New convolutions associated with the Mellin transform and their applications in integral equations*

L. P. Castro[†] A. S. Silva[‡] N. M. Tuan[§]

In honor of Frank-Olme Speck on the occasion of his 75th birthday

Abstract

In this paper, we introduce two new convolutions associated with the Mellin transform which exhibit factorization properties upon the use of certain weight functions. This is applied to the solvability analysis of classes of integral equations. In particular, we present sufficient conditions for the solvability of an integral equation and a system of integral equations of convolution type.

Keywords Convolution, Mellin transform, convolution operator, factorization, integral equations of the convolution type.

Mathematics Subject Classification (2010) 44A35, 42A85, 44A15, 45E10, 45P05.

1 Introduction

Integral transforms and its associated convolution operators have been studied and used for a long time to solve many problems in applied mathematics, mathematical physics and engineering science. A well-known and useful integral transform is the Mellin transform. It was H. Mellin (1854–1933) who first gave a systematic formulation of Mellin transform and its inverse. Although a change of variables shows that the Mellin transform is closely related to the Laplace and Fourier transforms, there are certain applications where it is convenient to operate directly with the Mellin transform. Besides the applications to other sciences, also in mathematics the Mellin transform is an important tool to the study of the behavior of many important functions such as the zeta function and Dirichlet series occurring in number theory, and also the gamma function occurring in the complex function theory (cf., e.g., [10, 13, 14]). The Mellin transform shows to be also important in many other different subareas, as e.g. it is the case of analysis of certain algorithms and probability theory (cf., e.g., [11]).

*Accepted author's manuscript (AAM) published in [J. Class. Anal. 16(2) (2020), 65–77] [DOI: dx.doi.org/10.7153/jca-2020-16-08]. The final authenticated version is available online at: `http:http://files.ele-math.com/articles/jca-16-08.pdf`.

†CIDMA–Center for Research and Development in Mathematics and Applications, University of Aveiro, Portugal, castro@ua.pt

‡CIDMA–Center for Research and Development in Mathematics and Applications, University of Aveiro, Portugal, anabela.silva@ua.pt

§Department of Mathematics, VNU University of Education, Viet Nam National University, Hanoi, Vietnam, nguyentuan@vnu.edu.vn

Convolutions are frequently used to help in the modeling of applied problems. If considered in different ways and types, they allow distinct possibilities of "multiplication", as well as different kinds of integral equations of convolution type. Additionally, the convolutions of mathematical physics are often represented in form of some integrals. Therefore, one of the important applications of convolutions is their association to corresponding integral equations of convolution type (cf., e.g., [1, 3, 4, 5, 6, 7, 8, 9]), and the consequent possibility to consider new integral classes (cf., e.g., [12]).

In this paper, we introduce two new convolutions and show how they can be applied to obtain the solutions of integral equations or systems of integral equations associated with them. Throughout this paper, we will operate with the Mellin transform as well as its classical associated convolution. Those will interact with our new convolutions and, altogether, will allow us to obtain the below results. In view of this, we start by presenting some topics from the theory of Mellin integral transforms which will be useful in our further analysis.

Let $\mathbb{R}_+ := (0, \infty)$ be the set of positive real numbers and let $L^1(\mathbb{R}_+)$ be the space of all (Lebesgue) measurable complex-valued functions $f : \mathbb{R}_+ \to \mathbb{C}$ with the finite norm

$$\|f\|_{L^1(\mathbb{R}_+)} := \int_0^\infty |f(u)| du.$$

In this sense, for some $a \in \mathbb{R}$, we denote by $L^1(\{a\} \times i\mathbb{R})$ the set of all functions $g : \{a\} \times i\mathbb{R} \to \mathbb{C}$ with $g(a + i\cdot) \in L^1(\mathbb{R})$.

In what follows, for $s \in \mathbb{C}$, we always denote such complex numbers by $s = a + ib$, with $a, b \in \mathbb{R}$.

Definition 1. *If $f : \mathbb{R}_+ \to \mathbb{C}$ is a function such that $f(x)x^{s-1} \in L^1(\mathbb{R}_+)$, for some $s \in \mathbb{C}$, then the Mellin transform is defined by the following identity*

$$(\mathcal{M}f)(s) = f^*(s) = \int_0^\infty f(t)t^{s-1}dt, \; s \in \{a\} \times i\mathbb{R}. \tag{1.1}$$

The inverse Mellin integral transform for a function $f^* \in L^1(\{a\} \times i\mathbb{R})$ is defined as

$$f(t) = \left(\mathcal{M}^{-1}f^*(s)\right)(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f^*(s)t^{-s}ds, \quad t > 0,$$

where the integral is understood in the sense of Cauchy principal value.

If $f(x)x^{a-1} \in L^1(\mathbb{R}_+)$, for some $a \in \mathbb{R}$, then the Mellin transform exists for all $s \in \mathbb{C}$, and the integral (1.1) is absolutely convergent. Moreover, the Mellin transform f^* is a continuous function on the line $\{a\} \times i\mathbb{R}$ (cf. [2]). In this sense, let us describe an appropriate spaces framework for which the Mellin transform exists.

