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Optimal leader-following consensus of fractional opinion formation models

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ABSTRACT

This paper deals with a control strategy enforcing consensus in a fractional opinion formation model with leadership, where the interaction rates between followers and the influence rate of the leader are functions of deviations of opinions between agents. The fractional-order derivative determines the impact of the memory during the opinion evolution. The problem of leader-following consensus control is cast in the framework of nonlinear optimal control theory. We study a finite horizon optimal control problem, in which deviations of opinions between agents and with respect to the leader are penalized along with the control that is applied only to the leader. The existence conditions for optimal consensus control are proved and necessary optimality conditions for the considered problem are derived. The results of the paper are illustrated by some examples.

1. Introduction

In many real-world systems, we observe that a number of individuals (animal or human) achieve a common objective without any central decision maker. Namely, individuals (also called agents) only through local interactions and computations reach an agreement upon a common state, referred to as consensus. Examples of such collective behavior include flocking of birds, swarming of bees and schooling of fish [1,2]. In all those systems, individuals are interconnected with and communicate some information each other in their neighborhood. For several years, group opinion formation has been a fertile ground for research. There are many approaches, for example, the French–DeGroot model [3], the Abelson model [4], the bounded confidence model of Vicek [5], the Hegelsmann–Krause model [6], the Cucker–Smale model [7], and their further modifications [8,9] including those with a leader [10,11]. In this paper, we deal with the last mentioned case of opinion formation models. Namely, we consider a fractional-order system consisting of N agents, plus one additional agent—the group's leader. The weights quantifying the way the agents influence each other are functions of deviations of opinions between them, but there is no information flow from agents to the leader. Since the formation of consensus in the model strongly depends on the communication rate function and the initial configuration of the system, it is relevant to consider external control strategies that will facilitate the consensus. Motivated by [11], we investigate

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"non-invasive" optimal control for the system, that is, the control function is applied only to the leader and it is designed in such a way that the average deviation of opinions between agents and the deviations of agents' opinions with respect to the leader are minimized along with a quadratic control term (the cost of control). However, compared with [11], our optimal control problems are governed by a system with fractional derivatives. Fractional calculus is a generalization of differentiation and integration to the arbitrary (real or even complex) order. Despite the fact that the concept has a long, more than three hundred years, history only since the second half of the twentieth century, we have observed great strides in the study of fractional-order systems [12–15].

Fractional-order derivatives are nonlocal operators and therefore they could be used to describe memory-dependent and hereditary phenomena [16,17], systems with long range interactions in space and/or time (memory) and processes with many scales of space and/or time involved [18–20]. Thanks to those properties and advances in the pure mathematical theory of fractional calculus, fractional operators have been applied in many areas of science and engineering: physics (anomalous diffusion, electrical spectroscopy impedance), control (fractional-order PID), signal and image processing, biology, environmental science, economics, and many others. For a recent overview of applications of fractional calculus, we refer the reader to the review paper [21].

It turned out that fractional operators are also excellent tools for modeling the behavior of human beings, especially psychological processes that depend on the experience in the past [22,23]. This brings us to consider opinion formation models in the framework of fractional calculus. By introducing the memory parameter (which is the order of the fractional derivative), we take into account the influence of the past on interactions among the agents. In other words, the fractional-order derivative determines the impact of the memory during the opinion evolution.

Summing up, the originality of this paper lies in the following:

- A new formulation of the opinion formation model is proposed. We replace the usual time derivative by a fractional derivative in the system dynamics and in this way the memory factor is incorporated into the model.
- The weights quantifying the way the agents influence each other are functions of deviations of opinions between them. Therefore, the considered problem is nonlinear.
- We propose "non-invasive" external control strategy, that is the control function is applied only to the leader and it may be applied as soon as the leader is available.

Contributions of this work may be summarized as follows:

- Necessary optimality conditions for the optimal control problem governed by a fractional opinion formation model with leadership are derived.
- The existence conditions for optimal "non-invasive" consensus control are proved.

This paper is organized as follows. In the next section, we review basic definitions and theorems that are used for proving our main results. Section 3 is devoted to the analysis of our proposed fractional opinion formation model with leadership. The model is formulated as a fractional optimal control problem. In Section 3.1, we prove existence and uniqueness of a solution to the control system (Theorem 12). Then, in Section 3.2, necessary optimality conditions (Theorem 14) and sufficient existence conditions (Theorem 17) for optimal "non-invasive" control are derived. Those conditions ensure optimal leader-following consensus of considered fractional opinion formation model. Numerical examples are given in Section 4 to illustrate the obtained theoretical results. Conclusions are drawn in Section 5.

2. Preliminaries

In this section, we present definitions and properties concerning fractional differential and integral operators (cf. [13,24]). Moreover, we formulate a general Caputo-type optimal control problem and recall theorems concerning existence and uniqueness of solutions to the control system as well as necessary and sufficient optimality conditions [25].

2.1. Basics of fractional calculus

Suppose that $[a,b] \subset \mathbb{R}$ is any bounded interval. Let $\alpha > 0$ and $f \in L^1([a,b],\mathbb{R}^n)$. By the left and the right-sided Riemann–Liouville fractional integrals of the function f of order α we understand

$$(I^{\alpha}_{a+}f)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \quad t \in [a,b] \ a.e.,$$

$$(I^{\alpha}_{b-}f)(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}}d\tau, \quad t \in [a,b] \ a.e.,$$

respectively. Moreover, for $1 \le p < \infty$, we define spaces

$$I^{\alpha}_{a+}(L^p) := \left\{ f : [a,b] \to \mathbb{R}^n : \quad f = I^{\alpha}_{a+}g \ a.e. \text{ on } [a,b], \quad g \in L^p([a,b],\mathbb{R}^n) \right\}$$

and

$$I^{\alpha}_{b-}(L^p) := \left\{ f : [a,b] \to \mathbb{R}^n : \quad f = I^{\alpha}_{b-}g \ a.e. \text{ on } [a,b], \quad g \in L^p([a,b],\mathbb{R}^n) \right\}.$$

Note that, Property 4 in [26] implies that if $\frac{1}{p} < \alpha < 1$, then $I^{\alpha}_{a+}(L^p) \subset C_a([a,b],\mathbb{R}^n)$, where

$$C_a([a,b],\mathbb{R}^n) := \{f \in C([a,b],\mathbb{R}^n) : f(a) = 0\}.$$

In addition, using similar arguments as in the proof of Property 4 in [26], one can obtain an analogous result for the space $I^{\alpha}_{b-}(L^p)$.

Proposition 1. *If $\frac{1}{p} < \alpha < 1$, then $I^{\alpha}_{b-}(L^p) \subset C_b([a,b],\mathbb{R}^n)$, where*

$$C_b([a,b],\mathbb{R}^n) := \{f \in C([a,b],\mathbb{R}^n) : f(b) = 0\}.$$

Proof. Let $a \le t_1 < t_2 < b$. Using the Hölder inequality and [26, relation (13)] we obtain

$$\begin{aligned}
|(I^{\alpha}_{b-}f)(t_1) - (I^{\alpha}_{b-}f)(t_2)| &= \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{b} f(s)(s-t_1)^{\alpha-1}ds - \int_{t_2}^{b} f(s)(s-t_2)^{\alpha-1}ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left(\left|\int_{t_1}^{t_2} f(s)(s-t_1)^{\alpha-1}ds\right| + \left|\int_{t_2}^{b} f(s)((s-t_1)^{\alpha-1} - (s-t_2)^{\alpha-1})ds\right|\right) \\
&\le \frac{\|f\|_{L^p}}{\Gamma(\alpha)}\left[\left(\int_{t_1}^{t_2} (s-t_1)^{(\alpha-1)q}ds\right)^{\frac{1}{q}} + \left(\int_{t_2}^{b} ((s-t_2)^{\alpha-1} - (s-t_1)^{\alpha-1})^q ds\right)^{\frac{1}{q}}\right] \\
&\le \frac{\|f\|_{L^p}}{\Gamma(\alpha)}\left[\left(\int_{t_1}^{t_2} (s-t_1)^{(\alpha-1)q}ds\right)^{\frac{1}{q}} + \left(\int_{t_2}^{b} ((s-t_2)^{q(\alpha-1)} - (s-t_1)^{q(\alpha-1)})ds\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\left[(t_2-t_1)^{(1+(\alpha-1)q)\frac{1}{q}} + (b-t_2)^{(1+(\alpha-1)q)\frac{1}{q}}\right] \\
&\quad + \frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\left[(t_2-t_1)^{(1+(\alpha-1)q)\frac{1}{q}} - (b-t_1)^{(1+(\alpha-1)q)\frac{1}{q}}\right] \\
&\le \frac{2\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(t_2-t_1)^{\alpha-\frac{1}{p}},
\end{aligned}$$

whereby $\frac{1}{p} + \frac{1}{q} = 1$. This means that $I^{\alpha}_{b-}f$ is continuous on $[a,b)$ (even Hölder continuous). Moreover, for $t \in [a,b)$ we have

$$|(I^{\alpha}_{b-}f)(t)| \le \frac{1}{\Gamma(\alpha)}\int_{t}^{b} |f(s)|(s-t)^{\alpha-1}ds \le \frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(b-t)^{\alpha-\frac{1}{p}} \underset{t\to b^-}{\longrightarrow} 0.$$

