



DIGITAL TOOLS AND PAPER-AND-PENCIL IN SOLVING-AND- EXPRESSING: HOW TECHNOLOGY EXPANDS A STUDENT'S CONCEPTUAL MODEL OF A COVARIATION PROBLEM

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Abstract

This study aims at understanding the role of the tools chosen throughout the processes of solving a non-routine mathematical problem and communicating its solution. In assuming that problem-solving is a synchronous activity of mathematization and expression of mathematical thinking we take our proposed Mathematical Problem Solving with Technology (MPST) model to analyze the processes of solving-and-expressing-problems. Resorting to qualitative methods for data collection and analysis, we report on the case of an 8th grader working on a covariation problem to examine the role that paper-and-pencil and digital tools play in the development of a conceptual model of the situation. We found that the resources used throughout the solving-and-expressing activity influenced the depth of the conceptual model developed, within a process of progressive mathematization. Whereas paper-and-pencil led to the emergence of a conceptual model based on exploring particular cases, the digital transformation of the solution was triggered by the process of communicating its mathematical justification and expanded the previous model. Moreover, the complexity of this activity is evidenced by its multiple sequences of processes. Finally, the integration process seems crucial as the concomitant use of technological and mathematical resources precedes major advancements in the expansion of the conceptual model.

Keywords: mathematical problem-solving, conceptual model, covariation, paper-and-pencil, digital technology, techno-mathematical fluency

Abstrak

Penelitian ini bertujuan untuk memberikan pemahaman tentang peran pemilihan media teknologi dalam proses pemecahan masalah matematika non-rutin dan mengkomunikasikan solusinya. Diasumsikan bahwa pemecahan masalah adalah sinkronisasi aktivitas matematika dan ekspresi pemikiran matematis, kami menawarkan model *Mathematical Problem Solving with Technology* (MPST) untuk menganalisis proses pemecahan-dan-pengungkapan-masalah. Berdasarkan penggunaan metode kualitatif dalam pengumpulan dan analisis data, kami melaporkan kasus seorang siswa kelas 8 yang mengerjakan masalah kovarian untuk mengidentifikasi peran penggunaan kertas dan pensil serta alat digital dalam pengembangan model konseptual dari aktivitas ini. Kami menemukan bahwa sumber daya yang digunakan selama aktivitas pemecahan dan pengungkapan mempengaruhi kedalaman model konseptual yang dikembangkan dalam proses matematisasi progresif. Penggunaan kertas-dan-pensil menyebabkan munculnya model konseptual berdasarkan eksplorasi kasus-kasus tertentu, sedangkan solusi dari transformasi digital dipicu oleh proses mengkomunikasikan justifikasi matematisnya dan memperluas model sebelumnya. Selain itu, kompleksitas aktivitas ini dibuktikan dengan beberapa urutan prosesnya. Akhirnya, proses integrasi tampaknya penting, dikarenakan penggunaan sumber daya teknologi dan matematika secara bersamaan mengakibatkan kemajuan besar dalam perluasan model konseptual.

Kata kunci: pemecahan masalah matematis, model konseptual, kovarian, kertas dan pensil, teknologi digital, techno-mathematical fluency

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On the verge of the 4th industrial revolution, our highly mathematized digital world is prompting new discussions about how technologies affect perceptions of reality, social interactions, and even our cognition. For the past twenty years, researchers have set the alarm on the need to study the mathematical

understandings, abilities and practices developed by means of technological tools, required for thriving beyond school in a 21st century digitally-based society (Lesh, 2000; Noss, Healy, & Hoyles, 1997). However, research in other areas suggests that digital reading impairs students' understanding when compared to reading on paper (Halamish & Elbaz, 2020), and there is a lack of evidence for deciding to replace paper-and-pencil by digital writing in early writing instruction (Wollscheid, Sjaastad, & Tømte, 2016).

Within the mathematics education research area, results on the divide 'digital media' and 'paper-and-pencil use' seem controversial. Chan and Leung (2014) concluded that students working with a dynamic geometry environment (DGE) reached better results than those with paper-and-pencil. Barrera-Mora and Reyes-Rodriguez (2013) concluded that DGEs act as reorganizers of mathematical thinking, allowing cognitive activities only possible due to the dynamism embedded in the tool, leading to explore variation and covariation, and to different forms of justifying a conjecture and expressing it with the tool. Other studies point that the mathematics used by students working on a paper-and-pencil task is different from the one they apply when using a touchscreen device (Bairral, Arzarello, & Assis, 2017); students use similar strategies both with paper-and-pencil and an applet in solving equations (Jupri, Drijvers, & van den Heuvel-Panhuizen, 2016); there are benefits in both environments (Koyuncu, Akyuz, & Cakiroglu, 2015), or the two environments play a complementary role in mathematical learning, particularly in conjecturing and proving activities (Komatsu & Jones, 2020). Some researchers call for a "partnership of paper and digital settings" (Usiskin, 2018, p. 861), since the 'electronic partner' provides new affordances for real-world problem-solving, such as dynamic representations of variation.

Thus, the question on the effects of using paper-and-pencil and digital technology for solving mathematical problems remains unanswered. The need to know how technologies facilitate connections between seeing, doing and expressing (Noss, Healy, & Hoyles, 1997) is still on the agenda. Technology offers a wider range of possibilities for developing mathematical thinking, such as exploring, manipulating, identifying variants or invariants, triggering conjectures, fostering justifications and generalizations (Yao & Manouchehri, 2019). Moreover, the successful experiences with particular digital tools and their affordances integrate and reshape cognition, reconfigure the essence of mathematical thinking and activity, and can be transferred to other situations (Barrera-Mora & Reyes-Rodriguez, 2013; Borba & Villarreal, 1998).

