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Mapification of n-dimensional abstract polytopes and hypertopes

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Abstract

The n-dimensional abstract polytopes and hypertopes, particularly the regular ones, have gained great popularity over recent years. The main focus of research has been their symmetries and regularity. The planification of a polyhedron helps its spatial construction, yet it destroys symmetries. No "planification" of n-dimensional polytopes do exist, however it is possible to make a "mapification" of an n-dimensional polytope; in other words it is possible to construct a restrictedly-marked map representation of an abstract polytope on some surface that describes its combinatorial structures as well as all of its symmetries. There are infinitely many ways to do this, yet there is one that is more natural that describes reflections on the sides of $(n-1)$-simplices (flags or n-flags) with reflections on the sides of n-gons. The restrictedly-marked map representation of an abstract polytope is a cellular embedding of the flag graph of a polytope. We illustrate this construction with the 4-cube, a regular 4-polytope with automorphism group of size 384. This paper pays a tribute to Lynne James' last work on map representations.

Keywords: Maps, hypermaps, representation, polytopes, hypertopes.

Math. Subj. Class. (2020): 05C10, 05C25, 05C65, 05E18, 20F65

1 Introduction

This paper stands to be a tribute to Lynne James' last, and unfinished, work [9], where she outlines a method of representing topological categories, such as the categories of cell decompositions of n-manifolds, by other categories, for example the category of cell decompositions of oriented surfaces.

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A map is a cellular embedding of a graph (possibly with multiple edges, loops and/or free edges) on connected surfaces with or without boundary. Algebraically a map $\mathcal{M} = (\Omega; r_0, r_1, r_2)$ is a set Ω of triangular pieces of surface called *flags*, and 3 involutory permutations r_0, r_1, r_2 on Ω satisfying $(r_0 r_2)^2 = 1$ and generating a transitive group on Ω called the *monodromy group* of the map.

An abstract n-polytope $\mathcal{P}$ is a partially ordered set (poset) of faces with a strictly monotone rank function of range $\{-1, 0, \ldots, n\}$, represented by a Hasse diagram with $n + 1$ layers, where the poset obey the diamond condition and the set of flags are strongly flag-connected. *Flags* are maximal chains of faces, that is, vectors consisting of $n + 2$ faces of rank $-1, 0, 1, \ldots, n$ respectively. There is a unique least face, the (-1)-face F_{-1} and a unique greatest face the n-face F_n. Faces of rank $0, 1$ and $n - 1$ are called *vertices*, *edges* and *facets*, respectively. Two flags are *adjacent* if they differ only by one face (entry). Flags are *strongly flag-connected* means that any two flags Ψ_1, Ψ_2 are connected by a sequence of flags $\Phi_0 = \Psi_1, \Phi_1, \ldots, \Phi_m = \Psi_2$ such that two successive flags Φ_i, Φ_{i+1} are adjacent and for any i, j, $\Phi_i \cap \Phi_j \supseteq \Psi_1 \cap \Psi_2$. The *diamond condition* says that whenever F_{i-1} and F_{i+1} are faces of ranks $i-1$ and $i+1$ for some i, with $F_{i-1} < F_{i+1}$, then there are exactly two faces F_i of rank i containing F_{i-1} and contained in F_{i+1}, that is, $F_{i-1} < F_i < F_{i+1}$. In other words, the poset of the section $F_{i+1}/F_{i-1} = \{F \in \mathcal{P} \mid F_{i-1} \leq F \leq F_{i+1}\}$ is like a diamond.

An abstract 2-polytope is just a polygon while a 3-polytope is a non-degenerate map (cellular embedding of a loopless graph on some compact connected (i.e. closed) surface), with the property that every edge is incident with exactly two faces, and every vertex on a face is incident with two edges of that face.

A n-hypertope is an extension of an n-polytope by eliminating the partial order set condition [6]. All n-polytopes are finite in this paper and $n > 2$ everywhere. For a further reading on polytopes we address the reader to the classical book by McMullen and Schulte [13].

2 Restrictedly-marked maps

Lynne James in [9] introduced maps representations and associate it to a non-commutative multiplication operation between map type objects. Although restrictedly-marked map representations [4] lie in a different category, they represent the same topological objects with a different perspective and semantics.

Consider the "right triangle" group $\Gamma = \langle R_0, R_2 \rangle * \langle R_1 \rangle \cong (C_2 \times C_2) * C_2$ generated by the three reflections R_0, R_1, R_2 in the sides of a hyperbolic right triangle with two zero internal angles.

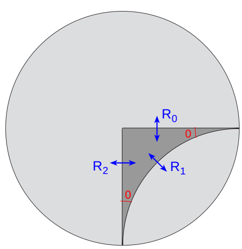


Figure 1: Hyperbolic right triangle on the Poincaré disc.

Every finite index subgroup $M < \Gamma$ determines a finite map $\mathcal{M} = (\Gamma/_r M; M^* R_0, M^* R_1, M^* R_2)$, where M^* is the core of M in Γ and each $M^* R_i$ acts as a permutation on the right cosets $\Gamma/_r M$ of M in Γ by right multiplication. M is called the *fundamental map subgroup* of $\mathcal{M}$ (or just "map subgroup"). Let Θ be a normal subgroup of Γ with finite index n. A map is Θ-*conservative* if M is a subgroup of Θ. In this case the flags of $\mathcal{M}$ are n coloured under the action of Θ, each colour determined by an orbit (the Θ-orbit) under the action of Θ. By the Kurosh Subgroup Theorem [11, Proposition 3.6, p. 120], Θ freely decomposes into a free product $C_2 * \cdots * C_2 * D_2 * \cdots * D_2 * C_\infty * \cdots * C_\infty = \langle Z_1, \ldots, Z_m \rangle$ for some finite number (possibly zero) of factors $C_2, D_2 = C_2 \times C_2$ and C_∞. This decomposition is unique up to a permutation of the factors [12, p. 245]. A Θ-conservative map can then be represented by a Θ-*marked map* $\mathcal{Q} = (\Omega; z_1, \ldots, z_m)$, where Ω is the set of right cosets $\Theta/_r M$ of M in Θ, and each $z_i = M_\Theta Z_i \in \Theta/M_\Theta$ (where M_Θ is the core of M in Θ). The geometric construction described in [2], which can be adapted to Γ [4], uses Θ-slices, polygonal regions determined by a Schreier transversal for Θ in Γ. Θ-slices represent the elements of Ω. For example, a Γ-slice is a "flag" and a Γ^+-slice is a "dart", where Γ^+ is the normal subgroup of index 2 in Γ consisting of the words of even length on R_0, R_1, R_2. The group generated by $z_1, \ldots, z_m$, called the *monodromy group* of $\mathcal{Q}$ (denoted $\mathrm{Mon}(\mathcal{Q})$), or the Θ-monodromy group of $\mathcal{M}$, acts transitively on the set of the Θ-slices Ω.

