_2) = Pr[X_1 \leq t_1 \text{ and } X_2 \leq t_2]$, where $t_i = \Phi^{-1}(\frac{1-\mu_i}{2})$, and $X_1, X_2 \in \mathcal{N}(0,1)$ with covariance c (in other words, Γ_c is the bivariate normal distribution function with covariance c, with a transformation on the input).

Let $\rho = w_i w_j$ and $\tilde{\rho} = \tilde{w}_i\tilde{w}_j = \frac{\rho-\xi_i\xi_j}{\sqrt{1-\xi_i^2}\sqrt{1-\xi_j^2}}$. Observe that the products $\tilde{w}_i \cdot r$ and $\tilde{w}_j \cdot r$ are $\mathcal{N}(0,1)$ random variables with covariance $\tilde{\rho}$. Thus, the probability that $\tilde{w}_i \cdot r \leq T(\xi_i)$ and $\tilde{w}_j \cdot r \leq T(\xi_j)$ (i.e., both x_i, x_j are set to -1) is exactly $\Gamma_{\tilde{\rho}}(\nu(\xi_i), \nu(\xi_j))$. The probability that $x_i = x_j = 1$ is equal to $\Gamma_{\tilde{\rho}}(-\nu(\xi_i), -\nu(\xi_j))$. Austrin [2, Proposition 2.1] shows that $\Gamma_c(-\mu_1,-\mu_2) = \Gamma_c(\mu_1,\mu_2) + \mu_1/2 + \mu_2/2$. Using this fact we can calculate the probability that $x_i = x_j$, which, in turn, gives that

$$\mathbb{E}\left[x_i x_j\right] = 4\Gamma_{\tilde{\rho}}(\nu(\xi_i), \nu(\xi_j)) + \nu(\xi_i) + \nu(\xi_j) - 1 \ . \tag{7}$$

With Equations (6) and (7) at hand we can calculate the expected value of our rounding algorithm (i.e., the expected value of (4a)) for every choice of ν, and compare it against the value of the SDP in (5a). Specifically, we will aim for a term-by-term approximation. Define the following quantities:

$$\ell_\nu^{OR}(\xi_i,\xi_j,\rho) = \frac{3-\xi_i-\xi_j-\rho}{4-2\nu(\xi_i)-2\nu(\xi_j)-4\Gamma_{\tilde{\rho}}(\nu(\xi_i),\nu(\xi_j))}$$
$$\ell_\nu^{AND}(\xi_i,\xi_j,\rho) = \frac{1+\xi_i+\xi_j+\rho}{2\nu(\xi_i)+2\nu(\xi_j)+4\Gamma_{\tilde{\rho}}(\nu(\xi_i),\nu(\xi_j))} \ ,$$

and let

$$\ell^{OR}(\nu) = \min_{\xi_i,\xi_j,\rho} \ell_\nu^{OR}(\xi_i,\xi_j,\rho) \qquad \text{and} \qquad \ell^{AND}(\nu) = \min_{\xi_i,\xi_j,\rho} \ell_\nu^{AND}(\xi_i,\xi_j,\rho) \ ,$$

where the minimization is over all choices of $\xi_i, \xi_j, \rho \in [-1,1]$ that satisfy the triangle inequalities (Constraints (5c)-(5e)). It is now straightforward to see that the term-by-term analysis implies that, for any choice of ν, our approximation ratio is at most $\max\{\ell^{OR}(\nu), \ell^{AND}(\nu)\}$.

Choosing ν and Putting Things Together

We are left to choose the function ν that results in the smallest approximation ratio $\max\{\ell^{OR}(\nu), \ell^{AND}(\nu)\}$. We consider a rounding function of the form $\nu(y) = \alpha \cdot y + \beta$ for parameters α, β to be chosen. Using extensive computational effort, we found that $\alpha = 0.8825$ and $\beta = 0.0384$ perform well. Once we have a choice for α and β, it remains to prove the approximation ratio.

We have a computer-assisted proof showing that the approximation ratio is at most 1.1; our computer-based proof approach is similar to that of [26]. Roughly speaking, we divide the cube $(\xi_i, \xi_j, \rho) \in [-1,1]^3$ into a certain number of subcubes. For each subcube, we (numerically) compute an upper bound to $\max\{\ell_\nu^{OR}(\xi_i,\xi_j,\rho), \ell_\nu^{AND}(\xi_i,\xi_j,\rho)\}$. If this upper bound is already at most 1.1, then we are finished with the subcube. Otherwise, we divide it further into a certain number of subcubes. By continuing this process, we eventually manage to show that for the whole region $[-1,1]^3$ that satisfies the triangle inequalities, the ratio must be at most 1.1, as desired. (The smallest subcube our proof considers has edge length 0.00078.)

