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Games with Incomplete Information: a Framework Based on Possibility Theory

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Summary

Probabilistic games with incomplete information, called Bayesian games, offer a suitable framework for games where the utility degrees are additive in essence. This approach does not apply to ordinal games where the utility degrees capture no more than a ranking, nor to situations of decision under qualitative uncertainty.

In the first part of this thesis, we propose a representation framework for ordinal games under possibilistic incomplete information (Π-games). These games constitute a suitable framework for the representation of ordinal games under incomplete knowledge. We extend the fundamental notions of secure strategy, pure Nash equilibrium, and mixed Nash equilibrium to this framework. Furthermore, we show that any possibilistic game with incomplete information can be transformed into an equivalent normal form game with complete information. The fundamental notions such Nash equilibria (pure and mixed) and secure strategies are in bijection in both frameworks. This representation result is a qualitative counterpart of Harsanyi results about the representation of Bayesian games by normal form games under complete information. It is more of a representation result than the premise of a solving tool. We show that deciding whether a pure Nash equilibrium exists in a Π-game is a difficult task (NP-hard) and propose a Mixed Integer Linear Programming (MILP) encoding of this problem. We also propose a polynomial-time algorithm to find a secure strategy in a Π-game and show that a possibilistic mixed equilibrium can be computed in polynomial time (w.r.t., the size of the game), which contrasts with probabilistic mixed equilibrium computation in cardinal game theory. To confirm the feasibility of the MILP formulation and the polynomial-time algorithms, we introduce a novel generator for Π-games based on the well-known standard normal form game generator: GAMUT.

Representing a Π-game in standard normal form requires an extensive expression of the utility functions and the possibility distribution on the product spaces of actions and types. This is the concern of the second part of this thesis where we propose a less costly view of Π-games, namely min-based polymatrix Π-games, which allows to concisely specify Π-games with local interactions, i.e., the interactions between players are pairwise and the utility of a player depends on her neighbors and not on all other players in the Π-game. This framework allows, for instance, the compact representation of coordination games under uncertainty where the satisfaction of a player is high if and only if her strategy is coherent with all of her neighbors, the game being possibly only incompletely known to the players. We show that any 2-player Π-game can be transformed into an equivalent min-based polymatrix game. This result is the qualitative counterpart of Howson and Rosenthal's theorem linking Bayesian games to polymatrix games. Furthermore, as soon as a simple condition on the coherence of the players' knowledge about the world is satisfied, any polymatrix

Π-game can be transformed in polynomial time into an equivalent min-based and complete information polymatrix game. We show that the existence of a pure Nash equilibrium in a polymatrix Π-game is an NP-complete problem but no harder than deciding the existence of a pure Nash equilibrium in a Π-game. Finally, we show that the latter family of games can be solved through a MILP formulation. We introduce a novel generator for min-based polymatrix Π-games based on the Π-game generator. Experiments confirm the feasibility of this approach.

Keywords

Game theory, games with incomplete information, ordinal games, possibility theory, pure Nash equilibrium, mixed Nash equilibrium, secure strategy, polymatrix games, games under qualitative uncertainty.

Resumé

Les jeux probabilistes à information incomplète, appelés jeux Bayesiens, offrent un cadre adapté au traitement de jeux à utilités cardinales sous incertitude. Ce type d'approche ne peut pas être utilisé dans des jeux ordinaux, où l'utilité capture un ordre de préférence, ni dans des situations de décision sous incertitude qualitative.

Dans la premiere partie de cette thèse, nous proposons un modèle de jeux à information incomplète basé sur la théorie de l'utilité qualitative possibiliste: les jeux possibiliste à information incomplete, nommés Π-games. Ces jeux constituent un cadre approprié pour la représentation des jeux ordinaux sous connaissance incomplète. Nous étendons les notions fondamentales de stratégie de sécurité et d'équilibres de Nash (pur et mixte). De plus, nous montrons que tout jeu possibiliste à information incomplète peut être transformé en un jeu à information complète sous la forme normale équivalent au jeu initial, dont les stratégies de sécurité, les équilibres de Nash purs et mixtes sont en bijection dans les deux jeux. Ce résultat de représentation est une contrepartie qualitative de celui de Harsanyi sur la représentation des jeux Bayésiens par des jeux sous forme normale à information complète. Cela est plus un résultat de représentation qu'un outil de résolution. Nous montrons que décider si un équilibre de Nash pur existe dans un Π-game est un problème NP-complet et proposons un codage de programmation linéaire mixte en nombres entiers (PLNE) du problème. Nous proposons également un algorithme en temps polynomial pour trouver une stratégie de sécurité dans un Π-game et montrons qu'un équilibre mixte possibiliste peut être également calculé en temps polynomial (en fonction de la taille du jeu). Pour confirmer la faisabilité de la formulation de programmation linéaire en nombres entiers mixtes et des algorithmes en temps polynomial, nous introduisons aussi un nouveau générateur pour les Π-games basé sur le génerateur de jeux sous la forme normale: GAMUT.

Représenter un Π-game sous forme normale standard nécessite une expression extensive des fonctions d'utilité et de la distribution des possibilités, à savoir sur les espaces produits des actions et des types. La deuxième partie de cette thèse propose une vue moins coûteuse des Π-games, à savoir la polymatrix Π-games basée sur min, qui permet de spécifier de manière concise les Π-games avec des interactions locales, en d'autre termes, lorsque les interactions entre les joueurs sont par paires et l'utilité d'un joueur dépend de son voisinage et non de tous les autres joueurs du Π-game. Ce cadre permet, par exemple, la représentation compacte des jeux de coordination sous incertitude où la satisfaction d'un joueur est élevée si et seulement si sa stratégie est cohérente avec celles de l'ensemble de ses voisins. Dans cette thèse, nous montrons que n'importe quel Π-game à 2 joueurs peut être transformé en un jeu polymatriciel équivalent basé sur le min. Ce résultat est la contrepartie qualitative du théorème de Howson et Rosenthal reliant les jeux Bayésiens aux jeux polymatriciels. De plus, dès qu'une simple condition de cohérence des connaissances des joueurs sur le monde est satisfaite, tout polymatrix Π-game peut être transformé en temps polynomial en un jeu polymatriciel, basé sur le min, à information complète équivalent. Nous montrons que l'existence d'un équilibre de Nash pur dans un polymatrix Π-game est un problème NP-complet mais pas plus difficile que de décider si un équilibre de Nash pur existe dans un Π-game. Enfin, nous montrons que cette dernière famille de jeux peut être résolue grâce à une formulation de programmation linéaire en nombres entiers mixtes. Nous introduisons un nouveau générateur pour les polymatrix Π-games basés sur le générateur de Π-game. Les experimentations confirments la faisabilité de cette approche.

Mots clés

Théorie des jeux, jeux à information incomplète, jeux ordinaux, théorie des possibilités, equilibre de Nash pur, equilibre de Nash mixte, stratégie de sécurité, jeux polymatriciels, jeux sous uncertitude qualitative.

Contents

Introduction

Game theory has been studied for a long time (Von Neumann and Morgenstern, 1944). Game theory proposes a very simple but powerful framework to capture decision problems involving several agents: in a game with complete information, each agent called "player", can be an individual, a company, a country, an organization, etc. She has to choose a decision, called "action", from a set of possible actions. The final outcome of the game depends on the actions chosen by all the players of the game. The term "payoff" is often used to designate their utility. The classical model indeed suits problems where the satisfaction can be expressed on a cardinal scale, typically a monetary scale.

Despite their capacity to model many problems in different domains, cardinal games are not able to adequately address situations where players do not have appropriate payoff functions for each outcome. For these problems, players nevertheless have some preferences and are able to rate their outcomes from the worst to the best or the contrary. Thus, players' preferences may be ordinal rather than cardinal. To model such situations, a specific theory based on the rank ordering of players' preferences was proposed, called "*ordinal games*" (Xu, 2000, Cruz and Simaan, 2000, Ouenniche et al., 2016).

When making decisions, players are supposed to be rational in the sense that they aim to maximize their utilities. Therefore, solution concepts based on ranking outcomes were proposed such as pure Nash equilibrium (Nash, 1950), Pareto optimal solutions, dominance. Note that they do not require any cardinality assumption. They are basically "ordinal".

In classic non-cooperative games under complete information[1], the players cannot coordinate their actions but each of them knows everything about the game: the players, their available actions, and all their utilities. This assumption of complete knowledge

[1]In *non-cooperative games*, it is assumed that each agent acts independently, without collaboration or communication with any of her opponents. In this thesis, we focus on non-cooperative games.

cannot always be satisfied. In the real world, players are not so well informed and have only limited knowledge about the game. That is why, incomplete information games and more particularly Bayesian games (Harsanyi, 1967a) have been proposed. They model problems where the utility degrees are additive and the knowledge of the players is quantified in a probabilistic way.

Harsanyi has proposed the notion of type to present the beliefs of players in order to capture the beliefs about the game, the knowledge about the other players, the hierarchical beliefs, etc. In incomplete information games, it is assumed that every player eventually knows her own type but not the types of the other players. In Bayesian games, the knowledge is represented by a probability distribution over the joint types. This latter is common to all players but each player may receive some private information, hence a personal view of the game: Bayes' rule of conditioning is used to derive the knowledge of each agent, hence the denomination "Bayesian game" (Harsanyi, 1967a).

Bayesian games do not apply to ordinal games where the utility degrees capture no more than a ranking, and nor to situations of decision under qualitative uncertainty.

