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Numerical approximation of second-order boundary value problems via hybrid boundary value method

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Abstract. Hybrid Boundary Value Method (HyBVM) is a new scheme, which is based on Linear Multistep Method (LMM). The HyBVM is the hybrid version of the Boundary Value Methods (BVMs) which are methods derived to overcome the limitations of the LMMs. This new scheme shares the same characteristic with the Runge Kutta method as data are utilized at off-step points. In this work, we apply this method to two second order Boundary Value Problems (BVPs) with mixed boundary conditions and the results are efficient when compared to other BVMs in literature.

Keywords: Hybrid BVM; Linear Multistep Method; Boundary Value Problem; Boundary Value Method

1. Introduction

Boundary Value Problems (BVPs) arise from the modelization of real world phenomena and are applicable in Sciences and Engineering. They are more difficult to handle compare to the Initial Value Problems (IVPs) and are usually solved by reducing the BVP to IVP.

Boundary Value Methods (BVMs) are methods based on Linear Multistep Methods (LMMs) used for the numerical approximation of Differential problems. They were proposed to overcome the limitations of the LMMs. That is, the application of the shooting method to BVP, which requires that the BVP is first converted to IVP before solving [1].

In this work, a new class of BVMs called Hybrid Boundary Value Methods (HyBVMs) is introduced and also applied to second-order linear and nonlinear BVPs. These methods are also based on LMMs, where data are used at and off-step points.

The derivation is achieved by the generalization of the Numerov method by interpolating and collocation procedures. These methods are then applied and implemented as BVM and used as a numerical integrator for the BVP of the form:

$$\left.\begin{aligned} &y''(x) = f\left(x, y(x), y'(x)\right) \\ &a_0 y(0) - b_0 y(0) = \alpha_0 \\ &a_1 y(1) - b_1 y(1) = \alpha_1 \end{aligned}\right\} \tag{1}$$

where $a_0,\ a_1,\ b_0,\ b_1,\ \alpha_0,\ \alpha_1$ are constants and f is a continuous function which satisfies the conditions for existence and uniqueness of solutions.

Many researchers have proposed different hybrid methods for the approximation of differential equations and their properties have also been fully discussed [2 – 6]. The application of BVMs for the numerical integration of BVPs was first proposed by Brugnano and Trigiante with their first two symmetric schemes: the Extended Trapezoidal Rule (ETR) and the Top Order Method (TOM). BVMs have been applied to different differential equations and their properties fully discussed [7 – 21].

2. Derivation of Methods [20]

In this section, the objective is to derive a LMM and derivative formula of the form:

$$y_{n+k}+\sum_{i=0}^{k-1}\alpha_i y_{n+i}=h^2\sum_{i=0}^{k}\beta_i f_{n+i}+h^2\sum_{v_i}\beta_{v_i} f_{n+v_i} \tag{2}$$

$$hy'_{n+k}+\sum_{i=0}^{k-1}\alpha'_i y_{n+i}=h^2\sum_{i=0}^{k}\beta'_i f_{n+i}+h^2\sum_{v_i}\beta'_{v_i} f_{n+v_i} \tag{3}$$

We start the process of derivation by seeking to approximate the analytical solution $y(x)$ by a continuous method $Y(x)$:

$$Y(x)=\sum_{i=0}^{r+s-1}\beta_i P_i(x) \tag{4}$$

where $r,\ s$ are the number of interpolation and collocation points, $P_i(x)$ are the polynomial basis of degree $r+s-1$. A k-step multistep collocation method is then constructed from:

$$Y(x)=V^T\left(M^{-1}\right)P(x) \tag{5}$$

where

$$P(x_n)=\left[P_0(x),P_1(x),P_2(x),\ldots,P_{r+s-1}(x)\right] \tag{6}$$

$$V=\left[y_n,\ldots,y_{n+r-1},f_n,\ldots,f_{n+s-1}\right] \tag{7}$$

$$M=\begin{pmatrix} P_0(x_n) & P_1(x_n) & \cdots & P_{r+s-1}(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ P_0(x_{n+r-1}) & P_1(x_{n+r-1}) & \cdots & P_{r+s-1}(x_{n+r-1}) \\ P''_0(x_n) & P''_1(x_n) & \cdots & P''_{r+s-1}(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ P''_0(x_{n+s-1}) & P''_1(x_{n+r-1}) & \cdots & P''_{r+s-1}(x_{n+s-1}) \end{pmatrix} \tag{8}$$