This means that $I^{\alpha}_{b-}f$ can be continuously extended by 0 at $t = b$, so $I^{\alpha}_{b-}f \in C_b([a,b],\mathbb{R}^n)$. The proof is completed. □

Let $\alpha \in (0,1)$, $f \in L^1([a,b],\mathbb{R}^n)$ and suppose that f is such that functions $I^{1-\alpha}_{a+}f$ and $I^{1-\alpha}_{b-}f$ are absolutely continuous on $[a,b]$. The left-sided Riemann–Liouville fractional derivatives $D^{\alpha}_{a+}f$ and the right-sided Riemann–Liouville fractional derivatives $D^{\alpha}_{b-}f$ of order α of f are defined by

$$(D^{\alpha}_{a+}f)(t) := \frac{d}{dt}(I^{1-\alpha}_{a+}f)(t), \quad t \in [a,b] \text{ a.e.}$$

and

$$(D^{\alpha}_{b-}f)(t) := -\frac{d}{dt}(I^{1-\alpha}_{b-}f)(t), \quad t \in [a,b] \text{ a.e.},$$

respectively.

The left and the right Caputo fractional derivatives of order $\alpha \in (0,1)$ of a continuous function f on the interval $[a,b]$ are respectively given by

$$({}^{C}D^{\alpha}_{a+}f)(t) := D^{\alpha}_{a+}(f(\cdot) - f(a))(t),\ t \in [a,b] \text{ a.e.}$$

and

$$({}^{C}D^{\alpha}_{b-}f)(t) := D^{\alpha}_{b-}(f(\cdot) - f(b))(t),\ t \in [a,b] \text{ a.e.},$$

provided that derivatives in the Riemann–Liouville sense exist.

2.2. A nonlinear optimal control problem with the Caputo derivative

Let us consider the following optimal control problem

$$\text{minimize} \quad J(y,u) = \int_a^b g_0(t, y(t), u(t))dt, \tag{1}$$

subject to

$$\left({}^C D^{\alpha}_{a+} y\right)(t) = g(t, y(t), u(t)), \quad t \in [a,b] \text{ a.e.}, \tag{2}$$

$$y(a) = y_0, \tag{3}$$

$$u(t) \in M \subset \mathbb{R}^m, \quad t \in [a,b], \tag{4}$$

where $0 < \alpha < 1$, $g : [a,b] \times \mathbb{R}^n \times M \to \mathbb{R}^n$, $g_0 : [a,b] \times \mathbb{R}^n \times M \to \mathbb{R}$, $y_0 \in \mathbb{R}^n$.

For $1 \le p < \infty$ we define the following set of controls:

$$\mathcal{U}_M := \{u \in L^p([a,b], \mathbb{R}^m) : u(t) \in M,\ t \in [a,b]\}.$$

Let $p > \frac{1}{\alpha}$. By a solution to the control system (2)–(4), corresponding to any fixed control $u \in \mathcal{U}_M$, we mean a function

$$y \in K^{\alpha,p}_{a+} := I^{\alpha}_{a+}(L^p) + \{c : c \in \mathbb{R}^n\}$$

satisfying equation (2) a.e. on $[a,b]$ and initial condition (3).

Remark 2. Let us note that if $y \in K^{\alpha,p}_{a+}$ with $p > \frac{1}{\alpha}$, then there exist a function $\varphi \in L^p([a,b], \mathbb{R}^n)$ and a constant $c \in \mathbb{R}^n$ such that

$$y(t) = c + (I^{\alpha}_{a+}\varphi)(t), \quad t \in [a,b] \text{ a.e.}$$

Moreover, since $p > \frac{1}{\alpha}$, the function $\psi : [a,b] \to \mathbb{R}^n$ given by $\psi(\cdot) = (I^{\alpha}_{a+}\varphi)(\cdot)$ is continuous on $[a,b]$ and $\psi(a) = 0$ (cf. [26, Property 4]). Consequently, y is continuous on $[a,b]$, $y(a) = c$ and $y(\cdot) - y(a) \in I^{\alpha}_{a+}(L^p)$. This means that there exists the Caputo derivative ${}^C D^{\alpha}_{a+} y$ and

$$({}^C D^{\alpha}_{a+} y)(t) = D^{\alpha}_{a+}(y(\cdot) - y(a))(t) = (D^{\alpha}_{a+} I^{\alpha}_{a+}\varphi)(t) = \varphi(t), \quad t \in [a,b] \text{ a.e.}$$

Definition 3. We say that a pair $(y_*, u_*) \in K^{\alpha,p}_{a+} \times \mathcal{U}_M$ is a locally optimal solution to problem (1)–(4), if y_* is a solution to system (2)–(4), corresponding to the control u_* and there exists a neighborhood V of the point y_* in $K^{\alpha,p}_{a+}$ such that

$$J(y_*, u_*) \le J(y, u)$$

for all pairs $(y,u) \in V \times \mathcal{U}_M$ satisfying (2)–(4).

Definition 4. We say that a pair $(y_*, u_*) \in K^{\alpha,p}_{a+} \times \mathcal{U}_M$ is a globally optimal solution to problem (1)–(4), if y_* is a solution to system (2)–(4), corresponding to the control u_* and

$$J(y_*, u_*) \le J(y, u)$$

for all pairs $(y,u) \in K^{\alpha,p}_{a+} \times \mathcal{U}_M$ satisfying (2)–(4).

Now, we recall results concerning problem (1)–(4) obtained in [25]. Along the work, by $\|\cdot\|_{l_2^k}$ we shall denote the Euclidean norm on $\mathbb{R}^k$. We start with a theorem ensuring existence and uniqueness of solutions to control system (2)–(4) (cf. [25, Theorem 3.6 for the case $\beta = 1$]).

Theorem 5 (*cf. [25, Theorem 3.6]*)**.** *Let $p > \frac{1}{\alpha}$. If*

(1_g) *$g(\cdot, y, u)$ is measurable on $[a,b]$ for all $y \in \mathbb{R}^n$, $u \in M$, $g(t, y, \cdot)$ is continuous on M for $t \in [a,b]$ a.e. and all $y \in \mathbb{R}^n$;*

(2_g) *there exists $L > 0$ such that*

$$\|g(t, y_1, u) - g(t, y_2, u)\|_{l_2^n} \le L\|y_1 - y_2\|_{l_2^n}$$

for $t \in [a,b]$ a.e. and all $y_1, y_2 \in \mathbb{R}^n$, $u \in M$;

(3_g) *there exist $v \in L^p([a,b], \mathbb{R})$ and $\theta \geqslant 0$ such that*

$$\|g(t, 0, u)\|_{l_2^n} \le v(t) + \theta\|u\|_{l_2^m}$$

for $t \in [a,b]$ a.e. and all $u \in M$;

then, for any fixed $u \in \mathcal{U}_M$, there exists a unique solution $y_u \in K^{\alpha,p}_{a+}$ to control system (2)–(4).

The second result (cf. [25, Theorem 3.7 for the case $\beta = 1$]) regarding necessary optimality conditions is given as follows.