Ordinal games (Xu, 2000, Cruz and Simaan, 2000, Ouenniche et al., 2016) have been recently extended to model games with incomplete information. In possibilistic Boolean games (De Clercq et al., 2018), the knowledge of each player is expressed in the framework of possibilistic logic and the players do not receive any private information before playing. Therefore, the authors consider the problem from the external point of view of an observer who proceeds to a fusion of these pieces of knowledge and computes the possibility and the necessity of a given profile of actions being a Nash equilibrium in the usual sense.

In this thesis, we propose a framework of games with incomplete information where the utility of the players captures an order of preference and the knowledge is qualitative. Players' preferences are modeled by ordinal utilities whereas qualitative knowledge is modeled by possibility theory. Thus, our framework is called "*possibilistic games with incomplete information*" (Π-games). They constitute an appropriate framework for the representation of ordinal games under incomplete qualitative knowledge. We study several notions of game theory such secure strategy, pure Nash equilibria and mixed Nash equilibrium in the sens of (Ben Amor et al., 2017, Hosni and Marchioni, 2019).

In addition to that, in this thesis, we study Π-games where the interactions between players are pairwise and the utility of a player depends on her neighbors and not on all other players in the Π-game. We propose a new framework of *min-based polymatrix* Π-*games*, which allows us to concisely specify Π-games with local interactions. This framework allows, for instance, the compact representation of coordination games under possibilistic uncertainty.

This thesis is decomposed into two main parts:

I. The first part focuses on the notions on which the thesis relies:

   - **Chapter 1** presents the basic concepts related to game theory especially on games with complete information;

   - **Chapter 2** presents ordinal games and recalls the basics of possibility theory. This chapter also presents possibilistic mixed Nash equilibria in these games;

   - **Chapter 3** details games with incomplete information.

II. The second part of the thesis presents our main contributions. It is structured as follows:

   - **Chapter 4** is the core of this thesis and proposes a representation framework for ordinal games under possibilistic incomplete information (Π-games). It extends several solution concepts to this framework namely Nash equilibria (pure and mixed) and secure strategy;

   - **Chapter 5** focuses on solving possibilistic games with incomplete information. It presents a polynomial-time algorithm to compute a secure strategy in a Π-game. As to pure Nash equilibria are concerned, a MILP formulation is proposed. Then, a polynomial-time algorithm to compute a mixed Nash equilibrium was proposed. Experiments are reported in the final part of this chapter;

   - **Chapter 6** defines the new framework of *min-based polymatrix* Π-*games* which allows us to concisely specify Π-games with local interactions. It shows that a *min-based polymatrix* Π-*game* can be transformed, in polynomial time, into a (complete information) min-based polymatrix game with equivalent pure Nash equilibria. This chapter also studies the complexity to check the existence of a pure Nash equilibrium in these games and proposes a MILP formulation for this problem. Experiments are reported in the final part of this chapter.

The main results of this thesis are published in:

- (Ben Amor et al., 2018): Nahla Ben Amor, Hélène Fargier, Régis Sabbadin, and Meriem Trabelsi. Possibilistic games with incomplete information. Actes de la

Rencontres Francophones sur la Logique Floue et ses Applications, pages 216-226 (LFA 2018), Arras, France. *This paper is presented in Chapter 4*;

- (Ben Amor et al., 2019a): Nahla Ben Amor, Hélène Fargier, Régis Sabbadin, and Meriem Trabelsi. Possibilistic games with incomplete information. In Proceedings of the International Joint Conference on Artificial Intelligence, pages 1544–1550, (IJCAI 2019) Macao, China. *This paper is presented in Chapter 4 and Chapter 5*;

- (Ben Amor et al., 2020a): Nahla Ben Amor, Hélène Fargier, Régis Sabbadin, and Meriem Trabelsi. Ordinal Polymatrix Games with Incomplete Information. In Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning, pages 99–108, (KR 2020) Rhodes, Greece, and Actes de la Rencontres Francophones sur la Logique Floue et ses Applications, pages 75-82 (LFA 2020), Sète, France. *This paper is presented in Chapter 6.*

Part I

Background

Chapter 1

Games with Complete Information

1.1 Introduction

Game theory (Von Neumann and Morgenstern, 1944) has been studied for a long time. It is essentially used to model and analyze strategic real-life situations where an agent, called "player", can be an individual, a company, a country, an organization, etc. She has to choose a decision, called "action", from a set of possible actions. The final outcome of the game depends on the actions chosen by all the players of the game.

Game theory was firstly studied by (Von Neumann and Morgenstern, 1944) then developed by (Nash, 1950). First, in the game theory studies, the outcomes of the games were assumed to be represented in a numeric way. In these situations, every player has complete information about the game, i.e., the players, their actions, and all the outcomes. Such games are called "*games with complete information*".

This chapter focuses on the basics on "*games with complete information*", and it is organized as follows: Section 1.2 focuses on the form of representations of games with complete information. Sections 1.3 and 1.4 present the different strategies in games with complete information and the well-known solution concepts. Section 1.5 focuses on the well known cardinal games classes. Finally, Section 1.6 presents different ways to succinctly model games with complete information.

1.2 Games Representations

We can distinguish two kinds of games representation: "*Normal form games*" used to model "simultaneous games", also called "static games", where all players play at the same time and "*extensive form games*" used to model "sequential games", also called "dynamic games", where players play sequentially: one after another. In the following, we will detail these forms of representation.

1.2.1 Normal Form Games

In a "*static game*", players choose their actions simultaneously. No player is informed of the action chosen by any other player, she does not receive any information before choosing her action. The game ends when all players have played. After that, no player has the opportunity to change her action. At this stage, the outcome of the game is immediately visible (Von Neumann and Morgenstern, 1944, Owen, 1982).

A static game is generally represented in *"standard normal form"* (Von Neumann and Morgenstern, 1944). A *"standard normal form game"*, also known as the *"strategic"* or *"matrix"* form game, is the general model for all classes of games. It is the most familiar representation of strategic interactions in game theory.

More formally, a standard normal form game (SNF) is defined as follows:

Definition 1.1 (Standard Normal Form Game)**.** *A standard normal form game (SNF) is a tuple $G = \langle N, A, \mu \rangle$, where:*

- $N = \{1, \ldots, n\}$ *is a finite set of n players;*
- $A = \times_{i \in N} A_i$, *where A_i is a finite set of actions available to player $i \in N$;*
- $\mu = \{(\mu_i)_{i \in N}\}$ *is a set of utility functions. $\mu_i(a)$ captures the utility of player i for the joint action $a \in A$.*

For a joint action $a \in A$, a_i is the action of player i in a, $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in A_{-i} = \times_{j \neq i} A_j$ is its restriction to all the players but i and "." denotes the concatenation, e.g., $\forall (a'_i, a_{-i})$, $a'_i.a_{-i} = (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n)$.

In SNF games, each utility function is represented explicitly, by an n-dimensional table having an entry for each joint action. For instance, a 2-player normal form game is represented by a table where each row corresponds to an available action for player 1. Each column corresponds to a possible action for player 2 and each cell corresponds to one possible outcome, i.e., the first one corresponds to player 1 and the second one corresponds to player 2 as shown in Example 1.1.

Example 1.1 (Prisoner's dilemma)**.** *Two members of a criminal gang are arrested and imprisoned in isolated rooms. Both of them want to minimize their prison punishment. They can cover (C) or denounce (D). The utilities are displayed in Table 1.1. The utility of a player is equal to the length of a prison term that she gets.*

		Player 2 C	Player 2 D
Player 1	C	-1 , -1	-3 , 0
	D	0 , -3	-2 , -2

Table 1.1: "Prisoner's dilemma" game.

1.2.2 Extensive Form Games

In "*dynamic games*", also called "*sequential-move games*", players decide what to play sequentially by following an explicit time-schedule. Every player knows the past play. In other words, every player knows what other players did in the past, i.e., all moves before each stage of the game.

The normal form game representation does not allow to incorporate the notion of time or the sequence of actions of the players. For that, an alternative representation, called "*extensive form game*", allows explicitly to represent both the actions and the information over time of all players in the game. The extensive form is modeled as a tree where the nodes represent the choices of the players, the edges represent the actions and the leaves are the outcomes of the game (Von Neumann and Morgenstern, 1944).

There are two cases of dynamic games: "*perfect information games*" and "*imperfect information games*":

- **Perfect information game** is a game in which every player is aware of the moves (all actions) of all other players that have already taken place. In other words, every player knows the actions previously played by all other players and knows exactly where she is in the game tree (Gale, 1953);

- **Imperfect information game** is a game in which at least one player does not know the previous actions, i.e., movement, taken by the other players, or even players with limited memory of their past actions (Blair et al., 1993).

Example 1.2. *Two firms share the market, already colluding, and maintaining high prices. Each firm can decide to stop colluding and decrease the price, in order to increase their market share, even force the other to quit the market. Firm 1 can either keep colluding with Firm 2 or decrease the price. If Firm 1 decides to keep colluding, Firm 2 will need to make a decision. If they both agree to collude, they will get (5\$,5\$). However, if one of them decides to decrease the price, the set of utilities will be either (4\$,3\$) or (3\$,4\$), depending on which one starts the game. In other words, if Firm 1 (resp. Firm 2) first chooses to decrease the price she will get 4\$ and thus Firm 2 (resp. Firm 1) gets 3\$. This game can be represented by an extensive form game as shown in Figure 1.1.*

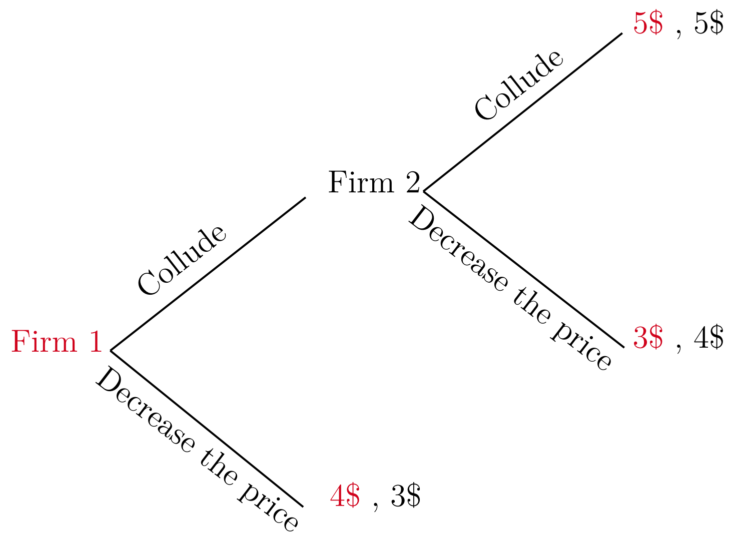

Figure 1.1: An extensive form game.