Comparison to Prior Work

As stated earlier, our algorithm, with the exception of the choice of ν, is similar to [21] and the follow-up works (e.g. [2, 26]). However, perhaps surprisingly, we end up with a better approximation ratio than the Max 2-AND problem[4], whose approximation ratio is known to be at least 1.143 assuming the UGC [3]. To understand the difference, recall that Max 2-AND can be written as $\max \frac{1}{4}\sum_{(i,j,b_i,b_j)}(1 + b_i x_i + b_j x_j + b_i b_j x_i x_j)$ where $b_i, b_j \in \{\pm 1\}$ (representing whether the variable is negated in the clause). This is very similar to our problem (4a), except that Max 2-AND has the aforementioned b_i, b_j-terms for negation. It turns out that this is also the cause that we can achieve better approximation ratio. Specifically, these negation terms led previous works [21, 2, 26, 3] to only consider ν that is an odd function, i.e., $\nu(y) = \nu(-y)$ for all $x \in [-1, 1]$. For example, Austrin [2] considers a function of the form $\nu(y) = \alpha \cdot y$. We note here that, due to the aforementioned UGC-hardness of Max 2-AND, we cannot hope to get an approximation ratio smaller than 1.143 using odd ν. Nonetheless, since we do not have "negation" in our problem, we are not only restricted to odd ν, allowing us to consider a more general family of the form $\nu(y) = \alpha \cdot y + \beta$ for $\beta \neq 0$. This ultimately leads to our better approximation ratio.

5 Constrained Persuasion

Let us now turn to the constrained persuasion problem. The sender first commits to a signaling scheme φ, which she then uses to transmit information to the receiver, once the state of nature is revealed. Given that the sender has commitment power and the receiver knows φ, the receiver picks action A if and only if conditioned on receiving signal σ, the expected utility of A is more than R, i.e.,

$$\sum_{\theta \in N(\sigma) \cap \Theta_A} \varphi(\theta, \sigma) \geq \sum_{\theta \in N(\sigma) \cap \Theta_R} \varphi(\theta, \sigma)$$

or, equivalently, $2 \cdot \sum_{\theta \in N(\sigma) \cap \Theta_A} \varphi(\theta, \sigma) \geq \sum_{\theta \in N(\sigma)} \varphi(\theta, \sigma)$.

In this case, we say that σ is an *accept signal*, otherwise we call σ a *reject signal*. An optimal signaling scheme φ^* maximizes the expected utility of the sender, i.e., the total probability associated with accept signals. Note that if both accepting and rejecting are optimal actions for the receiver, we assume that she breaks ties in favor of the sender (so, in our case, accept). This mild assumption is standard in economic bilevel problems (e.g., when indifferent between buying and not buying, a potential customer is usually assumed to buy) and is often without loss of generality. This way we avoid obfuscating technicalities in the definition of optimal schemes φ^*.

We study the computational complexity of finding φ^* and polynomial-time approximation algorithms. In general, approximating φ^* can be an extremely hard problem, even in the constrained persuasion problem. Our first insight in Section 5.1 is that the main source of hardness in the problem is deciding the optimal set of accept signals. We then provide a simple $2n$-approximation algorithm and a $n^{1-\varepsilon}$-hardness in Section 5.2. The PTAS and the matching strong NP-hardness for instances with unique accepts is discussed in Section 5.3.

[4] This is the problem where we are given a set of clauses, each of which is an AND of two literals. The goal is to assign the variables as to maximize the number of satisfied clauses.

5.1 Signal Partitions

A signaling scheme φ partitions the signal space Σ into (Σ_A, Σ_R), in the sense that the receiver takes action A if and only if she gets signal $\sigma \in \Sigma_A$ (and R for Σ_R). Determining this partition of the signal set turns out to be the main source of computational hardness of finding φ^*: Given an optimal partition of the signal set, the reduced problem of computing appropriate optimal signaling probabilities is solved with a linear program.

We prove this result in a general case of the persuasion problem, in which the receiver has an arbitrary finite set $\mathcal{A}$ of actions. Moreover, sender and receiver can have utilities $u_s, u_r : \mathcal{A} \times \Theta \to \mathbb{R}$ that yield arbitrary positive or negative values for every (state of nature, action)-pair.