Theorem 6 (*cf. [25, Theorem 3.7]*).
Let $p > \frac{1}{\alpha}$. We assume that M is compact and

(A_g) *$g \in C^1$ with respect to $y \in \mathbb{R}^n$ and satisfies assumptions (1_g)–(3_g) of Theorem 5;*
(B_g) *$g_0(\cdot, y, u)$ is measurable on $[a, b]$ for all $y \in \mathbb{R}^n$, $u \in M$ and $g_0(t, y, \cdot)$ is continuous on M for a.e. $t \in [a, b]$ and all $y \in \mathbb{R}^n$;*
(C_g) *$g_0 \in C^1$ with respect to $y \in \mathbb{R}^n$ and*

$$|g_0(t, y, u)| \leqslant a_1(t) + C_1 \|y\|^p_{l_2^n}, \tag{5}$$

$$\left\| (g_0)_x (t, y, u) \right\|_{l_2^n} \leqslant a_2(t) + C_2 \|y\|^{p-1}_{l_2^n} \tag{6}$$

for a.e. $t \in [a, b]$ and all $y \in \mathbb{R}^n$, $u \in M$, where $a_2 \in L^{p'}([a, b], \mathbb{R}_0^+)$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$, $a_1 \in L^1([a, b], \mathbb{R}_0^+)$, $C_1, C_2 \geqslant 0$;
(D_g) *$g_y(\cdot, y, u)$, $(g_0)_y(\cdot, y, u)$ are measurable on $[a, b]$ for all $y \in \mathbb{R}^n$, $u \in M$;*
(E_g) *$g_y(t, y, \cdot)$, $(g_0)_y(t, y, \cdot)$ are continuous on M for a.e. $t \in [a, b]$ and all $y \in \mathbb{R}^n$;*
(F_g) *for a.e. $t \in [a, b]$ and all $y \in \mathbb{R}^n$ the set*

$$Z := \left\{ (g_0(t, y, u), g(t, y, u)) \in \mathbb{R}^{n+1}, \quad u \in M \right\}$$

is convex.

If the pair $(y_*, u_*) \in K^{\alpha,p}_{a+} \times \mathcal{U}_M$ is a locally optimal solution to problem (1)–(4), then there exists a function $\lambda \in I^{\alpha}_{b-}(L^{p'})$ such that

$$\left(D^{\alpha}_{b-}\lambda\right)(t) = g_y^T(t, y_*(t), u_*(t))\lambda(t) - (g_0)_y (t, y_*(t), u_*(t))$$

for a.e. $t \in [a, b]$ and

$$\left(I^{1-\alpha}_{b-}\lambda\right)(b) = 0.$$

Moreover,

$$g_0(t, y_*(t), u_*(t)) - \lambda(t) g(t, y_*(t), u_*(t)) = \min_{u \in M} \{g_0(t, y_*(t), u) - \lambda(t) g(t, y_*(t), u)\}, \quad t \in [a, b] \text{ a.e.}$$

Now, we formulate the following theorem on the existence of optimal solutions (cf. [25, Theorem 3.8 for the case $\beta = 1$]).

Theorem 7 (*cf. [25, Theorem 3.8]*).
Let $p > \frac{1}{\alpha}$. If

(A) *the set M is compact;*
(B) *g satisfies assumptions $(1_g) - (2_g)$ of Theorem 5;*
(C) *the function g is such that*

$$\|g(t, y_0, u)\|_{l_2^n} \leqslant c_1 + c_2(t-a)^{\lambda}$$

for a.e. $t \in [a, b]$ and all $u \in M$, where $c_1 \geqslant 0$, $c_2 \geqslant 0$, $\lambda > -\frac{1}{p}$;
(D) *$g_0(\cdot, y, u)$ is measurable on $[a, b]$ for all $y \in \mathbb{R}^n$ and $u \in M$;*
(E) *$g_0(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^n \times M$ for $t \in [a, b]$ a.e.;*
(F) *assumption (F_g) from Theorem 6 holds;*
(G) *for any function $\kappa \in L^p([a, b], \mathbb{R}^+)$ there exists a function $\psi \in L^1([a, b], \mathbb{R}_0^+)$ such that*

$$|g_0(t, y + y_0, u)| \leqslant \psi(t)$$

for $t \in [a, b]$ a.e., $\|y\|_{l_2^n} \leqslant \kappa(t)$ and all $u \in M$.

Then problem (1)–(4) possesses a globally optimal solution $(y_*, u_*) \in K^{\alpha,p}_{a+} \times \mathcal{U}_M$.

Remark 8. In [25] the set $K^{\alpha,p}_{a+}$ is denoted by $K^{\alpha,\beta,p}_{a+}$ with $\beta = 1$.

3. Optimal control of the fractional opinion formation model with leadership

Let us consider the following system with one leader, labeled as 0, and N agents (followers), labeled as 1 to N:

$$\begin{cases} \left({}^C D^{\alpha}_{0+} x_0\right)(t) = u(t) \\ \left({}^C D^{\alpha}_{0+} x_i\right)(t) = \sum_{j\neq 0,i}^{N} a_{ij}(x_j(t) - x_i(t)) + c_i(x_0(t) - x_i(t)), \\ (x_0(0), x_i(0)) = (x_{00}, x_{i0}), \qquad i = 1, \ldots, N \\ u(t) \in M \subset \mathbb{R}^d, \end{cases} \quad t \in [0,T] \ a.e. \tag{7}$$

where $\alpha \in (0,1)$ and the state $x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{(N+1)d}$ represents opinions of the leader and agents. Moreover, $u : [0,T] \to \mathbb{R}^d$ is a control function acting only on the leader and $M := \left\{ z \in \mathbb{R}^d : \|z\|_{l_2^d} \le K \right\}$ for a given $K > 0$. The coefficients a_{ij} describe the way the agents interact and are given by

$$a_{ij} = a\left(\|x_i - x_j\|^2_{l_2^d}\right), \tag{8}$$

while the coefficients c_i describe the influence of the leader on the ith agent and are given by

$$c_i = \gamma \phi\left(\|x_i - x_0\|^2_{l_2^d}\right). \tag{9}$$

We shall focus on two types of system (7). The first of them is called a fractional Cucker–Smale (FCS) type model and involves coefficients (8)–(9) with $\gamma > 0$, $a(r) = \frac{R}{(1+r)^{\theta}}$, $\phi(r) = \frac{1}{(1+r)^{\zeta}}$, $r \ge 0$, where $R > 0$ and $\theta, \zeta \ge 0$ are fixed reals. The second one, called a fractional Hegselmann–Krause (FHK) type model, consists of coefficients (8)–(9), where $a : [0,+\infty) \to [0,1]$ is the following smooth cut off function

$$a(r) = a(r,\delta,\varepsilon) = \begin{cases} 1, & 0 \le r \le \delta, \\ \varphi(r), & \delta < r < \delta + \varepsilon, \\ 0, & \delta + \varepsilon \le r, \end{cases} \tag{10}$$

with function $\varphi : [\delta, \delta+\varepsilon] \to [0,1]$ satisfying the following properties:

(1_φ) φ is a decreasing and smooth function,
(2_φ) $\varphi(\delta) = 1$ and $\varphi(\delta+\varepsilon) = 0$,

and $\phi : [0,+\infty) \to (0,1]$ satisfying conditions:

(1_ϕ) ϕ is a smooth, non-increasing function,
(2_ϕ) $\phi(0) = 1$ and $\lim_{r\to\infty} \phi(r) = 0$.

It is worth pointing out that the weights (a_{ij} and c_i) quantifying the way the agents influence each other are functions of deviations of opinions between agents. By definition, those weights are nonnegative and the inter-agent connectivity intensity decreasing or even fading as the difference in opinions in pairs increases. In the FCS model, we superimposing the interactions between all agents, while in the FHK model agents interact only with those whose opinions are close enough to their opinions.

Remark 9. Let us note that we modified slightly FHK type model. In the original model (cf. [11]) coefficients a_{ij} and c_i are given by

$$a_{ij} = \tilde{a}\left(\|x_i - x_j\|_{l_2^d}\right) \quad \textit{and} \quad c_i = \gamma\tilde{\phi}\left(\|x_i - x_0\|_{l_2^d}\right), \tag{11}$$

where function $\tilde{a}$ is of a type (10), with function $\tilde{\varphi}$ satisfying conditions (1_φ)–(2_φ), and $\tilde{\phi}$ satisfying (1_ϕ)–(2_ϕ). Nevertheless, proposed model (7) can be still called the fractional Hegselmann–Krause type model, because, putting

$$\tilde{a}(r) := a(r^2) \quad \textit{and} \quad \tilde{\phi}(r) := \phi(r^2), \quad r \ge 0,$$

it is easy to check that functions $\tilde{a}$ and $\tilde{\phi}$ have properties (1_φ)–(2_φ) and (1_ϕ)–(2_ϕ), respectively. We shall see that such modification is crucial in our approach.