Definition 2. *Let us consider the weight function $w(x) = x^{a-1}$ for some $a \in \mathbb{R}$. The weighted Lebesgue space $L^1(\mathbb{R}_+, w)$ is defined by*

$$L^1(\mathbb{R}_+, w) := \{f : \mathbb{R}_+ \to \mathbb{C} : f(x)x^{a-1} \in L^1(\mathbb{R}_+)\} \tag{1.2}$$

with the associated norm given by

$$\|f\|_{L^1(\mathbb{R}_+,w)} := \|f(x)x^{a-1}\|_{L^1(\mathbb{R}_+)} = \int_0^\infty |f(u)|u^{a-1}du.$$

For $f : \mathbb{R}_+ \to \mathbb{C}$, $a \in \mathbb{R}$, $h \in \mathbb{R}_+$, we define the Mellin translation operator by

$$(\tau_h^c f)(x) = h^c f(hx), \qquad x \in \mathbb{R}_+.$$

The Mellin translation operator $\tau_h^{\tilde{a}} : L^1(\mathbb{R}_+, w) \to L^1(\mathbb{R}_+, w)$, for $a, \tilde{a} \in \mathbb{R}$, $h \in \mathbb{R}_+$ is an isomorphism with $(\tau_h^{\tilde{a}})^{-1} = \tau_{1/h}^{\tilde{a}}$ and

$$\|\tau_h^{\tilde{a}} f\|_{L^1(\mathbb{R}_+,w)} = h^{\tilde{a}-a} \|f\|_{L^1(\mathbb{R}_+,w)}, \qquad f \in L^1(\mathbb{R}_+, w) \tag{1.3}$$

(cf. [2]).

Let $f * g$ be the classic Mellin convolution of two functions $f, g : \mathbb{R}_+ \to \mathbb{C}$ defined by

$$(f * g)(x) = \int_0^\infty f\left(\frac{x}{u}\right) g(u) \frac{du}{u}, \qquad x \in \mathbb{R}_+$$

whenever the integral exists. There hold the following properties.

Theorem 1. [2, Theorem 3]

(a) *If $f, g \in L^1(\mathbb{R}_+, w)$, then the convolution $f * g$ exists (a.e.) on $\mathbb{R}_+$, belongs to $L^1(\mathbb{R}_+, w)$ and one has*

$$\|f * g\|_{L^1(\mathbb{R}_+,w)} \le \|f\|_{L^1(\mathbb{R}_+,w)} \|g\|_{L^1(\mathbb{R}_+,w)}.$$

In addition, $x^a f(x)$ is uniformly continuous and bounded on $\mathbb{R}_+$, then $f * g$ is continuous on $\mathbb{R}_+$.

(b) **(Convolution Theorem)** *If $f, g \in L^1(\mathbb{R}_+, w)$, then*

$$[\mathcal{M}(f * g)](s) = (\mathcal{M}f)(s)(\mathcal{M}g)(s), \ s \in \{a\} \times i\mathbb{R}.$$

(c) *The convolution product is associative and commutative. In particular, $(L^1(\mathbb{R}_+, w), +, *)$ is a Banach algebra.*

(d) *The Parseval equality*

$$\int_0^\infty f\left(\frac{x}{t}\right) g(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) g^*(s) x^{-s} ds$$

holds true.

Note that the Mellin integral transform can be obtained from the Fourier integral transform by the exponential substitution and rotating the complex plane by a right angle:

$$(\mathcal{M}f)(s) = \int_0^\infty f(t) t^{s-1} dt = \int_{-\infty}^\infty f(e^{-x}) e^{-xs} dx = \int_{-\infty}^\infty f(e^{-x}) e^{-ax} e^{-ibx} dx = (\mathcal{F}g)(b),$$

with $g(x) = f(e^{-x}) e^{-ax}$. In the same way, the inverse of the Mellin transform and its classical associated convolution can be obtained by the same substitutions from the inverse of the Fourier transform and its associated convolution.

2 New convolutions

In this section, we propose two new convolutions associated with Mellin integral transform. One of the most important properties of a convolution is to satisfy a factorization property which is typically associated with one or more than one integral operators (i.e., a Convolution Theorem). In most of the cases, such factorization property is fundamental to solve consequent integral equations which can be characterized by those convolutions. In this sense, we show that the introduced convolutions exhibit certain factorization identities when considering the integral operator under study. The Mellin convolution operator plays an essential role in the further discussion.

Definition 3. *For any f and $g \in L^1(\mathbb{R}_+, w)$ $(w(x) = x^{a-1})$, we define the convolution operator $\oplus$ by*

$$(f \oplus g)(x) := \int_0^\infty \int_0^\infty e^{-2\frac{x^2}{v^2}} f\left(\frac{v}{u}\right) g(y) \frac{dv}{v} \frac{du}{u}. \tag{2.1}$$

Theorem 2. *Let f and $g \in L^1(\mathbb{R}_+, w)$ $(w(x) = x^{a-1})$. Then the convolution $\oplus$ of functions f and g belongs to $L^1(\mathbb{R}_+, w)$ and satisfies the following weighted factorization identity associated with Mellin integral transform*

$$\mathcal{M}(f \oplus g)(s) = \psi(s)(\mathcal{M}f)(s)(\mathcal{M}g)(s), \tag{2.2}$$

where $\psi(s) = 2^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right)$, $s \in \mathbb{C}$.