▶ **Proposition 12.** *Given a partition $P = (\Sigma_a)_{a\in\mathcal{A}}$ of the signal space such that the receiver's best action for a signal $\sigma \in \Sigma_a$ is action a, an optimal signaling scheme φ^*_P for the general persuasion problem that (1) implements these receiver preferences and (2) maximizes the sender utility, can be computed by solving a linear program of polynomial size.*

Proof. Given $P = (\Sigma_a)_{a\in\mathcal{A}}$, consider the following linear program (8).

$$\begin{array}{llcll}
\text{Max.} & \sum_{a\in\mathcal{A}} \sum_{\sigma\in\Sigma_a} \sum_{\theta\in N(\sigma)} x_{\theta,\sigma} \cdot u_s(a,\theta) & & & \\
\text{s.t.} & \sum_{\theta\in N(\sigma)} x_{\theta,\sigma} \cdot u_r(a,\theta) & \geq & \sum_{\theta\in N(\sigma)} x_{\theta,\sigma} \cdot u_r(a',\theta) & \text{for all } a \in \mathcal{A}, \sigma \in \Sigma_a, a' \in \mathcal{A} \\
& \sum_{\sigma\in N(\theta)} x_{\theta,\sigma} & = & q_\theta & \text{for all } \theta \in \Theta \\
& x_{\theta,\sigma} & \geq & 0 & \text{for all } \sigma \in \Sigma, \theta \in N(\sigma)
\end{array} \tag{8}$$

For each $\sigma \in \Sigma_a$ and every action $a' \neq a$ we must satisfy that $\mathbb{E}[u_r(a,\theta) \mid \sigma] \geq \mathbb{E}[u_r(a',\theta) \mid \sigma]$, encoded by the first constraint. The other two constraints encode the feasibility of the scheme. Subject to these constraints, the objective is to maximize the expected utility of the sender. An optimal LP-solution x^* directly implies an optimal signaling scheme $\varphi^*_P(\theta,\sigma) = x^*_{\theta,\sigma}$. ◀

5.2 A $2n$-Approximation Algorithm and Hardness

Going back to constrained persuasion with binary actions, we start by giving a simple $2n$-approximation algorithm. First, we give a useful benchmark for the optimal scheme.

▶ **Lemma 13.** *An optimal signaling scheme φ^* yields a sender utility of at most* $\min\{1, 2q_A\}$.

Proof. The upper bound of 1 is trivial. φ^* partitions the signal space into (Σ_A, Σ_R), the accept and reject signals, respectively. The expected utility of the sender is

$$\sum_{\sigma\in\Sigma_A} \sum_{\theta\in N(\sigma)} \varphi^*(\theta,\sigma) \leq \sum_{\sigma\in\Sigma_A} \sum_{\theta\in N(\sigma)\cap\Theta_A} 2 \cdot \varphi^*(\theta,\sigma) \leq 2 \sum_{\theta\in\Theta_A} q_\theta = 2 \cdot q_A \ . \qquad ◀$$

Our simple algorithm considers the m partitions with a single accept signal $\Sigma_A = \{\sigma\}$, for every $\sigma \in \Sigma$. For each such partition, the algorithm determines an optimal scheme and then picks the best one, among all m partitions. Instead of solving the LP of Proposition 12, given a proposed partition we proceed as follows. Assign as much probability mass from $\Theta_A \cap N(\sigma)$ to σ and at most the same amount from $\Theta_R \cap N(\sigma)$ – this ensures that σ is an accept signal. The remaining probability mass is assigned arbitrarily to other signals. Note that if this is impossible, there is no scheme that makes σ an accept signal.

▶ **Proposition 14.** *For constrained persuasion there is a* $2n$*-approximation algorithm that runs in polynomial time.*

Proof. Suppose $\theta' \in \Theta_A$ is an acceptable state from which φ^* assigns the largest amount to accept signals, i.e., $\theta' = \arg\max_{\theta \in \Theta_A} \sum_{\sigma \in \Sigma_A \cap N(\theta)} \varphi^*(\theta, \sigma)$. Clearly, the optimum accumulates on the accept signals at most n times this probability mass from the set of acceptable states, and at most the same from rejectable states. Hence, $\sum_{\sigma \in \Sigma_A \cap N(\theta')} \varphi^*(\theta', \sigma) < q_{\theta'}$ is at least a $1/(2n)$-fraction of the optimal sender utility.

