The Waring Distribution as a Low-Frequency Prediction Model: A Study of Organic Livestock Farms in Andalusia

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Abstract: Although the numbers are relatively small with respect to non-organic livestock, the importance of organic livestock farms lies in their sustainable coexistence with the natural environment and in the high-quality food products obtained. In this type of production, no artificial chemicals or genetically modified organisms are used, therefore there will be less impact on the environment and, in most cases, native breeds are employed. This paper describes a geostatistical study of organic livestock farms in Andalusia (southern Spain), conducted using information from the 2009 Agricultural Census, by classes of livestock. This region currently records the highest output in Spain for organic livestock farming. The number of farms was fitted according to the univariate generalized Waring distribution, which is presented as a means of analyzing this type of discrete measurement, using agricultural or livestock data. The Waring distribution is used when the frequency of occurrence of a phenomenon is very low and allows one to divide the variance. The most important outcome of this study is the finding that livestock data variability is mainly due to external factors such as the proneness component of the variance.

Keywords: agricultural census; modelling; organic livestock farming; Waring distribution

1. Introduction

Organic livestock farming is a model of sustainable development in rural areas, based on techniques that can have less impacts to the environment [1], promote animal welfare [2] and are beneficial to human health [3,4]. This type of farming reduces atmospheric emissions [5], preserves soil fertility, and through the appropriate use of natural resources, reduces production costs and avoids the use of chemicals and genetically modified organisms [6].

Andalusia is the Spanish region with the highest output of organically-reared livestock [7]. It has 48% of the cattle farms and 54% of the sheep farms in the country [8] and their numbers have risen considerably in recent years. However, the majority of the population remains unaware of the reality of ecological production. A study of consumption patterns [9] revealed that most individuals who claim to consume organic products (22%) do not actually do so, although they believe they do. This highlights the importance of conveying a good understanding of what organic production actually means and of promoting the recognition of eco-certified products (in Andalusia, the body responsible for certification is the Directorate General for Industry and Food Quality, together with the Directorate General for Organic Agriculture).

Organic livestock farming has significant advantages over the non-organic approach, including environmental protection, concern for animal welfare and greater consumer confidence

in the quality of the food produced [10]. Organic livestock farming also plays an important role in heightening the biodiversity of breeds [11]. Moreover, it benefits rural development [12], since to a large extent it is the small and medium-sized producers who have adopted this type of production.

To ensure high-quality organic production, the EU has issued general rules on organic production, processing, distribution, labelling and controls. Farmers, processors, intermediaries and retailers must meet strict EU requirements to label their products as organic, under Regulation (EC) 834/2007 of the EU Council of 28 June 2007 on organic production and labelling of organic products. Under this regulation, under organic farming, the animals' health must be maintained through "the application of animal husbandry practices, which enhance the immune system and strengthen natural defences against diseases, in particular including regular exercise and access to open air areas and pastureland".

Other organizations, too, have published requirements or recommendations for the implementation of efficient organic production. A WWF report sets out proposals for improvements in organic production [13], to reduce its environmental impact. For the region examined in our study, the Andalusian Organic Farming Plan sets out a series of measures to encourage this type of production, such as support for organic products, support for the handling and processing of organic products, encouraging the domestic consumption of organic food, promoting training and research, ensuring the protection and transparency of the organic sector, increasing the participation of women in the organic sector and achieving effective horizontal strategic planning.

1.1. Objectives

In this paper, we describe how the discrete variable "number of organic livestock farms" can be fitted using the univariate generalized Waring distribution [14]. The paper has the following main aims. On the one hand, we propose a statistical distribution for fitting discrete variables which present very low frequencies of appearance. In addition, the variance can be split to reveal the intrinsic randomness in the individuals, the internal differences between individuals (liability) and, finally, the external factors that may influence this behavior (proneness). This approach can be very useful because it highlights the origin of the observed variability in the data. Finally, the paper presents a new tool that may be of great interest in future research, since this distribution has never been used before in the field of livestock farming.

There are three goals of our research. The first is to consider the suitability of the Waring distribution as a means of investigating phenomena presenting very low frequencies of appearance (i.e., asymmetric distributions), as is the case of organic livestock farms. The second is to show how the Waring distribution can be used to decompose the variance, thus highlighting the origin of the variability observed in the data. The last goal is to establish the best method for estimating the parameters of the Waring distribution.