Our work has been addressing the mathematical problem-solving with technology activity of 12-13-year-old students participating in a beyond-school web-based mathematical competition (<http://fctec.ualg.pt/matematica/5estrelas/subs/sub14.html>). Participants in the competition SUB14 (organized by the University of Algarve) had access to a mathematical problem posted online every two weeks and had to develop and submit a solution to the problem using the tools of their choice. Each solution should be submitted electronically and include a complete explanation of the processes followed. A selection of exemplary solutions was published on the website, portraying different approaches or representations, or showing the use of different tools (Carreira et al., 2016). Within this empirical context, our research aimed to understand the role of the tools chosen to solve and express mathematical problems, particularly when developing a conceptual model of the problem situation. The research questions that inform this study are: a) What is the role of the tool in

the processes of problem solving and expressing? and b) How does the tool affect mathematical thinking? The analysis presented in this paper draws on the case of one participant and focuses on the role and the effects of paper-and-pencil and digital tools in solving and explaining a covariation problem.

Problem-Solving as Mathematization and Expression of Mathematical Thinking

We endorse the idea that “mathematical problem-solving is about seeing (interpreting, describing, explaining) situations mathematically, and not simply about executing rules, procedures, or skills expertly” (Lesh & Zawojewski, 2007, p. 782). We claim that productive ways of thinking mathematically lead to developing procedures in a cyclic and iterative way (Carlson & Bloom, 2005) which incorporate an approach to find the solution and may be called as a conceptual model of the problem situation (Ekawati et al., 2019; Lesh & Zawojewski, 2007; Zawojewski & Lesh, 2003).

The non-routine word problems proposed by SUB14 are the type of tasks that encourage students to transform and extend earlier conceptual understandings and generate successful interpretations of the situations (Zawojewski & Lesh, 2003). The participants develop models of the situations integrating informal, experience-based strategies as a way of giving them meaning, shifting from the real world to the world of mathematical ideas (Brady, Eames, & Lesh, 2015) in a horizontal mathematization process. These models evolve as the focus shifts to the mathematical objects, relationships and procedures typical of vertical mathematization (Gravemeijer, 2005; Treffers, 1987). This transition involves moving away from the context, placing an emphasis on symbolization, looking for more formal strategies and relationships that support mathematical reasoning. The conceptual model assumes, gradually and progressively, the characteristics of a mathematical object. Conceptual models, that convey mathematical understandings of the situations, are usually “expressed using a variety of interacting representational media” (Lesh & Harel, 2003, p. 159), thus bringing about diverse representations such as paper-based diagrams or graphs, tables, written text, symbols, drawings, images, computer-based graphics, dynamic or hands-on figures.

As producing an explanation is paramount for this problem-solving activity, the descriptions reveal the ways the students interpret the situations and how their conceptual models work (Carreira et al., 2016). Obtaining solutions includes creating explanatory descriptions that “are not simply postscripts that students give after ‘the answer’ has been produced. They ARE the most important components of the responses that are needed” (Lesh & Doerr, 2003, p. 3, emphasis in the original). Mathematical problem-solving is conceptualized as the simultaneous activity of mathematization and expression of mathematical thinking (Carreira et al., 2016), which means that the metaphor ‘solving-and-expressing-problems’ refers to the development of a conceptual model of the situation and the explanation of the inherent reasoning processes.

Processes of Mathematical Problem Solving with Technology

Some models that focus on solving technology-based problems tend to disregard vital aspects of mathematical thinking, and lack components of mathematical problem-solving. Similarly, mathematical problem-solving models were mainly developed from paper-and-pencil work and do not account for the

affordances of digital tools or do not provide ways of explaining their role in the processes. But, “[w]hat happens when subjects use systematically computational tools to make sense of problem statement, represent, explore and solve problems?” (Santos-Trigo & Camacho-Machín, 2013, p. 282). And to what extent do previous frameworks still account for the mathematical problem-solving proficiency in the presence of digital tools (Barrera-Mora & Reyes-Rodríguez, 2013)?

Looking at the two complementary insights, we worked on a combination of two theoretical models: Martin and Grudziecki’s (2006) model for approaching a technological task or problem, and Schoenfeld’s (1985) model of mathematical problem-solving. The former addressed the meaning and operationalization of ‘digital literacy’ by describing the desired activity of a digital literate person when dealing with a digital problem. This model’s processes are: stating, identifying, accessing, evaluating, interpreting, organizing, integrating, analyzing, synthesizing, creating, communicating, disseminating and reflecting. The latter expanded Polya’s work by proposing a model for mathematical problem-solving that includes five stages: reading, analysis, exploration, planning and implementation, and verification. Based on our comprehensive knowledge about students’ problem-solving with digital technologies (Carreira et al., 2016), we explored the links between the processes of solving a digital problem and the stages of solving a mathematical problem, and tested it with a paradigmatic case (Jacinto & Carreira, 2017a). Table 1 presents the resultant model of Mathematical Problem Solving with Technology, comprising ten processes, where a description of the actions that characterize each process is offered as well as the backbone contributions from the two models used (signaled within the parenthesis). The MPST model looks at mathematical problem-solving as a flexible activity in that the set of processes may have clear, well-defined boundaries at their core, but are flexible enough to be considered in different phases, as recognized by Schoenfeld (1985).

Table 1. Processes of Mathematical Problem Solving with Technology (Jacinto & Carreira, 2017a)

Grasp	Appropriation of the situation and the conditions in the problem, and early ideas on what it involves. (Read ^a ; Statement ^b).	Communicate - Interact with relevant others whilst dealing with the problem or task. (Communication ^b).
Notice	Initial attempt to comprehend what is at stake, namely the mathematics that may be relevant and the digital tools that may be necessary. (Analysis ^a ; Identification ^b , Accession ^b).	
Interpret	Placing affordances in the technological resources in pondering mathematical ways of approaching the solution. (Analysis ^a ; Evaluation ^b , Interpretation ^b).	
Integrate	Combining technological and mathematical resources within an exploratory approach. (Exploration ^a ; Organisation ^b , Integration ^b).	
Explore	Using technological and mathematical resources to explore conceptual models that may enable the solution. (Exploration ^a ; Analysis ^b).	
Plan	Outlining an approach to achieve the solution based on the analysis of the conjectures explored. (Planning and Implementation ^a ; Synthesis ^b).	
Create	Carrying out the outlined approach, recombining resources in new ways which will enable the solution and create new knowledge objects, units of information or other outputs which will contribute to solve-and-express the problem. (Planning and Implementation ^a ; Creation ^b).	
Verify	Engaging in activities to explain or justify the solution achieved based on the mathematical and technological resources. (Verification ^a).	
Disseminate	Present the solutions or outputs to relevant others and consider the success of the problem-solving process. (Verification ^a ; Reflection ^b , Dissemination ^b).	