A covering, or morphism, ψ from a Θ-marked map $\mathcal{Q}_1 = (\Omega_1; z_1, \ldots, z_m)$ to another Θ-marked map $\mathcal{Q}_2 = (\Omega_2; z'_1, \ldots, z'_m)$ is a function $\psi\colon \Omega_1 \longrightarrow \Omega_2$ that commutes the diagram

$$\begin{tikzcd}
\Omega_1 \times \mathrm{Mon}(\mathcal{Q}_1) \arrow[r] \arrow[d, "\psi"', "\iota\colon z_i \mapsto z'_i"] & \Omega_1 \arrow[d, "\psi"] \\
\Omega_2 \times \mathrm{Mon}(\mathcal{Q}_2) \arrow[r] & \Omega_2
\end{tikzcd}$$

An automorphism of $\mathcal{Q}$ is just a bijective covering from $\mathcal{Q}$ to $\mathcal{Q}$. A Θ-marked map $\mathcal{Q}$ is *regular*, or the Γ-marked map $\mathcal{M}$ is Θ-regular, if M is a normal subgroup of Θ. In this case the automorphism group of $\mathcal{Q}$, which is the automorphism group of $\mathcal{M}$ preserving each Θ-orbit, coincides with the monodromy group $\mathrm{Mon}(\mathcal{Q})$, but with different action on Ω. For a more detailed exposition see [2]; though the focus here has been hypermaps the results are easily adapted to maps (see [4]).

By a *restrictedly-regular* (or resctrictly-regular) map we mean a map that is Θ-regular for some (finite index) normal subgroup $\Theta \triangleleft \Gamma$. In a similar way as done in [2], not all maps are restrictedly-regular. However, any group G is the monodromy group (and hence the automorphism group) of a restrictedly-regular map ([3, Lemma 2.2] easily adapted to Γ).

3 Algebraic representation of finite n-polytopes

A Coxeter group is a group with presentation $\langle s_0, s_1, \ldots, s_{n-1} \mid s_i^2 = (s_i s_j)^{p_{ij}} = 1 \rangle$ where $p_{ij} \geq 2$ is a positive integer possibly ∞. If $p_{ij} = \infty$ then the relation $(s_i s_j)^{p_{ij}}$ is vacuous and is not considered in the above presentation. Let $\mathcal{P}$ be an abstract n-polytope, and denote by $\Omega_\mathcal{P}$ the set of flags of $\mathcal{P}$. As an immediate consequence of the diamond condition, for any flag $\Phi \in \Omega_\mathcal{P}$ and for any $0 \leq i \leq n-1$, the set $\{\Phi' \in \Omega_\mathcal{P} \mid F_j(\Phi') = F_j(\Phi), \forall j \neq i\}$, where $F_j(\Phi)$ is the face of rank j of Φ, contains exactly two elements, being Φ one of them. Denote by $\Phi r_i = \Phi'$ the other flag of this set. Then we have n permutations $r_i = \prod_{\Phi\in\Omega_{\mathcal{P}}}(\Phi, \Phi r_i)$ for $i \in \{0, 1, \ldots, n-1\}$, giving rise to a transitive permutation group $G(\mathcal{P}) = \langle r_0, r_1, \ldots, r_{n-1}\rangle$ on $\Omega_{\mathcal{P}}$, called the *connection group* (or *monodromy group*) of $\mathcal{P}$, that describes the polytope $\mathcal{P}$: each rank i face F_i for $i \in \{0, 1, \ldots, n-1\}$, corresponds to an orbit of $\langle r_0, \ldots, \hat{r_i}, \ldots, r_{n-1}\rangle$ on $\Omega_{\mathcal{P}}$, where $\hat{r_i}$ denotes the absence of r_i. In fact, if F_i is a face of rank i and Φ and Ψ are two flags containing F_i, then by strong connectedness

$$\Phi\langle r_0, \ldots, \hat{r_i}, \ldots, r_{n-1}\rangle = \Psi\langle r_0, \ldots, \hat{r_i}, \ldots, r_{n-1}\rangle.$$

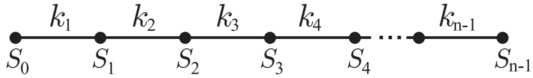
So $\Phi\langle r_0, \ldots, \hat{r_i}, \ldots, r_{n-1}\rangle$ is the set of all flags containing the common i-face F_i. An i-face $\Phi\langle r_0, \ldots, \hat{r_i}, \ldots, r_{n-1}\rangle$ is incident to a j-face $\Psi\langle r_0, \ldots, \hat{r_j}, \ldots, r_{n-1}\rangle$ $(i \neq j)$ if and only if $\Phi\langle r_0, \ldots, \hat{r_i}, \ldots, r_{n-1}\rangle \cap \Psi\langle r_0, \ldots, \hat{r_j}, \ldots, r_{n-1}\rangle \neq \emptyset$; so incidence corresponds to non-empty intersection.