1.2. Waring Distribution

The generalized Waring distribution [15,16] is a discrete distribution on nonnegative integers. It is infinitely divisible, self-compensating as defined by Steutel and Van Harn [17], and is, "in limit terms", complete [18]. The generalized Waring is a particular case of equilibrium distribution [19] and in fact, there are different efficient algorithms for it that generate Sibuya's digamma and trigamma distributions [20]. Furthermore, it is used in stochastic aggregation models [21]. Sarabia and Castillo [22] presented two multivariate extensions of the distribution of Waring and Rodríguez-Avi et al. [23] published an analysis of different parameter estimation methods for hypergeometric Gaussian distributions. In Rodríguez-Avi et al. [24] the maximum likelihood method was applied to the generalized univariate Waring distribution, to the negative binomial distribution and the extended Waring distribution.

The generalized Waring distribution has been applied in many scientific fields. Newbold [25] fitted a negative binomial distribution to the number of accidents of factory workers. Later, Irwin (1968) used these data to improve the fit through univariate generalized Waring distribution. Since then,

the Waring distribution has been used, in addition to the theory of accidents [15,26], in other fields such as biology [27], reliability theory [26], library science [28], computer science [29], psychiatry [30], medicine [31], linguistics, economics [32] and agriculture [33]. However, in this research it is applied in studies related to livestock, which has never been done.

2. Materials and Methods

The data used were obtained from the Agricultural Census [34] conducted by the Spanish National Statistics Institute and from the interpretation of this information by the Andalusian Institute of Statistics and Cartography [35]. Maps were created using R free software together with SPSS for data cleaning.

In the following, we define the Waring distribution. A random variable X follows a univariate generalized Waring distribution, with parameters a, k and ρ, $UGWD(a, k; \rho)$, when its probability mass function is:

$$P\{X = r\} = \frac{\rho_{(k)}}{(a + \rho)_{(k)}} \frac{a_{(r)} k_{(r)}}{(a + k + \rho)_{(r)}} \frac{1}{r!} \quad r = 0, 1, \cdots a, k, \rho > 0$$

where $x_{(r)} = \frac{\Gamma(x+r)}{\Gamma(x)}$, $x > 0$, $r \in \Re$, and where $\Gamma(.)$ is the gamma function.

The mean of the Waring distribution is infinite if $\rho \leq 1$ and takes the following form:

$$\mu_1 = \frac{ak}{\rho - 1}, \quad \rho > 1$$

and the variance is infinite if $\rho \leq 2$:

$$\sigma^2 = \mu_2 = \frac{ak(\rho + a - 1)(\rho + k - 1)}{(\rho - 1)^2(\rho - 2)}, \quad \rho > 2$$

when the variance is finite, that is when $\rho > 2$, it can be divided into three components: the first, σ_R^2, contains random factors, the second, σ_λ^2, reflects the internal variability between individuals (liability) and the third, σ_v^2, reflects the variability caused by external factors affecting the population (proneness):

$$\sigma^2 = \sigma_R^2 + \sigma_\lambda^2 + \sigma_v^2 = \frac{ak}{\rho - 1} + \frac{ak(k + 1)}{(\rho - 1)(\rho - 2)} + \frac{a^2 k(\rho + k - 1)}{(\rho - 1)^2(\rho - 2)}, \quad \rho > 2$$

Various methods can be used to estimate the parameters of this distribution [36] including maximum likelihood estimation (MLE), the method of moments (MM3) (which relates the non-centred moments of the distribution and in which a three-equation system must be resolved) and "mixed" methods (MF21 and MF12) with two equations describing the moments, and one, the relation between frequencies (MF21) or otherwise one equation for the moments and two for the frequencies (MF12).