Consider the accept signals Σ_A in φ^* and any such signal $\sigma' \in N(\theta') \cap \Sigma_A$. When our algorithm checks the partition with σ' as the unique accept signal, it finds a feasible scheme, since the optimum scheme makes σ' an accept signal and the algorithm only assigns more probability from Θ_A to σ'. The value of this solution is at least $q_{\theta'}$. ◀

In addition to this simple algorithm, we show a number of strong hardness results for constrained persuasion. The proofs of the following two theorems can be found in the full version.

▶ **Theorem 15.** *For any constant* $\varepsilon > 0$*, constrained persuasion is* NP*-hard to approximate within a factor of* $n^{1-\varepsilon}$*, even for instances with degree-2 states and degree-1 accepts.*

For instances with degree-1 rejects a similar result follows with a slightly different reduction.

▶ **Theorem 16.** *For any constant* $\varepsilon > 0$*, constrained persuasion is* NP*-hard to approximate within a factor of* $n^{1-\varepsilon}$*, even for instances with degree-1 rejects.*

5.3 Unique Accepts

In this section, we examine instances in which there is only a single acceptable state, for which we prove NP-hardness and give a PTAS. It will be convenient to state a lemma which allows us to get a better handle on the sender utility in an optimal signaling scheme for a given signal partition. This lemma will be helpful in both our hardness and algorithm analyses.

To state this lemma, we need some additional notation: for every subset $\tilde{\Sigma} \subseteq \Sigma$, we use $\Theta_R(\tilde{\Sigma})$ to denote $\{\theta \in \Theta_R \mid N(\theta) \subseteq \tilde{\Sigma}\}$; when $\tilde{\Sigma} = \{\sigma\}$ is a singleton, we write $\Theta_R(\sigma)$ in place of $\Theta_R(\{\sigma\})$ for brevity. Moreover, let $N(\tilde{\Sigma})$ denote $\bigcup_{\sigma \in \tilde{\Sigma}} N(\sigma)$. The lemma can now be stated as follows.

▶ **Lemma 17.** *Suppose that there exists a unique accept state* θ_a*. For any partition* $P = (\Sigma_A, \Sigma_R)$ *of the signal space such that* $\Sigma_A \neq \emptyset$*, we have*

1. *There exists a signaling scheme* φ *such that every signal in* Σ_A *is accepted and every signal in* Σ_R *is rejected by the receiver if and only if* $\Sigma_A \subseteq N(\theta_a)$ *and* $\sum_{\theta \in \Theta_R(\Sigma_A)} q_\theta \leq q_{\theta_a}$*.*
2. *When the above condition holds, any optimal signaling scheme* φ^* *for the sender has utility equal to* $\min\{2q_{\theta_a}, \sum_{\theta \in N(\Sigma_A)} q_\theta\}$*, and, such a signaling scheme can be computed in polynomial time.*

We remark that the algorithm for finding φ^* in the above lemma is a simple greedy algorithm that tries to "put as much probability mass from rejectable states as possible" in Σ_A and then use the probability mass of the acceptable state θ_a to "balance out" the mass from the rejectable states, so that eventually the signals in Σ_A are accepted. This is in contrast to the generic linear program-based algorithm in Proposition 12. The simpler greedy algorithm allows us to consider more concrete conditions and exactly compute the utility as stated in Lemma 17.

Proof of Lemma 17.

1. ($\Rightarrow$) First, assume that there is such a signaling scheme φ. Clearly, every signal not in $N(\theta_a)$ must be rejected, which implies that $\Sigma_A \subseteq N(\theta_a)$. Furthermore, for all $\sigma \in \Sigma_A$, we must have $\varphi(\theta_a, \sigma) \geq \sum_{\theta \in N(\sigma) \cap \Theta_R} \varphi(\theta, \sigma)$. Summing up over all $\sigma \in \Sigma_A$ gives

$$\begin{aligned} q_{\theta_a} &\geq \sum_{\sigma \in \Sigma_A} \sum_{\theta \in N(\sigma) \cap \Theta_R} \varphi(\theta, \sigma) \\ &\geq \sum_{\sigma \in \Sigma_A} \sum_{\theta \in \Theta_R(\Sigma_A)} \varphi(\theta, \sigma) = \sum_{\theta \in \Theta_R(\Sigma_A)} \sum_{\sigma \in \Sigma_A} \varphi(\theta, \sigma) = \sum_{\theta \in \Theta_R(\Sigma_A)} q_\theta. \end{aligned}$$