^a stage of mathematical problem solving as proposed by Schoenfeld (1985)

^b process of digital technology problem solving as proposed by Martin & Grudziecki (2006)

METHOD

This study aimed at examining the role of the tools in the development of a conceptual model of a covariance problem. Taking an interpretative stance that involved qualitative techniques for data collection and analysis (Merriam, 2009), we developed several cases of participants who usually resorted to a variety of digital tools to solve and express the problems of the competition.

Data collection included the participants' solutions produced throughout two annual editions of the competition, a video-recorded session of mathematical problem-solving, and an in-depth interview. The use of the think aloud protocol was pondered, since it has been used in studying mathematical problem-solving to gain insights into thinking processes (Kurniati et al., 2018; Montague & Applegate, 1993; Pugalee, 2004). It allows to retrace procedures otherwise hindered, i.e., failed attempts, hypothesis posed, the concepts or the technological tools used along the process. Even though it calls for the least interference possible from the researcher, the participants communicate with others fairly often during their activity. Still, obtaining verbal reports concurrent with the task was a priority, hence the participants in our research were overtly instructed to verbalize everything that came to mind, related to the problem, to mathematics or the tools while solving the problem. Inspired by Schoenfeld's (1985) method, questions such as 'what exactly are you doing?', 'why are you doing it?', or 'how does it help you?' were also considered and posed whenever further clarification was felt necessary. The moments of silence were overcome with prompts such as 'what are you thinking right now?', 'which path are you considering following?'

The Case of Beatrice

In this paper we examine the problem-solving-and-expressing activity of Beatrice (pseudonym), selected as a case of interest not only because she had experience in several editions of the competition but also due to her ability and willingness in using iconical representational affordances of digital tools, usually translating her thinking into visual and pictorial representations. Our analysis draws on the observation of her activity, at home, with the formal consent of her parents. Beatrice was asked to choose and solve one problem from three options posted at the competition's website; she was encouraged to perform as closely as possible to her usual problem-solving activity in SUB14, and to explain out loud her actions and thinking. Besides the observation and video recording of the problem-solving activity, including the voicing of her thinking and acting, we also gathered her written notes and recorded her computer screen.

Beatrice chose to solve the covariation problem 'Switching balls' (Figure 1). The ability to deal with such problems involves understanding the simultaneous change of two variables, usually time and distance, thus entailing cognitive activities that allow the coordination of the variables, changing in tandem, while assessing the ways in which each variable is changing in relation to the other (Carlson et al., 2002). The notion of covariational reasoning "holds a consistent connection with the creation of dynamic mental images, metaphorical reasoning, physical enactment and bodily referents" (Carreira et al., 2016, p. 176). It has a strong connection with visualization as it involves the construction of dynamic images of phenomena in motion and that requires a certain ability in translating the implicit dynamism in the situation to a static model

(such as an algebraic expression).


<p>Afonso and Bernardo live in the opposite ends of the same street. Afonso had a ball borrowed from Bernardo and Bernardo had another ball belonging to Afonso. They both left their homes, at the same time, to switch the balls. Bernardo's speed was twice the speed of Afonso until they meet on the street. As soon as they switch the balls, they go back to their homes, but Afonso's speed is now twice the speed of Bernardo. When Afonso got to his house, Bernardo was still 120m away from home. How long is the street?</p>	
<p>Do not forget to explain your problem-solving process!</p>	

Figure 1. Statement of the problem “Switching balls”

Data Analysis

The collected data was organized and analyzed using the qualitative data analysis software NVivo: the interview was transcribed and the overall data was segmented and coded. Data coding consisted in reading the transcripts and watching the videos repeatedly to inspect the participant's processes of problem-solving-and-expressing. Triangulation was carried out by comparing data from different sources (e.g., the interview's transcript, the video recording and the screen capture) in order to gain good understanding of the processes. Each author independently analyzed the data in light of our theoretical framework; the outcomes were then reviewed by both authors looking for consistency in the coding and interpretation, until reaching full agreement. To analyze the concomitant use of the technological and mathematical resources, while developing the conceptual model, we used the processes identified in the MPST model (grasp, notice, interpret, integrate, explore, plan, create, verify, disseminate, and communicate), where each process corresponded to a coding category. We continued our analysis by writing the case of Beatrice, which we consider to be a descriptive research product that resorts to a narrative form of speech (Dooley, 2002) to expose her processes of problem-solving-and-expressing.

RESULTS AND DISCUSSION

Beatrice, aged 13, usually showed a great concern with clarity, organization and completeness of the solutions submitted, by combining diagrams, images, colors and text within a presentation editor. She solved and expressed the problems of the competition in two steps, usually starting with paper-and-pencil and, afterwards, building a 'replica' of the solution with a digital tool.

B: First, I made a sketch on paper of how I was going to solve [the problem]. I started doodling there until I got an idea of how I left things ... then I went to the computer, opened PowerPoint and did it there, I copied it to the computer.

Beatrice seems eager to find a solution to the problems on the first day they are posted online, but she spends the remaining time working on the best way to present her solution. She often discusses her hesitations with friends, at school (*“I had the idea in a maths class ... I wasn't listening to the teacher”*)

[laughs]) or even during out-of-school meetings (“*On the day I started, I went out with my friends ... they gave me an idea, but I ended up with another one*”), thus supporting the idea that communication is a process that occurs throughout the problem-solving-and-expressing activity.