Hence the polytope $\mathcal{P}$ can be identified with the $n+1$ tuple $(\Omega_{\mathcal{P}}; r_0, r_1, \ldots, r_{n-1})$. Two such $n+1$ tuples $(\Omega_1; r_0, r_1, \ldots, r_{n-1})$ and $(\Omega_2; s_0, s_1, \ldots, s_{n-1})$ are *isomorphic* if there is a bijection f from Ω_1 to Ω_2 that satisfy $\omega r_i f = \omega f s_i$ for every $\omega \in \Omega_1$ and $i \in \{0, 1, \ldots, n-1\}$.

Denote by Δ_{n-1} the Coxeter group $\langle S_0, S_1, \ldots, S_{n-1} \mid S_i^2 = 1\rangle$. Then we have a natural epimorphism $\pi\colon \Delta_{n-1} \to G(\mathcal{P})$, mapping each S_i to r_i, inducing an action $\Phi d := \Phi d\pi$ of Δ_{n-1} on $\Omega_{\mathcal{P}}$. Similarly to [2, §1.2], fixing a flag $\Phi \in \Omega_{\mathcal{P}}$ and letting P be the stabiliser of Φ in Δ_{n-1}, then Δ_{n-1} acts on $\Delta_{n-1}/_{r}P$ by right multiplication, inducing a bijective function $\pi_\Phi\colon \Delta_{n-1}/_{r}P \to \Omega_{\mathcal{P}}$, $Pd \mapsto \Phi d\pi$. The kernel of π is the core P^* of P in Δ_{n-1}[1] and the group Δ_{n-1}/P^* acts transitively on $\Delta_{n-1}/_{r}P$ by right multiplication in a similar way as $G(\mathcal{P})$ acts on $\Omega_{\mathcal{P}}$. Hence the polytope $(\Omega_{\mathcal{P}}; r_0, r_1, \ldots, r_{n-1})$ is isomorphic to $(\Delta_{n-1}/_{r}P; P^*S_0, P^*S_1, \ldots, P^*S_{n-1})$. Every polytope $\mathcal{P}$ is described by such $(n+1)$-tuples; the converse is false. The set of all such $(n+1)$-tuples will be called for the moment the set of $(n-1)$-*hypermaps* (see also Section 7). So both n-polytopes and n-hypertopes are $(n-1)$-hypermaps, the converse is false. The subgroup P will be called a *fundamental subgroup* of $\mathcal{P}$. This is unique up to a conjugacy in Δ_{n-1}.

A $(n-1)$-hypermap $\mathcal{H} = (\Omega_{\mathcal{P}}; r_0, r_1, \ldots, r_{n-1})$ is regular if the connection group acts regularly on Ω; this is equivalent to say that the fundamental subgroup P is normal in Δ_{n-1}. In such case $P^* = P$ and, up to an $(n-1)$-hypermap isomorphism, $\Omega = \Delta_{n-1}/_{r}P = G$ is the connection group, which coincides with the automorphism group of $\mathcal{H}$. The action of G on $\Omega = G$ as a connection group is done by right multiplication, while as automorphism group is done by left multiplication.

A string Coxeter group of type $[k_1, k_2, \ldots, k_{n-1}]$ $(k_1 > 2, k_2 > 2, \ldots, k_{n-1} > 2)$ is a Coxeter group $\langle S_0, S_1, \ldots, S_{n-1} \mid S_i^2 = 1\rangle$ satisfying the Dynking diagram (or string diagram) of type $[k_1, k_2, \ldots, k_{n-1}]$:



A regular polytope $\mathcal{P}$ is of type $[k_1, k_2, \ldots, k_{n-1}]$ if its automorphism group (or connection group) is a smooth quotient of a string Coxeter group Δ_{n-1} of type $[k_1, k_2, \ldots, k_{n-1}]$.

For additional details on polytopes and related subjects we address the reader to [14].

[1]Let $\alpha\colon G(\mathcal{P}) \to S_\Omega$ be the action homomorphism of $G(\mathcal{P})$ on Ω, then $\pi\alpha\colon \Delta_{n-1} \to S_\Omega$ (right action notation) is the action homomorphism of Δ_{n-1} on Ω. As $G(\mathcal{P})$ acts faithfully on Ω, then $P^* = \mathrm{Ker}(\pi\alpha) = \pi^{-1}(\mathrm{Ker}(\alpha)) = \pi^{-1}(1) = \mathrm{Ker}(\pi)$.

4 Topological approach to n-polytopes and n-hypertopes

Polytopes appear in the literature both as abstract and geometric. Hypertopes have been essentially introduced as abstract constructions. There is a topological construction that is relevant for what follows later in Section 8. From the example below (Section 6) one see that a flag of the hypercube can be associated to a tetrahedron, that is, a 3-simplex. Replacing faces in the poset of a polytope (hypertope) by simplices and the rank function by the dimension function, under the same conditions, we get an abstract simplicial complex model of a polytope (hypertope).

Towards a more topological approach, let a n-*flag* be an $(n-1)$-simplex with its n vertices labelled $0, 1, \ldots, n-1$ and its n facets labelled by the opposite vertex label. Let also Ω be a set of n-flags.

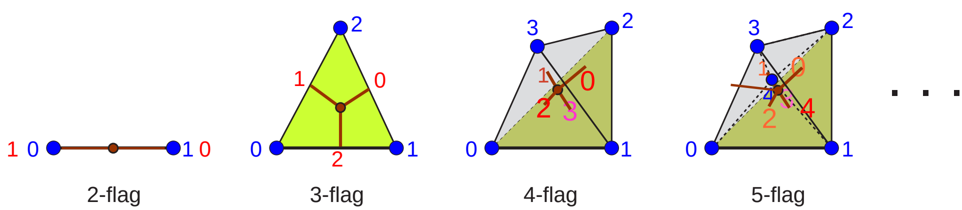


Figure 2: Example of n-flags.