($\Leftarrow$) Assume that $\emptyset \neq \Sigma_A \subseteq N(\theta_a)$ and $\sum_{\theta \in \Theta_R(\Sigma_A)} q_\theta \leq q_{\theta_a}$. We may construct a desired signaling scheme φ as follows. First, we assign $\varphi(\theta, \sigma)$ arbitrarily for all $\theta \in \Theta_R(\Sigma_A)$. Then, we assign $\varphi(\theta_a, \sigma)$ such that $\varphi(\theta_a, \sigma) = 0$ for all $\sigma \notin \Sigma_A$ and that $\varphi(\theta_a, \sigma) \geq \sum_{\theta \in \Theta_R(\Sigma_A)} \varphi(\theta, \sigma)$ for all $\sigma \in \Sigma_A$. The former is possible because $\Sigma_A \neq \emptyset$ and the latter possible because $\sum_{\theta \in \Theta_R(\Sigma_A)} q_\theta \leq q_{\theta_a}$. Finally, for each $\theta \in \Theta_R \setminus \Theta_R(\Sigma_A)$, assign $\varphi(\theta, \sigma) = 0$ for all $\sigma \in \Sigma_A$. It is straightforward from the construction that this φ is a desired signaling scheme.

2. First, we will show that any signaling scheme φ has utility at most $\min\{2q_{\theta_a}, \sum_{\theta \in N(\Sigma_A)} q_\theta\}$ for the sender. Observe that the upper bound $2q_{\theta_a}$ follows trivially from Lemma 13. Thus, it suffices for us to prove that the utility is at most $\sum_{\theta \in N(\Sigma_A)} q_\theta$. To do so, let us rearrange the utility as follows:

$$\sum_{\sigma \in \Sigma_A} \sum_{\theta \in N(\sigma)} \varphi(\theta, \sigma) \leq \sum_{\theta \in N(\Sigma_A)} \sum_{\sigma \in N(\theta)} \varphi(\theta, \sigma) = \sum_{\theta \in N(\Sigma_A)} q_\theta.$$

Finally, we construct a signaling scheme φ^* with utility equal to $\min\{2q_{\theta_a}, \sum_{\theta \in N(\Sigma_A)} q_\theta\}$. The algorithm is a modification of the algorithm from the first part, and it works in four steps:

- For every $\theta \in \Theta_R(\Sigma_A)$, assign $\varphi(\theta, \sigma)$ arbitrarily.
- For every $\theta \in (N(\Sigma_A) \cap \Theta_R) \setminus \Theta_R(\Sigma_A)$, assign $\varphi(\theta, \sigma)$ so that

$$\sum_{\sigma \in \Sigma_A} \sum_{\theta \in N(\sigma) \cap \Theta_R} \varphi(\theta, \sigma) = \min\{q_{\theta_a}, \sum_{\theta \in N(\Sigma_A) \cap \Theta_R} q_\theta\}.$$

  Note that this step is possible because $\sum_{\theta \in \Theta_R(\Sigma_A)} q_\theta \leq q_{\theta_a}$.
- Assign $\varphi(\theta_a, \sigma)$ so that $\varphi(\theta_a, \sigma) = 0$ for all $\sigma \notin \Sigma_A$, and that

$$\varphi(\theta_a, \sigma) \geq \sum_{\theta \in N(\sigma) \cap \Theta_R} \varphi(\theta, \sigma)$$

  for all $\sigma \in \Sigma_A$. Note that this is possible because, from the previous step, we must have $\sum_{\sigma \in \Sigma_A} \sum_{\theta \in N(\sigma) \cap \Theta_R} \varphi(\theta, \sigma) \leq q_{\theta_a}$.
- All other remaining assignments are made arbitrarily in order to turn φ into a feasible signaling scheme.

It is straightforward to check that φ^* is the desired signaling scheme with utility equal to $q_{\theta_a} + \min\{q_{\theta_a}, \sum_{\theta \in N(\Sigma_A) \cap \Theta_R} q_\theta\} = \min\{2q_{\theta_a}, \sum_{\theta \in N(\Sigma_A)} q_\theta\}$. ◀

With Lemma 17 ready, we now prove NP-hardness of the problem.