Proof. Let $f, g \in L^1(\mathbb{R}_+, w)$. Using the identity $\int_0^\infty e^{-2t^2} t^{s-1} dt = 2^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right)$ (for $\Re e(s) > 0$), we obtain

$$\begin{aligned}
\psi(s)(\mathcal{M}f)(s)(\mathcal{M}g)(s) &= 2^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right) \int_0^\infty f(t)t^{s-1}dt \int_0^\infty g(u)u^{s-1}du \\
&= \int_0^\infty e^{-2v^2} v^{s-1} dv \int_0^\infty f(t)t^{s-1}dt \int_0^\infty g(u)u^{s-1}du \\
&= \int_0^\infty e^{-2y^2} * (f * g)(y) y^{s-1} dy \\
&= \int_0^\infty \left(\int_0^\infty e^{-2\frac{y^2}{z^2}} \int_0^\infty f\left(\frac{z}{w}\right) g(w) \frac{dw}{w}\frac{dz}{z} \right) y^{s-1} dy \\
&= \int_0^\infty \left(\int_0^\infty \int_0^\infty e^{-2\frac{y^2}{z^2}} f\left(\frac{z}{w}\right) g(w) \frac{dw}{w}\frac{dz}{z} \right) y^{s-1} dy \\
&= [\mathcal{M}(f \oplus g)](s).
\end{aligned}$$

Let us now prove that $f \oplus g \in L^1(\mathbb{R}_+, w)$. For that purpose, using the change of variable $z = \frac{x}{v}$

and considering (1.3), we obtain that

$$\begin{aligned}
\|(f \oplus g)(x)\|_{L^1(\mathbb{R}_+,w)} &= \int_0^\infty |(f \oplus g)(x)| x^{a-1} dx \\
&\leq \int_0^\infty |g(y)| \int_0^\infty \left| f\left(\frac{v}{y}\right)\right| v^a \int_0^\infty e^{-2z^2} z^{a-1} dz \frac{dv}{v}\frac{dy}{y} \\
&= \int_0^\infty |g(y)| \int_0^\infty \left| f\left(\frac{v}{y}\right)\right| v^a \frac{dv}{v}\frac{dy}{y} \int_0^\infty e^{-2z^2} z^{a-1} dz \\
&= \int_0^\infty |g(y)| \int_0^\infty \left| f\left(\frac{v}{y}\right)\right| v^a \frac{dv}{v}\frac{dy}{y} \int_0^\infty e^{-2z^2} z^{a-1} dz \\
&= \int_0^\infty |g(y)| y^{a-1} dy \int_0^\infty y^{-a} \left| f\left(\frac{v}{y}\right)\right| v^{a-1} dv \int_0^\infty e^{-2z^2} z^{a-1} dz \\
&= \|g\|_{L^1(\mathbb{R}_+,w)} \|\tau^a_{u^{-1}} f\|_{L^1(\mathbb{R}_+,w)} \int_0^\infty e^{-2z^2} z^{a-1} dz \\
&= \|g\|_{L^1(\mathbb{R}_+,w)} \|f\|_{L^1(\mathbb{R}_+,w)} \|p\|_{L^1(\mathbb{R}_+,w)}
\end{aligned}$$

where $p(z) = e^{-2z^2}$, $z \in \mathbb{R}_+$. Since $p \in L^1(\mathbb{R}_+, w)$, we conclude that $f \oplus g \in L^1(\mathbb{R}_+, w)$. □

Let us proceed with an additional convolution and its respective factorization property.

Definition 4. *For any f and $g \in L^1(\mathbb{R}_+, w)$ ($w(x) = x^{a-1}$), we define the convolution operator $\odot$ by*

$$(f \odot g)(x) := \int_0^\infty \int_0^\infty e^{-2\frac{x}{u}} f\left(\frac{u}{v}\right) g(u) \frac{dv}{v}\frac{du}{u}. \tag{2.3}$$

Theorem 3. *Let f and $g \in L^1(\mathbb{R}_+, w)$ ($w(x) = x^{a-1}$). Then the convolution $\odot$ of functions f and g belong to $L^1(\mathbb{R}_+, w)$ and satisfies the following weighted factorization identity associated with Mellin integral transform*

$$[\mathcal{M}(f \odot g)](s) = \varphi(s)(\mathcal{M}f)(s)(\mathcal{M}g)(s), \tag{2.4}$$

where $\varphi(s) = 2^{-s}\Gamma(s)$.

Proof. Let $f, g \in L^1(\mathbb{R}_+, w)$. Using the identity $\int_0^\infty e^{-2t} t^{s-1} dt = 2^{-s}\Gamma(s)$ (for $\Re e(s) > 0$) and changing variables $z = \frac{x}{v}$, we obtain

$$\begin{aligned}
\varphi(s)(\mathcal{M}f)(s)(\mathcal{M}g)(s) &= 2^{-s}\Gamma(s) \int_0^\infty f(t) t^{s-1} dt \int_0^\infty g(y) y^{s-1} dy \\
&= \int_0^\infty e^{-2v} v^{s-1} dv \int_0^\infty f(t) t^{s-1} dt \int_0^\infty g(y) y^{s-1} dy \\
&= \int_0^\infty e^{-2x} * (f * g)(x) x^{s-1} dx \\
&= \int_0^\infty \left[\int_0^\infty e^{-2\frac{z}{u}} \int_0^\infty f\left(\frac{u}{v}\right) g(v) \frac{dv}{v}\frac{du}{u} \right] z^{s-1} dz \\
&= \int_0^\infty \left[\int_0^\infty \int_0^\infty e^{-2\frac{z}{u}} f\left(\frac{u}{v}\right) g(v) \frac{dv}{v}\frac{du}{u} \right] z^{s-1} dz \\
&= [\mathcal{M}(f \odot g)](s).
\end{aligned}$$