For each $i \in \{0, 1, \ldots, n-1\}$ we denote by the transposition $\tau_i = (a, b)$ the joining of two n-flags $a, b \in \Omega$ along their facet labelled i so that its facet's vertex numbers match up, and call it an i-transposition. Denote by r_i the product of i-transpositions recording those pairs of n-flags that are joined by their i-labelled facets; $r_i = 1$ just means that no pairs of n-flags are joined by their facets labelled i. The group $G = \langle r_0, r_1, \ldots, r_{n-1} \rangle$ records all the existing joining between the n-flags. We call it a connection group (or monodromy group). If this group acts transitively on Ω then $(\Omega; r_0, r_1, \ldots, r_{n-1})$ describes a topological/algebraic object isomorphic[2] to an $(n-1)$-hypermap. Call it n-*hyperplex*[3]. Since n-flags are only connected by their facets, no k-face, for $k < n-1$, can occur as the intersection of two consecutive n-flags. Moreover, if an n-flag is fixed by some r_i this means that the facet labelled i of the flag is on the boundary. Thus boundary cannot be made of k-faces for $k < n-1$. Thus if two n-flags have in common a k-face ($k < n-1$) then necessarily there must be a sequence of n-flags that intersect two by two on a facet containing the k-face. Hence by construction, if this connection group acts transitively on Ω it does so strongly transitively.

This bring polytopes (hypertopes) close to Piecewise Linear Manifolds (PL-manifolds).

5 Regular representation of n-polytopes by restrictedly-marked maps

Following Lynne's ideas [9], and more explicitly the notations and definitions expressed in [4], a regular representation of $(n-1)$-hypermaps by restrictedly-marked maps is a $(m+1)$-tuple $(\Theta; X_0, X_1, \ldots, X_{m-1})$, consisting of a normal subgroup Θ of Γ freely generated by $X_0, X_1, \ldots, X_{m-1}$ for some $m \geq n$, together with an epimorphism ρ from Θ to Δ_{n-1}.

[2] Isomorphisms taken in the same sense as isomorphisms between Θ-marked maps defined in Section 3.

[3] Terminology introduced by Steve Wilson in BIRS Workshop 17w5162, Canada, 2017.

Such representation gives rise to a bijection between the set of $(n-1)$-hypermaps $\mathcal{P}$ with fundamental subgroup H to the set of regular Θ-marked maps with fundamental subgroup $H\rho^{-1}$, henceforth a representation of n-polytopes (or n-hypertopes).

$$\begin{tikzcd}
 & \Gamma = C_2 * D_2 = \langle R_1\rangle * \langle R_0, R_2\rangle \arrow[d, equal] & & \\
Q & \Theta \arrow[r, two heads, "\rho"] \arrow[d, dash] & \Delta_{n-1} = \langle S_0, S_1, \dots, S_{n-1}\rangle \arrow[d, dash] & P \\
\Theta\text{-marked map} & H\rho^{-1} \arrow[r, two heads] & H &
\end{tikzcd}$$

Theorem 5.1. *There is a regular restrictedly-marked representation of n-polytopes such that:*

(1) n-flags ($(n-1)$-simplices for n-polytopes and -hypertopes) correspond to n-gons;

(2) local reflections about facets of an n-flag corresponds to local reflections on the sides of the n-gon;

(3) the (full) automorphism group of the n-polytope (-hypertope) is the (full) automorphism group of the restrictedly marked map;

(4) the n-polytope (-hypertope) is orientable if and only if the restrictedly marked map is orientable.

Proof. Lynne James's first example [9], more specifically the example given by the alternative construction, gives an answer to this question for $n = 4$. The proof could be resumed to find a normal subgroup Θ of Γ which is freely generated by reflections. However there are only four subgroups that are freely generated by, and only by, reflections, namely

$$\begin{aligned}
\Gamma_{2.1} &= \langle R_0, R_1, R_2R_1R_2\rangle = C_2 * C_2 * C_2,\\
\Gamma_{2.4} &= \langle R_1, R_2, R_0R_1R_0\rangle = C_2 * C_2 * C_2,\\
\Gamma_{2.5} &= \langle R_1, R_2R_0, R_0R_1R_0\rangle = C_2 * C_2 * C_2, \text{ and}\\
\Gamma_{4.2} &= \langle R_1, R_0R_1R_0, R_2R_1R_2, R_0R_2R_1R_2R_0\rangle = C_2 * C_2 * C_2 * C_2.
\end{aligned}$$

These solve the problem for $n = 3$ and 4. The following approach gives a general construction for all $n \geq 3$.

Denote by $\prod_k(R_i, R_j)$ the product $R_iR_jR_iR_jR_i\dots$ of R_i and R_j in alternate form, starting from R_i and counting k total factors. If $k = 0$ then put $\prod_0(R_i, R_j) = 1$. Now take the normal subgroup[4]

$$\Gamma_n = \langle R_0, R_0^{R_1}, R_0^{R_1R_2}, \dots, R_0^{\prod_{n-1}(R_1,R_2)}, (R_1R_2)^n\rangle$$

of rank $n+1$ and index $2n$ in Γ (Γ/Γ_n is a dihedral group of order $2n$). By the Kurosh's Subgroup Theorem [11, Proposition 3.6], these generators decompose Γ_n as a free product

[4]There is another subgroup generated by reflections and one rotation with the same decomposition as a free product $C_2 * C_2 * C_2 * \cdots * C_\infty$, it is the dual resulting from swapping R_0 with R_2. Another subgroup also appears with such free product decomposition $C_2 * C_2 * C_2 * \cdots * C_\infty$, however one of the C_2 is generated by the rotation R_0R_2, instead of a reflection.