▶ **Theorem 18.** *Constrained persuasion with unique accepts is NP-hard.*

Proof. We reduce from the MAX-K-VERTEX-COVER problem, where we have a graph $G = (V, E)$. The goal is to choose a set V' of k vertices in order to maximize the number of edges incident to at least one vertex in V'. For every vertex $v \in V$, let $E(v)$ be the set of incident edges, then we try to pick a subset V' of k vertices to maximize $|\bigcup_{v \in V'} E(v)|$.

For each edge $e \in E$, we introduce a rejectable state θ_e with $q_{\theta_e} = \frac{1}{(|V|+k)(|E|+1)+2|E|}$. For each vertex v we introduce a signal σ_v. The graph H between states and signals expresses the incident property of edges and vertices. In addition, for each signal σ, we introduce an auxiliary rejectable states that have σ as their unique signal. Each auxiliary state θ has $q_\theta = \frac{|E|+1}{(|V|+k)(|E|+1)+2|E|}$. Finally, the unique acceptable state θ_a is incident to all signals and has probability

$$q_{\theta_a} = \frac{k(|E|+1) + |E|}{(|V|+k)(|E|+1) + 2|E|} \ .$$

From Lemma 17, the optimal signaling scheme has sender utility equal to

$$\max_{\Sigma_A} \min \left\{ 2q_{\theta_a}, \sum_{\theta \in N(\Sigma_A)} q_\theta \right\},$$

where the maximum is over non-empty $\Sigma_A \subseteq \Sigma$ such that $\sum_{\theta \in \Theta_R(\Sigma_A)} q_\theta \le q_{\theta_a}$. Notice that, in our construction, this condition is satisfied iff $|\Sigma_A| \le k$. This means that $\Sigma_A = \{\sigma_v\}_{v \in V'}$ for some subset V' of size at most k. It is also not hard to see that

$$\min \left\{ 2q_{\theta_a}, \sum_{\theta \in N(\Sigma_A)} q_\theta \right\} = \sum_{\theta \in N(\Sigma_A)} q_\theta = \frac{(|V'| + k)(|E|+1) + |\bigcup_{v \in V'} E(v)|}{(|V|+k)(|E|+1) + |E|}.$$

In other words, the utility is maximized iff V' is an optimal solution to the instance of MAX-K-VERTEX-COVER. Since the latter is NP-hard, we can conclude that constrained persuasion with unique accepts is also NP-hard. ◀

We next give a PTAS for the problem. Before we formalize our PTAS, let us give an informal intuition. Observe that the condition in Lemma 17 implies that $q_{\theta_a} \ge \sum_{\sigma \in \Sigma_A} \left(\sum_{\theta \in \Theta_R(\sigma)} q_\theta \right)$. This latter constraint is a *knapsack constraint*. One generic strategy to solve knapsack problems is to first brute-force enumerate all possibilities of selecting "heavy items", which in our case are the signals with large $\sum_{\theta \in \Theta_R(\sigma)} q_\theta$. Then, use a simple greedy algorithm for the remaining "light items". Our PTAS follows this blueprint. However, since neither our constraints nor our objective function are exactly the same as in knapsack problems, we cannot use results from there directly and have to carefully argue the approximation guarantee ourselves.

▶ **Theorem 19.** *For constrained persuasion with unique accepts, for every fixed $\varepsilon \in (0, 1]$, Algorithm 1 runs in time $m^{O(1/\varepsilon)} n^{O(1)}$ and outputs a $(1+\varepsilon)$ approximate solution.*

Proof. It is clear that our algorithm runs in time $m^{O(1/\varepsilon)} n^{O(1)}$. Let φ^* be any optimal signaling scheme, with utility OPT for the sender. We prove that the utility of φ_{ALG} is at least $(1 - 0.5\epsilon)$OPT.

Without loss of generality we assume that the utility of φ^* is non-zero. Now, let (Σ_A^*, Σ_R^*) denote the signal partition of φ^*; since the utility of φ^* is non-zero, we must have $\Sigma_A^* \neq \emptyset$. Furthermore, the first item of Lemma 17 implies that $\Sigma_A^* \cap \Sigma_{\ge \varepsilon}$ must be of size at most $1/\varepsilon$. As a result, our algorithm must consider $S = (\Sigma_A^* \cap \Sigma_{\ge \varepsilon})$ in the for-loop (3). For this particular S, let T' denote the largest T for which Line (6) is executed. We next consider two cases, based on whether or not we have $T' = S \cup (\Sigma_{< \varepsilon} \cap N(\theta_a))$.