When analyzing opinion formation models one of the most crucial questions that has to be answered is whether the agents reach consensus of opinions, i.e., if there exists $x^* \in \mathbb{R}^{(N+1)d}$, with $x_0^* = x_1^* = \cdots = x_N^*$, such that $\lim_{t\to\infty} x(t) = x^*$. In this work, we aim to steer all agents in system (7) to reach a consensus by using an optimal control strategy, i.e., we shall minimize the following cost functional

$$J(x,u) = \int_0^T \left(\frac{1}{2N^2} \sum_{i,j=1}^{N} \left\|x_i(t) - x_j(t)\right\|^2_{l_2^d} + \frac{1}{2}\sum_{i=1}^{N} \|x_0(t) - x_i(t)\|^2_{l_2^d} + \frac{\nu}{2}\|u(t)\|^2_{l_2^d} \right) dt \tag{12}$$

subject to system (7). The first term in J is the average deviation of opinions between agents, the second term is the deviations of agents' opinions with respect to leader, whereas the third term represents the cost of the control with $\nu > 0$ that is a weight constant.

3.1. Existence and uniqueness of a solution to the control system

In this section, we prove theorem ensuring existence of a unique solution to system (7), corresponding to any fixed control $u \in \mathcal{U}_M$. In the proof of this fact we use the following lemma.

Lemma 10. *Let $h : [0, \infty) \to (0, \infty)$ be a non-increasing and continuously differentiable function on $[0, \infty)$. Then $\lim_{r\to\infty}(rh'(r)) \neq \pm\infty$.*

Proof. It is clear that $\lim_{r\to\infty}(rh'(r)) \neq +\infty$. Let us suppose that $\lim_{r\to\infty}(rh'(r)) = -\infty$. This means that, for any fixed $R > 0$, there exists $A > 0$ such that, for all $r \geq A$,

$$h'(r) < -\frac{R}{r}.$$

Let $s > A$ and $\lim_{r\to\infty} h(r) = g \geq 0$ (the existence of this limit is obvious). Then,

$$h(s) - h(A) = \int_A^s h'(r)dr < -R\int_A^s \frac{1}{r}dr = -R\ln s + R\ln A.$$

Thus

$$g - h(A) \underset{s\to\infty}{\longleftarrow} h(s) - h(A) < R\ln s + R\ln A \underset{s\to\infty}{\longrightarrow} -\infty.$$

The obtained contradiction completes the proof. □

Corollary 11. *If the limit $\lim_{r\to\infty}(rh'(r))$ exists, then the function $(\cdot)h'(\cdot)$ is bounded on $[0, \infty)$.*

Now, we are in condition to formulate and prove the existence and uniqueness theorem.

Theorem 12. *Let $p > \frac{1}{\alpha}$. Assume that*

(A) functions $(\cdot)\phi'(\cdot)$ and $(\cdot)a'(\cdot)$ are bounded on $[0, \infty)$ provided that limit $\lim_{r\to\infty}(r\phi'(r))$ and $\lim_{r\to\infty}(ra'(r))$, respectively, do not exist.

Then, for any fixed control $u \in \mathcal{U}_M$, there exists a unique solution $x \in K^{\alpha,p}_{0+}([0, T], \mathbb{R}^{(N+1)d})$ to system (7).

Proof. First, let us note that system (7) can be written as (2)–(4), where $g : [0, T] \times \mathbb{R}^{(N+1)d} \times M \to \mathbb{R}^{(N+1)d}$ is given by

$$g(t, x, u) = \left(g^0(t, x, u), g^1(t, x, u), \ldots, g^N(t, x, u)\right), \tag{13}$$

whereby

$$g^0(t, x, u) = u \tag{14}$$

and

$$g^i(t, x, u) = \sum_{j\neq 0,i}^{N} a_{ij}(x_j - x_i) + c_i(x_0 - x_i), \quad i = 1, \ldots, N. \tag{15}$$

It is sufficient to check that g satisfies all assumptions of Theorem 5. It is clear that conditions (1_g) and (3_g) are satisfied. Moreover, the partial derivative g_x is given by

$$g_x(t, x, u) = \begin{bmatrix} \frac{\partial g^0}{\partial x_0}(t, x, u) & \frac{\partial g^0}{\partial x_1}(t, x, u) & \ldots & \frac{\partial g^0}{\partial x_N}(t, x, u) \\ \frac{\partial g^1}{\partial x_0}(t, x, u) & \frac{\partial g^1}{\partial x_1}(t, x, u) & \ldots & \frac{\partial g^1}{\partial x_N}(t, x, u) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^N}{\partial x_0}(t, x, u) & \frac{\partial g^N}{\partial x_1}(t, x, u) & \ldots & \frac{\partial g^N}{\partial x_N}(t, x, u) \end{bmatrix}_{(N+1)d\times(N+1)d}, \tag{16}$$

where

$$\frac{\partial g^0}{\partial x_k}(t,x,u) = \mathbf{0}_d, \tag{17}$$

$$\frac{\partial g^i}{\partial x_k}(t,x,u) = \begin{bmatrix} \frac{\partial g_1^i}{\partial x_k^1}(t,x,u) & \frac{\partial g_1^i}{\partial x_k^2}(t,x,u) & \dots & \frac{\partial g_1^i}{\partial x_k^d}(t,x,u) \\ \frac{\partial g_2^i}{\partial x_k^1}(t,x,u) & \frac{\partial g_2^i}{\partial x_k^2}(t,x,u) & \dots & \frac{\partial g_2^i}{\partial x_k^d}(t,x,u) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_d^i}{\partial x_k^1}(t,x,u) & \frac{\partial g_d^i}{\partial x_k^2}(t,x,u) & \dots & \frac{\partial g_d^i}{\partial x_k^d}(t,x,u) \end{bmatrix}_{d\times d} \tag{18}$$

for $i = 1,\dots,N$, $k = 0,1,\dots,N$, whereby $x_k = (x_k^1,\dots,x_k^d)$, $g^i = (g_1^i,\dots,g_d^i)$, $i = 1,\dots,N$,

$$g_l^i(t,x,u) = \sum_{j\neq 0,i}^{N} a_{ij}(x_j^l - x_i^l) + c_i(x_0^l - x_i^l), \quad i = 1,\dots,N,\; l = 1,\dots,d,$$

and

$$\begin{aligned} \frac{\partial g^i}{\partial x_0} &= \left[\frac{\partial g_m^i}{\partial x_0^n}\right]_{m\times n} \\ &= \left(\gamma\phi\left(\|x_i - x_0\|^2_{l_2^d}\right)\right)\mathbf{I}_d + \left(2\gamma(x_i^n - x_0^n)(x_i^m - x_0^m)\frac{d\phi}{dr}\left(\|x_i - x_0\|^2_{l_2^d}\right)\right)\mathbf{1}_d, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial g^i}{\partial x_j} &= \left[\frac{\partial g_m^i}{\partial x_j^n}\right]_{m\times n} \\ &= \left(a\left(\|x_i - x_j\|^2_{l_2^d}\right)\right)\mathbf{I}_d + \left(2\frac{da}{dr}\left(\|x_i - x_j\|^2_{l_2^d}\right)(x_i^n - x_j^n)(x_i^m - x_j^m)\right)\mathbf{1}_d, \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{\partial g^i}{\partial x_i} &= \left[\frac{\partial g_m^i}{\partial x_i^n}\right]_{m\times n} \\ &= \sum_{j\neq 0,i}^{N}\left(\left(-2\frac{da}{dr}\left(\|x_i - x_j\|^2_{l_2^d}\right)(x_i^n - x_j^n)(x_i^m - x_j^m)\right)\mathbf{1}_d - \left(a\left(\|x_i - x_j\|^2_{l_2^d}\right)\right)\mathbf{I}_d\right) \\ &\quad - \left(\gamma\phi\left(\|x_i - x_0\|^2_{l_2^d}\right)\right)\mathbf{I}_d - \left(2\gamma(x_i^n - x_0^n)(x_i^m - x_0^m)\frac{d\phi}{dr}\left(\|x_i - x_0\|^2_{l_2^d}\right)\right)\mathbf{1}_d \end{aligned} \tag{21}$$

for $m,n = 1,\dots,d$, $i = 1,\dots,N$ and $j = 1,\dots,N$, $j \neq i$. Here symbols $\mathbf{I}_d$, $\mathbf{0}_d$ denote a d-dimensional identity and zero square matrices, respectively, whereas $\mathbf{1}_d$ is a d-dimensional square matrix consisting of ones. Note that

$$2|(x_i^n - x_j^n)(x_i^m - x_j^m)| \leq (x_i^n - x_j^n)^2 + (x_i^m - x_j^m)^2 \leq \|x_i - x_j\|^2_{l_2^d},$$

for $m,n = 1,\dots,d$, $i = 1,\dots,N$ and $j = 0,1,\dots,N$, $j \neq i$. Therefore, using assumption (A) and Corollary 11, we assert that there exist constants $C_1, C_2 > 0$ such that

$$\left|-2(x_i^n - x_j^n)(x_i^m - x_j^m)\frac{da}{dr}\left(\|x_i - x_j\|^2_{l_2^d}\right)\right| \leq \|x_i - x_j\|^2_{l_2^d}\left|\frac{da}{dr}\left(\|x_i - x_j\|^2_{l_2^d}\right)\right| \leq C_1$$

and

$$\left|-2(x_i^n - x_0^n)(x_i^m - x_0^m)\frac{d\phi}{dr}\left(\|x_i - x_0\|^2_{l_2^d}\right)\right| \leq \|x_i - x_0\|^2_{l_2^d}\left|\frac{d\phi}{dr}\left(\|x_i - x_0\|^2_{l_2^d}\right)\right| \leq C_2.$$