$C_2 * C_2 * \cdots * C_2 * C_\infty$. We take the epimorphism $\rho\colon \Gamma_n \to \Delta_{n-1}$ by mapping each $R_0^{\prod_k(R_1,R_2)}$ to S_k, for $k = 0, 1, \ldots, n-1$, and $(R_1R_2)^n$ to 1. Then the regular map with dihedral automorphism group of size $2n$ corresponding to the quotient Γ/Γ_n, called *trivial* Γ_n*-map*, is a star graph cellular embedded in the disk, thus a boundary map with one vertex and n edges. We need to cut open this disk to its centre to create a Γ_n-slice (see [4] for the constructing example of such a Γ_n-slice) for the restricted Γ_n-marked map, however we need to join it back to accomplish $(R_1R_2)^n = 1$, satisfied by the epimorphism ρ, to create a Γ_n-slice for this representation ρ. Each $(n-1)$-hypermap $\mathcal{P}$, and hence each n-polytope (and each n-hypertope), corresponding to a fundamental subgroup P, is isomorphic to a Γ_n-marked map $\mathcal{Q}$ with fundamental subgroup the inverse image $Q = P\rho^{-1}$. The rooted Γ_n-slice for $\mathcal{Q}$ is the above n-gon with a distinguished flag (in black) as shown in Figure 3.

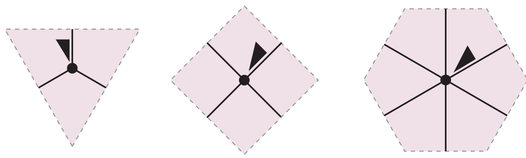


Figure 3: Rooted Γ_3-slice, Γ_4-slice and Γ_6-slice.

The monodromy group (which corresponds to the connection group of the $(n-1)$-hypermap, n-polytope or n-hypertope) is generated by the reflections on the sides of this n-gon. The isomorphism $\bar{\rho}$ between the restricted Γ_n-marked map $\mathcal{Q}$ and $\mathcal{P}$ establishes the third statement. A $(n-1)$-hypermap is orientable (and so is an n-polytope and an n-hypertope) if and only if any word on $r_0, r_1, \ldots, r_{n-1}$ that turns to be the identity has even length, that is, it can be expressed as a word on the rotations $r_1r_2, r_2r_3, \ldots, r_{n-2}r_{n-1}$. Given that the isomorphism $\bar{\rho}$ sends each odd length word $R_0^{\prod_i(R_1,R_2)}$ to r_i, that is also true in the restrictedly marked map representation, that is, the representation word will also be a word of even length on $R_0, R_1, \ldots, R_n$. This establishes the last statement. □

As it turns out from the rooted Γ_n-slice, a restrictedly-marked map representation of an abstract polytope is a cellular embedding of the flag graph of the polytope on a connected surface without boundary. In [4] we dealt only with *clean* restrictedly-marked representations (of hypermaps), that is, regular restricted Θ-marked map representations of n-ranked objects where the generators of Θ give rise to free product decompositions of Θ of equal rank n; this translates, for instance, to 3 generators for representations of hypermaps (rank 3 polytopes). As $(n-1)$-hypermaps (n-polytopes, n-hypertopes) have rank n and Γ_n has rank $n+1$, the above restricted Γ_n-marked map representations are not clean. As a consequence of this fact we have,

Proposition 5.2. *There are infinitely many regular restricted* Γ_n*-marked map representations of* $(n-1)$*-hypermaps. Thus in general, there are infinitely many regular restrictedly-marked representations of* $(n-1)$*-hypermaps, and so of* n*-polytopes and* n*-hypertopes.*

Proof. In order to get distinct epimorphisms $\rho\colon \Gamma_n \to \Delta_{n-1} = \langle S_0, \ldots, S_{n-1}\rangle = C_2 * C_2 * \cdots * C_2$ we just assign $R_0^{\prod_k(R_1,R_2)}$ to S_k as before, for $k = 0, 1, \ldots, n-1$, and $(R_1R_2)^n$ to different, and non conjugate, elements in Δ_{n-1}. In this way we get infinitely many distinct epimorphism and hence infinitely many regular restricted Γ_n-marked map representations of $(n-1)$-hypermaps. □

6 Example: The hypercube

As an illustration we take the hypercube, an orientable and regular 4-polytope with 384 flags. The rooted Γ_4-slice of the restricted Γ_4-marked map representation is illustrated in the picture above (Figure 3). To construct the regular restricted Γ_4-map $\mathcal{Q}$ that represents the hypercube, we need to join the 384 rooted Γ_4-slices through their four sides according to the rule dictated by the side reflections $r_0 = R_0, r_1 = R_0^{R_1}, r_2 = R_0^{R_1 R_2}$ and $r_3 = R_0^{R_1 R_2 R_1}$.

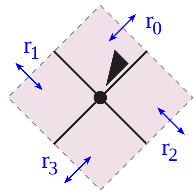


Figure 4: Identification sides on the rooted Γ_4-slice.

The automorphism group G of the hypercube is a Coxeter group of type $[4, 3, 3]$ with presentation

$$\langle r_0, r_1, r_2, r_3 \mid r_0^2, r_1^2, r_2^2, r_3^2, (r_0 r_2)^2, (r_0 r_3)^2, (r_1 r_3)^2, (r_0 r_1)^4, (r_1 r_2)^3, (r_2 r_3)^3 \rangle.$$

Since it is regular, its connection group coincide with its automorphism group (only its action on the flags is different), and since the automorphism group acts regularly on the set of flags its size is the number of flags. So we may replace the set of flags by the automorphism group, in which case the action of the automorphism group on the flags is done by left multiplication while the action of the connection group is done by right multiplication. For the constructing we use the group as a connection group and automatically label its elements $1, 2, 3, \ldots,$ being the identity element the element labelled 1, followed by the elements $2 = r_0 = R_0, 3 = r_1 = R_0^{R_1}, 4 = r_0 r_1, 5 = r_0 r_1 r_0$ etc, so that the first 8 label all the elements of the dihedral subgroup $\langle r_0, r_1 \rangle$ (the central 8-gon). As the hypercube is symmetric around the central 8-gon, we only need to construct one sector, being the rest of the 7 sectors obtained by reflections and rotations about this central 8-gon. So we only need to figure out how to arrange the 48 Γ_4-slices and the final labelling of the outside border of this sector. This is done (with the help of GAP [15]) in the figure below (Figure 5). GAP was used twice:

(i) to ensure that each rooted Γ_4-slice placed inside the sector does not appear when reflecting or rotating around the central 8-gon,

(ii) to get the side-pairings between the labelled sides of the sector with the sides of the rest of the picture.