Algorithm 1 A PTAS for constrained persuasion with unique accepts.

Input: Graphs H with a single acceptable state θ_a, and $\varepsilon > 0$.

1 Let $\Sigma_{\geq\varepsilon}$ be the set of all signals $\sigma \in \Sigma$ such that $\sum_{\theta\in\Theta_R(\sigma)} q_\theta \geq \varepsilon q_{\theta_a}$. Then, let $\Sigma_{<\varepsilon} = \Sigma \setminus \Sigma_{\geq\varepsilon}$;

2 Let φ_{ALG} be an arbitrary signaling scheme;

3 **for** *every (possibly empty) subset* $S \subseteq \Sigma_{\geq\varepsilon}$ *of size at most* $1/\varepsilon$ **do**

4   Let $T = S$;

5   **while** $\sum_{\theta\in\Theta_R(T)} q_\theta \leq q_{\theta_a}$ **do**

6   If the utility of φ_{ALG} is less than $\min\{2q_{\theta_a}, \sum_{\theta\in N(T)} q_\theta\}$, then let φ_{ALG} be the optimal signaling scheme consistent with signaling partition $\Sigma_A = T$, which can be computed in polynomial time due to Lemma 17 ;

7   If $T = \Sigma_{<\varepsilon} \cap N(\theta_a)$, break from the loop;

8   Otherwise, add an arbitrary signal from $(\Sigma_{<\varepsilon} \cap N(\theta_a)) \setminus T$ to T;

9   **end**

10 **end**

Output: φ_{ALG}.

- Case I: $T' = S \cup (\Sigma_{<\varepsilon} \cap N(\theta_a))$. Notice that $T' \supseteq \Sigma_A^*$. Lemma 17, implies that the utility of φ_{ALG} must be at least OPT.
- Case II: $T' \neq S \cup (\Sigma_{<\varepsilon} \cap N(\theta_a))$. This means that there exists a signal $\sigma^* \in (\Sigma_{<\varepsilon} \cap N(\theta_a))$ whose addition to T' breaks the condition of the while-loop (5), i.e., $q_{\theta_a} < \sum_{\theta\in\Theta_R(T'\cup\{\sigma^*\})} q_\theta$. The right hand side of this inequality is equal to

$$\begin{aligned}\sum_{\substack{\theta\in\Theta_R\\ N(\theta)\subseteq(T'\cup\{\sigma^*\})}} q_\theta &\leq \sum_{\substack{\theta\in\Theta_R\\ N(\theta)\cap T'\neq\emptyset}} q_\theta + \sum_{\substack{\theta\in\Theta_R\\ N(\theta)=\{\sigma^*\}}} q_\theta\\ &= \sum_{\theta\in N(T')\cap\Theta_R} q_\theta + \sum_{\theta\in\Theta_R(\sigma^*)} q_\theta\\ &< \sum_{\theta\in N(T')\cap\Theta_R} q_\theta + \varepsilon q_{\theta_a} \ ,\end{aligned}$$

where the last inequality since σ belongs to $\Sigma_{<\varepsilon}$. Combining the two inequalities we have

$$\sum_{\theta\in N(T')\cap\Theta_R} q_\theta > (1-\varepsilon) q_{\theta_a} \ . \tag{9}$$

On the other hand, from Lemma 17, when we execute line Line (6) for $T = T'$, it must result in a signaling scheme of utility

$$\min\left\{2q_{\theta_a}, \sum_{\theta\in N(T')} q_\theta\right\} = \min\left\{2q_{\theta_a}, q_{\theta_a} + \sum_{\theta\in N(T')\cap\Theta_R} q_\theta\right\} \overset{(9)}{>} (2-\epsilon) q_{\theta_a} \ ,$$

which is at least $(1-0.5\varepsilon)$OPT due to Lemma 13.

Hence, we can conclude that our algorithm always outputs a signaling scheme with sender utility at least $(1-0.5\varepsilon)$OPT. In other words, its approximation ratio is at most $\frac{1}{1-0.5\varepsilon} \leq 1+\varepsilon$. ◀

References

1 Farid Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Opt.*, 5(1):13–51, 1995.

2 Per Austrin. Balanced MAX 2-SAT might not be the hardest. In *Proc. 39th Symp. Theor. Comput. (STOC)*, pages 189–197, 2007.