Consequently all elements of matrix (16) are bounded (the fact that the functions $\phi\left(\|x_i - x_0\|^2_{l_2^d}\right)$ and $a\left(\|x_i - x_j\|^2_{l_2^d}\right)$ are bounded is clear). Using Mean Value Theorem, we assert that g satisfies a global Lipschitz condition with respect to x. The proof is completed. □

Remark 13. Assumption (A) in the above result is necessary. One can define the function ϕ (or the function a) for which the limit $\lim_{r\to\infty}(r\phi'(r))$ does not exist and $(\cdot)\phi'(\cdot)$ is unbounded on $[0,\infty)$.

3.2. Optimality conditions

In this section, we consider the optimal control problem defined by Eqs. (7) and (12), that is problem **P** of the form:

$$\text{minimize} \quad J(x,u) = \int_0^T \left(\frac{1}{2N^2} \sum_{i,j=1}^N \left\| x_i(t) - x_j(t) \right\|_{l_2^d}^2 + \frac{1}{2} \sum_{i=1}^N \|x_0(t) - x_i(t)\|_{l_2^d}^2 + \frac{\nu}{2} \|u(t)\|_{l_2^d}^2 \right) dt$$

subject to

$$\begin{cases} \left({}^C D_{0+}^{\alpha} x_0\right)(t) = u(t) \\ \left({}^C D_{0+}^{\alpha} x_i\right)(t) = \sum_{j\neq 0,i}^N a_{ij}(x_j(t) - x_i(t)) + c_i(x_0(t) - x_i(t)), \\ (x_0(0), x_i(0)) = (x_{00}, x_{i0}), \qquad i = 1, \dots, N \\ u(t) \in M = \left\{ z \in \mathbb{R}^d : \|z\|_{l_2^d} \leq K \right\} \subset \mathbb{R}^d, \end{cases} \qquad t \in [0,T] \ a.e.$$

We start by proving necessary optimality conditions for problem **P**.

Theorem 14. *Suppose that* $p \geq 2$, $\alpha > \frac{1}{p}$ *and condition (A) from Theorem 12 holds. If* $(x_*, u_*) \in K_{0+}^{\alpha,p}([0,T], \mathbb{R}^{(N+1)d}) \times \mathcal{U}_M$ *is a locally optimal solution to problem* **P**, *then there exists a function* $\lambda \in I_{T-}^{\alpha}(L^{p'}) \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ *such that*

$$\left(D_{T-}^{\alpha} \lambda\right)(t) = g_x^T(t, x_*(t), u_*(t))\lambda(t) - (g_0)_x(t, x_*(t), u_*(t)) \tag{22}$$

for a.e. $t \in [0,T]$ *and*

$$\left(I_{T-}^{1-\alpha}\lambda\right)(T) = 0. \tag{23}$$

Moreover,

$$\begin{aligned} &g_0(t, x_*(t), u_*(t)) - \lambda(t) g(t, x_*(t), u_*(t)) \\ &\quad = \min_{u\in M} \left\{ g_0(t, x_*(t), u) - \lambda(t) g(t, x_*(t), u) \right\}, \quad t \in [0,T] \ a.e., \end{aligned}$$

where g, g_x *are given by* (13)–(15) and (16)–(21), *respectively*, $g_0 : [0,T] \times \mathbb{R}^{(N+1)d} \times M \to \mathbb{R}$, $(g_0)_x : [0,T] \times \mathbb{R}^{(N+1)d} \times M \to \mathbb{R}^{(N+1)d}$ *are given by*

$$g_0(t,x,u) = \frac{1}{2N^2} \sum_{i,j=1}^N \left\| x_i - x_j \right\|_{l_2^d}^2 + \frac{1}{2} \sum_{i=1}^N \|x_0 - x_i\|_{l_2^d}^2 + \frac{\nu}{2} \|u\|_{l_2^d}^2 \tag{24}$$

and

$$(g_0)_x(t,x,u) = \left(N x_0 - \sum_{i=1}^N x_i, \frac{2}{N}\left(x_i - \frac{1}{N}\sum_{j=1}^N x_j \right) + (x_i - x_0) \right), \quad i = 1, \dots, N,$$

respectively.

Proof. It is sufficient to check that functions g and g_0 given by (13)–(15) and (24), respectively, satisfy all assumptions of Theorem 6. It is clear that conditions (A_g), (B_g), (D_g) and (E_g) hold. Moreover, since the set M is convex, the function g is linear and g_0 is convex with respect to u, assumption (F_g) is also fulfilled. Now, we check growth conditions in assumption (C_g). We have

$$\begin{aligned} |g_0(t,x,u)| \leq & \frac{1}{2N^2} \sum_{i,j=1}^N \left(\|x_i\|_{l_2^d}^2 + 2\|x_i\|_{l_2^d} \|x_j\|_{l_2^d} + \|x_j\|_{l_2^d}^2 \right) \\ & + \frac{1}{2} \sum_{i=1}^N \left(\|x_0\|_{l_2^d}^2 + 2\|x_0\|_{l_2^d} \|x_i\|_{l_2^d} + \|x_i\|_{l_2^d}^2 \right) + \frac{\nu}{2} \|u\|_{l_2^d}^2 \\ \leq & \frac{1}{N^2} \sum_{i,j=1}^N \left(\|x_i\|_{l_2^d}^2 + \|x_j\|_{l_2^d}^2 \right) + \sum_{i=1}^N \left(\|x_0\|_{l_2^d}^2 + \|x_i\|_{l_2^d}^2 \right) + \frac{\nu}{2} \|u\|_{l_2^d}^2 \\ = & \frac{N+1}{N^2} \sum_{i=1}^N \|x_i\|_{l_2^d}^2 + \sum_{i=1}^N \|x_i\|_{l_2^d}^2 + N\|x_0\|_{l_2^d}^2 + \frac{\nu}{2} \|u\|_{l_2^d}^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2+N}{N}\sum_{i=1}^{N}\|x_i\|_{l_2^d}^2 + N\sum_{i=0}^{N}\|x_i\|_{l_2^d}^2 + \frac{\nu}{2}\|u\|_{l_2^d}^2 \\
&\leq 3\sum_{i=1}^{N}\|x_i\|_{l_2^d}^2 + N\sum_{i=0}^{N}\|x_i\|_{l_2^d}^2 + \frac{\nu}{2}\|u\|_{l_2^d}^2 \\
&\leq (3+N)\|x\|_{l_2^{(N+1)d}}^2 + \frac{\nu}{2}K^2 \leq (3+N)\|x\|_{l_2^{(N+1)d}}^p + \frac{\nu}{2}K^2
\end{aligned}
$$

for a.e. $t \in [0, T]$, all $u \in M$ and $x \in \mathbb{R}^{(N+1)d}$ such that $\|x\|_{l_2^{(N+1)d}} > 1$, where K is a constant from a definition of the set M. If $\|x\|_{l_2^{(N+1)d}} \leq 1$, then

$$
|g_0(t, x, u)| \leq 3 + N + \frac{\nu}{2}K^2 \leq 3 + N + \frac{\nu}{2}K^2 + C_1\|x\|_{l_2^{(N+1)d}}^p, \quad t \in [0.T] \ a.e., \ \ u \in M.
$$