1.3 Solutions Concepts in Normal Form Games

In this section, we will start by defining the notion of "*joint pure strategy*". A "*pure strategy*" of player i is represented by her action $a_i \in A_i$. A "*joint pure strategy*" $a = (a_1, \ldots, a_i, \ldots, a_n) \in A$ is a selection of an action for each player. The utility of a player for a joint pure strategy is directly given by the utility table.

Every player acts independently and determines the action that gives her the best utility taking into consideration the possible decisions of her opponents. Hence, the use of solution concepts helps players to find the action to play for a given situation. Solution concepts are defined as a subset of joint pure strategies that constitute the results of the game. In the following, we will present the most fundamental solution concepts: Pareto optimally, secure strategy, dominant strategy, and pure Nash equilibrium.

1.3.1 Pareto Optimal Joint Strategies

Given the actions of the other players, every player can be sure that one action is better than another one for her. Formally, this intuition is called "*Pareto dominance*". A joint strategy is dominated by another one if some players can be made better off without making any other player worse off. Formally:

Definition 1.2 (Pareto Domination)**.** *Let $G = \langle N, A, \mu \rangle$ be a normal form game. A joint pure strategy $a \in A$ Pareto dominates a joint pure strategy $a' \in A$ iff:*

$$\forall i \in N,\ \mu_i(a) \geq \mu_i(a') \ \textit{and}\ \exists j \in N,\ \textit{s.t.},\ \mu_j(a) > \mu_j(a').$$

A joint pure strategy a is called "*Pareto optimal*" if it does not exist a joint pure strategy a' such that a' Pareto dominates a. Formally:

Definition 1.3 (Pareto optimal). *Let $G = \langle N, A, \mu \rangle$ be a normal form game. A joint strategy a is called Pareto optimal, iff no strategy $a' \in A$ Pareto dominates a.*

Example 1.3 (Cont. Example 1.1). *The prisoner's dilemma game contains three Pareto optimal outcomes: $C.C$, $C.D$ and $D.C$. $D.D$ is the only outcome that cannot be a Pareto optimal since outcome $C.C$ dominates it:*

$$\mu_1(C.C) > \mu_1(D.D) \text{ and } \mu_2(C.C) > \mu_2(D.D).$$

1.3.2 Maximin Strategy: Secure Strategy

For every player, a "*secure strategy*" or a "*maximin strategy*" is an action that guarantees the best outcome under the worst conditions, i.e., whatever the actions of remaining players. Therefore, the secure strategy can be seen as a reasonable alternative for a cautious player who always prefers maximizing her worst-case utility whichever the actions chosen by the other players. Formally:

Definition 1.4 (Level of Security, Secure Strategy). *Let $G = \langle N, A, \mu \rangle$ be a normal form game. The level of security of player $i \in N$ for an action $a_i \in A_i$ is:*

$$\mu_i^{secure}(a_i) = \min_{a_{-i} \in A_{-i}} \mu_i(a_i.a_{-i}). \tag{1.1}$$

A joint action $a = a_i.a_{-i}$ is a secure strategy iff $\forall i \in N$, $\forall a_i' \in A_i$:

$$\mu_i^{secure}(a_i) \geq \mu_i^{secure}(a_i'). \tag{1.2}$$

Example 1.4 (Cont. Example 1.1). *In the Prisoner's dilemma game depicted in Table 1.1. We have:*

- $\mu_1^{secure}(D) = \min\big(\mu_1(D.C), \mu_1(D.D)\big) = -2$;
- $\mu_1^{secure}(C) = \min\big(\mu_1(C.C), \mu_1(C.D)\big) = -3$;
- $\mu_2^{secure}(D) = \min\big(\mu_2(D.C), \mu_2(D.D)\big) = -2$;
- $\mu_2^{secure}(C) = \min\big(\mu_2(C.C), \mu_2(C.D)\big) = -3$.

Thus, the secure strategy of both players is D since:

$$\mu_1^{secure}(D) \geq \mu_1^{secure}(C) \text{ and } \mu_2^{secure}(D) \geq \mu_2^{secure}(C).$$

1.3.3 Dominated Strategies

To analyze a game, players have to check if there exist some actions to eliminate. Indeed, a player may have an action that brings her a better utility whatever the actions of the other players. In other words, a player can have an action that dominates another. The gradations of dominance can be defined as follows:

Definition 1.5 (Domination)**.** *Let $G = \langle N, A, \mu \rangle$ be a normal form game. Let a_i and $a_i^{'}$ be two actions of player i and A_{-i} the set of the joint strategies of the remaining players. Then:*

- *a_i **strictly dominates** $a_i^{'}$: $\forall a_{-i} \in A_{-i}$, $\mu_i(a_i.a_{-i}) > \mu_i(a_i^{'}.a_{-i})$;*
- *a_i **weakly dominates** $a_i^{'}$: $\forall a_{-i} \in A_{-i}$, $\mu_i(a_i.a_{-i}) \geq \mu_i(a_i^{'}.a_{-i})$ and $\exists a_{-i} \in A_{-i}$, s.t., $\mu_i(a_i.a_{-i}) > \mu_i(a_i^{'}.a_{-i})$.*

If one action dominates all others, we say that it is (strongly or weakly) dominant:

Definition 1.6 (Dominant Strategy)**.** *Let $G = \langle N, A, \mu \rangle$ be a normal form game. An action a_i is strictly (resp. weakly) dominant for player i iff it strictly (resp. weakly) dominates any other actions $a_i^{'} \in A_i$, s.t., $a_i^{'} \neq a_i$.*

Example 1.5 (Cont. Example 1.1)**.** *According to the above definition, the strictly dominant strategy for both players in the prisoner's dilemma game is D since:* $\mu_1(D.C) > \mu_1(C.C)$, $\mu_1(D.D) > \mu_1(C.D)$, $\mu_2(D.C) > \mu_2(C.C)$ *and* $\mu_2(D.D) > \mu_2(C.D)$.

When playing a game, players consider their dominant strategies. To solve a game, each player eliminates the action that gives her a utility lower than all others. This action is called "*dominated strategy*", it is dominated by all other possible actions.

It is obvious that if a player finds a dominated strategy, she will ignore it from her choices. Formally, a dominated strategy is defined as follows:

Definition 1.7 (Dominated Strategy)**.** *Let $G = \langle N, A, \mu \rangle$ be a normal form game. An action a_i is strictly (resp. weakly) dominated for player i iff $\exists a_i^{'} \in A_i$, s.t., $a_i^{'} \neq a_i$ strictly (resp. weakly) dominates action a_i.*

To solve a game, we can find its solution by removing dominated pure strategies, since these latter give an utility less or equal to other actions. If one dominated strategy is removed, it can be possible to find a new dominated strategy that was not dominated before. The following example explains the process.

Example 1.6. *Let us take an example of a two-player normal form game. Each one has three actions: actions E, F and D are available to player 1 whereas actions X, Y, and Z are available to player 2. The utilities are represented in Table 1.2. We can check that player 1 would never play action D since it is dominated by action E. In this case, we remove line D. Second, player 2 would never play action Z since it is dominated by actions X and Y (this domination appears after removing line D). Thus, we eliminate column Z. This process is repeated. Dominated strategies are*

		Player 2		
		X	Y	Z
Player 1	E	10 , 4	5 , 3	3 , 2
	F	0 , 1	4 , 6	6 , 0
	D	2 , 1	3 , 5	2 , 8

Table 1.2: A normal form game with dominated strategies.

		Player 2		
		X	Y	Z
Player 1	E	10 , 4	5 , 3	3 , 2
	F	0 , 1	4 , 6	6 , 0
	D	2 , 1	3 , 5	2 , 8

		Player 2		
		X	Y	Z
Player 1	E	10 , 4	5 , 3	3 , 2
	F	0 , 1	4 , 6	6 , 0

		Player 2	
		X	Y
Player 1	E	10 , 4	5 , 3
	F	0 , 1	4 , 6

		Player 2	
		X	Y
Player 1	E	10 , 4	5 , 3

		Player 2
		X
Player 1	E	10 , 4

Table 1.3: Iterative removal of dominated strategies.

removed (F then Y) until no player has a dominated strategy. Finally, we find just one joint strategy $(E.X)$. Therefore, the solution of this game $(E.X)$ is obtained using the iterative removal.

1.3.4 Pure Nash Equilibrium

Pure Nash equilibrium is the most known and fundamental solution concept in non-cooperative games. It corresponds to a joint pure strategy in which no player has the incentive to change unilaterally her action.