Beatrice started by reading carefully the problems posted online, trying to get some sense of the mathematical topics and possible approaches to each one, and looked for similarities with previous experiences in the competition. She chose the problem ‘Switching balls’ by noting that “*this one, kind of resembles the first one*” she had solved in the competition (*grasp*).

Solving-and-Expressing with Paper-and-Pencil

When asked about the approach she was imagining, she laughed, holding a paper: “*Drawings! ... Yes, drawings! I’m going to draw a house... very cute!*”, and went on reading the problem.

B: It says here they... live in opposite streets, right? Oh, on the opposite ends of the same street. So, one is here and the other one here ... hum, they had a ball, switched balls, there! Hum... they then got back... Bernardo’s speed was twice the speed of Afonso.

This out loud reading is an attempt to make sense of the essential conditions in the problem: the two friends live on opposite ends of the street and not on opposite streets, and there is a relation between the speeds at which they move (*notice*). While reading slowly, Beatrice begins to draw a first ‘diagram’ with paper-and-pencil, as if to bring the situation to a concrete form (*interpret*). Shortly after, she made another sketch (Figure 2) and decided to try out ‘arbitrary’ but plausible pairs of numbers for Afonso and Bernardo’s speeds, preserving the relationship, in a quest to make sense of the situation (*integrate*).

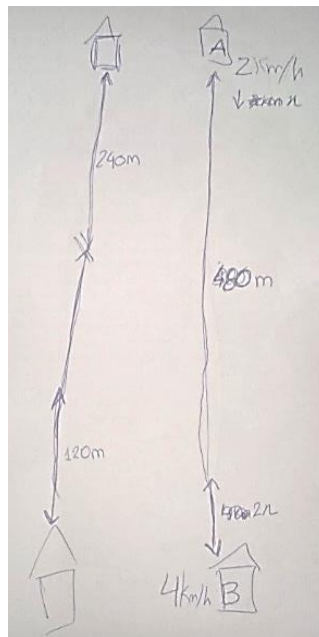


Figure 2. The two diagrams produced by Beatrice

She then picked a squared sheet of paper as it would be easier “*to compare, [as] the parts are more equal*”, while signaling with her thumb that she was comparing two lengths (*integrate*). After verbalizing a

certain discomfort with the problem, she kept thinking out loud on the statement: “*I only know the speed ... when they go get the balls, the speed of Bernardo is twice the speed of Afonso. And when they go back home it is the other way around*” (interpret). By then, Beatrice was already making sense of the first condition, connecting the speeds of Afonso and Bernardo, but she did not yet fully understand what ‘the other way around’ meant. Next, she constructed another diagram, on the bottom in [Figure 2](#).

Focusing on her drawings, Beatrice continued experimenting with a pair of chosen values, according to a given relationship to see if they made sense in the diagram (integrate). Realizing that these values didn’t work, yet not explaining why, she kept struggling to go beyond the first condition and consider what happens with the friends’ speeds on their way back home, independently from the third condition. She kept using a trial and error approach: “*now I will use 480m instead of 360 because... I can’t explain why... I’m using trials because I know they changed [speeds] but I can’t explain*”. In fact, these were not randomly choices as the numbers were multiples of the only value available in the statement: 120m.

B: It says here that when going back home... the speed... they changed, right? ... when Afonso arrived, Bernardo still had 120m to walk. Hum, while he walks 120, this one walked 240 ... But I think this isn’t it, yet, there’s something missing in between. But I’ll try it anyway ... I’ll consider 360 as ... the length of the street.

R: Why did you choose that value?

B: Because... Bernardo still had 120 meters to walk and while he walks 120, the other one walks 240... I don’t know... This is how I’m thinking.

R: Where does the 360 come from?

B: From adding those two. But this way I’m pretty sure there is something missing in between, because this would mean Bernardo hasn’t started to walk. So, this can’t be right.

Beatrice looked for realistic values to be able to determine their plausibility as a solution. Hesitant about the use of the speed of 4km/h, she referred to a similar problem that she had solved in the competition.

B: [Time] is important if it were as in the other [problem of SUB14] ... one walks twice as fast as the other, right? Well, it would be cool to know the time. Actually, time is important. Oh my, I don’t know where the other problem is. I saved it...

She kept insisting on the first condition of the problem, deepening her understanding about the displacements in the same time interval. As she couldn’t find her own solution to the similar problem mentioned, she decided to analyze other students’ solutions that had been published online (communicate). Although showing some anxiety, as the problems were rather different, she reminded herself of using the least common multiple, and tried to use it in this new situation (integrate).

B: I’m now thinking of something else. In that problem, the first one ... I used the least common multiple of the speeds ... It was 5 and 4, wasn’t it? I calculated the least common multiple of 5 and 4, which is 20. It could work now. I don’t know... 20km is quite a bit for a street, so I think I will use meters instead.

The least common multiple emerged as an important mathematical resource. Beatrice tried the pairs 20 and 40 and after some time observed that the street could be 480m long, but then Bernardo would not be

120m away from home when Afonso arrived at his. About to give up, Beatrice tried another pair of values, 80 and 160, which would lead to the solution of the problem if she had managed to simulate the inbound journey. However, she was unable to establish a relationship between the results obtained and the expected one, so she continued to pursue another hypothesis. At some point, Beatrice explained “*I am doing the same as always, that is, when there is the least common multiple, I double [the value] and add them, and that is the [street’s] length*”. These are evidences that a conceptual structure of the situation is being generated, although at a very early stage due to the lack of coordination of the two parts of the journey.

About 1h30 after, Beatrice gave up and tried to solve another problem of the options presented but quickly concluded that they were also difficult. After reassuring Beatrice it would be alright if she would like to end the session, the interviewer presented a diagram, similar to the ones she had been experimenting with, and invited her to analyze it (*communicate*).