The advantages and disadvantages of each method can be summarized as follows. The three moments method (MM3) usually provides relatively high values for the parameters and the chi-square statistic, due to the multiplicative nature of the calculations needed to resolve the system. The mixed methods (moments and frequencies) usually improve these results considerably. With the MF21 method, we obtain finite values for the statistic and reasonably valid values for the parameters. On the other hand, with one equation for the moments and two for frequencies (MF12), we again obtain infinite values for the goodness-of-fit statistic, for the same reasons given above. The maximum likelihood method has the disadvantage that numerical methods must be used to obtain estimates of the parameters. In the present study, the MM3 and MF21 methods were applied in R-project, and the GWRM package in R [37] was used for optimisation by maximum likelihood.

3. Results

We analyzed the number of farms dedicated to "organic cattle farming", "organic sheep and goat farming", "organic pig farming" and "organic poultry farming" and recorded the total number of farms. Andalusia has a total of 1819 farms with 339,859 animals. This represents an average of 2.36 farms per municipality, with the municipality of Tarifa containing the highest number (87 farms, with a total of 8562 animals, mostly cattle). The spatial distribution of the number of farms per municipality is shown in Figure 1, where it can be seen that the majority of the organic production of livestock takes place in the eastern part of the region, especially in the provinces of Huelva and Cádiz and, to a lesser extent, Córdoba. With respect to the number of animals reared in organic farms, the spatial pattern is similar (Figure 2).
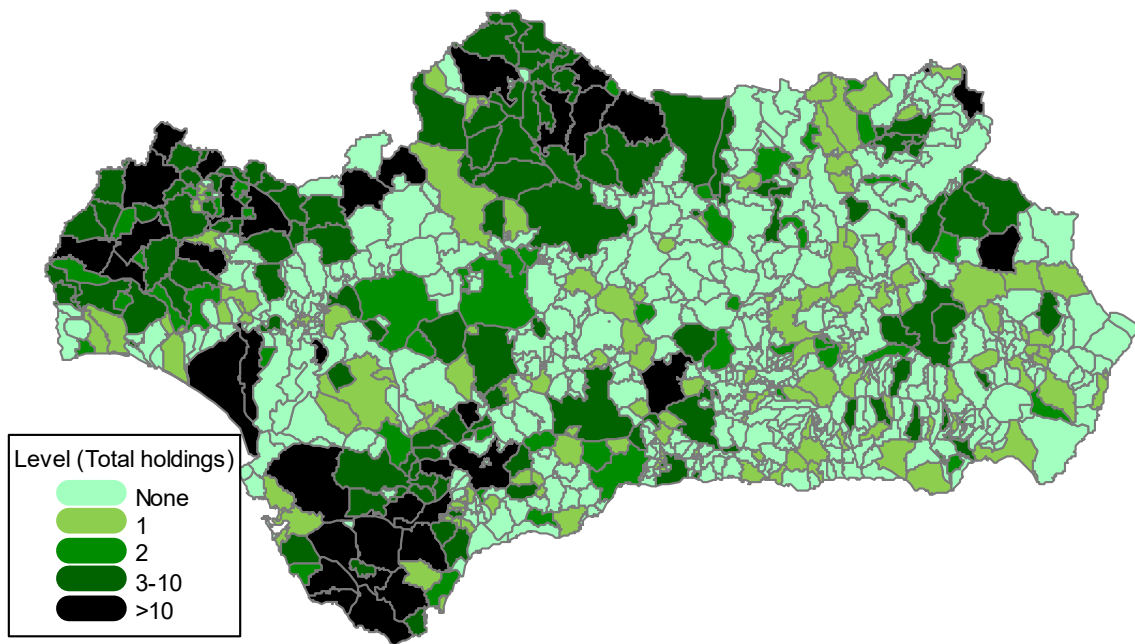

Figure 1. Spatial distribution of total organic livestock farms by municipalities. As the frequencies of appearance are very low, presenting asymmetry, with a standardized skewness of 63.65, fitting with the Waring distribution is appropriate.

Analysis by each kind of animal (Figure 3) shows that on average there are 1.21 cattle farms per municipality and 1.07 for sheep and goats. The values for pigs and poultry are negligible (0.05 and 0.02, respectively). There are 934 organic cattle farms in Andalusia (with 59,339 animals), representing 51.35% of the total holdings and 17.46% of the total head of cattle. By municipality, Tarifa is at the forefront, with 65 farms and 6989 animals, followed by Barbate (49 farms with 3777 animals) and Almonte (47 farms with 994 animals). Also significant is the municipality of Los Barrios, where although there are only 42 farms, they have 3437 animals.