Moreover,

$$
\begin{aligned}
\|(g_0)_x(t, x, u)\|_{l_2^{(N+1)d}} &= \sqrt{\left\|Nx_0 - \sum_{i=1}^{N}x_i\right\|_{l_2^d}^2 + \sum_{i=1}^{N}\left\|\frac{2}{N}\left(x_i - \frac{1}{N}\sum_{j=1}^{N}x_j\right) + (x_i - x_0)\right\|_{l_2^d}^2} \\
&\leq N\|x_0\|_{l_2^d} + \sum_{i=1}^{N}\|x_i\|_{l_2^d} + \sum_{i=1}^{N}\left\|\frac{2}{N}\left(x_i - \frac{1}{N}\sum_{j=1}^{N}x_j\right) + (x_i - x_0)\right\|_{l_2^d} \\
&\leq \frac{2N+4}{N}\sum_{i=1}^{N}\|x_i\|_{l_2^d} + 2N\|x_0\|_{l_2^d} \leq (2N+4)\sqrt{\left(\sum_{i=0}^{N}\|x_i\|_{l_2^d}\right)^2} \\
&\leq (2N+4)\sqrt{N}\sqrt{\sum_{i=0}^{N}\|x_i\|_{l_2^d}^2} = (2N+4)\sqrt{N}\|x\|_{l_2^{(N+1)d}} \\
&\leq \begin{cases} (2N+4)\sqrt{N}\|x\|_{l_2^{(N+1)d}}^{p-1} & \text{if} \quad \|x\|_{l_2^{(N+1)d}} > 1 \\ (2N+4)\sqrt{N} + C_2\|x\|_{l_2^{(N+1)d}}^{p-1} & \text{if} \quad \|x\|_{l_2^{(N+1)d}} \leq 1. \end{cases}
\end{aligned}
$$

This means that growth conditions (5)–(6) are satisfied. The proof is completed. □

Remark 15. From Proposition 1 it follows that, if $p = 2$, then system (22)–(23) can be written as

$$
\left({}^{C}D_{T-}^{\alpha}\lambda\right)(t) = g_x^T(t, x_*(t), u_*(t))\lambda(t) - (g_0)_x(t, x_*(t), u_*(t)), \quad t \in [0, T] \ a.e.,
$$

$$
\lambda(T) = 0.
$$

Remark 16. Let us note that in Theorem 6 the assumption of continuous differentiability of g with respect to x is required. It is the main reason why we have modified FHK model. Due to such a modification and assumption (A) the function g, given by (13)–(15), satisfies also a global Lipschitz condition with respect to x.

Now, we prove the existence of an optimal solution to problem **P**.

Theorem 17. *Let $p \geq 2$, $\alpha > \frac{1}{p}$ and condition (A) from Theorem 12 holds. Then problem* **P** *has a globally optimal solution* $(x_*, u_*) \in K_{0+}^{\alpha,p}([0, T], \mathbb{R}^{(N+1)d}) \times \mathcal{U}_M$.

Proof. The proof is based on Theorem 7. Obviously, the function g given by (13)–(15) satisfies assumption (C) and g_0 of the form (24) fulfills condition (G). Therefore, we start with a verification of condition (C). Let $y_0 = (x_{00}, x_{10}, \ldots, x_{N0})$. Since functions a and ϕ are bounded on $[0, +\infty)$, therefore all components of the function g given by (13) are also bounded on $\mathbb{R}^d$. Consequently, the function g is bounded on $\mathbb{R}^{(N+1)d}$ and condition (C) holds. Moreover, let us note that since g_0 satisfies growth condition (5), for any fixed function $\kappa \in L^2([0, T], \mathbb{R}^+)$ we obtain

$$
\begin{aligned}
|g_0(t, x + y_0, u)| &\leq a_1(t) + C_1\|x + y_0\|_{l_2^{(N+1)d}}^2 \leq a_1(t) + 2C_1(\|x\|_{l_2^{(N+1)d}}^2 + \|y_0\|_{l_2^{(N+1)d}}^2) \\
&\leq a_1(t) + 2C_1\|y_0\|_{l_2^{(N+1)d}}^2 + 2C_1\kappa^2(t)
\end{aligned}
$$

for a.e. $t \in [0,T]$ and all $\|x\|_{l_2^{(N+1)d}} \leq \kappa(t)$, $u \in M$ (here a_1, C_1 are data from (5)). Putting

$$\psi(\cdot) = a_1(\cdot) + 2C_1\|y_0\|^2_{l_2^{(N+1)d}} + 2C_1\kappa^2(\cdot) \in L^1([0,T], \mathbb{R}_0^+),$$

we assert that assumption (G) is fulfilled. The proof is completed. □

4. Illustrative examples

In the previous section, we have proved the existence conditions for a globally optimal solution and the necessary optimality conditions for the optimal control problem **P**. Those last conditions allow determining a candidate solution to the considered problem. On the other hand, it is well known that fractional differential equations mostly have to be solved numerically. However, we know that an optimal solution to the considered problem exists. Therefore, also numerical methods for fractional optimal control problems could be used, see, e.g., [27]. In this section, we present two numerical examples that show that by optimal control of the fractional opinion formation model with leadership we obtain consensus on an initial configuration of the system that would otherwise diverge. First, in Example 18, we consider problem **P** with leader's initial opinion being placed between the opinions of other agents. In this case, bounds imposed on the control function do not influence the behavior of the group. However, if we place the initial opinion of the leader outside opinions of other agents, as it is in Example 19, we observe that depending on the amount of control, the leader follows the opinions of other agents or agents move in the direction of leader's opinion.

Example 18. Let us consider system (7) with $N = 4$, $d = 1$ and $M = \left\{z \in \mathbb{R} : |z| \leq 10^{-6}\right\}$, i.e.,

$$\begin{cases} \left({}^C D^{\alpha}_{0+}x_0\right)(t) = u(t) \\ \left({}^C D^{\alpha}_{0+}x_i\right)(t) = \sum_{j\neq 0,i}^{4} a_{ij}(x_j(t) - x_i(t)) + c_i(x_0(t) - x_i(t)), \\ (x_0(0), x_1(0), x_2(0), x_3(0), x_4(0)) = (4, 2, 3, 10, 13), \\ u(t) \in M \subset \mathbb{R}. \end{cases} \quad t \in [0,T] \text{ a.e.} \tag{25}$$

The coefficients a_{ij} and c_i are given by

$$a_{ij} = a\left(|x_i - x_j|^2\right), \quad c_i = \phi\left(|x_i - x_0|^2\right),$$

where

$$a(r) = a(r, 2, 0.5) = \begin{cases} 1, & 0 \leq r \leq 0.5, \\ \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{1}{r-2} + \frac{1}{r-(2+0.5)}\right), & 0.5 < r < 2 + 0.5, \\ 0, & 2 + 0.5 \leq r, \end{cases}$$

and $\phi(r) = \exp(-r^2)$. A solution to (25) has to minimize the following functional:

$$J(x,u) = \int_0^T \left(\frac{1}{32}\sum_{i,j=1}^{4} \left|x_i(t) - x_j(t)\right|^2 + \frac{1}{2}\sum_{i=1}^{4} |x_0(t) - x_i(t)|^2 + |u(t)|^2 \right) dt. \tag{26}$$

Note that $\lim_{r\to\infty} r\phi'(r) = \lim_{r\to\infty} ra'(r) = 0$ and, by Theorem 12, for any fixed control $u \in \mathcal{U}_M$ and for $p > \frac{1}{\alpha}$, there exists a unique solution $x \in K^{\alpha,p}_{0+}([0,T], \mathbb{R}^5)$ to system (25). Thereupon, for $\alpha = 0.2$ there exists a unique solution of (25) in $K^{0.2,6}_{0+}([0,T], \mathbb{R}^5)$, while for $\alpha = 0.9$ in $K^{0.9,2}_{0+}([0,T], \mathbb{R}^5)$. Moreover, by Theorem 17, a solution to optimal control problem (25)–(26) exists and has to satisfy necessary optimality conditions given in Theorem 14.

In Figs. 1 and 3 we report the approximate trajectory solutions to system (25) without a leader and control, i.e.,

$$\begin{cases} \left({}^C D^{\alpha}_{0+}x_i\right)(t) = \sum_{j\neq 0,i}^{4} a_{ij}(x_j(t) - x_i(t)), \\ (x_1(0), x_2(0), x_3(0), x_4(0)) = (2, 3, 10, 13), \end{cases} \quad t \in [0,T] \text{ a.e.} \tag{27}$$

while in Figs. 2 and 4 we present approximate trajectory solutions to the optimal control problem (25)–(26).