Beatrice appeared quite surprised by the simplicity of the diagram (Figure 3), drawn with a pencil on a sheet of paper: “*it is more or less what I was doing, isn’t it?... only... there are no numbers, right?*”. By looking attentively to the diagram, she identified A and B as the houses of Afonso and Bernardo, understood that the arrows above the squares referred to their displacements in the first part of the journey, and identified the 120m of the street that Bernardo had to walk at the end of his journey (*interpret*). Surprisingly, however, Beatrice wasn’t able to recognize this distance as $\frac{1}{2}$ of the street’s length, hence still couldn’t see the solution: “*But now how I get to the numbers... I don’t know*”.

Recognizing that Afonso kept the same speed in the inbound journey as in the outbound, Beatrice was making sense of the second condition. After a while she realized that if each arrow corresponded to a displacement of 3, on the inbound journey, Afonso’s travel was represented by one arrow and Bernardo’s by an arrow with half of that size. As the diagram depicts the displacement unit as an arrow (corresponding to 3 squares), Beatrice was struggling to give meaning to ‘one displacement and a half’ (*interpret*), and decided to resort to experimentations with numbers (Figure 3, in blue). She then engaged in building an enlargement of the diagram (*integrate*), which lead her to a more inquiring approach since she already had access to a resource she could fairly understand but needed to recreate on her own.

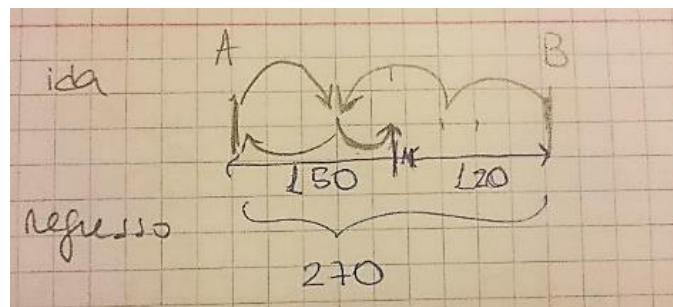


Figure 3. Diagram presented (pencil) with values registered by Beatrice (blue ink)

After building a new diagram (Figure 4), she found a flaw when trying to explain the result of 270m (*explore*): Afonso’s inbound journey corresponded to two arrows to the left, while Bernardo’s was not

completely represented. By completing the diagram (*integrate*), she observed that the 120m missing corresponded to three units in a total of six, thus finding the solution (*explore*).

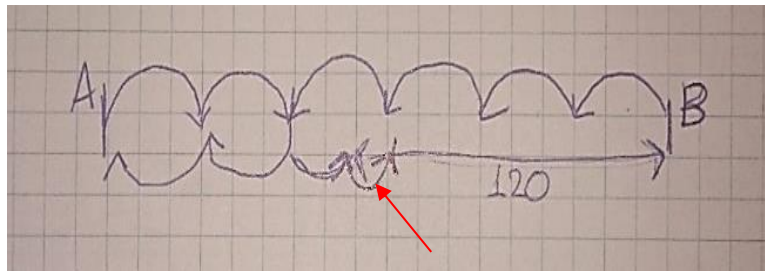


Figure 4. Larger diagram built by Beatrice (signaled the arrow drawn afterwards)

After this experiment, with the speeds of 2 and 4, Beatrice went back to the previous diagram (with the relation 1:2) and was then able to tell that when Afonso arrived at home, Bernardo was exactly at the middle of the street: “Yes, yes! It’s 240!”. This means she went back to confirm that the conceptual model obtained from the enlarged diagram was still accurate in that previous diagram. Thus, she realized the possibility of generalizing the result, since it was obtained based on the relationships and not on particular speeds or displacements (*explore*).

Solving-and-Expressing with a Presentation Editor

The construction of a digital solution was carried out using a presentation editor (PowerPoint), a text editor, and the Internet. Beatrice explained that what she was about to do “*is not very different from the diagram... I’ll do about the same [as with paper-and-pencil]*”, that is, she intended to replicate the solution constructed previously (*plan*). The first slides included a summary of the relevant information in the problem statement. She googled an image of a house, pasted it on the top left side of a new slide, reduced its size, duplicated it and placed the other image on the bottom right side, changed their color and labelled them A and B (Figure 5). The diagram reproduced the displacement of the two friends while walking towards each other following the initial conditions: as Bernardo walks twice as fast as Afonso, she inserts two arrows with equal direction and orientation from B to A; and another one with the same length and direction but orientation from A to B. As the two friends get together to switch the balls where the green arrow meets the blue one, Beatrice placed the beginning of the inbound journeys, roughly, at that point. Afonso reaches his house with a single displacement (black arrow) but as Bernardo’s speed is now half of Afonso’s, in the same amount of time, his displacement is half of the previous one (smaller black arrow). She inserted a purple dashed line to represent the 120m that Bernardo still had to travel (*integrate*).

Above, on the right, she included relevant data for a comprehensive reading of the diagram, namely, the speeds of Afonso and Bernardo, and the indication that they both left home at the same time. She also defined the equal displacements using the unknown x (*explore*) and included labels on the lines and arrows in the diagram using the Equation Editor tool (from Word) to insert the appropriate mathematical symbols (*integrate*). When asked about the things she was doing differently, she replied:

B: Yes, as I work, sometimes I use other things to explain better. I was thinking that it might be best to explain with an x , for instance, if Afonso walked x , then Bernardo walked $2x$. When they go back home ... if Afonso walked x , Bernardo walked half x . [If] by walking x the other one reached home... after Bernardo walked $\frac{1}{2}$ of x it was missing $120m$, this part [pointing at the screen], right?

Beatrice has embraced the paper-and-pencil diagram to such an extent that she was able to use it as a model to explain how she achieved the solution. But the digital diagram, that encloses technological entities that convey a mathematical meaning (e.g., lengths, colors), was also becoming part of her techno-mathematical lexicon (*integrate*).