3 Per Austrin. Towards sharp inapproximability for any 2-CSP. *SIAM J. Comput.*, 39(6):2430–2463, 2010.

4 Jesse Bull and Joel Watson. Hard evidence and mechanism design. *Games Econom. Behav.*, 58:75—93, 2007.

5 Gabriel Carroll and Georgy Egorov. Strategic communication with minimal verification. *Econometrica*, 87(6):1867–1892, 2019.

6 Shaddin Dughmi. On the hardness of designing public signals. *Games Econom. Behav.*, 118:609–625, 2019.

7 Shaddin Dughmi, David Kempe, and Ruixin Qiang. Persuasion with limited communication. In *Proc. 17th Conf. Economics and Computation (EC)*, pages 663–680, 2016.

8 Shaddin Dughmi, Rad Niazadeh, Alexandros Psomas, and Matthew Weinberg. Persuasion and incentives through the lens of duality. In *Proc. 15th Int. Conf. Web & Internet Econom. (WINE)*, pages 142–155, 2019.

9 Shaddin Dughmi and Haifeng Xu. Algorithmic Bayesian persuasion. In *Proc. 48th Symp. Theor. Comput. (STOC)*, pages 412–425, 2016.

10 Shaddin Dughmi and Haifeng Xu. Algorithmic persuasion with no externalities. In *Proc. 18th Conf. Economics and Computation (EC)*, pages 351–368, 2017.

11 Yuval Emek, Michal Feldman, Iftah Gamzu, Renato PaesLeme, and Moshe Tennenholtz. Signaling schemes for revenue maximization. *ACM Trans. Econom. Comput.*, 2(2):1–19, 2014.

12 Uriel Feige and Michel Goemans. Approximating the value of two power proof systems, with applications to MAX 2-SAT and MAX DI-CUT. In *Proc. 3rd Israel Symp. Theor. Comput. Syst.*, pages 182–189, 1995.

13 Jacob Glazer and Ariel Rubinstein. On optimal rules of persuasion. *Econometrica*, 72(6):1715–1736, 2004.

14 Jacob Glazer and Ariel Rubinstein. A study in the pragmatics of persuasion: a game theoretical approach. *Theor. Econom.*, 1(4):395–410, 2006.

15 Michel Goemans and David Williamson. .879-approximation algorithms for MAX CUT and MAX 2SAT. In *Proc. 26th Symp. Theor. Comput. (STOC)*, pages 422–431, 1994.

16 Ronen Gradwohl, Niklas Hahn, Martin Hoefer, and Rann Smorodinsky. Algorithms for persuasion with limited communication, 2020. CoRR abs/2007.12489. `arXiv:2007.12489`.

17 Niklas Hahn, Martin Hoefer, and Rann Smorodinsky. Prophet inequalities for bayesian persuasion. In *Proc. 29th Int. Joint Conf. Artificial Intelligence (IJCAI)*, pages 175–181, 2020.

18 Niklas Hahn, Martin Hoefer, and Rann Smorodinsky. The secretary recommendation problem. In *Proc. 21st Conf. Economics and Computation (EC)*, page 189, 2020.

19 Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *Amer. Econom. Rev.*, 101(6):2590–2615, 2011.

20 Subhash Khot and Rishi Saket. Hardness of bipartite expansion. In *Proc. 24th European Symp. Algorithms (ESA)*, pages 55:1–55:17, 2016.

21 Michael Lewin, Dror Livnat, and Uri Zwick. Improved rounding techniques for the MAX 2-SAT and MAX DI-CUT problems. In *Int. Conf. Integer Prog. Combinat. Opt. (IPCO)*, pages 67–82, 2002.

22 Prasad Raghavendra. Optimal algorithms and inapproximability results for every CSP? In *Proc. 40th Symp. Theor. Comput. (STOC)*, pages 245–254, 2008.

23 Prasad Raghavendra and David Steurer. How to round any CSP. In *Proc. 50th Symp. Foundations of Computer Science (FOCS)*, pages 586–594, 2009.

24 Itai Sher. Credibility and determinism in a game of persuasion. *Games Econom. Behav.*, 71(2):409–419, 2011.

25 Itai Sher. Persuasion and dynamic communication. *Theor. Econom.*, 9(1):99–136, 2014.
26 Henrik Sjögren. *Rigorous analysis of approximation algorithms for MAX 2-CSP*. Skolan för datavetenskap och kommunikation, Kungliga Tekniska högskolan, 2009.
27 Joel Sobel. Giving and receiving advice. *Adv. Econom. Econometrics*, 1:305–341, 2013.