Bold numbers and letters label the sides of this sector; the red labels signalize identifications inside the same sector, while the black ones label indentifications outside this sector.

Now copy reflecting this sector about the central 8-gon we get the final picture of the hypercube (Figure 6) which reflects a Γ_4-restrictedly regular map on an orientable surface of genus 41. Not all the sides were labelled. To complete the labelling we use the reflections and rotations about the central polygonal region. For example, the central bottom

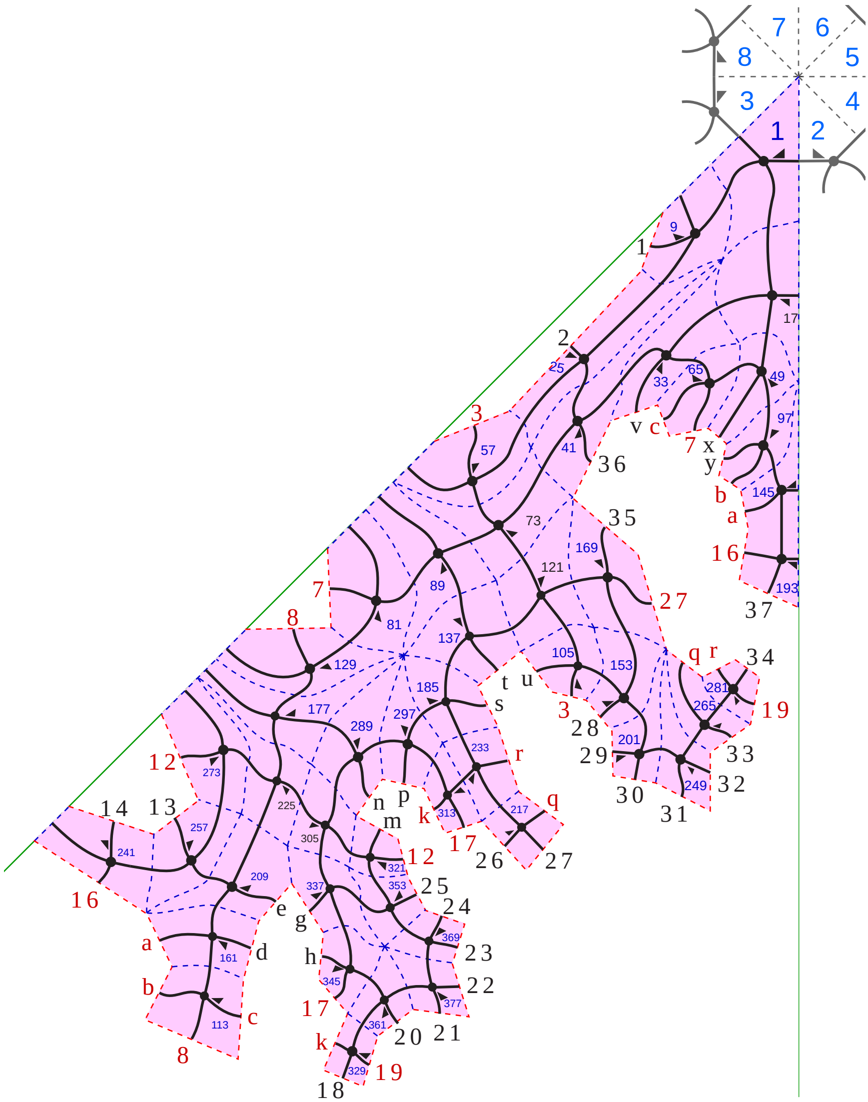


Figure 5: The first sector of the hypercube.

side labelled 37 has its right side unlabelled; label it x for a while. This x vertically mirror reflects to 37, so the x side should be identified to the side y that is the vertical mirror reflection of the identification pair of 37. There is also no arrows to instruct the identification side pairing; this is unnecessary as well since the identification is done similarly to the matching of the internal sides, which was done by following the words R_0, $R_0^{R_1}$, $R_0^{R_1R_2}$ and $R_0^{R_1R_2R_1}$ corresponding to the sides (Figure 4); any of these words will take a rooted Γ_4-slice to a neighbouring rooted Γ_4-slice.