The approach continued to develop as Beatrice wrote the relations in the text box, changed the color of the lines matching blue to Bernardo's journey, his house color, and green to the displacement of Afonso, and added a text box on the bottom left side to explain the inbound journey (Figure 6). Then, she signalled Bernardo's position when he is at $120m$ from his home (*explore*) and inserted a dashed line perpendicular to his displacement to show that it matches half of his second displacement in the outbound journey, thus arriving at the middle of the street when Afonso reached his house (*integrate*). The construction of the digital diagram triggered an expansion of the conceptual model of covariance, previously developed with paper-and-pencil, now improved with algebraic expressions (*explore*).

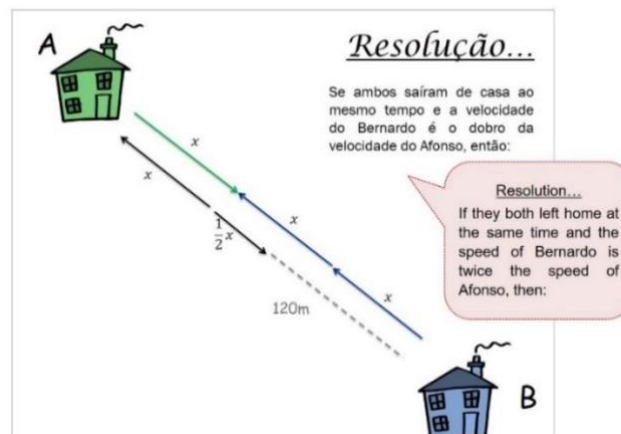


Figure 5. Intermediate phase of the construction of the digital solution (with translation)

Beatrice was developing her solution digitally, as this was not a mere reproduction of the approach rehearsed with paper-and-pencil; instead, she was coordinating technological and mathematical resources to visually communicate powerful ideas: the oriented segments with a particular length, direction, and orientation represent the displacements of the two friends on the outbound and inbound journeys, mathematized by associating them with a label containing algebraic expressions that exhibit their relationships (x and $\frac{1}{2}x$). These techno-mathematical entities are new knowledge objects that unveil Beatrice's way of thinking and understanding this conceptual model of the situation (*create*).

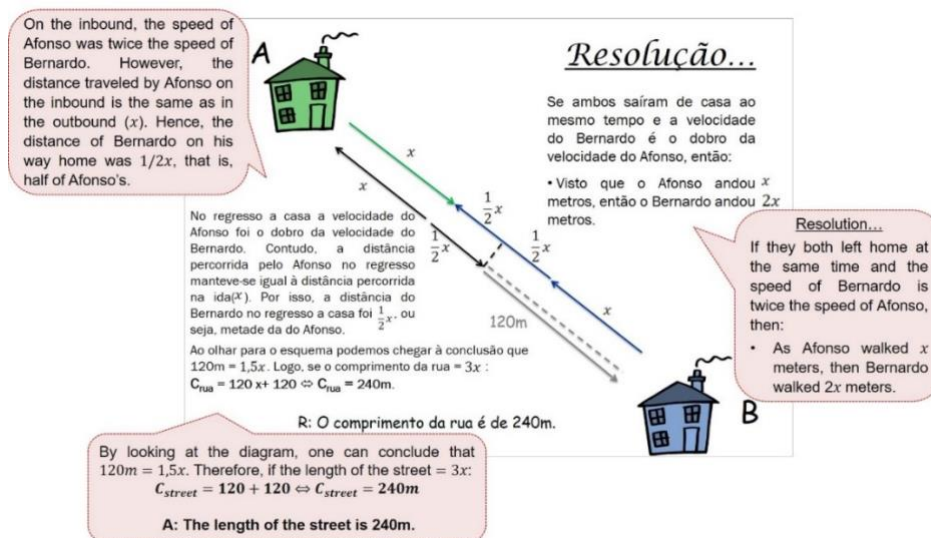


Figure 6. Solution submitted with translation of the written explanation

She combined text with algebraic expressions. From the analysis of this diagram she noted that $120m = 1,5x$ and that the total length of the street is $3x$. Considering this mathematization, she could have found the value of x from the first expression and, from there, the length of the street. Her choice of a different approach suggests that the strong visual perception afforded by the paper-and-pencil trials led her into recognizing the $120m$ left to walk as half the street. She continued by adding 120 with 120 , concluding that the street length was $240m$. With this process of composing a digital solution, using mathematical resources, Internet, PowerPoint and Word resources, Beatrice explained her reasoning through a diagram that evolves into a visual argument supported by the written justification (*verify*).

After almost 3 hours of work, Beatrice still insisted in formatting text and googling images related to the context to further elaborate her presentation (*dissemination*). Despite feeling tired, Beatrice explained her problem-solving-and-expressing activity based on the digital diagram, expressing with great clarity her understanding regarding the covariation situation, namely in explaining how the speeds of the two friends were changing. At a first glance, the second condition suggested new variables, which became a considerable obstacle to Beatrice. She explained:

B: Since Afonso walked x , then Bernardo walked two x , that is, two arrows. Afterwards, I dealt with the inbound journey. There was a switch here, the speed of Afonso becomes twice as that of Bernardo. Hum... but Bernardo kept his speed constant, that is, it didn't change, and he walked exactly the same distance, so it is still x . So we have to make changes in [the speed of] Bernardo, that is, he begins to walk at half of his [previous] speed, meaning that he walked half the distance than Afonso, hum, that is half x .

Beatrice's activity entailed: i) the experiences with numbers, which afforded a solid understanding of the relationship between the speeds and displacements on the outbound journey; ii) the construction of the enlarged diagram, which stabilized previous understandings and allowed to recognize that $120m$ is half of the length of the street; iii) the construction of the digital diagram, that

drove a vertical mathematization of the situation. Figure 7 summarizes schematically the processes of solving-and-expressing the problem, addressing the activity of Beatrice with paper-and-pencil in light orange, and with digital tools in blue.