Let us now prove that $f \odot g \in L^1(\mathbb{R}_+, w)$. For that purpose, using the change of variable $z = \frac{x}{u}$ and considering (1.3), we obtain that

$$\begin{aligned}
\|(f\odot g)(x)\|_{L^1(\mathbb{R}_+,w)} &= \int_0^\infty |(f\odot g)(x)| x^{a-1} dx \\
&\le \int_0^\infty |g(v)| \int_0^\infty \left| f\left(\frac{u}{v}\right)\right| u^a \int_0^\infty e^{-2z} z^{a-1} dz \frac{dv}{v}\frac{du}{u} \\
&= \int_0^\infty |g(v)| \int_0^\infty \left| f\left(\frac{u}{v}\right)\right| u^a \frac{du}{u}\frac{dv}{v} \int_0^\infty e^{-2z} z^{a-1} dz \\
&= \int_0^\infty |g(v)| \int_0^\infty \left| f\left(\frac{u}{v}\right)\right| u^{a-1} du \frac{dv}{v} \int_0^\infty e^{-2z} z^{a-1} dz \\
&= \int_0^\infty |g(v)| v^{a-1} dv \int_0^\infty v^{-a} \left| f\left(\frac{u}{v}\right)\right| u^{a-1} du \int_0^\infty e^{-2z} z^{c-1} dz \\
&= \|g\|_{L^1(\mathbb{R}_+,w)} \|\tau^a_{v^{-1}} f\|_{L^1(\mathbb{R}_+,w)} \int_0^\infty e^{-2z} z^{a-1} dz \\
&= \|g\|_{L^1(\mathbb{R}_+,w)} \|f\|_{L^1(\mathbb{R}_+,w)} \|q\|_{L^1(\mathbb{R}_+,w)}
\end{aligned}$$

where $q(z) = e^{-2z}$, $z \in \mathbb{R}_+$. Since $q \in L^1(\mathbb{R}_+, w)$, we conclude that $f \odot g \in L^1(\mathbb{R}_+, w)$. □

3 Convolution integral equations

In this section we will apply our new convolutions to solve integral equations in which those convolutions can be somehow considered. In view of this, let us consider the following integral equation in $L^1(\mathbb{R}_+, w)$:

$$\lambda\varphi(x) + \int_0^\infty \int_0^\infty e^{-\frac{2x^2}{v^2}} \varphi\left(\frac{v}{u}\right) g(u) \frac{dv}{v}\frac{du}{u} = h(x),$$

$\lambda \in \mathbb{C}$, $x \in \mathbb{R}_+$, $g, h \in L^1(\mathbb{R}_+, w)$ and $\varphi \in L^1(\mathbb{R}_+, w)$ is to be determined.

We will use the notation

$$A(s) := \lambda + \psi(s)(\mathcal{M}g)(s), \tag{3.1}$$

where $\psi(s) = 2^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right)$, $s \in \{a\} \times i\mathbb{R}$.

Proposition 4. *(a) If $\lambda \neq 0$, then $A(s) \neq 0$ for every $s \in \{a\} \times i\mathbb{R}$ outside a finite interval.*

(b) Assume that $\lambda \neq 0$ and $A(s) \neq 0$ for every $s \in \{a\} \times i\mathbb{R}$. Then $\mathcal{M}h \in L^1(\{a\} \times i\mathbb{R})$ if and only if $\frac{\mathcal{M}h}{A} \in L^1(\{a\} \times i\mathbb{R})$.

Proof. (a) Let us first observe that $(\mathcal{M}f)(s) = \int_0^\infty x^{a-1} e^{ib\log(x)} f(x) dx$. By the Riemann-Lebesgue Lemma, for any a which lies in the strip of analyticity of Mellin transform of $f(x)$,

$$\lim_{|b|\to\infty} (\mathcal{M}f)(a + ib) = 0,$$

i.e., the Mellin transform is of $f(x)$ vanishes at infinity in the strip of its analyticity (cf. [2, Theorem 2]). Consequently, $\lim_{|b|\to\infty} A(s) = \lambda$. Since $\lambda \neq 0$, the result follows from the analyticity of $A(s)$ along the vertical line $\Re e(s) = a$.

(b) Assume that $\mathcal{M}h \in L^1(\{a\} \times i\mathbb{R})$. By the analyticity of A and $\lim_{|b|\to\infty} A(a+ib) = \lambda \neq 0$, there is $R > 0$ such that $\inf_{|b|>R} |A(s)| > \epsilon_1$, for any $\epsilon_1 > 0$. Since A is analytic in $\{a\} \times i\mathbb{R}$ and does not vanish in the compact set $S(0, R) = \{a + ib \in \mathbb{C} : |b| < R\}$, there exists ϵ_2 such that $\inf_{|b|\leq R}|A(s)| > \epsilon_2$. We then have $\sup_{s\in\{a\}\times i\mathbb{R}}(\frac{1}{A(s)}) \leq \max\{1/\epsilon_1, 1/\epsilon_2\} < \infty$. It follows that the function $\frac{1}{|A(s)|}$ is bounded on $\{a\}\times i\mathbb{R}$. Since $\mathcal{M}h \in L^1(\{a\}\times i\mathbb{R})$, we have that $\frac{\mathcal{M}h}{A} \in L^1(\{a\}\times i\mathbb{R})$.