Which results into a continuous LMM

$$Y(x)=\sum_{i=0}^{r-1}\alpha_i(x)y_{n+i}+h^2\sum_{i=0\left(\frac{1}{2}\right)}^{s-1}\beta_i(x)f_{n+i} \tag{9}$$

Where $\alpha_i(x),\beta_i(x)$ are continuous coefficients to be determined. This is then used to generate the discrete LMMs of the form (2) and other additional methods. These equations are then applied simultaneously to solve (1).

2.1. *Specification of the Methods [20]*

Consider the case with the specification $r=2$ and $s=5$ using (4) we have the polynomial of degree $r+s-1$:

$$Y(x)=\sum_{i=0}^{6}\beta_i P_i(x) \tag{10}$$

which yield the following vectors and collocation/interpolation matrix:

$$P=\left[1,x,x^2,x^3,x^4,x^5,x^6\right] \tag{11}$$

$$V=\left[y_0,y_1,f_0,f_{\frac{1}{2}},f_1,f_{\frac{3}{2}},f_2\right] \tag{12}$$

$$M=\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 & x_0^5 & x_0^6 \\ 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & x_1^6 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 & 30x_0^4 \\ 0 & 0 & 2 & 6x_{\frac{1}{2}} & 12x_{\frac{1}{2}}^2 & 20x_{\frac{1}{2}}^3 & 30x_{\frac{1}{2}}^4 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 & 30x_1^4 \\ 0 & 0 & 2 & 6x_{\frac{3}{2}} & 12x_{\frac{3}{2}}^2 & 20x_{\frac{3}{2}}^3 & 30x_{\frac{1}{2}}^4 \\ 0 & 0 & 2 & 6x_2 & 12x_2^2 & 20x_2^3 & 30x_2^4 \end{pmatrix} \tag{13}$$

These are then substituted into the equation below:

$$Y(x)=V^T\left(M^{-1}\right)P(x) \tag{14}$$

Which results into a continuous LMM

$$Y(x)=\left\{\begin{array}{l} \dfrac{(h-x+x_0)y_0}{h}+\dfrac{(x-x_0)y_1}{h}-\dfrac{f_0}{360h^4}(x-x_0)(h-x+x_0) \\ \begin{pmatrix} 53h^4+123h^2(x-x_0)^2-52h(x-x_0)^3 \\ +8(x-x_0)^4+127h^3(-x+x_0) \end{pmatrix} \\ -\dfrac{f_{\frac{1}{2}}}{45h^4}\begin{pmatrix} 18h^5(x-x_0)-60h^3(x-x_0)^3+65h^2(x-x_0)^4 \\ -27h(x-x_0)^5+4(x-x_0)^6 \end{pmatrix} \\ +\dfrac{f_1}{60h^4}\begin{pmatrix} 5h^5(x-x_0)-60h^3(x-x_0)^3+95h^2(x-x_0)^4 \\ -48h(x-x_0)^5+8(x-x_0)^6 \end{pmatrix} \\ -\dfrac{f_{\frac{3}{2}}}{45h^4}\begin{pmatrix} 2h^5(x-x_0)-20h^3(x-x_0)^3+35h^2(x-x_0)^4 \\ -21h(x-x_0)^5+4(x-x_0)^6 \end{pmatrix} \\ +\dfrac{f_2}{360h^4}\begin{pmatrix} 3h^5(x-x0)-30h^3(x-x_0)^3+55h^2(x-x_0)^4 \\ -36h(x-x_0)^5+8(x-x_0)^6 \end{pmatrix} \end{array}\right\} \tag{15}$$

The main method (16) is then obtained by evaluating (15) at x_{n+2} :

$$y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{60}\left[f_n+26f_{n+1}+f_{n+2}+16\left(f_{n+\frac{1}{2}}+f_{n+\frac{3}{2}}\right)\right] \tag{16}$$