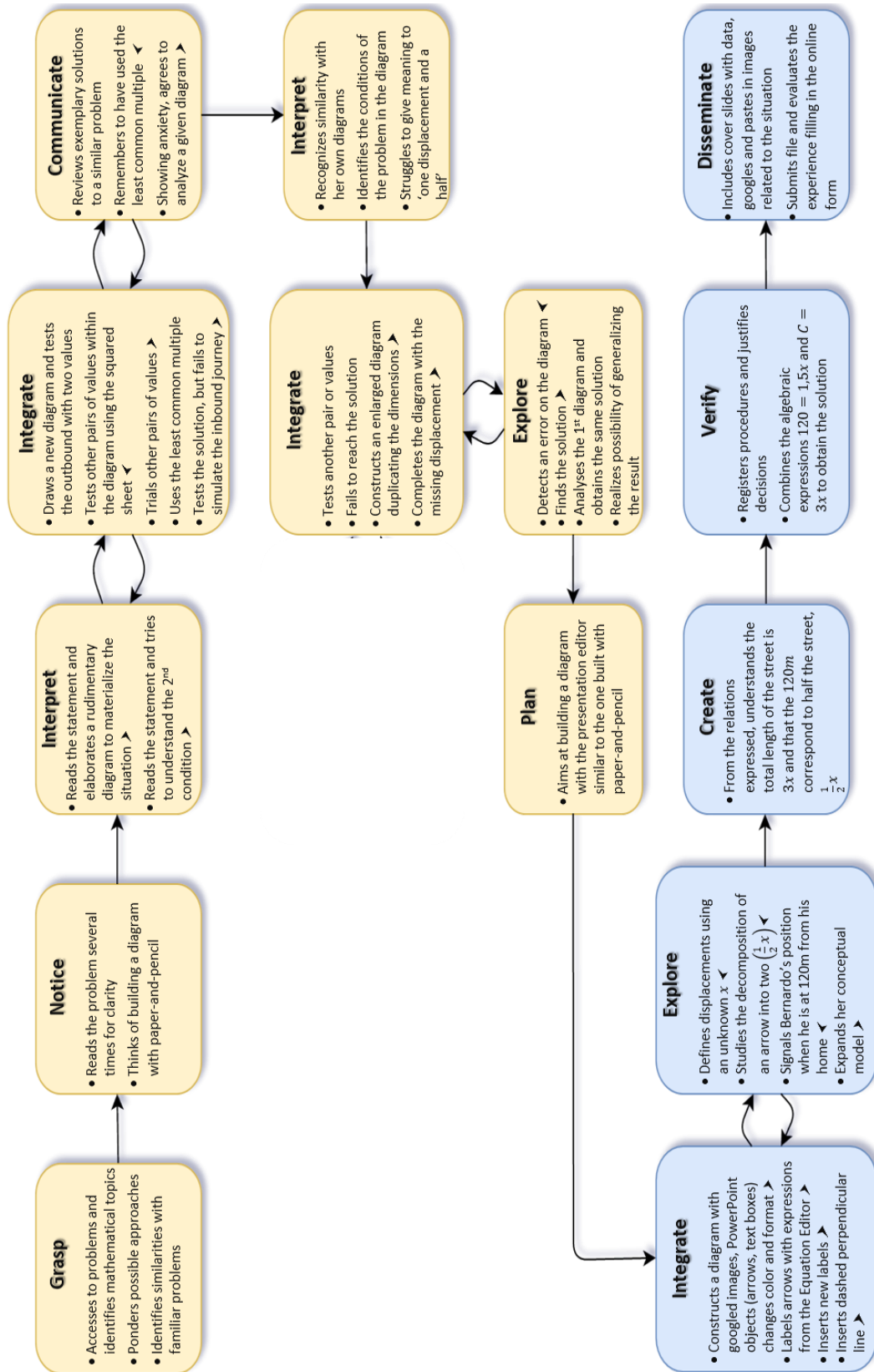


Figure 7. Processes of solving-and-expressing the problem "Switching balls"

CONCLUSION

Through a paradigmatic case we analyzed the emergence and development of a conceptual model of a covariation problem, focusing on the use of two tools: paper-and-pencil and digital technologies. In the following sections we address the main questions we sought to answer: the role of the tools in the processes of problem-solving-and-expressing and the ways in which the tools affect mathematical thinking.

Digital Tools and Paper-and-Pencil in Solving-and-Expressing: The Role of the Diagrams

We have identified two moments in the activity of Beatrice, discernible by the technology she uses: firstly, paper-and-pencil and, afterwards, a presentation editor combined with other digital resources. The case of Beatrice shows how mathematical thinking can be enhanced and deepened when timely expressed through digital technology, similarly to the findings of Barrera-Mora and Reyes-Rodríguez (2013). It also supports our claim that solving the problem and expressing the solution are often entangled processes. The trials with pairs of numbers and the paper-and-pencil diagrams allowed to deepen the understanding about the three conditions, triggering the development of a conceptual model within a horizontal mathematization activity (Gravemeijer, 2005; Treffers, 1987).

The need to construct a digital diagram drove a connection between mathematical knowledge (algebraic relations, solving equations) and technological knowledge (oriented segments, lines, colors with PowerPoint, use of Equation editor and the Internet) that provided robustness and formalism to the conceptual model, transforming it into an authentic mathematical model, within a vertical mathematization activity. This is consistent with the evolving nature of mathematical explanations, which Jones (2000) described as “progressive mathematization” processes.