Conversely, from the assumption $\frac{\mathcal{M}h}{A} \in L^1(\{a\}\times i\mathbb{R})$ and the function $1/A(s)$ is analytic on $\{a\}\times i\mathbb{R}$, we can deduce that $\mathcal{M}h \in L^1(\{a\}\times i\mathbb{R})$ and the proposition is proved. □

Theorem 5. *Assume that $A(s) \neq 0$ for every $s \in \{a\}\times i\mathbb{R}$ and one of the following conditions holds:*

(i) $\frac{\mathcal{M}h}{A} \in L^1(\{a\}\times i\mathbb{R})$*;*

(ii) $\lambda \neq 0$ *and* $\mathcal{M}h \in L^1(\{a\}\times i\mathbb{R})$*.*

Then, equation (3.1) *has a solution in* $L^1(\mathbb{R}_+, w)$ *if and only if* $\mathcal{M}^{-1}\left(\frac{\mathcal{M}h}{A}\right) \in L^1(\mathbb{R}_+, w)$*. Moreover, if the last condition holds, then the solution is given by*

$$\varphi = \mathcal{M}^{-1}\left(\frac{\mathcal{M}h}{A}\right) \in L^1(\mathbb{R}_+, w).$$

Proof. Let us first assume that (i) is fulfilled. Suppose that equation (3.1) has a solution $\varphi \in L^1(\mathbb{R}_+, w)$. Applying $\mathcal{M}$ to both sides of equation (3.1) and using the factorization property (2.2), we obtain

$$A(s)(\mathcal{M}\varphi)(s) = (\mathcal{M}h)(s),$$

where $\mathcal{M}\varphi$ is the unknown function. Since $A(s) \neq 0$ for $s \in \{a\}\times i\mathbb{R}$, it follows $(\mathcal{M}\varphi)(s) = \frac{(\mathcal{M}h)(s)}{A(s)}$. As $\frac{\mathcal{M}h}{A} \in L^1(\{a\}\times i\mathbb{R})$, we apply the inverse of Mellin transform to obtain

$$\varphi = \mathcal{M}^{-1}\left(\frac{\mathcal{M}h}{A}\right).$$

Suppose now that $\varphi = \mathcal{M}^{-1}\left(\frac{\mathcal{M}h}{A}\right)$ is the solution of equation (3.1). It implies that $\varphi \in L^1(\mathbb{R}_+, w)$. Applying the Mellin transform, we have $\mathcal{M}\varphi = \frac{\mathcal{M}h}{A}$, and thus, we have

$$A(s)(\mathcal{M}\varphi)(s) = (\mathcal{M}h)(s).$$

Using factorization identity (2.2), we obtain

$$\mathcal{M}\left(\lambda\varphi(s) + \int_0^\infty \int_0^\infty e^{-\frac{2x^2}{v^2}}\varphi\left(\frac{v}{u}\right)g(u)\frac{dv}{v}\right) = (\mathcal{M}h)(x).$$

By the uniqueness of $\mathcal{M}$, φ fulfills equation (3.1) for $s \in \{a\}\times i\mathbb{R}$.

Assume now that (ii) is fulfilled. In this case, the proof follows from Proposition 4. □

4 Systems of integral equations

In this section we will be considering systems of integral equations generated by the previously introduced convolutions. Namely, let us consider the following system of integral equations

$$\begin{cases} f(x) + \lambda_1(h \oplus g)(x) = p(x) \\ g(x) + \lambda_2(k \odot f)(x) = q(x), \end{cases} \tag{4.1}$$

or equivalently,

$$\begin{cases} f(x) + \lambda_1 \int_0^\infty \int_0^\infty e^{-2\frac{x^2}{v^2}} h\left(\frac{v}{u}\right) g(u) \frac{dv}{v}\frac{du}{u} = p(x) \\ g(x) + \lambda_2 \int_0^\infty \int_0^\infty e^{-2\frac{x}{v}} k\left(\frac{v}{u}\right) f(u) \frac{dv}{v}\frac{du}{u} = q(x), \end{cases}$$

where, $\lambda_1, \lambda_2 \in \mathbb{C}$, $h, k, p, q \in L^1(\mathbb{R}_+, w)$, f and $g \in L^1(\mathbb{R}_+, w)$ are the unknown functions.

Let us fix the notation

$$B(s) := 1 - \lambda_1\lambda_2\varphi(s)\psi(s)\mathcal{M}(k * h)(s), \tag{4.2}$$

where $\psi(s) = 2^{-s}\Gamma(s)$ and $\varphi(s) = 2^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right)$.

Proposition 6. *Assume that $B(s) \neq 0$ for any $s \in \{a\} + i\mathbb{R}$. Then $\mathcal{M}h \in L^1(s \in \{a\} + i\mathbb{R})$ if and only if $\frac{\mathcal{M}h}{B} \in L^1(\{a\} + i\mathbb{R})$, for any $h \in L^1(\mathbb{R}_+, w)$.*

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ and $h, k \in L^1(\mathbb{R}_+, w)$. We first observe that

$$B(s) = 1 - \mathcal{M}(\ell * (k \oplus h))(s),$$

where $\ell(x) = e^{-2x} \in L^1(\mathbb{R}_+, w)$. It follows that $\ell * (k \oplus h) \in L^1(\mathbb{R}_+, w)$.