If player i knows the chosen actions of all her opponents (i.e., if player i knows a_{-i}), she will choose the action that maximizes her utility. This latter is called the "*best response*" of player i to joint action a_{-i}. Formally:

Definition 1.8 (Best response)**.** *Let $G = \langle N, A, \mu \rangle$ be a normal form game. An action $a_i \in A_i$ is a best response (BR) of player i to a_{-i}, iff $\forall a'_i \in A_i$, s.t., $a'_i \neq a_i$:*

$$\mu_i(a_i.a_{-i}) \geq \mu_i(a'_i.a_{-i}). \tag{1.3}$$

Note that the best response of player i is not necessarily unique.

Generally, every player chooses her best response to other players' best responses. The corresponding joint strategy is called a "*pure Nash equilibrium*" (Nash, 1950).

A joint strategy $a^* \in A$ is a pure Nash equilibrium (PNE), defined as follows:

Definition 1.9 (Pure Nash Equilibrium)**.** *Let $G = \langle N, A, \mu \rangle$ be a normal form game. The joint pure strategy $a^* \in A$ is a pure Nash equilibrium (PNE), iff $\forall i \in N$, $\forall a'_i \in A_i$:*

$$\mu_i(a_i^*.a_{-i}^*) \geq \mu_i(a'_i.a_{-i}^*). \tag{1.4}$$

It is easily checked that the above definition is equivalent to writing that, in a pure Nash equilibrium, every player chooses her best response to other players' best responses.

Example 1.7 (Cont. Example 1.1)**.** *It can be checked that the joint strategy $a = D.D$ is the unique PNE in the Prisoner's dilemma game. Indeed, when player 1 denounces, player 2 has no incentive to deviate from D to C since: $\mu_2(D.D) = -2 > \mu_2(D.C) = -3$. Similarly, when player 2 denounces, player 1 has no interest to change from D to C since her utility would decrease from -2 to -3.*

It is not guaranteed to get a pure Nash equilibrium in a given standard normal form game as illustrated by the following example:

Example 1.8 (Matching Pennies Game)**.** *Given a matching pennies game with two players. Everyone has a penny. To play this game, each one chooses to display either heads or tails. Then, both of them compare their pennies. If the results are the same, then player 1 pockets both, otherwise, the second player gains and pocket the two pennies. For more details, Table 1.4 presents the normal form representation of a matching pennies game:*

		Player 2	
		Heads	Tails
Player 1	Heads	1 , -1	-1 , 1
	Tails	-1 , 1	1 , -1

Table 1.4: "Matching Pennies" game.

It can be checked that the above game does not admit a PNE.

Determining whether a game expressed in standard normal form has a pure Nash equilibrium is an polynomial time problem.

We note that any dominant strategy is always a Nash equilibrium. However, not all Nash equilibria are dominant strategies. Indeed, given a dominant strategy $a \in A$ where each player plays her dominant strategy, no player has the interest to deviate to another action since she plays a dominant strategy. Hence, $a \in A$ is a pure Nash equilibrium.

The above notions of *Pareto optimal*, *secure strategy*, *dominated strategies* and *pure Nash equilibrium* are based on ranking the utilities of players; they are basically purely *ordinal.* Nevertheless, in game theory literature, other solution concepts exist such as *minmax regret* and *mixed Nash equilibrium* which are *cardinal*: the utilities of players are computed using arithmetic operations (addition, subtraction, multiplication, etc.). These are detailed in the next section.

1.4 Solutions Concepts in Cardinal Games

Generally, in normal form games, the preferences of the agents among the outcomes are captured by utility functions. In *cardinal games*, the term "payoff" is often used to designate their utility. Every player is able to asses an utility function for each possible outcome. This suits problems where the satisfaction can be expressed on a cardinal scale, typically a monetary scale.

In the following, we present the cardinal solution concepts:

1.4.1 Minimax Regret

In Section 1.3.2, we argued that players can play maxmin strategies to maximize their worst-case utilities. However, in the case where the other agents are not believed to be malicious but instead entirely unpredictable, it can make sense that players care about minimizing their worst-case losses. The idea is to calculate the *regret* of the player if she chooses an action rather than the other one (Bell, 1982). The regret of player i can be described by the amount that the player loses by playing an action a_i rather than playing her best response to a_{-i}. Formally, the *regret* of a player is defined as follows:

Definition 1.10 (Regret)**.** *Let $G = \langle N, A, \mu \rangle$ be a cardinal normal form game, i.e., $\mu \to \mathbb{R}$. The regret of player i for the joint action $a \in A$ is:*

$$regret_i(a) = \Big(\max_{a'_i \in A_i} \mu_i(a'_i.a_{-i})\Big) - \mu_i(a_i.a_{-i}). \tag{1.5}$$

Naturally, a player would minimise her regret, hence, she will attempt to minimize her worst-case regret, i.e., minimize her maximum regret. Formally, the *minimax regret* actions for player i are defined as:

Definition 1.11 (Minimax regret)**.** *Let $G = \langle N, A, \mu \rangle$ be a cardinal normal form game.*

The minmax regret actions for player i are defined as:

$$regret(i) = \arg\min_{a_i \in A_i} \Big(\max_{a_{-i} \in A_{-i}} regret_i(a_i.a_{-i}) \Big). \tag{1.6}$$

1.4.2 Mixed Nash Equilibrium

1.4.2.1 Mixed strategy

A "*mixed strategy*" consists of randomizing the set of available actions according to some probability distribution. Formally, a mixed strategy in a normal form game is defined as follows:

Definition 1.12 (Mixed strategy in normal form game). *Let $G = \langle N, A, \mu \rangle$ be a cardinal normal form game. The set of mixed strategies for player $i \in N$ is the set of all probability distributions over the set of her actions A_i.*

Given a mixed strategy $s_i \in S_i$, $s_i(a_i)$ denotes the probability that an action a_i will be played by player $i \in N$ under the mixed strategy s_i.

As a joint pure strategy, a joint mixed strategy $s = (s_1, ..., s_n)$ is a selection of a (mixed) strategy for each player. However, unlike pure strategies, the utility of a player cannot be calculated directly from the utility table. If she chooses to randomize over her available actions with a certain probability distribution, then it leads to the calculation of her expected utility. The expected utility of player $i \in N$ for the joint mixed strategy $s \in S$ is defined by:

Definition 1.13 (Expected utility for a mixed strategy). *Let $G = \langle N, A, \mu \rangle$ be a cardinal normal form game and $s = (s_1, ..., s_n)$ be a joint mixed strategy. The expected utility EU_i of player i for the mixed strategy s is equal to:*

$$EU_i(s) = \sum_{a \in A} \mu_i(a) \prod_{j=1}^{n} s_j(a_j). \tag{1.7}$$

1.4.2.2 Mixed Equilibrium

If each player i randomizes over the set of her available actions A_i according to some probability distribution in the form of $s_i : A_i \mapsto [0, 1]$, the mixed Nash equilibrium appears as a solution where no player can improve her expected utility by changing her mixed strategy. Unlike pure Nash equilibrium, a mixed Nash equilibrium is assumed to exist in any game (Nash, 1950). Formally, a mixed Nash equilibrium is defined as follows:

Definition 1.14 (Mixed Nash Equilibrium). *Let $G = \langle N, A, \mu \rangle$ be a cardinal normal form game. The mixed strategy $s^* \in S$ is a mixed Nash equilibrium (MNE), iff $\forall i \in$*

N, $\forall s_i' \in S_i$:

$$EU_i(s_i^*.s_{-i}^*) \geq EU_i(s_i'.s_{-i}^*). \tag{1.8}$$

Example 1.9 (Cont. Example 1.8)**.** *Let us take a matching pennies game depicted in Table 1.4. We have to compute a mixed equilibrium. Suppose that:*

- *player 2 plays heads H with probability p and tails with probability $1-p$ then:*
  $EU_1(H) = EU_1(T)$
  $1p - 1(1-p) = -1p + 1(1-p)$
  $p = 0.5$*, hence, player's 2 mixed strategy is* $(0.5, 0.5)$*;*

- *player 1 plays heads H with probability q and tails with probability $1-q$ then:*
  $EU_2(H) = EU_2(T)$
  $-1q + 1(1-q) = 1q - 1(1-q)$
  $q = 0.5$*, hence, player's 1 mixed strategy is* $(0.5, 0.5)$*.*

It can be checked that the joint mixed strategy $s^* = (s_1^*.s_2^*)$, such that:
$s_1^*(H) = 0.5$*,* $s_1^*(T) = 0.5$*,* $s_2^*(H) = 0.5$ *and* $s_2^*(T) = 0.5$ *is an MNE for the matching pennies game.*

The computation of a mixed Nash equilibrium is a fundamental problem for algorithmic game theory. (Nash, 1950) has shown that any game has an MNE. (Chen and Deng, 2005, Daskalakis et al., 2009) have studied the complexity of finding an MNE in a game and have proved that computing an MNE is PPAD-complete (Polynomial Parity Arguments on Directed graphs)[1]. Furthermore, (Conitzer and Sandholm, 2008) has proved that determining whether a game contains more than one MNE is an NP-Complete problem.

1.5 Cardinal Games Classes

In the following, we present the main games classes which are cardinal in essence.

1.5.1 Zero-Sum Games

One of known games classes is "*constant-sum games*" where, for a joint strategy $a \in A$, the sum of all players' payoffs is equal to a constant c. In game theory studies, the most used "term" of this class is "*zero-sum games*" also called "*competitive games*". Indeed, a zero-sum game is a constant-sum game if and only if the sum of all players' payoffs for a joint strategy $a \in A$ is equal to 0. Zero-sum games represent situations of competition between players. Formally, a zero-sum game is defined as follows:

[1] The name of this class is proposed by (Papadimitriou, 1994) and it refers to a class of computational problems in which solutions are guaranteed to exist due to a specific combinatorial principle.