In the case of the system without a leader and control (see Figs. 1 and 3) consensus is not attained and the convergence to clusters can be observed. However, if we introduce the leader and some level of control (see Figs. 2 and 4), then followers' opinions approach leader's opinion.

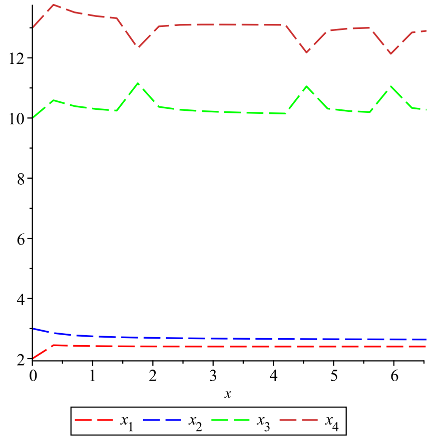

Fig. 1. Uncontrolled case for $\alpha = 0.2$: time evolution of the FHK type system without a leader.

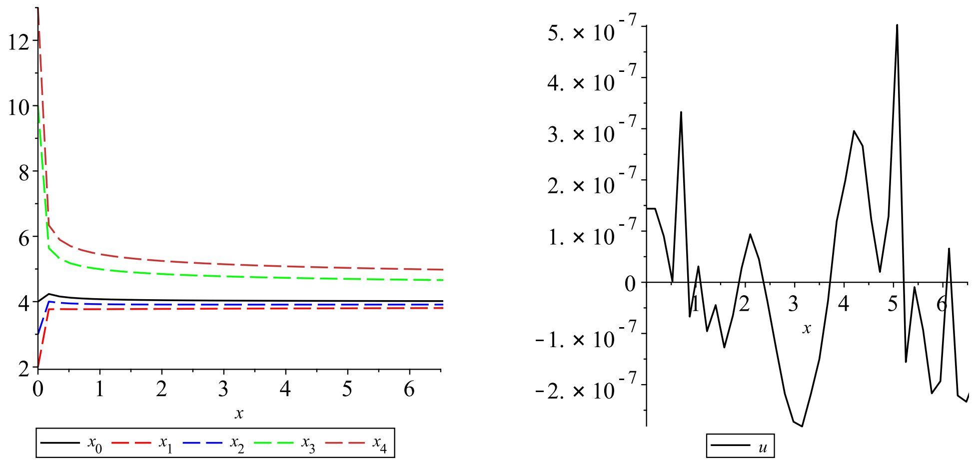

Fig. 2. Optimal control for $\alpha = 0.2$: time evolution of the FHK type system with the leader (left) and control (right).

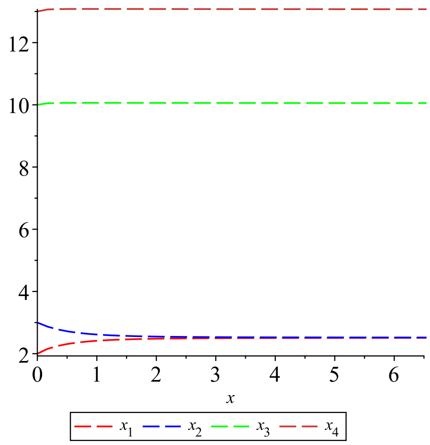

Fig. 3. Uncontrolled case for $\alpha = 0.9$: time evolution of the FHK type system without a leader.

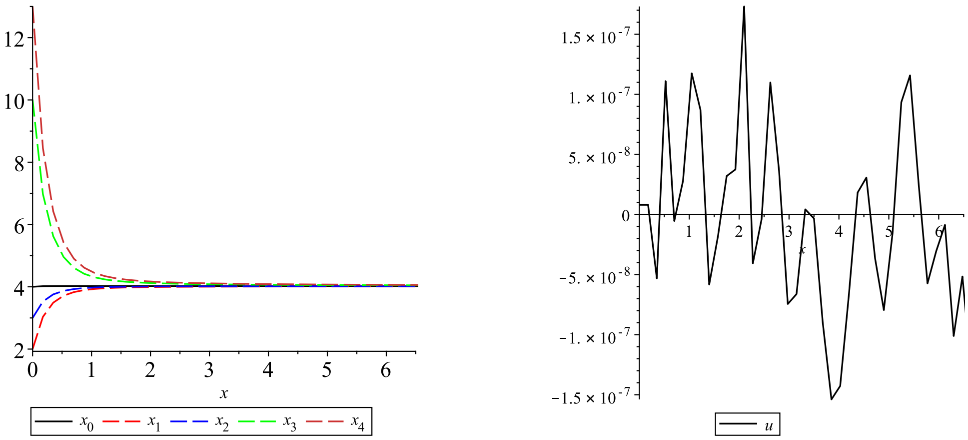

Fig. 4. Optimal control for $\alpha = 0.9$: time evolution of the FHK type system with the leader (left) and control (right).

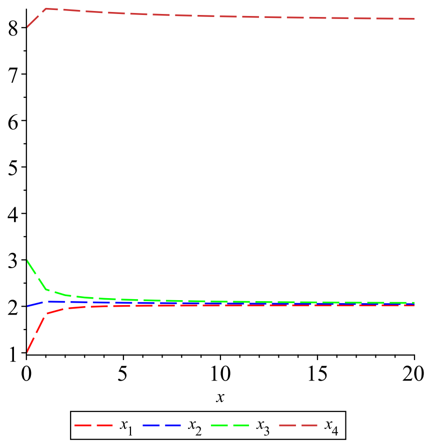

Fig. 5. Uncontrolled case for $\alpha = 0.6$: time evolution of the FHK type system without a leader.

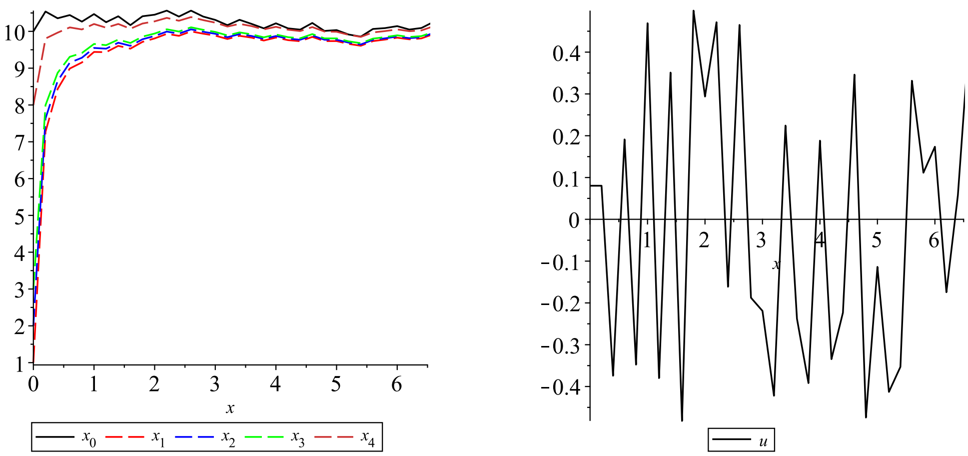

Fig. 6. Optimal control for $\alpha = 0.6$: time evolution of the FHK type system with the leader (left) and control (right) for $K = 0.5$.

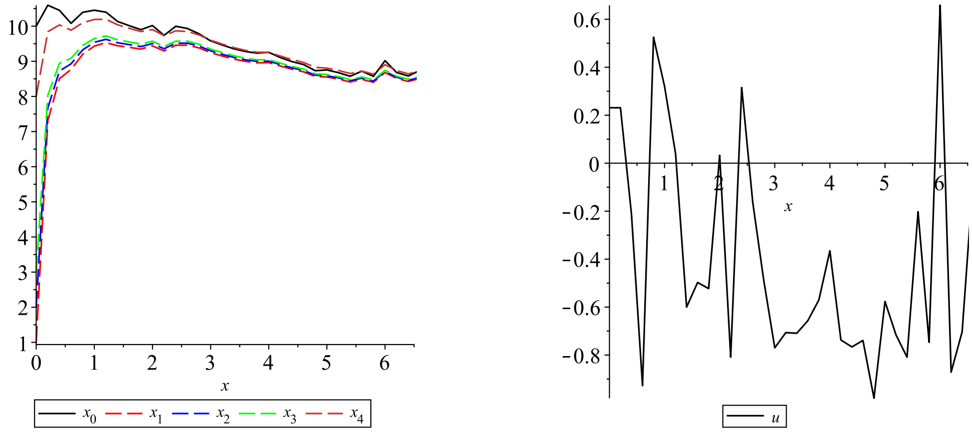

Fig. 7. Optimal control for $\alpha = 0.6$: time evolution of the FHK type system with the leader (left) and control (right) for $K = 1$.