With respect to sheep and goats, there are 829 farms with 275,826 animals; these values represent 45.27% of the total number of farms and 81.16% of the animals. The leading municipalities are Santiago-Pontones (33 farms with 16,263 animals), Alanís (32 farms with 17,602 animals) and Cazalla de la Sierra (24 farms with 11,196 animals). There are 43 pig farms (2.36% of the total), with 3760 animals (1.11% of the total). By municipalities, the most important are Tarifa (five farms with 134 animals), Villaluenga del Rosario (three farms with 215 animals and Alosno (three farms with 71 animals). As is also noteworthy, not by number of farms but by the number of animals, is Pulpí with 1909 animals on just one farm. With respect to poultry, there are 13 farms with 934 animals, representing 0.71% and 0.27% of the total, respectively. The leading producers are in Tarifa (three farms with 80 animals) and

Torre Alháquime (three farms with 56 animals). Jimena de la Frontera and the city of Córdoba each have only one farm, but 300 and 140 animals, respectively.
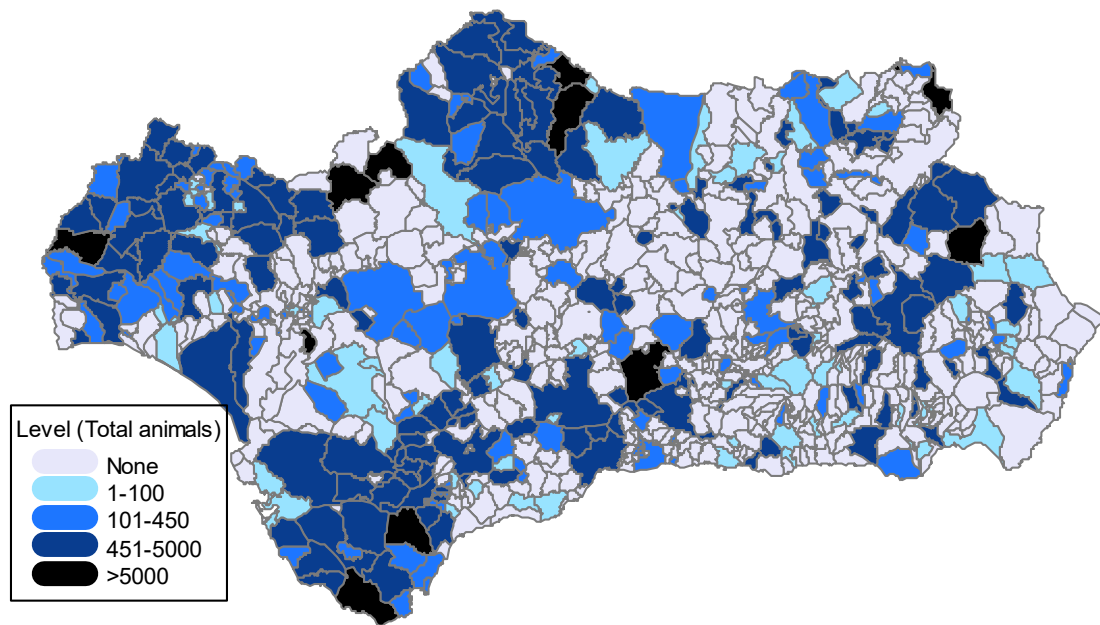