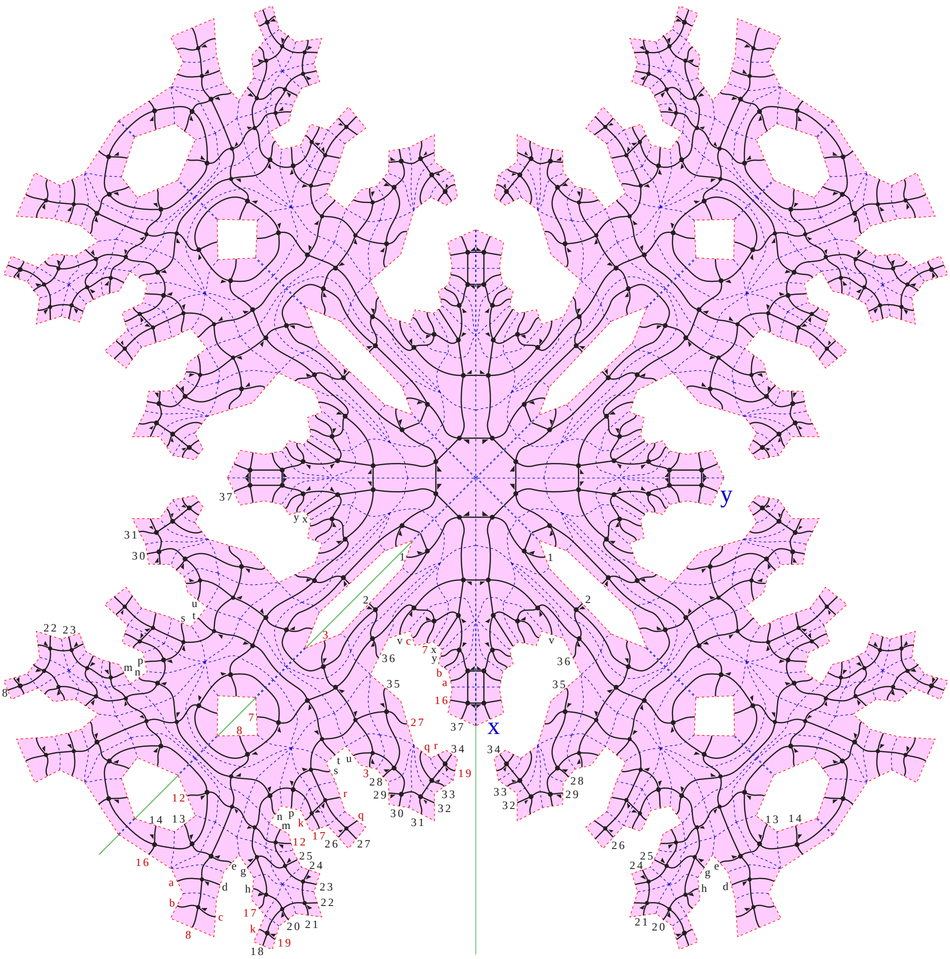


Figure 6: The hypercube.

7 Genus of a regular orientable n-polytope and n-hypertope

The genus $\mathfrak{g}$ of an orientable n-polytope (resp. orientable n-hypertope) can be defined to be the genus of the orientable $(n-1)$-hypermap it corresponds to, which is the genus of the regular Γ_n-marked map representation $\mathcal{Q}$ without boundary. Recall that $\mathfrak{g} = \frac{2-\chi}{z}$, where $z = 2$ if $\mathcal{Q}$ is orientable and 1 otherwise. The formula of the characteristic χ (i.e. the Euler characteristic of the underlying surface of the regular Γ_n-marked map representation) presented below is derived from the characteristic formula in [2] taking into account that the trivial Γ_n-map is a boundary map with edges and faces on the boundary. A direct calculation can go as follows: we see from a rooted Γ_n-slice (Figure 7) that the embedding of the n-coloured graph (flag graph in the case of polytopes and hypertopes) produces n type of faces $f_1, f_2, \dots, f_n$ determined by

$$
\begin{aligned}
\rho_1 &= r_0 r_1 = (R_0R_1)^2,\\
\rho_2 &= r_0 r_2 = (R_0R_1)^{R_2},\\
\rho_3 &= r_1 r_3 = ((R_0R_1)^2)^{R_2R_1},\\
&\ \vdots\\
\rho_{n-1} &= r_{n-3}r_{n-1} = ((R_0R_1)^2)^{\prod_{n-2}(R_2,R_1)}, \text{ and}\\
\rho_n &= r_{n-2}r_{n-1} = ((R_0R_1)^2)^{\prod_{n-1}(R_2,R_1)}.
\end{aligned}
$$

Then $F_i = \frac{|G|}{2m_i}$ is the number of faces of type i, where $m_i = |\rho_i|$ and $G = \langle r_0, \dots, r_{n-1}\rangle$ is the set of n-flags. It also produces n types of edges, one for each r_i, being the number E_i of edges of type i given by $E_i = \frac{|G|}{2}$.

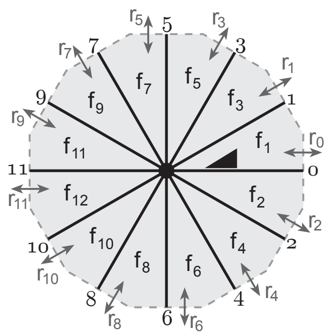


Figure 7: Γ_{12}-slice showing the i-labelled edges and the i-labelled faces.

Finally, as the number V of vertices is $|G|$, the number of edges is $E = E_0 + E_1 + \cdots + E_{n-1}$ and the number of faces is $F = F_1 + F_2 + \cdots + F_n$, then the characteristic $\chi = V - E + F$ of a regular $(n-1)$-hypermap (or a regular n-hypertope) $\mathcal{H} = (G; r_0, r_1, \dots, r_{n-1})$ is given by

$$\chi = \frac{|G|}{2}\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_{n-1}} + \frac{1}{m_n} + 2 - n\right).$$

In regular $(n-1)$-hypermaps we may have $m_i = 1$, so writing $N = -\chi$, we have

$$|G| = \frac{2N}{n-2-(\frac{1}{m_1}+\frac{1}{m_2}+\cdots+\frac{1}{m_n})} \leq \frac{2N}{n-2-(\frac{1}{1}+\cdots+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{7})} = 84N,$$

the usual Hurwitz bound.