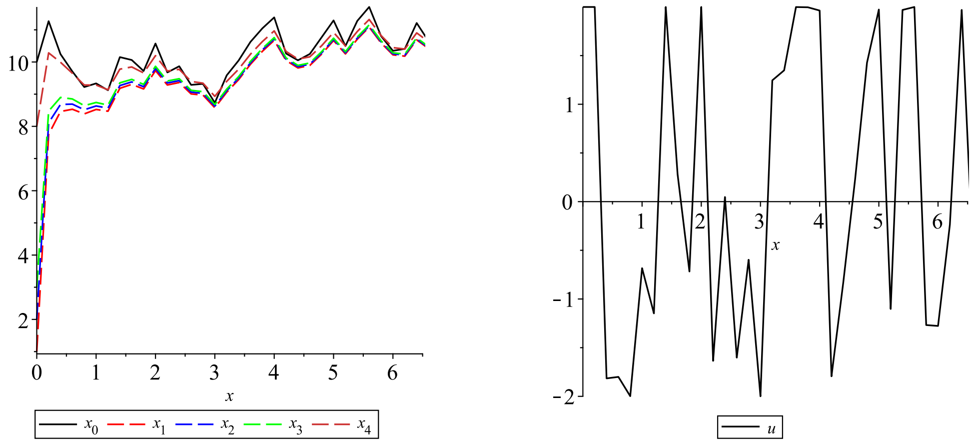

Fig. 8. Optimal control for $\alpha = 0.6$: time evolution of the FHK type system with the leader (left) and control (right) for $K = 2$.

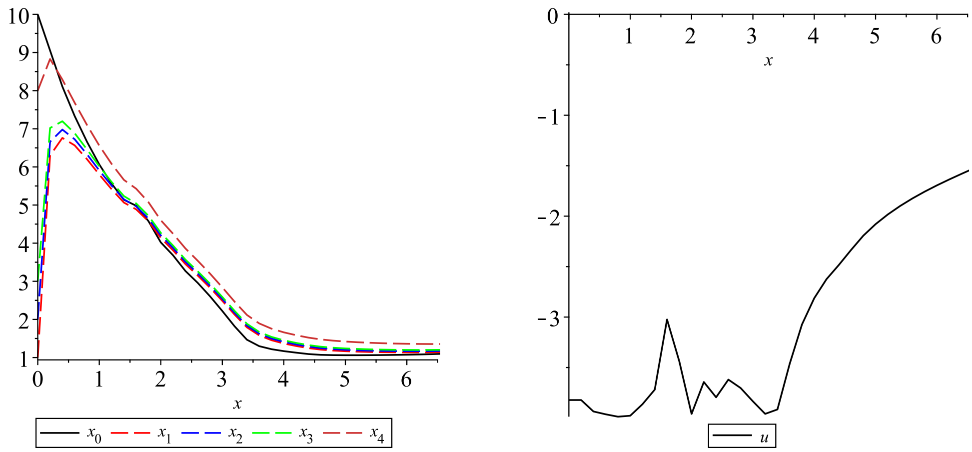

Fig. 9. Optimal control for $\alpha = 0.6$: time evolution of the FHK type system with the leader (left) and control (right) for $K = 4$.

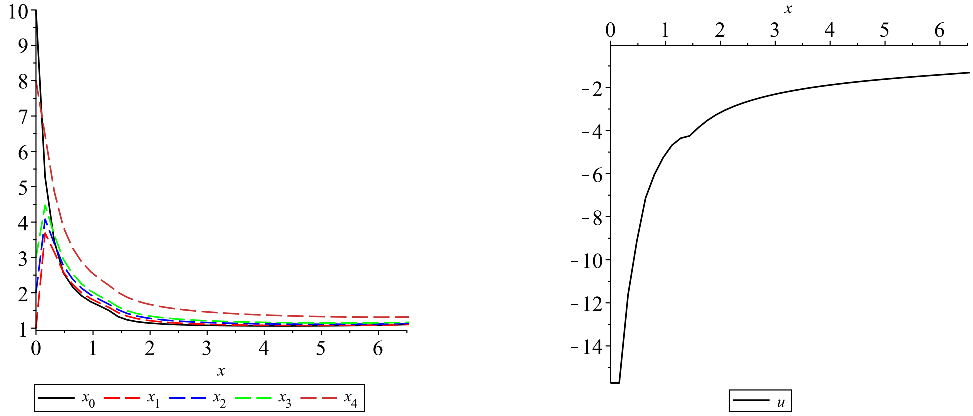

Fig. 10. Optimal control for $\alpha = 0.6$: time evolution of the FHK type system with the leader (left) and unbounded control (right).

Example 19. In this example, we take $N = 4$, $d = 1$, $\alpha = 0.6$, $M = \{z \in \mathbb{R} : |z| \leq K\}$ and consider the following system:

$$\begin{cases} \left({}^C D^{0.6}_{0+} x_0\right)(t) = u(t) \\ \left({}^C D^{0.6}_{0+} x_i\right)(t) = \sum\limits_{j \neq 0, i}^{4} a_{ij}(x_j(t) - x_i(t)) + c_i(x_0(t) - x_i(t)), & t \in [0, T] \ a.e. \\ (x_0(0), x_1(0), x_2(0), x_3(0), x_4(0)) = (10, 1, 2, 3, 8), \\ u(t) \in M \subset \mathbb{R}, \end{cases} \tag{28}$$

where a_{ij} and c_i are given by

$$a_{ij} = a\left(|x_i - x_j|^2\right), \quad c_i = \phi\left(|x_i - x_0|^2\right),$$

$$a(r) = a(r, 3, 2) = \begin{cases} 1, & 0 \leq r \leq 1, \\ \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{1}{r-1} + \frac{1}{r-2}\right), & 1 < r < 2, \\ 0, & 2 \leq r, \end{cases}$$

and $\phi(r) = \exp\left(\dfrac{-r^2}{10^2}\right)$. In addition, we want solution of (28) to minimize the following cost functional:

$$J(x, u) = \int_0^T \left(\frac{1}{32} \sum_{i,j=1}^{4} \left|x_i(t) - x_j(t)\right|^2 + \frac{1}{2} \sum_{i=1}^{4} |x_0(t) - x_i(t)|^2 + \frac{1}{2 \cdot 10^{10}} |u(t)|^2 \right) dt. \tag{29}$$

Clearly $\lim_{r\to\infty} r\phi'(r) = \lim_{r\to\infty} ra'(r) = 0$ and, by Theorems 12 and 17, solutions to system (27) and optimal control problem (27)–(29) exist. Moreover, a solution of (27)–(29) has to satisfy necessary optimality conditions presented in Theorem 14.

In Fig. 5 we can observe that approximate trajectory solutions to the system without leader and control, i.e.,

$$\begin{cases} \left({}^C D^{0.6}_{0+} x_i\right)(t) = \sum\limits_{j \neq 0, i}^{4} a_{ij}(x_j(t) - x_i(t)), & t \in [0, T] \ a.e. \\ (x_1(0), x_2(0), x_3(0), x_4(0)) = (1, 2, 3, 8), \end{cases} \tag{30}$$

do not converge to consensus. While in the presence of the leader and control (see Figs. 6–10) we observe convergence to leader-following consensus. Note that initial opinions of the agents and bounds imposed on control function influence the opinion evolution of agents. Figs. 6–10 show that in the case of the small amount of control leader's opinion is almost constant and followers' opinions approach its opinion, while with unbounded control or greater value of K, leader's opinion is time-varying and converges to opinions' of agents.

5. Conclusions

In this paper, we have studied the leader-following consensus problem for fractional opinion formation models with the weights quantifying the way the agents influence each other being functions of deviations of opinions between agents. In virtue of the fractional-order derivative, the memory effect was included during the evolution of opinions dynamics.

Based on the optimal control theory, we have provided necessary optimality conditions and existence conditions for "non-invasive" controls. Some numerical examples have demonstrated the validity of the proposed control strategy.

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