Figure 2. Spatial distribution of total animals in organic livestock farms by municipalities.
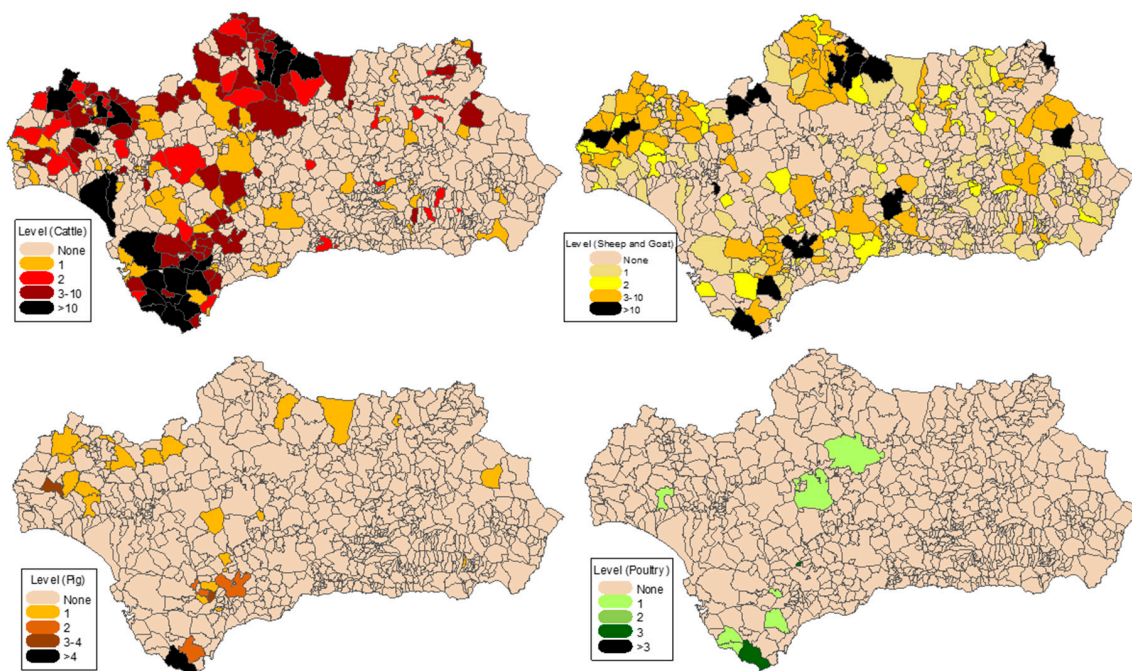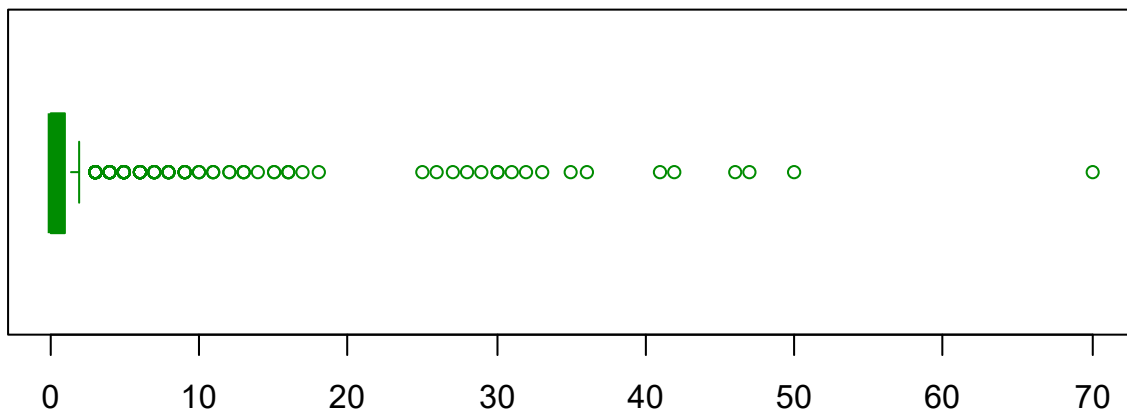

Figure 3. Spatial distribution of total organic livestock farms by municipality and class. This spatial distribution shows how the frequencies of pig and poultry farms are very low (too many zeros); however, this is not the case for the number of cattle farms and for sheep and goat farms.

As the aim of this analysis is to apply the univariate Waring distribution, we examine the shape of the variable "number of farms". The corresponding Box plots (Figures 4 and 5) reveal the severe asymmetry of the distributions, in which the value 0 has a high density, as do the initial values. However, not all the skewed distributions will correctly fit a Waring distribution. Figure 5 shows that the distributions of "pig" and "poultry" have a very small range of values; in these cases, the Waring

distribution is not at all appropriate. For this reason, we model the distribution of all organic farms (without distinguishing by type of livestock).

Number of organic livestock farms in each municipality in Andalusia

Figure 4. Box plot of total organic livestock farms. This distribution has an appropriate degree of asymmetry for the Waring distribution to be used with the standardized skewness of 63.65.