However, for regular n-polytopes, of type $[m_1, k_2, k_3, \dots, k_{n-1}, m_n]$, we have $m_2 = m_3 = \dots = m_{n-1} = 2$, which gives the formula

$$\chi = \frac{|G|}{2}\left(\frac{1}{m_1} + \frac{1}{m_n} + \frac{2-n}{2}\right).$$

We have also $m_1 \geq 3$ and $m_n \geq 3$, so that

$$|G| = \frac{2N}{\frac{n-2}{2} - (\frac{1}{m_1} + \frac{1}{m_n})} \leq \frac{12N}{3n-10}.$$

In particular if $n > 3$, then

$$|G| < \frac{4N}{n-4} \quad \text{and} \quad N \geq \frac{|G|}{12}(3n-10) > \frac{|G|}{4}(n-4).$$

For $n > 8$ the minimum size of a regular polytope is $|G| = 2.4^{n-1}$ (Conder [5]), which gives $N > 2.4^{n-2}(n-4)$. A better refinement for n in $\{3, 4, 5, 6, 7, 8\}$ can be made by taking the Propositions 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6 of [5] into account:

Table 1: Leasted values for $|G|$ and negative characteristic N for regular n-polytopes.

n	$\min \lvert G\rvert$ (from [5])	$\min N = -\chi$	inf
3	$\lvert G\rvert \geq 24$	$N \geq -2$	from $\lvert G\rvert = 24$, type $[3,3]$
4	$\lvert G\rvert \geq 96$	$N \geq 18$	from $\lvert G\rvert = 108$, type $[3,6,3]$
5	$\lvert G\rvert \geq 432$	$N \geq 198$	from $\lvert G\rvert = 432$, type $[3,6,3,4]$
6	$\lvert G\rvert \geq 1728$	$N \geq 1296$	from $\lvert G\rvert = 1728$, type $[4,3,6,3,4]$
7	$\lvert G\rvert \geq 7776$	$N \geq 7452$	from $\lvert G\rvert = 7776$, type $[3,6,3,6,3,4]$
8	$\lvert G\rvert \geq 31104$	$N \geq 38234$	estimated with $\lvert G\rvert = 32772$, type $[3,\dots,3]$
n	$\lvert G\rvert \geq 2.4^{n-1}$	$N > 2.4^{n-2}(n-4)$	

Regular n-polytopes and their duals have the same genus. The restricted Γ_4-map pictured in Figure 6 that represents the hypercube has genus 41 ($N = 80$). If we have done the same for the hypertetrahedron, an orientable and regular 4-polytope with 120 flags and automorphism group the Coxeter group of type $[3,3,3]$, we would end up with a regular restricted Γ_4-map of genus 11 ($N = 20$).

The above formulae do not take into account the smallest dimension that a n-polytope (or n-hypertope) might be realised as a complex simplicial manifold. For this we have the genus $g(M)$ of a piecewise linear manifold M introduced by Gagliardi [8] as being the minimum genus of any colour-graph that induces the same piecewise linear manifold M, where the genus of a coloured-graph is the minimum genus of its strongly-regular embeddings. Despite $g(M)$ be a topological invariant, $g(M) = 0$ if and only if M is the $(n-1)$-sphere, and $g(M)$ coincides with usual surface genus if $\dim(M) = 2$ and with Heegaard genus if $\dim(M) = 3$, to calculate $g(M)$ one needs to apply dipoles operations of addition and/or subtraction consecutively on the n-graph in order to transform it into a minimal coloured-graph embedding for M, called a *crystallisation* of M [7]. The genus of a crystallisation is a topological invariant and coincides with $g(M)$. However it has been shown to be difficult to get a transition from a crystallisation to another one [16].

8 n-hypermaps as generalisation of hypermaps

Hypermap are generalisations of maps by allowing edges to join more than two vertices. They are accomplished by cellular embeddings of hypergraphs (bipartite graphs) on connected surfaces. Now n-hypermaps can also be seen as a further generalisation of hypermaps. A map is a cellular embedding of a (1-partite) graph on a connected surface (so to speak a 1-partite map). A *n-hypermap*, $n > 1$, is a cellular embedding of an n-partite graph on a connected surface (that is an n-partite map or n-coloured map) such that each vertex coloured k with $1 < k < n$, is alternately surrounded by vertices coloured $k-1$ and $k+1$, while vertices coloured 1 (resp. n) are surrounded by vertices coloured 2 (resp. $n-1$). They arise as quotients of "$(n+1)$-gonal" groups. Take a hyperbolic $(n+1)$-gon with zero internal angles (Figure 8 left). The dual is a cellular embedding tree (Figure 8 right) and so the Coxeter group generated by the reflections on the sides of this hyperbolic $(n+1)$-gon is a free product $C_2 * C_2 * \cdots * C_2$. Each conjugacy class of a subgroup H in Δ_n determines, up to isomorphism, a n-hypermap $\mathcal{H} = (\Delta_n/_r H; H^* r_0, H^* r_1, \ldots, H^* r_n)$ with monodromy group $\mathrm{Mon}(\mathcal{H}) = \Delta_n/H^*$, where H^* is the core of H in Δ_n, and

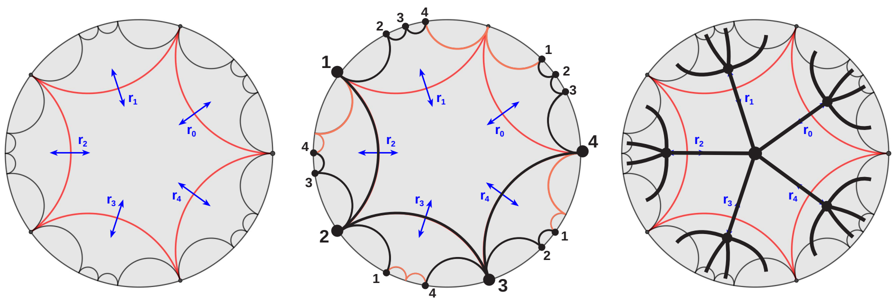


Figure 8: Hyperbolic 5-gonal tesselation (left), 4-hypermap (centre), 5-valency tree (right).

$\mathrm{Aut}(\mathcal{H}) = N_{\Delta_n}(H)/H$. It occurs as an orbifold of the universal hyperbolic n-hypermap illustrated in Figure 8 centre (for $n = 4$). A map is a 1-hypermap and a hypermap is a 2-hypermap.

These $(n+1)$-gonal hyperbolic tessellations on the Poincaré disc are maps. Their duals are maps whose edges are $(n+1)$ coloured (Figure 8 right) representing hyperbolic cellular embeddings of universal $(n+1)$-coloured graphs [1] or $(n+1)$-GEMs ($(n+1)$-graph encoding manifolds) [10].

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