Definition 1.15 (Zero-sum Game). *A cardinal normal form game $G = \langle N, A, \mu \rangle$ is a zero-sum game iff:* $\forall a \in A$

$$\sum_{i \in N} \mu_i(a) = 0. \tag{1.9}$$

The most well-known example of a zero-sum game is the matching pennies game (Example 1.8).

1.5.2 Repeated Games

In many strategic situations, players have to react many times over time. This kind of games is called *"repeated game"* where a given game (generally in normal form) is played many times by the same set of players. At a specific time, the game being repeated is called "stage game". Two kinds of repeated games exist:

- **Finitely repeated games:** the number of iteration is finite. At each stage game, players play a normal form game, i.e., no player knows what the other players are playing but afterward they do. This means that, after the stage game, all players will have information about the actions of the other players. Then move to the next stage. Note that the utility function of every player is additive. In other words, the outcome of the game is the sum of the results of all stage games. A finitely repeated game can be represented as an extensive form game where the outcomes are in the terminal nodes (Benoit et al., 1984);

- **Infinitely repeated games:** are games that continue forever, this is no limit number of the stage game. The representative tree is infinite. Unlike finitely repeated games, the final outcome cannot be attached to the terminal nodes nor can they be defined as the sum of the payoffs on the stage games. For that, two possible ways exist to compute the final payoff of each player: *average reward* and *discounted reward* (Abreu et al., 1990).

1.5.3 Stochastic Games

"*Stochastic games*" have been introduced by (Shapley, 1953). A stochastic game is a dynamic game with probabilistic transitions. It has possible states (each state is given by a normal form game) and it is played in stages. The move from one stage to another depends on transition probabilities. This latter depends on the actions chosen by all players at the current stage.

In other words, at each stage, the game is in one state, the players choose their actions and receive their utilities which depend on the chosen actions and the current state. The next stage is chosen based on *(i)* the previous stage and *(ii)* the actions played by all players. This procedure is repeated for a finite or infinite number of stages. Finally, the total utility of each player is equal to the sum of the stage utilities if the number of stages is finite and equal to the limit inferior of the averages of the stage payoffs if the game is infinite (Shapley, 1953).

1.6 Succinct Games

Representing a game is a challenge in terms of memory. Researchers have proposed representations of a multiple players games more succinctly than the normal form.

Without placing constraints on the players' utilities, describing a game in SNF involving n players, each facing d possible actions, requires listing n utility tables of size d^n. Therefore, as mentioned by (Gottlob et al., 2005), "for large population games (modeling, for instance, agents interactions over the internet), the SNF is practically unfeasible, while the more succinct graphical normal form works very well, and is actually a more natural representation". In many cases indeed, the utility of a player does not depend on the actions of all the other ones – the influence of what the other players decide is often *local*.

To represent such games, *polymatrix games* (Yanovskaya, 1968), *graphical games* (Kearns et al., 2001), *hypergraphical games* (Papadimitriou and Roughgarden, 2008), etc. have been proposed as a convenient way to represent games with multiple players and local interactions, e.g., coordination games.

1.6.1 Polymatrix Games

Polymatrix games have been proposed in the late 60's (Yanovskaya, 1968) as a convenient way to represent games with multiple players and pairwise interactions. Polymatrix games are defined as:

Definition 1.16 (Polymatrix game). *A* polymatrix game *is a tuple* $G = \langle N, E, A, \mu \rangle$ *where:*

- $N = \{1, \dots, n\}$ *is a finite set of* n *players;*
- E *is a set of pairs of distinct players of* N*;*
- $A = \times_{i \in N} A_i$, *where* A_i *is the set of actions available to player* i*;*
- $\mu = \{(\mu_{i,j}, \mu_{j,i}), \{i, j\} \in E\}$ *is a set of pairs of utility functions on* $A_i \times A_j$*:* $\mu_{i,j}(a_i.a_j)$ *is the local utility for player* i *of the joint action* $(a_i.a_j) \in A_i \times A_j$*.*

(N, E) is a graph where nodes N represent the players and edges E capture the interactions between players. An absence of an edge between players i and j corresponds to the pair of players i and j in which utilities are independent of the actions of the other. In other words, the utility of player i does not depend on the utility of player j and vice versa. Each edge $\{i,j\} \in E$ corresponds to a local two-player game $G_{i,j} = \langle \{i,j\}, A_i \times A_j, \{u_{i,j}, \mu_{j,i}\} \rangle$. $G_{i,j}$ is a game in SNF, i.e., represented by a matrix – hence the name "polymatrix game".

Classical polymatrix games are sum-based: the global utility function of a player is the sum of the utilities gathered by this player in the local games she is involved in.

Definition 1.17 (Sum-based polymatrix game)**.** *A* sum-based polymatrix game *is a game $G = \langle N, E, A, \mu \rangle$ where the utility of each player $i \in N$ for the joint action $a \in A$ is:*

$$\mu_i(a) = \sum_{j \in N, \{i,j\} \in E} \mu_{i,j}(a_i.a_j). \tag{1.10}$$

In other terms, if G is sum-based, its equivalent standard normal form is the game $\langle N, A, \mu \rangle$ where utilities are computed using Equation (1.10).

Polymatrix games can be much more frugal in memory space than SNF games – a polymatrix game indeed involves at most $2 \cdot n \cdot (n-1)$ utility tables of size d^2 (d being the maximum number of actions available to one player) – to be compared to the n utility tables of size d^n required by its equivalent standard normal form.

1.6.2 Graphical Games

"*Graphical games*" represents the useful interactions between players (Kearns et al., 2001). Each player's utility depends only on the chosen actions of the players in her neighborhood. A graphical game is represented by a graph (N, E) such that each node $i \in N$ represents a player i and an edge $e_{i,j} \in E$, between two nodes i and j, exists if the utility of player i depends on the actions of player j. Every node of the graph, i.e., every player, has a local matrix that depends on the interactions between the neighbors. Formally, a graphical game is defined as follows:

Definition 1.18 (Graphical Game)**.** *A graphical game is a tuple $G = \langle N, E, A, M \rangle$ where:*

- *$N = \{1, ..., n\}$ is a finite set of n players;*
- *E is a set of subsets of distinct players of N;*
- *$A = \times_{i \in N} A_i$, where A_i is a finite set of actions available to player i;*
- *$M = \{(M_i)_{i \in N}\}$ is a set of local matrices, where $M_i(a_e)$, $e \in E$ is a local matrix of player i for the joint action a_e.*

Given a joint action $a \in A$, $M_i(a_e)$ specifies the utility of player i for the joint action a_e, which depends only on the actions taken by her neighbors in e.

Note that, any SNF game can be represented by a graphical game where the graph (N, E) is complete. The representation of a game by a graphical game can be more compact than the representation in standard normal form. It requires n utility functions of size d^{Neigh} where d is the number of actions per player and $Neigh$ is the maximal number of neighbors of any player. In general, $Neigh << n$, that is, the number of neighborhood of each player is much smaller than the overall players.

1.6.3 Hypergraphical Games

To generalize the two previous classes of games, (Papadimitriou and Roughgarden, 2008) has proposed "*Hypergraphical games*". In an hypergraphical game, every player can be involved in several multiple players subgames. Formally an hypergraphical game is defined as follows:

Definition 1.19 (Hypergraphical game)**.** *An Hypergraphical game is a tuple* $G = \langle N, E, A, \mu \rangle$ *where:*

- $N = \{1, \ldots, n\}$ *is a finite set of* n *players;*
- E *is a set of subsets of distinct players of* N*;*
- $A = \times_{i \in N} A_i$*, where* A_i *is the set of actions available to player* i*;*
- $\mu = \{\mu_i^e, e \in E\}$ *is a set of utility functions.* $\mu_i^e(a_e)$ *is the local utility for player* i *of the joint action* a_e *in the local SNF game between the set of* e *players.*

(N, E) is a hypergraph where nodes N represent the players and hyperedges $e \in E$ capture interactions between players. An absence of an hyperedge corresponds to a subset of players in which utilities are independent of the actions of the other. Each hyperedge $e \in E$ corresponds to a local $|e|$-player SNF $G_e = \langle e, A, \mu \rangle$.

Originally, hypergraphical games are sum based: the global utility function of a player is the sum of the utilities gathered by this player in the local games she is involved in.

Definition 1.20 (Sum-based hypergraphical game)**.** Hypergraphical game *is a game* $G = \langle N, E, A, \mu \rangle$ *where the utility of any player* $i \in N$ *for the joint action* $a \in A$ *is defined as:*

$$\mu_i(a) = \sum_{e \in E, i \in e} \mu_i^e(a_e). \tag{1.11}$$

Hypergraphical game generalizes both polymatrix games and graphical games. Indeed, if the graph is a clique the hypergraphical game is a polymatrix game. However, if each edge has nonzero utility in only one hyperedge, and these nonzero hyperedges have a certain symmetry property, the hypergraphical game can be seen as a graphical game.

1.6.4 Boolean Games

Boolean games are firstly defined by (Harrenstein et al., 2001) then by (Bonzon et al., 2006). In a boolean game, every player has a set of propositional variables, and their actions consist of a set of "prioritized goal bases". The utility functions are binary and described by a single propositional formula. The utility of a player is equal to 1 if the propositional formula is True and 0 otherwise. In other words, a player is satisfied (get a utility equal to 1) iff her goal is satisfied and she is unsatisfied otherwise.