By applying the maximum likelihood method, we obtain $\widehat{a} = 0.3608$, $\hat{k} = 4.3361$ and $\hat{\rho} = 1.6262$ (*p*-value $\chi^2 = 0.8628$), which indicates that the fit obtained is correct. The data observed and the fitted values are shown in Table 1. By the method of the three relations between moments (MM3), we find that $\widehat{a} = 2382.7443$, $\hat{k} = 0.1304$ and $\hat{\rho} = 144.6804$ (*p*-value $\chi^2 < 0.001$), and by the MF21 method, the corresponding results are $\widehat{a} = 12.5354$, $\hat{k} = 0.3379$ and $\hat{\rho} = 2.9589$ (*p*-value $\chi^2 = 0.008$). Therefore, the best estimates in this case are obtained with the maximum likelihood method; nevertheless, the variance cannot be decomposed, because $\hat{\rho} < 2$.

Table 1. Observed frequencies (Freq) for the number of organic livestock farms and estimated frequencies (Fit) by the maximum likelihood estimation (MLE), three moments method (MM3) and moments and frequencies method (MF21).

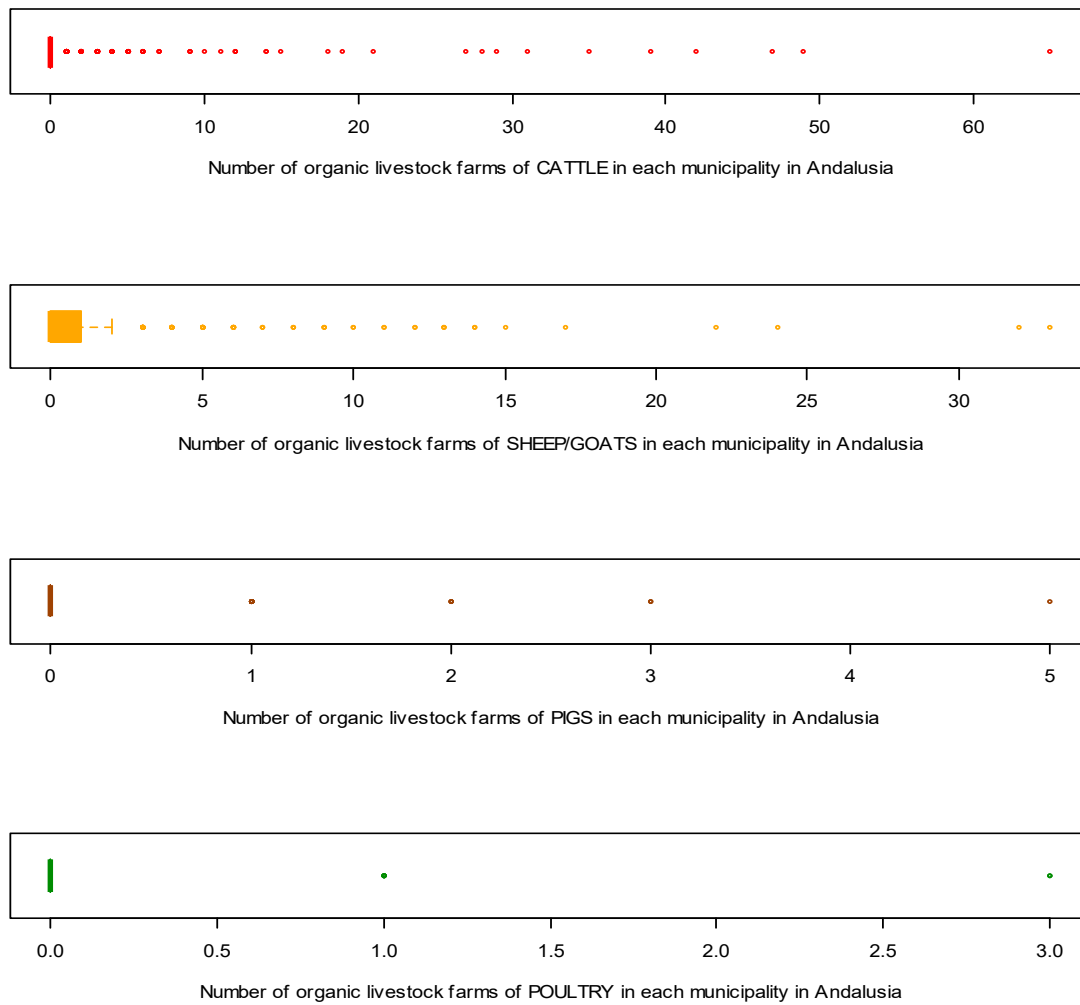| N°	Freq	Fit MLE	Fit MM3	Fit MF21	N°	Freq	Fit MLE	Fit MM3	Fit MF21
0	456	457	530	427	17	1	1	3	2
1	122	113	65	114	18	1	1	2	2
2	49	56	35	61	19	0	1	2	1
3	32	34	23	39	20	0	1	2	1
4	19	22	17	27	21	0	1	2	1
5	17	16	13	19	22	0	1	2	1
6	13	12	11	15	23	0	1	1	1
7	9	9	9	11	24	0	1	1	1
8	8	7	7	9	25	1	1	1	1
9	8	6	6	7	26	1	1	1	1
10	3	4	5	6	27	1	1	1	1
11	3	4	5	5	28	1	0	1	0
12	2	3	4	4	29	1	0	1	0
13	3	3	4	3	30	2	0	1	0
14	1	2	3	3	31–40	5	0	3	0
15	2	2	3	2	41–50	5	0	0	0
16	3	2	3	2	+50	1	0	0	0