Suppose that $B(s) \neq 0$ for any $s \in \{a\} + i\mathbb{R}$ and assume that $\mathcal{M}h \in L^1(\{a\} + i\mathbb{R})$. By the Riemann-Lebesgue lemma for Mellin transform (cf. [2, Theorem 2]), for any a which lies in the strip of analicity of Mellin transform of $\ell * (k \oplus h)$,

$$\lim_{|b|\to\infty} \mathcal{M}(\ell * (k \oplus h))(a + ib) = 0.$$

Consequently,

$$\lim_{|b|\to\infty} B(a + ib) = 1.$$

Thus, since B is analytic, there is an $R > 0$ such that

$$\inf_{|B|>R} |B(s)| > \delta_1, \quad \delta_1 > 0.$$

Since B is continuous in $\{a\} + i\mathbb{R}$ and does not vanishes in the compact set $S(0, R) = \{a + ib \in \mathbb{C} : |b| \leq R\}$, there exists $\delta_2 > 0$ such that

$$\inf_{|B|\leq R} |B(s)| \leq \delta_2.$$

Then, we have that

$$\sup_{s\in\{a\}\times i\mathbb{R}} \left(\frac{1}{B(s)}\right) \leq \max\left\{\frac{1}{\delta_1}, \frac{1}{\delta_2}\right\} < \infty.$$

It follows that $\frac{1}{B(s)}$ is bounded in $\{a\} + i\mathbb{R}$. Since $\mathcal{M}h \in L^1(\{a\} + i\mathbb{R})$, we have that $\frac{\mathcal{M}h}{B} \in L^1(\{a\} + i\mathbb{R})$.

Reciprocally, assume that $\frac{\mathcal{M}h}{B} \in L^1(\{a\} + i\mathbb{R})$ and the function $\frac{1}{B(s)}$ is bounded in $\{a\} + i\mathbb{R}$. We can deduce that $\mathcal{M}h \in L^1(\{a\} + i\mathbb{R})$ and the proposition is proved.

□

Theorem 7. *Suppose that the following conditions are verified:*

(i) $1 - \lambda_1\lambda_2\varphi(s)\psi(s)\mathcal{M}(k * h)(s) \neq 0,$

(ii) $\mathcal{M}(p-\lambda_1(h\oplus q))(s)\in L^1(\{a\}\times i\mathbb{R})$,

(iii) $\mathcal{M}(q-\lambda_2(k\odot p))(s)\in L^1(\{a\}\times i\mathbb{R})$,

where $\psi(s)=2^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right)$ *and* $\varphi(s)=2^{-s}\Gamma(s)$ *for* $s\in\{a\}\times i\mathbb{R}$.

Then, the system (4.1) *has the solution given in the form*

$$\begin{aligned} f(x) &= \mathcal{M}^{-1}\left(\frac{\mathcal{M}(p-\lambda_1(h\oplus q))(s)}{1-\lambda_1\lambda_2\varphi(s)\psi(s)\mathcal{M}(k*h)(s)}\right)(x),\\ g(x) &= \mathcal{M}^{-1}\left(\frac{\mathcal{M}(q-\lambda_2(k\odot p))(s)}{1-\lambda_1\lambda_2\psi(s)\varphi(s)\mathcal{M}(k*h)(s)}\right)(x),\ x\in\mathbb{R}_+. \end{aligned}$$

Proof. Let $p,q\in L^1(\mathbb{R}_+,w)$. Thus, $f(x)+\lambda_1(h\oplus g)(x)\in L^1(\mathbb{R}_+,w)$ and $g(x)+\lambda_2(k\odot f)(x)\in L^1(\mathbb{R}_+,w)$. Thus, we can apply the Mellin transform to both sides of the two equations and use the factorization properties (2.2) and (2.4). We obtain

$$\begin{cases} (\mathcal{M}f)(s)+\lambda_1\psi(s)(\mathcal{M}h)(s)(\mathcal{M}g)(s)=(\mathcal{M}p)(s)\\ (\mathcal{M}g)(s)+\lambda_2\varphi(s)(\mathcal{M}k)(s)(\mathcal{M}f)(s)=(\mathcal{M}q)(s) \end{cases}, \tag{4.3}$$

where $\psi(s)=2^{-s}\Gamma(s)$ and $\varphi(s)=2^{-\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right)$.

We have that the determinant

$$D=\begin{vmatrix} 1 & \lambda_1\psi(s)(\mathcal{M}h)(s)\\ \lambda_2\varphi(s)(\mathcal{M}k)(s) & 1 \end{vmatrix}=1-\lambda_1\lambda_2\varphi(s)\psi(s)(\mathcal{M}k)(s)(\mathcal{M}h)(s)\neq 0,$$

for all $s\in\{a\}\times i\mathbb{R}$. Thus, we conclude that there exists a unique solution of the linear system (4.3). Moreover, we have that

$$\begin{aligned} (\mathcal{M}f)(s) &= \frac{(\mathcal{M}p)(s)-\lambda_1\psi(s)(\mathcal{M}h)(s)(\mathcal{M}q)(s)}{1-\lambda_1\lambda_2\psi(s)\varphi(s)(\mathcal{M}k)(s)(\mathcal{M}h)(s)}\\ &= \frac{\mathcal{M}(p-\lambda_1(h\oplus q))(s)}{1-\lambda_1\lambda_2\varphi(s)\psi(s)\mathcal{M}(k*h)(s)} \end{aligned}$$

and

$$\begin{aligned} (\mathcal{M}g)(s) &= \frac{(\mathcal{M}q)(s)-\lambda_2\varphi(s)(\mathcal{M}k)(s)(\mathcal{M}p)(s)}{1-\lambda_1\lambda_2\psi(s)\varphi(s)(\mathcal{M}k)(s)(\mathcal{M}h)(s)}\\ &= \frac{\mathcal{M}(q-\lambda_2(k\odot p))(s)}{1-\lambda_1\lambda_2\psi(s)\varphi(s)\mathcal{M}(k*h)(s)}, \end{aligned}$$