In game theory studies, several other forms of succinct games exist such: "*Action graph games*" (Jiang et al., 2011) which are represented by a *directed graph* where each node corresponds to an action that is available to one or more players and "*Sparse games*" (Chen et al., 2006) in which most of the players' utilities are equal to zero.

1.7 Conclusion

This chapter presented the different representations of games with complete information: the standard normal form and extensive form. It presented several ordinal solution concepts: secure strategy, Pareto optimal, dominated strategies, and pure Nash equilibrium. Then, cardinal solution concepts: minmax regret and mixed Nash equilibrium. After that, this chapter presented some known classes of cardinal games and finally, it presented different cardinal succinct games.

In some situations like planning a vacation, players can only order (in an ordinal scale) their preferences over the outcomes: from the worst to the best or the contrary. The next chapter focuses on *ordinal games* to model these situations.

Chapter 2

Possibilistic and Ordinal Games

2.1 Introduction

As detailed in Chapter 1, several fundamental notions of game theory such pure *Nash equilibrium*, *secure strategy* and *dominance*, do not require the cardinal-payoff assumption. They are basically "*ordinal*" notions. Nevertheless, (probabilistic) *mixed Nash equilibrium*, repeated games, sum based polymatrix games rely on the idea that players assess the relative performance of their decisions by evaluating a payoff in a cardinal way.

Despite their capacity to model many problems in different domains, cardinal games are not able to adequately address problems in fields where human expertise is required such as military applications[1]. The main reason is the inability to formulate appropriate payoff functions for all players.

Players may have certain preferences and can list their outcomes, easily, in a specific order, i.e., from the worst to the best or the contrary. Hence, players' preferences are ordinal rather than cardinal. To model such situations, a specific theory based on the ranking ordering of players' preferences arises, called "*ordinal games*" (Ouenniche et al., 2016, Cruz and Simaan, 2000, Xu, 2000).

Roughly speaking, ordinal games (Ouenniche et al., 2016, Cruz and Simaan, 2000, Xu, 2000) can be identified as the qualitative counterpart of cardinal games, defined by (Von Neumann and Morgenstern, 1944), in which players can order their preferences instead of computing a payoff function.

This chapter focuses on ordinal games and it is organized as follows: Section 2.2

[1]For example, planning an air operation in the presence of an intelligent adversary is extremely difficult.

presents ordinal games. Section 2.3 shows the limits of probability theory to compute a mixed Nash equilibria in ordinal games. Section 2.4 gives the brief concepts of possibility theory to model possibilistic mixed Nash equilibrium (in Section 2.5). In the end, Section 2.6 presents the qualitative counterpart of succinct games presented in Chapter 1.

2.2 Ordinal Games

In some real-life situations, like planning a vacation (e.g., choosing between going to the beach or the mountain), the main difficulty is defining an adequate utility function for each player to evaluate her preferences (Ouenniche et al., 2016, Xu, 2000, Cruz and Simaan, 2000). However, players can easily list their outcomes on an ordered scale, i.e., from the worst to the best or the contrary. For example, players cannot evaluate their preferences using an utility function to choose between going to the beach, cinema, or restaurant. Nevertheless, they can easily express their preferences on a ranked scale such: a player can prefers the beach to the cinema and cinema to a restaurant.

Hence, *ordinal games* (Ouenniche et al., 2016, Cruz and Simaan, 2000, Xu, 2000) arose to model situations where players are easily able to express and order the situations through the game outcomes in an ordinal manner as illustrated in the following "coordination game".

Example 2.1 (Coordination game)**.** *Let us consider a game where several agents have to choose between multiple competing offers, e.g., choosing an internet provider. This game is a kind of "coordination game" inspired from (Simon and Wojtczak, 2017). An agent is satisfied at a high degree if and only if* all *her neighbors choose the same service as she does. For instance because the satisfaction of the agent is relative to the security of her communications with her neighbors and the security level of the network is not guarantee when different services are used.*

Consider a coordination game between n players, each choosing between two actions, x, and y. A player is satisfied iff all her neighbors play the same action and is satisfied to a lower level otherwise. Of course, players may have a prior preference for x or for y (e.g., a preference for some provider).
Let $Neigh(i)$ denotes the set of neighbors of player i.

In this game $N = \{1, \dots, n\}$ and $\forall i$: $A_i = \{x, y\}$ and μ_i is defined as follows:

- *if players i and j, s.t., $j \in Neigh(i)$ coordinate:*
  - $\mu_i(x.a_{-i}) = \alpha_{i,x}$ *if* $\forall j \in Neigh(i)$, $a_i = a_j = x$;
  - $\mu_i(y.a_{-i}) = \alpha_{i,y}$ *if* $\forall j \in Neigh(i)$, $a_i = a_j = y$;

- *if players i and j, s.t., $j \in Neigh(i)$ do not coordinate:*

  - $\mu_i(x.a_{-i}) = \beta_{i,x}$ *if* $a_i = x$ *and* $\exists j \in Neigh(i)$, *s.t.,* $a_i \neq a_j$;

  - $\mu_i(y.a_{-i}) = \beta_{i,y}$ *if* $a_i = y$ *and* $\exists j \in Neigh(i)$, *s.t.,* $a_i \neq a_j$.

Typically, in a coordination game, all β are lower than the α's. For players who prefer to coordinate on x rather than to coordinate on y, we have $\alpha_{i,x} > \alpha_{i,y}$ and $\beta_{i,x} > \beta_{i,y}$ and the contrary for players who prefer a coordination on y than on x we have$\alpha_{i,y} > \alpha_{i,x}$ and $\beta_{i,y} > \beta_{i,x}$. Table 2.1 presents such a game for the two players case.

		Player 2	
		x	y
Player 1	x	$\alpha_{1,x}$, $\alpha_{2,x}$	$\beta_{1,x}$, $\beta_{2,x}$
	y	$\beta_{1,x}$, $\beta_{2,x}$	$\alpha_{1,y}$, $\alpha_{2,y}$

Table 2.1: A coordination game with two players.

When only two players are involved in N, $(x.x)$ and $(y.y)$ are the two PNE of the game (because $\forall i \in N$, $\beta_{i,x}$ and $\beta_{i,y}$ are low).

The most fundamental core of an ordinal game is the player's preferences over the alternatives. Each player i can list her preferences from the best to the worst or the contrary. She can say that she strictly prefers a situation to another one, or she weakly prefers a situation to another one or she is indifferent between two situations.

As mentioned in the previous chapter, several fundamental notions of game theory such pure *Nash equilibrium*, *secure strategy* and *dominance*, do not require the cardinal-payoff assumption. They are basically *ordinal* notions. In other words, whether the game is ordinal or cardinal the latter notions have the same definition.

Several works have studied pure Nash equilibrium in ordinal games. (Cruz and Simaan, 2000) studied Nash equilibrium in ordinal game. They defined the optimal Nash as the most highest ranked Nash equilibrium for the game. They also studied generalised Stackelberg solution for ordinal games where one player announces her action before the other players in the game.

Moreover, (Durieu et al., 2008) shown that potential ordinal games[2] always admit ordinal Nash equilibria. This result is a qualitative counterpart of cardinal potential games, which always admit a pure Nash equilibrium (Monderer and Shapley, 1996).

Since an ordinal game does not always admit ordinal Nash equilibrium, (Cruz and Simaan, 2000, Ouenniche et al., 2016) proposed an approach which consists of listing

[2]In game theory, a game is said to be a potential game (Monderer and Shapley, 1996) if the incentive of all players to change their strategy can be expressed using a single global function called the potential function.

all combinations of pure strategies and use the list of ranks of pure strategies of each player to analyze all possible pure equilibria in the game.

The notion of (probabilistic) mixed Nash equilibrium cannot solve ordinal games. In the following, we will show the limits of probabilistic mixed equilibrium to solve ordinal games.

2.3 Limits of Probabilistic Mixed Equilibria in Ordinal Games

Every cardinal game guarantees at least one *probabilistic mixed Nash equilibrium* (Definition 1.14) (Nash, 1950). In this section, we show the impact of probabilistic mixed equilibrium in ordinal games in case of non existence of a pure Nash equilibrium.

Example 2.2 (Firm Competition)**.** *Let E be an established firm and N be a newcomer firm. They have to fix the packaging of a similar product. Two different packagings exist X and Y. The established producer prefers the newcomer's product looks like its own while the newcomer prefers that the products look different. If the established chooses to play Y, the utility of the newcomer playing X is greater than her utility if she plays Y. However, her utility playing X will be greater than her utility playing Y if E plays X. E is indifferent between X and Y. This situation can be modeled by an ordinal game. The following table illustrates players' utilities (😵 refers to the worst-case whereas 😄 refers to the best case). The ordinal scale is as follows: 😵 < ☹ < 😐 < 🙂 < 😄.*

		N: X	N: Y
E	X	😐 , 😐	☹ , 😄
	Y	☹ , 🙂	😐 , 😵

Table 2.2: Firm Competition.

The above game can be translated into an equivalent ordinal game where utilities are represented in numerical ordered scale where 😵 refers to 0, ☹ refers to 1, 😐 refers to 2, 🙂 refers to 3 and 😄 refers to 4. Hence the game is as follows:

		N: X	N: Y
E	X	2 , 2	1 , 4
	Y	1 , 3	2 , 0

Table 2.3: Firm Competition.

If we translate ordinal utilities to their corresponding real values, this cardinal game admits a single mixed Nash equilibrium,: $s_E^* = [0.6, 0.4]$ *and* $s_N^* = [0.5, 0.5]$.