Figure 5. Box plot of total organic livestock farms by class of animal. All distributions are highly asymmetric, especially those for pig and poultry farms. The standardized skewness is 66.59 for sheep and goat farms, 86.24 for cattle farms, 96.35 for pig farms and 152.53 for poultry farms.

4. Discussion

The generalized Waring process is a non-homogeneous stationary Markov process. Two models lead to the generalized Waring process: the Cox process and the compound Poisson process. The Poisson and Pólya processes arise as particular cases of the Generalized Waring process [38,39].

Various methods can be used to estimate the parameters of $UGWD(a,k;\rho)$ in addition to the maximum likelihood method, the method of moments and methods based on relations between moments and/or frequencies. In this respect, Rodríguez-Avi et al. observed that the minimum estimator χ^2 based on the relations between moments produced a relative asymptotic efficiency result close to 1. It can be seen in the result of the method of moments that the Waring distribution is practically biparametric as the value of k is very close to zero. All the analyzed methods produce similar results, but the method of two relations between moments and one for frequencies not only produces a good fit, in addition it has an important advantage over maximum likelihood in the speed of calculation, as it does not require the use of numerical methods, because the equations that must be solved in this approach do not require them. For this reason, many authors reject the use of this method to estimate the parameters of Waring-family distributions.

Not all distributions with significant asymmetry can be fitted by the Waring distribution. Our analysis shows that the fit for the variable "organic livestock holdings" is significant, and we believe this finding is of interest to those wishing to use this distribution to fit discrete observations.

The estimators of the parameters using maximum likelihood estimation (MLE) and the method based on two equations of moments and one of frequencies (MF21) are described in Section 2. It is important to note that $\rho > 2$ for the method of moments (MM3) and for the method with two equations for the moments and one for frequencies (MF21). Therefore, in this case the variance will be finite and can be decomposed as described by Irwin (1968). This is not the case, however, with maximum likelihood ($\hat{\rho} = 1.6262$) as we show for the variable studied.

On the other hand, the estimators of the parameters obtained by maximum likelihood, by the method of moments and by the method based on the first two relations between moments and the first relation between probabilities enable us to calculate the value of the $\chi2$ statistic. This is another reason for rejecting the method of one equation of moments and two of frequencies (MF12), apart from the fact that it does not produce good estimators. Therefore, we do not recommend using this method to estimate the parameters of the univariate generalized Waring distribution.

The estimators obtained by the method of moments (MM3) are considerably higher ($\hat{a} = 2382.7443$, $\hat{k} = 0.1304$ and $\hat{\rho} = 144.6804$), an outcome that is totally logical in view of the multiplicative nature of the calculations performed. This is apparent from the expression of the mass probability function.