each one belonging to $L^1(\{a\}\times i\mathbb{R})$ from conditions (ii) and (iii) and Proposition 6. Therefore, we can apply the inverse of the Mellin transform and obtain

$$\begin{aligned} f(x) &= \mathcal{M}^{-1}\left(\frac{\mathcal{M}(p-\lambda_1(h\oplus q))(s)}{1-\lambda_1\lambda_2\varphi(s)\psi(s)\mathcal{M}(k*h)(s)}\right)(x),\\ g(x) &= \mathcal{M}^{-1}\left(\frac{\mathcal{M}(q-\lambda_2(k\odot p))(s)}{1-\lambda_1\lambda_2\psi(s)\varphi(s)\mathcal{M}(k*h)(s)}\right)(x),\ x\in\mathbb{R}_+, \end{aligned}$$

and f and $g\in L^1(\mathbb{R}_+,w)$. □

5 Examples

In this section we will exemplify the above achievements with specific examples.

5.1 Example 1

Let us consider the integral equation

$$3if(x) + \int_0^\infty \int_0^\infty e^{-2\frac{x^2}{v^2}} f\left(\frac{v}{u}\right) g(u) \frac{dv}{v}\frac{du}{u} = h(x) \tag{5.1}$$

where $h(x) = e^{-\frac{x^2}{4}}$, $g(x) = 2e^{-2x^2} \in L^1(\mathbb{R}_+, w)$ and $f(x)$ is the unknown function. We have that $h, g \in L^1(\mathbb{R}_+)$.

We have that

$$A(s) = 3i + 2^{-s-1}\Gamma^2\left(\frac{s}{2}\right) \neq 0$$

for all $s \in \{1\} + i\mathbb{R}$ (cf. fig 1).

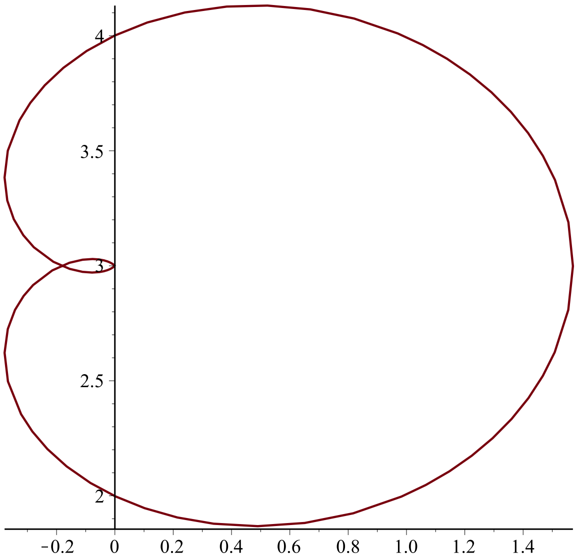

Figure 1: The graph of $A(s)$ for $s \in \{1\} + i\mathbb{R}$.

Moreover, $\lambda = 3i \neq 0$ and $(\mathcal{M}h)(s) = \frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right) \in L^1(\{1\} + i\mathbb{R})$. In fact,

$$\int_0^\infty \left| \frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}(1+ix)} \Gamma\left(\frac{1}{2}(1+ix)\right) \right| dx < 3.$$

Thus, applying the inverse of Mellin transform, we obtain the solution

$$f(x) = \mathcal{M}^{-1}\left(\frac{\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)}{3i + 2^{-s-1}\Gamma^2\left(\frac{s}{2}\right)}\right).$$

5.2 Example 2

Let us now consider the system of integral equations (4.1) with $\lambda_1 = \lambda_2 = 1$, $h(x) = e^{-x}$, $k(x) = e^{-x^2}$, $p(x) = 2e^{-x^4}$ and $q(x) = 0$, i.e.,

$$\begin{cases} f(x) + \int_0^\infty \int_0^\infty e^{-2\frac{x^2}{v^2} - \frac{v^2}{u^2}} g(u) \frac{dv}{v}\frac{du}{u} = 2e^{-x^4} \\ g(x) + \int_0^\infty \int_0^\infty e^{-2\frac{x}{v} - \frac{v}{u}} f(u) \frac{dv}{v}\frac{du}{u} = 0 \end{cases},$$

where f and g are the unknown functions.

We first observe that $B(s) = 1 - \varphi(s)\psi(s)\mathcal{M}(k * h)(s) = 1 - 2^{-\frac{3}{2}s-2}\Gamma^2\left(\frac{s}{2}\right)\Gamma^2(s) \neq 0$ for $s \in \{2\} + i\mathbb{R}$ (cf. fig 2).

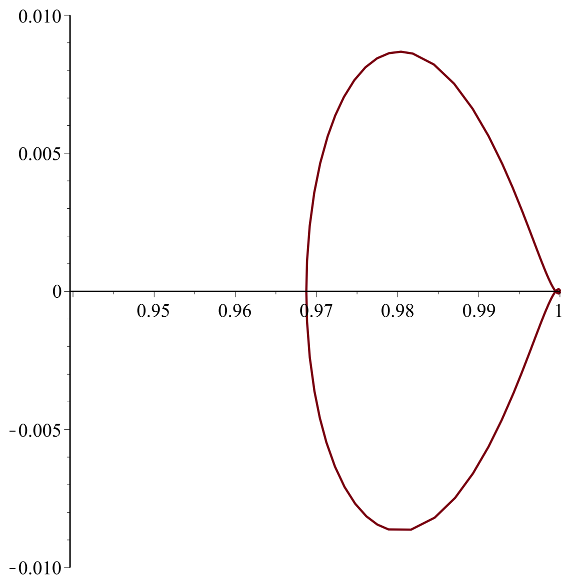

Figure 2: The graph of $1 - 2^{-\frac{3}{2}s-2}\Gamma^2\left(\frac{s}{2}\right)\Gamma^2(s) \neq 0$ for $s \in \{2\} + i\mathbb{R}$.