Let us transform this ordinal game into two different cardinal games detailed in Tables 2.4 and 2.5. Note that, these ordinal games are equivalent since the order of preferences is the same. The game detailed in Table 2.4 admits a new mixed equilibrium $s_E^* = [0.5, 0.5]$ *and* $s_N^* = [0.5, 0.5]$. *Whereas, the game detailed in Table 2.5 admits another mixed equilibrium* $s_E^* = [0.43, 0.57]$ *and* $s_N^* = [0.5, 0.5]$.

		N: X	N: Y
E	X	2 , 2	1, **5**
	Y	1 , 3	2 , 0

Table 2.4: Firm Competition.

		N: X	N: Y
E	X	2 , 2	1, **6**
	Y	1 , 3	2 , 0

Table 2.5: Firm Competition.

In this example, there exist different equilibria for the same game: $s_E^*(X) < s_E^*(Y)$, $s_E^*(X) = s_E^*(Y)$ *or* $s_E^*(X) > s_E^*(Y)$.

The previous example shows that the transformation of ordinal utilities to cardinal ones presents a problem in the definition of mixed equilibrium. Hence, it is recommended to use directly the original game instead of transforming it into a cardinal game to avoid the bias linked to ordinal-cardinal utility transition.

For that, (Hosni and Marchioni, 2013, Ben Amor et al., 2017) and (Hosni and Marchioni, 2019) introduce the notion of "*possibilistic mixed strategy*" to compute mixed Nash equilibrium in ordinal games. This approach is based on a qualitative uncertainty theory: *Possibilility theory* detailed bellow.

2.4 Basics on Possibility Theory

2.4.1 Possibility Distribution

Possibility theory offers a natural and flexible model to represent and handle uncertain information, especially qualitative uncertainty, and total ignorance. It was, first, introduced by (Zadeh, 1978) and further developed by (Dubois and Prade, 1988).

The basic building block in possibility theory (Dubois and Prade, 1988) is the notion of *possibility distribution.* A possibility distribution π is a mapping from a set of states S (also called the "states of the world" or "domain of discourse") to an ordered scale Δ (in the remaining, we consider $\Delta = [0, 1]$, but any ordered scale $\Delta = \{a < b < c < d < ...\}$ can be used). Formally, a possibility distribution is a function: $\pi : S \mapsto \Delta$.

For each state $s \in S$, $\pi(s) = 1$ means that s is totally possible, $\pi(s) = 0$ means that s is impossible and $\pi(s) > \pi(s')$ means that s is more plausible than s'. π is assumed to be normalized: there is at least one totally possible state. Formally, $\exists s \in S$ such $\pi(s) = 1$.

Given a possibility distribution, two external situations can be captured:

- **complete knowledge:** only one state is totally possible and the remaining ones are impossible. Formally, $\exists s \in S$, s.t., $\pi(s) = 1$ and $\forall s' \in S$, s.t., $s' \neq s$, $\pi(s') = 0$;

- **total ignorance:** all states have a possibility 1. Formally: $\forall s \in S$, $\pi(s) = 1$.

A possibility distribution π can be *more specific* than π' denoted by $\pi \leq \pi'$. In other words, π is more informative than π', therefore, if a state s is possible for π thus it is at least as possible for π'. Formally:

Definition 2.1 (Specificity relation)**.** *Given two possibility distributions π and π', π is more specific than π' iff:*

$$\pi \leq \pi' \Leftrightarrow \forall s \in S \pi(s) \leq \pi'(s). \tag{2.1}$$

Example 2.3. *A group of doctors is discussing the symptoms of a patient in the domain of discourse $S = \{d_1, d_2, d_3, h\}$. Suppose that the ordinal scale $\Delta = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Two doctors express their analysis in the form of normalized possibility distributions π_1 and π_2 defined as follows:*

$$\pi_1(d_1) = 0.6, \quad \pi_1(d_2) = 1, \quad \pi_1(d_3) = 0.8, \quad \pi_1(h) = 0.4,$$
$$\pi_2(d_1) = 0.4, \quad \pi_2(d_2) = 1, \quad \pi_2(d_3) = 0.2, \quad \pi_2(h) = 0.$$

We can say that the possibility distribution π_2 is more specific than π_1 since:
$\pi_2(d_1) \leq \pi_1(d_1)$, $\pi_2(d_2) \leq \pi_1(d_2)$, $\pi_2(d_3) \leq \pi_1(d_3)$, $\pi_2(h) \leq \pi_1(h)$.

2.4.2 Possibility and Necessity Measures

In possibility theory, to measure the occurrence of any event $E \subseteq S$ there are two essential measures:

- **possibility measure:** $\Pi(E) = \max_{s \in E} \pi(s)$ evaluates to what extent E is *consistent* with the knowledge represented by π;

- **necessity measure:** $N(E) = 1 - \Pi(\bar{E}) = 1 - \max_{s \notin E} \pi(s)$ corresponds to the extent to which $\bar{E}$ is inconsistent and thus evaluates at which level E is certainly implied by the knowledge.

Example 2.4 (Cont. Example 2.3)**.** *Let us consider the possibility distribution* π_1*. Suppose that the patient suffers from* d_1 *or* d_3*. Then, the possibility* Π_1 *and the necessity* N_1*, according to doctor 1, associated to the possible event* $E = \{d_1, d_3\}$ *are equal to:*
$\Pi_1(E) = \max\big(\pi_1(d_1), \pi_1(d_3)\big) = \max(0.6, 0.8) = 0.8$*;*
$N_1(E) = 1 - \max\big(\pi_1(d_2), \pi_1(h)\big) = 1 - \max(1, 0.4) = 0$*.*

In possibility theory, several rules have been proposed to compute the conditional possibility measures or distributions from unconditional ones. (Cooman, 1997, Walley and Cooman, 1999) discussed the alternative definitions of conditional possibilities. In the following, we will present the ordinal ones. We denote E and F two states such $E \in S$ and $F \in S$:

- **Zadeh's rule** (Zadeh, 1978) is the first definition of conditioning for possibility measures. The idea is that conditional degrees are equal to the unconditional ones: $\Pi(E|F) = \Pi(E, F)$. This rule may leads to an unnormalised conditional possibility distributions;

- **Hisdal's equation** (Hisdal, 1978) is inspired by the Bayes' rule. It is given by:

$$\Pi(E|F) = \begin{cases} \Pi(E \cap F) & if \;\; \Pi(E \cap F) < \Pi(F) \\ [\Pi(E \cap F), 1] & if \;\; \Pi(E \cap F) = \Pi(F) \end{cases} \tag{2.2}$$

  If $\Pi(E \cap F) = \Pi(F)$ then $\Pi(E \cap F)$ can be any possible degree in $[\Pi(E \cap F), 1]$, i.e., $\Pi(E \cap F) \leq \Pi(E|F) \leq 1$. This equation has one solution if and only if $\Pi(F) = 1$. In the other cases, it may have different solutions if $\Pi(E \cap F) = \Pi(F)$ and $\Pi(F) \neq 1$;

- **Ramer's rule** (Ramer, 1989) consists on picking one state $X \in S$ such $\Pi(X \cap F) = \Pi(F)$, then affects $\Pi(X|F) = 1$ and $\Pi(E|F) = \Pi(E \cap F)$ for all states E in order to have a normalized coditional possibility distribution. The disadvantage of this rule is the arbitrary choice of X if there several X such such $\Pi(X \cap F) = \Pi(F)$;

- **Dubois-Prade rule** (Dubois et al., 1994, Dubois and Prade, 1990) generalizes Rames's rule and Hisdal's equation. It sets $\Pi(E \cap F) = 1$ for all state E such $\Pi(E \cap F) = \Pi(F)$. It is defined as follows:

$$\Pi(E|F) = \begin{cases} \Pi(E \cap F) & if \;\; \Pi(E \cap F) < \Pi(F) \\ 1 & if \;\; \Pi(E \cap F) = \Pi(F) \end{cases} \tag{2.3}$$

  The Dubois-Prade rule is a least specific solution of Hisdal's equation. It guarantees that the conditional possibility $\Pi(E|F)$ is always normalized.

In the remaining, we use the notion of *conditional possibility measure* proposed by Dubois and Prade (Dubois and Prade, 1990, Dubois et al., 1994) (Equation 2.3) to stay in the pure ordinal context.

Roughly speaking, state E is totally possible, knowing that F occurred, if at least one of the most plausible states making F true also makes E true. Otherwise, the possibility of E is that of the most plausible state making both E and F true.

2.4.3 Possibilistic Qualitative Utilities

Considering qualitative (possibilistic) problems of decision under uncertainty, where each decision is evaluated by an utility function $\mu : S \mapsto \Delta$, authors in (Dubois and Prade, 1995, Dubois et al., 2001) proposed qualitative utilities as counterparts to (Von Neumann and Morgenstern, 1944) expected utility:

- **pessimistic utility:** generalizes the Wald criterion and estimates to what extent it is certain (i.e., necessary according to measure N) that a possibility distribution π reaches a good utility. The pessimistic utility, denoted U^{pes}, of a decision π is expressed by:
$$U^{pes}(\pi) = \min_{s \in S} \max(1 - \pi(s), \mu(s)) \tag{2.4}$$

- **optimistic utility:** estimates to what extent it is possible that a possibility distribution π reaches a good utility. The optimistic utility, denoted U^{opt}, of a decision π is expressed by:
$$U^{opt}(\pi) = \max_{s \in S} \min(\pi(s), \mu(s)) \tag{2.5}$$

U^{opt} is rather unnatural (too adventurous), while U^{pes} conveniently models the behaviour of an *uncertainty adverse* decision maker. This model makes a commensurability assumption between the utility levels and the levels of likelihood. This assumption is common to all the models which consider that the agent's preference relation is *complete* and *transitive* (this is the case in many models, be they qualitative or quantitative, e.g., expected utility (Savage, 1954, Von Neumann and Morgenstern, 1944), multi-prior non expected utility (Itzhak and David, 1989) and Sugeno integrals (Dubois et al., 1998)).