When the decomposition of the variance is addressed using the MM3 method, of the total variability found in the data, $\sigma^2 = 38.323$, only $\sigma_R^2 = 2.162$ (5.6%) corresponds to the part of the variability that is truly random, while $\sigma_\lambda^2 = 0.017$ (<1%) is the part caused by internal differences within each municipality and which therefore cannot be modelled, and $\sigma_\nu^2 = 36.143$ (94.3%) is the variability produced by the influence of external factors. Therefore, this decomposition which is not possible with other distributions indicates that a high percentage of variability is caused by external factors such as those that are economic, environmental, etc. For this reason, an open line of research would be the consideration of other environmental variables in the model, performing a Waring regression [40,41]. Such an approach would reveal which factors cause the proneness component to be so high.

Another promising future line of research regarding organic livestock would be to consider the spatial correlation structure of the data collected. It would be very useful to obtain geostatistical models to estimate the spatial trend of organic farming through covariates, when the location (latitude, longitude) of the information is known. Another possibility would be to consider lattice or raster type models. Such an approach could include all the aspects that might promote or hamper the adoption of organic livestock farming methods, such as economic factors (tax benefits, market prices and production costs), physical factors (for example, the type of soil and land elevation) or environmental factors such as the quantity and quality of grassland area, average temperatures and rainfall.

5. Conclusions

The Waring distribution is used mainly when the variable under study has a low probability of occurrence and one of the most important characteristics of the Waring distribution is the division of its variance, which reveals the behavior of the distribution in relation to the intrinsic randomness in the individuals, the internal differences between individuals (liability) and, finally, the external factors that may influence this behavior (proneness). This knowledge can be very useful to researchers because it reveals the origin of the observed variability in the data. The Waring distribution converges to the normal one when the parameters *a* and *k* tend towards infinity. If any of the parameters *a* or *k* are sufficiently small (as is the case in the present study) the distribution presents a strong skewness to the right, which makes it valid for modelling real random situations in which this skewness occurs.

Due to the nature of the distribution, by including a significant number of factorial products, the maximum likelihood method presents a series of calculation problems (calculation errors even using powerful computer software). For this reason, we present some important alternatives: the MM3 moment method and "mixed" estimation methods using MF12 and MF21 moments and frequencies. This research clearly shows that the use of the method of MF21 moments and frequencies is the one with the best results. It is more suitable than all other methods considered, including the method of moments, which produces estimates that are too high. Our study shows that the MLE and MF12

methods do not allow the necessary decomposition of the variance since they do not offer values of ρ greater than 2.

This paper introduces a new field in which the Waring distribution can be studied and applied. This was done in an earlier study in the context of crop farming, and is now extended to that of livestock farming, with particular regard to its sustainable application, namely organic livestock farming.

A good fit of the Waring distribution to livestock data will allow other researchers to use this distribution to determine the origin of the variability in their data (i.e., the influence of external factors on the variance, or on randomness or responsibility), which is a great advantage versus other distributions. In the case discussed in this paper, we show that the variability in the data is mainly due to external factors (94.3%). This leads us to conclude that a Waring regression could be an ideal tool to reflect the variability provoked by external factors. In the field of livestock farming, for example, the costs of production, the extension of pasture or the sale prices of this meat could be included.

It is important to note that the Waring distribution will be valid when when the frequency of occurrence in the variable is very low. Using this distribution we can divide the variance when $\rho > 2$, thus highlighting its behavior in relation to the intrinsic randomness in the observations, the internal differences between individuals and the external factors that can influence this behavior. This property is very useful for researchers because it reveals the origin of the variability in the data. Therefore, using a distribution with the properties described in this paper makes it possible to detect and quantify the degree of influence of every source of variation.

Abbreviations

L (Liability), MF12 (method of one equation of moments and two relations between frequency), MF21 (method of two relations between moments and one equation of frequency), MLE (method of log-likelihood optimisation), MM3 (method of the three relations between moments), P (Proneness), R (Randomness), UGWD (Univariate Generalized Waring Distribution), a, k, ρ (Waring distribution parameters), μ_1 (mean of the Waring distribution), σ^2 (variance of the Waring distribution).

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