The Expansion of the Conceptual Model: The Mediating Role of the Digital Tools

The data revealed the mediational role of the tools (Hoyles & Noss, 2009) during the activity of solving and expressing, namely, in the mathematization processes they support and their contribution to the development of a conceptual model of the situation. Beatrice came across some constraints in recognizing the covariational relation embedded in the situation which blocked her activity, leading to failed attempts, struggling to coordinate the journeys, as Carlson et al. (2002) also reported. She experimented with numbers without proper understanding of how the variables were related, only realizing that the two journeys could be described in terms of Bernardo’s speed when constructing the paper-and-pencil enlarged diagram. The abstraction of concrete objects was achieved with the formalism of the algebraic language, within the digital environment. Moving away from a paper-and-pencil to a digital diagram resulted in a progressive mathematization of the situation (Jones, 2000), intrinsically supported by the techno-mathematical resources developed. The visual model carried out within the digital environment uncovers a mathematical structure less dependent on concrete dimensions and rather more algebraic and abstract, thus revealing the underlying conceptual

mathematical structure. Regardless of using her visualization abilities to express mathematical thinking with digital tools, her complete solution suggests that she recognized limitations to the visual approach, hence added an algebraic approach that offered a more robust model of the problem. Yet, as the solution emerges with the construction of the expanded diagram she begins to ‘coordinate the variables’ – the first level of covariational reasoning as defined by Carlson et al. (2002). Such difficulties are frequently overcome with the construction of diagrams that aim at reproducing the dynamical nature of situation (Carreira et al., 2016).

Techno-Mathematical Fluency: The Combined Use of Paper-and-Pencil and Digital Tools

The resources used along the processes of solving-and-expressing the problem seem to influence the depth of Beatrice’s conceptual model. As the techno-mathematical resources are being integrated, allowing her to think-with-diagrams, they become increasingly independent from particular dimensions, and the model evolves to a greater abstraction level. This is in line with results indicating that students’ abilities in finding appropriate representations affect their competency in problem-solving (Santia et al., 2019).

The progression in the level of techno-mathematical thinking reflects the relationship between the solver’s mathematical abilities and her capacity to understand the digital tools’ affordances, that is, her techno-mathematical fluency (Jacinto & Carreira, 2017b). Beatrice’s techno-mathematical fluency emerges from her problem-solving-and-expressing activity and the need to produce a techno-mathematical solution, leading her to perceive useful ways to combine knowledge about digital technology (search images online; insert and format arrows, lines, images; format labels, use of color) with knowledge of mathematical facts and procedures (diagrams; multiples, least common multiple; variable, algebraic expressions; linear equations). Their simultaneous and connected use increases her understanding of the problem: through testing hypothesis, formulating a general solution, and preparing a justification. Although the conceptual model arises from particular trials, it is the digital transformation of the solution that impels going beyond the concrete cases while looking for a mathematical explanation of the solution. Similar abilities have been recognized in the literature as ‘techno-mathematical literacies’ (Hoyles et al., 2010) or ‘mathematical digital competence’ (Geraniou & Jankvist, 2019), though in different research contexts.

The MPST model allows to understand how the development of a mathematical way of thinking about the problem and communicating the solution takes place, both by means of those tools. Several micro-cycles revealed by the MPST model were related to the tool and to the mathematization its use enables, adding to the results obtained by Carlson and Bloom (2005) who reported on the cyclic nature of problem solving. A first micro-cycle consists of *interpret-integrate-communicate* where paper-and-pencil is used for drawing a diagram and affords trying particular cases of multiple numbers. Paper-and-pencil continues to support the development of a conceptual model throughout the loop *integrate-explore*, with the construction of the enlarged diagram and verification of the guessed solution. Both

these micro-cycles are related with the construction of a concrete model of the situation, thus, within a horizontal mathematization. As the work is mediated by the digital tools, a third micro-cycle emerges revolving around the processes *integrate-explore*. Digital tools support an exploratory activity; the mathematical meaning is conveyed visually through the digital diagram and symbolically through the use of algebraic expressions.

To *integrate* – technological and mathematical resources – is decisive in the problem-solving-and-expressing activity: not only it is occurring in the micro-cycles but it also precedes major advancements in the expansion of the conceptual model. Being techno-mathematical fluent implies to know about a particular type of technology, to know certain mathematical ideas or procedures that can be mobilized with that technology, and to find productive ways of combining them to generate new objects of knowledge – which happens during the process *integrate*. Hence, techno-mathematically fluency plays a crucial role in the activity of mathematical problem-solving-and-expressing with technology.

Final Remarks and Future Work

This study shows that the resources used by an 8th grader throughout the solving-and-expressing activity influence the depth of the conceptual model developed, within a progressive mathematization process. While paper-and-pencil afforded the emergence of a conceptual model based on the exploration of particular cases, the use of digital tools triggered a mathematical justification and expanded the previous model. The case also illustrates the complexity of this activity by evidencing its cyclic nature, where the integration of technological and mathematical resources emerges as a fundamental aspect, as it precedes major advancements in the development of the conceptual model.

The MPST model proves to be a robust analytical tool to examine and understand the role of the resources used by the solvers throughout the several processes of the problem-solving activity. Still, it is important to acknowledge two methodological limitations to this study. Firstly, this paper is focused on one particular case of a student working on a mathematical problem with the tools of her choice. However, the results are in line with previous findings drawing on the MPST model: the cases of Jessica solving-and-expressing with GeoGebra (Jacinto & Carreira, 2017a) and of Marco solving-and-expressing on the screen (Carreira & Jacinto, 2019) explained how these youngsters develop their activity from the early and continuing interplay between mathematical skills and the tools' affordances. Secondly, the diagram presented was intended to unblock the situation but may have influenced Beatrice's thinking. Despite the fact that these interactions with the interviewer resemble the kind of communicational interactions that she reports to establish with her colleagues, it triggered an exploratory activity that led her into devising her personal understanding of the problem since the solution she produced with digital tools entails a conceptual model that is mathematized quite differently.

The case of Beatrice offers a contribution on the role of tools, exposing that both thinking with

paper-and-pencil and thinking with digital tools are fundamental for the progressive mathematization and the development of a conceptual model. Although previous findings found benefits in both kinds of tools and refer to a partnership between paper and digital technology, further research is needed. Firstly, it would be interesting to look closely at the mathematical problem-solving (with technologies) occurring in the classroom. Secondly, our research has been focused on cases of students participating individually in the competition but it would be useful to know if the MPST model still accounts for collaborative activity. And, thirdly, it may be argued that students engage in a techno-mathematical thinking when developing and expressing mathematical reasoning mediated by digital tools, which opens another promising research avenue.

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