Moreover, we have that

$$\begin{aligned}\mathcal{M}(p - \lambda_1(h \oplus q))(s) &= \frac{1}{2}\Gamma\left(\frac{s}{4}\right)\\ \mathcal{M}(q - \lambda_2(k \odot p))(s) &= -2^{-s-2}\Gamma(s)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{4}\right).\end{aligned}$$

Since

$$\int_0^\infty \left|\frac{1}{2}\Gamma\left(\frac{2+ix}{4}\right)\right| dx < 3$$

and

$$\int_0^\infty \left|-2^{-2-ix-2}\Gamma(s)\Gamma\left(\frac{2+ix}{2}\right)\Gamma\left(\frac{2+ix}{4}\right)\right| dx < 0,2,$$

we conclude that

$$\frac{1}{2}\Gamma\left(\frac{2+ix}{4}\right) \in L^1(\{2\} + i\mathbb{R})$$

and

$$-2^{-2-ix-2}\Gamma(s)\Gamma\left(\frac{2+ix}{2}\right)\Gamma\left(\frac{2+ix}{4}\right) \in L^1(\{2\} + i\mathbb{R}).$$

Thus, conditions (i)–(iii) of Theorem 7 are satisfied. Therefore, we can conclude that

$$\begin{aligned}f(x) &= \mathcal{M}^{-1}\left(\frac{\frac{1}{2}\Gamma\left(\frac{s}{4}\right)}{1 - 2^{-\frac{3}{2}s-2}\Gamma^2\left(\frac{s}{2}\right)\Gamma^2(s)}\right)(x),\\ g(x) &= \mathcal{M}^{-1}\left(\frac{-2^{-s-2}\Gamma(s)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{4}\right)}{1 - 2^{-\frac{3}{2}s-2}\Gamma^2\left(\frac{s}{2}\right)\Gamma^2(s)}\right)(x),\ x \in \mathbb{R}_+.\end{aligned}$$

Acknowledgments

This work was supported by Fundação para a Ciência e a Tecnologia (FCT), within project UIDB/04106/2020 (CIDMA) and by national funds (OE), through FCT, I.P., in the scope of the framework contract foreseen in the numbers 4, 5 and 6 of the article 23, of the Decree-Law 57/2016, of August 29, changed by Law 57/2017, of July 19.

References

[1] G. Bogveradze, L.P. Castro, Toeplitz plus Hankel operators with infinite index, *Integral Equations Operator Theory* 62 (2008), no. 1, 43–63.

[2] P.L. Butzer, S. Jansche, A direct approach to the Mellin transform, *J. Fourier Anal.* 3 (1997), 325–376.

[3] L.P. Castro, R. Duduchava, F.-O. Speck, Finite interval convolution operators with transmission property, *Integral Equations Operator Theory* 52 (2005), no. 2, 165–179.

[4] L.P. Castro, R.C. Guerra, N.M. Tuan, On Wiener's Tauberian theorems and convolution for oscillatory integral operators, *Turkish J. Math.* 43 (2019), no. 3, 1124–1147.

[5] L.P. Castro, R.C. Guerra, N.M. Tuan, Convolution theorems related with the solvability of Wiener-Hopf plus Hankel integral equations and Shannon's sampling formula, *Math. Slovaca* 69 (2019), no. 5, 1149–1164.

[6] L.P. Castro, R.C. Guerra, N.M. Tuan, New convolutions weighted by Hermite functions and their applications, *Math. Inequal. Appl.* 22 (2019), no. 2, 719–745.

[7] L.P. Castro, L.T. Minh, N.M. Tuan, New convolutions for quadratic-phase Fourier integral operators and their applications, *Mediterr. J. Math.* 15 (2018), no. 13, 17pp. https://doi.org/10.1007/s00009-017-1063-y

[8] L.P. Castro, A.S. Silva, Invertibility of matrix Wiener-Hopf plus Hankel operators with symbols producing a positive numerical range, *Z. Anal. Anwend.* 28 (2009), no. 1, 119–127.

[9] L.P. Castro, F.-O. Speck, F.S. Teixeira, A direct approach to convolution type operators with symmetry. *Math. Nachr.* 269/270 (2004), 73–85.

[10] L. Debnath, D. Bhatta, *Integral Transforms and Their Applications*, 2nd ed. Chapman and Hall/CRC, New York, 2006.

[11] S.M. Khairnar, R.M. Pise, J.N. Salunkhe, Study of the mellin integral transform with applications in statistics and probability, *Arch. Appl. Sci. Res.* 4 (2012), no. 3, 1294–1310.

[12] Y. Luchko, Integral transforms of the Mellin convolution type and their generating operators, *Integral Transforms Spec. Funct.* 19 (2008), 809–851.

[13] O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions.* Theory and Algorithmic Tables. Ellis Horwood Ltd., Chichester, 1983.

[14] W. Szpankowski, *Average Case Analysis of Algorithms on Sequences.* John Wiley & Sons, Inc., New York, 2001

[15] E.C. Titchmarsh, *An Introduction to the Theory of Fourier Integrals*. Clarendon Press, Oxford, 1937.