This main method is then used together with the initial conditions (17 – 19):

$$y_{\frac{1}{2}} - \tfrac{1}{2} y_0 - \tfrac{1}{2} y_1 = \frac{h^2}{1920}\left[-19 f_0 - 14 f_1 + f_2 - 204 f_{\frac{1}{2}} - 204 f_{\frac{3}{2}}\right] \tag{17}$$

$$y_{\frac{3}{2}} + \tfrac{1}{2} y_0 - \tfrac{3}{2} y_1 = \frac{h^2}{1920}\left[17 f_0 + 402 f_1 - 3 f_2 + 252 f_{\frac{1}{2}} + 52 f_{\frac{3}{2}}\right] \tag{18}$$

$$y_{\frac{3}{2}} + \tfrac{1}{2} y_0 - \tfrac{3}{2} y_1 = \frac{h^2}{1920}\left[17 f_0 + 402 f_1 - 3 f_2 + 252 f_{\frac{1}{2}} + 52 f_{\frac{3}{2}}\right] \tag{19}$$

and with the following derivative formulas (20 – 23):

$$hy'_{\frac{1}{2}} + y_0 - y_1 = h^2\left[\frac{13 f_0}{480} - \frac{f_1}{10} - \frac{7 f_2}{1440} + \frac{7 f_{\frac{1}{2}}}{144} + \frac{7 f_{\frac{3}{2}}}{240}\right] \tag{20}$$

$$hy'_1 + y_0 - y_1 = h^2\left[\frac{f_0}{72} + \frac{13 f_1}{60} + \frac{f_2}{360} + \frac{13 f_{\frac{1}{2}}}{45} - \frac{f_{\frac{3}{2}}}{45}\right] \tag{21}$$

$$hy'_{\frac{3}{2}} + y_0 - y_1 = h^2\left[\frac{31 f_0}{1440} + \frac{8 f_1}{15} - \frac{f_2}{96} + \frac{19 f_{\frac{1}{2}}}{80} + \frac{157 f_{\frac{3}{2}}}{720}\right] \tag{22}$$

$$hy'_{n+2} + y_n - y_{n+1} = h^2\left[\frac{f_n}{120} + \frac{7 f_{n+1}}{20} + \frac{59 f_{n+2}}{360} + \frac{14 f_{n+\frac{1}{2}}}{45} + \frac{2 f_{n+\frac{3}{2}}}{3}\right] \tag{23}$$

3. Numerical Examples

In this section, the HyBVM derived in section 2 is applied to two second-order boundary value problems (linear and nonlinear). Their maximum error, CPU time and efficiency curves are compared to the ones in literature [21]. These are shown in the graphs and tables below:

Case 3.1: Consider the linear second-order BVP [22]:

$$y'' = \frac{y + xy'}{1 + x} \quad , \qquad x \in (0,1) \tag{24}$$

with initial and boundary conditions:

$$y(0) - 2y'(0) = -1$$
$$y(1) + 2y'(1) = 3e \tag{25}$$

with exact solution:

$$y(x) = e^x \tag{26}$$

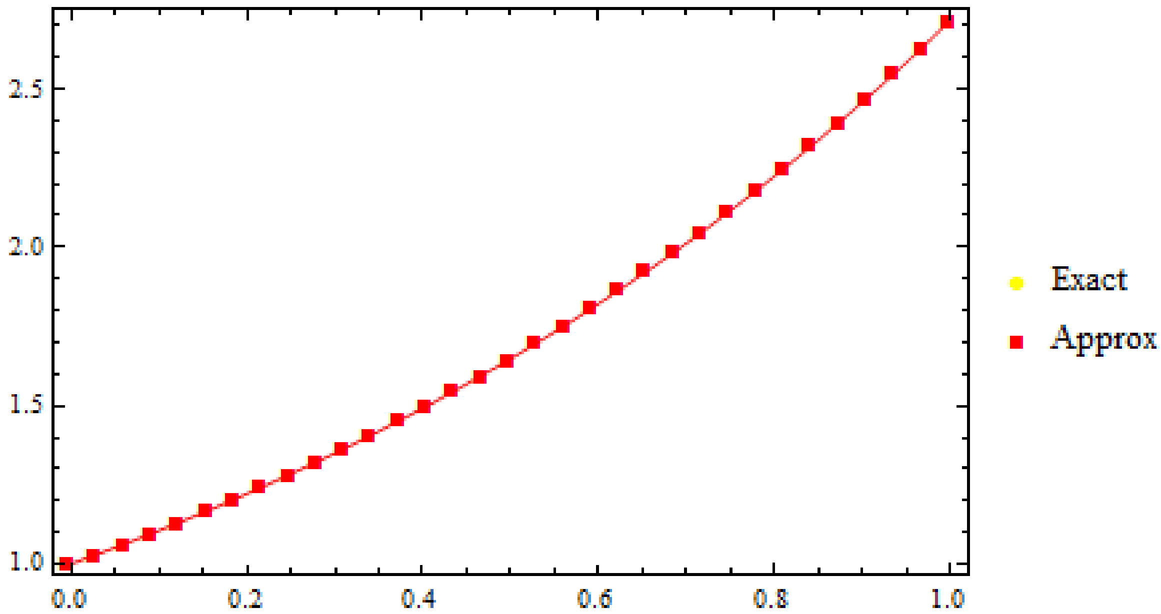

Fig. 1: Solution of Case 3.1

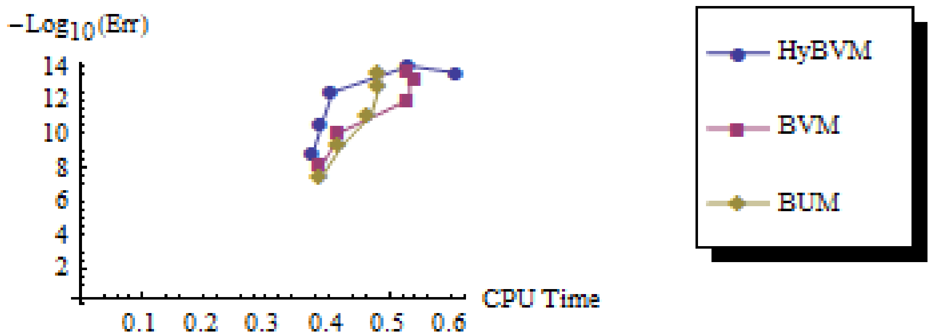

Fig. 2: Efficiency Curve for Case 3.1

Table I: Maximum errors and CPU time for HyBVM, BVM and BUM for Case 3.1

N	HyBVM		BVM [21]		BUM [21]	
	$\|e\|_\infty$	CPU Time	$\|e\|_\infty$	CPU Time	$\|e\|_\infty$	CPU Time
8	1.193e-9	0.376	6.368e-9	0.391	3.348e-8	0.390
16	2.050e-11	0.391	7.092e-11	0.422	5.211e-10	0.422
32	3.335e-13	0.407	9.117e-13	0.531	8.126e-12	0.469
64	9.104e-15	0.531	1.865e-14	0.532	1.459e-13	0.483
128	2.398e-14	0.608	5.418e-14	0.547	2.576e-14	0.485

Case 3.2: Consider the nonlinear second-order BVP [22]:

$$y'' = \frac{e^{2y} + (y')^2}{2} \quad , \qquad x \in (0,1) \tag{27}$$

with initial conditions:

$$y(0) - y'(0) = 1$$
$$y(1) + y'(1) = -ln\,2 - \frac{1}{2} \tag{28}$$

with exact solution:

$$y(x) = \log \frac{1}{1+x} \tag{29}$$

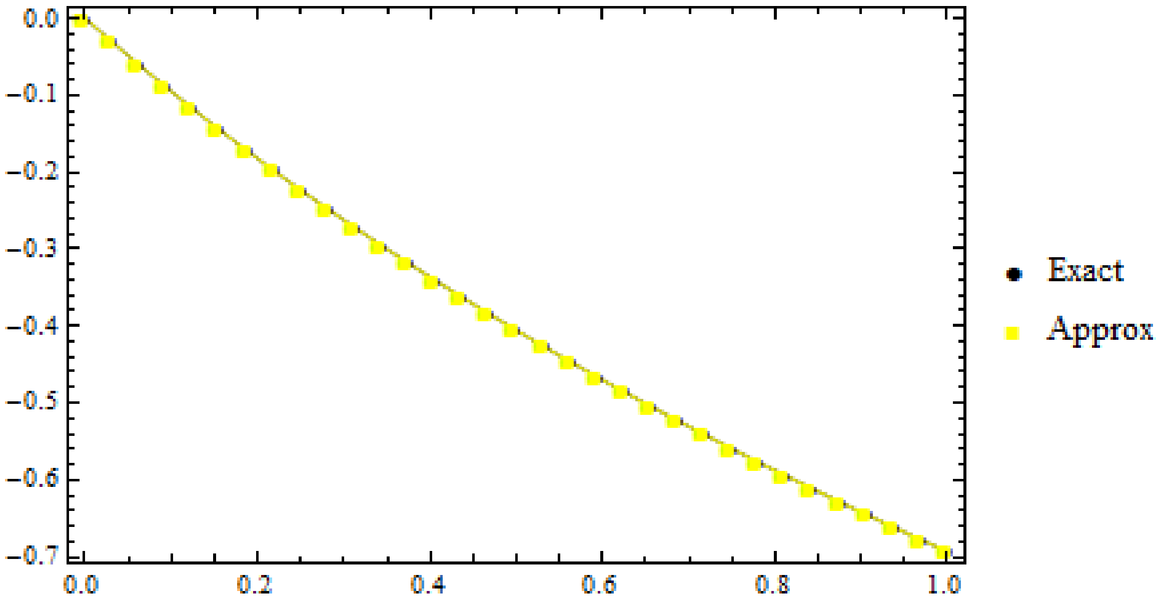

Fig. 3: Solution of Case 3.2

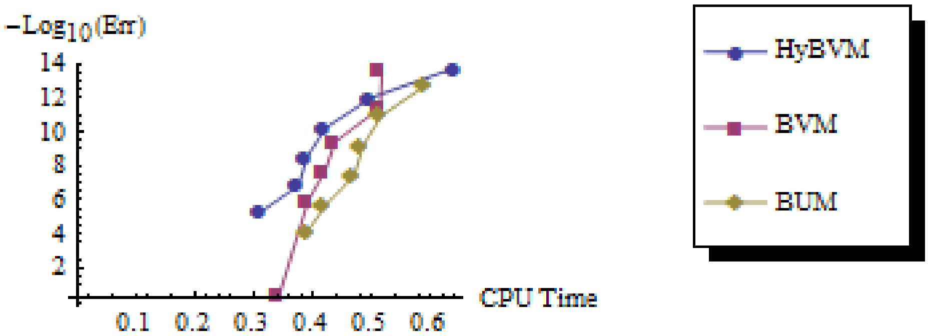

Fig. 4: Efficiency Curve for Case 3.2

Table II: Maximum errors and CPU time for HyBVM, BVM and BUM for Case 3.2

N	HyBVM $\|e\|_\infty$	CPU Time	BVM [21] $\|e\|_\infty$	CPU Time	BUM [21] $\|e\|_\infty$	CPU Time
4	4.640e-6	0.313	2.876e-1	0.344	6.970e-5	0.391
8	1.368e-7	0.376	1.122e-6	0.392	1.967e-6	0.422
16	3.025e-9	0.390	1.596e-8	0.422	3.892e-8	0.470
32	5.684e-11	0.422	4.118e-10	0.437	6.546e-10	0.485
64	9.756e-13	0.499	3.625e-12	0.515	1.043e-11	0.517
128	1.757e-14	0.640	1.983e-14	0.517	1.639e-13	0.594

4. Conclusion

In this work, we have applied the Hybrid Boundary Value Method (HyBVM) to two second order BVPs with boundary conditions and compared their maximum error and efficiencies with BVM and BUM in [21]; and the results are efficient when compared to other BVMs in literature In constructing these methods, the Numerov method was adopted as the LMM while utilizing data at both normal and off-step points.

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