2.5 Possibilistic Mixed Nash Equilibrium

The notion of pure Nash equilibrium is similar in both ordinal and cardinal games. However, the mixed Nash equilibrium differs from cardinal to ordinal games. We have shown in Section 2.3 that the probabilistic mixed Nash equilibrium in ordinal games cannot be computed using probability theory. In this section, we present the definition of possibilistic mixed strategies (Hosni and Marchioni, 2013, Ben Amor et al., 2017) as well as the notion of least specific possibilistic mixed equilibrium, recently defined by (Ben Amor et al., 2017).

2.5.1 Possibilistic Mixed Strategies

A possibilistic mixed strategy for player i is defined by (Hosni and Marchioni, 2013, Ben Amor et al., 2017) and (Hosni and Marchioni, 2019) as a *normalized possibility distribution* π_i, i.e., a ranking, on her set of actions. This ranking has a dual interpretation in terms of *preference* and *likelihood*. Indeed, for player i, distribution π_i models the ranking of alternatives in terms of preference or commitment. Under this interpretation, $\pi_i(a_i) = 1$ means that a_i is fully satisfactory/conceivable to player i to play, while $\pi_i(a_i) = 0$ means that it is absolutely not an option for player i. However, for all other players, π_i measures the likelihood of play: $\pi_i(a_i) = 1$ means that action a_i is a completely plausible play of player i while $\pi_i(a_i) = 0$ means that the action a_i is an impossible play of player i. This dual preference/likelihood interpretation is natural in game theory since, according to the other players, the most preferred alternatives of player i should be the most likely to be played.

The joint mixed strategy $\pi = (\pi_1, \dots, \pi_n)$ defines a possibility distribution over the action profiles: $a = (a_1, \dots, a_n)$ is played if and only if each player $i \in N$ plays action $a_i \in A_i$, the possibility that $a \in A$ is played is computed in a conjunctive way, i.e., as the *minimum* of the $\pi_i(a_i)$:

$$\pi(a) = \min_{i \in N} \pi_i(a_i). \tag{2.6}$$

By abuse of notations π designates both the above possibility distribution and the vector $(\pi_1, \dots, \pi_n)$. Since we assume that all the π_i are normalized, so π will be also normalized, i.e., there exists a joint action $a^* = (a_1^*, \dots, a_n^*)$ such $\pi(a^*) = 1$.

Following (Ben Amor et al., 2017), the global utility of a mixed strategy π is given by its pessimistic utility:

$$U_i^{pes}(\pi) = \min_{a \in A} \max\big(1 - \pi(a), \mu_i(a)\big). \tag{2.7}$$

Recently, possibility theory was used by (Hosni and Marchioni, 2019). Authors have proposed a qualitative approach based on Sugeno expectation with respect to a possibility measure. They studied the case of two-player games.

The Sugeno expectation of player i is the Sugeno integral of the utility function μ_i for the mixed strategy π is defined as:

$$E_i(\pi) = \bigvee_{a \in A} (\mu_i(a) \wedge \pi(a)). \tag{2.8}$$

$\bigvee$ corresponds to the max and $\wedge$ corresponds to the min thus:

$$\begin{aligned} E_i(\pi) &= \max_{a \in A} \min(\mu_i(a), \pi(a)) \\ &= U_i^{opt}(\pi). \end{aligned}$$

2.5.2 Least Specific Possibilistic Mixed Equilibrium

The notion of possibilistic mixed equilibrium (ΠME) introduced in (Ben Amor et al., 2017) in ordinal games is similar to the notion of probabilistic mixed equilibrium in cardinal games in the sense that it is a possibilistic mixed strategy $\pi^* = (\pi_1^*, \ldots, \pi_n^*)$ where no player has the incentive to deviate from her π_i^*. Formally:

Definition 2.2 (Possibilistic Mixed Equilibrium). *Let $G = \langle N, A, \mu \rangle$ be an ordinal game. $\pi^* = (\pi_1^*, \ldots, \pi_n^*)$ is a possibilistic mixed equilibrium (ΠME) iff, $\forall i \in N$, $\forall \pi_i'$ on A_i:*

$$U_i^{pes}(\pi_i^*.\pi_{-i}^*) \geq U_i^{pes}(\pi_i'.\pi_{-i}^*). \tag{2.9}$$

where $\pi_{-i}^* = (\pi_1^*, \ldots, \pi_{i-1}^*, \pi_{i+1}^*, \ldots, \pi_n^*)$.

Example 2.5 (Cont. Example 2.2). *Let us consider the firm competition game where the utility degrees are in $\Delta = \{0, 0.25, 0.5, 0.75, 1\}$:*

		N: X	N: Y
E	X	0.5 , 0.5	0.25 , 1
	Y	0.25 , 0.75	0.5 , 0

Table 2.6: Firm competition.

Let $\pi^* = (\pi_E^*.\pi_N^*)$ be a possibilistic mixed strategy such:

$$\pi_E^*(X) = 1,\ \pi_E^*(Y) = 1,\ \pi_N^*(X) = 1 \text{ and } \pi_N^*(Y) = 0.75.$$

It can be checked that π^* is a ΠME for the above game:

- $\forall \pi_E' \neq \pi_E^*$: $U_E^{pes}(\pi_E'.\pi_N^*) \leq U_E^{pes}(\pi_E^*.\pi_N^*)$;
- $\forall \pi_N' \neq \pi_N^*$: $U_N^{pes}(\pi_N'.\pi_E^*) \leq U_N^{pes}(\pi_N^*.\pi_E^*)$.

It can be shown that an ordinal game always admits such an equilibrium. (Radul, 2019) shows that when a possibilistic strategy is used in the context of (Hosni and Marchioni, 2019), the existence of mixed Nash equilibria is guaranteed.

As shown by (Ben Amor et al., 2017), every ordinal game admits a ΠME that can be found using a polynomial-time algorithm. In the following, we will detail how this algorithm works to find a least specific ΠME.

2.5.3 A Polynomial Time Algorithm for the Construction of Possibilistic Mixed Equilibria

In ordinal games, possibilistic mixed strategies should be interpreted as successive *commitments* in an ongoing negotiation process - at a given point of the negotiation process, each player indicates which options she may consider in actual play and which actions she prefers than others. The negotiation process is iterative. At each stage, given the joint mixed strategy, each player aims to improve her pessimistic utilities by changing her mixed strategy. Indeed, given a possibilistic mixed strategy π, if a player i changes π_i into a more specific one π'_i (i.e., a π'_i such $\forall a_i \in A_i, \pi'_i(a_i) \leq \pi_i(a_i)$), she can only become better off ($U_i^{pes}(\pi_i^{'}.\pi_{-i}) \geq U_i^{pes}(\pi_i.\pi_{-i})$). When no player has any more incentive to make her strategy more specific, the result of the negotiation leads to a "*least specific* mixed equilibrium". This was shown in (Ben Amor et al., 2017), where they proposed to focus on the *least specific* ΠME. Formally, a least-specific possibilistic mixed equilibrium is defined as follows:

Definition 2.3 (Least-specific possibilistic mixed equilibrium)**.**
Let $G = \langle N, A, \mu \rangle$ be an ordinal game. $\pi^* = (\pi_1^*, \dots, \pi_n^*)$ is a least-specific *possibilistic mixed equilibrium for G iff:*

- *π^* is a possibilistic mixed equilibrium for G;*
- *there exists no π', s.t., $\pi^* < \pi'$ and π' is a ΠME.*

To build the least specific MNE, at each stage of the negotiation process, an *improvement procedure* is proposed to check if a player can improve her mixed strategy unilaterally by making it more specific. Therefore, if a player i changes her mixed strategy π_i to a $\pi_i^{'}$ such $\pi_i^{'}$ is more specific than π_i, she increases her pessimistic utility.

Formally, the goal is to find a mixed strategy $\pi_i^{'} < \pi_i$ which strictly improves U_i^{pes}:

$$U_i^{pes}\left(\pi_i^{'}.\pi_{-i}\right) > U_i^{pes}(\pi).$$

Furthermore, $\pi_i^{'}$ has to be a least-specific such distribution:

$$U_i^{pes}\left(\pi_i^{''}.\pi_{-i}\right) \leq U_i^{pes}(\pi), \forall \pi_i^{'} < \pi_i^{''} \leq \pi_i$$

Given an ordinal game G, a player i and a joint mixed strategy π, the improvement procedure $\textsc{Improve}(G, \pi, i)$ has three steps:

1. Compute the pessimistic utility of player i given her mixed strategy π_i and the

mixed strategy π_{-i} of her opponents:

$$U_i^{pes}(\pi) = \min_{a \in A} \max\left(\max_{j \in N}\left(1 - \pi_j\left(a_j\right)\right), \mu_i(a)\right) = \min_{a_i \in A_i} \max\left(1 - \pi_j\left(a_j\right), U_i^{pes}\left(a_i, \pi_{-i}\right)\right). \tag{2.10}$$

where $U_i^{pes}\left(a_i, \pi_{-i}\right)$ is the utility of player i when she plays a_i and the other players play the mixed strategy π_{-i}:

$$U_i^{pes}\left(a_i, \pi_{-i}\right) = \min_{a_{-i} \in A_{-i}} \max\left(\max_{j \in N \setminus i}\left(1 - \pi_j\left(a_j\right)\right), \mu_i(a)\right). \tag{2.11}$$

2. Compute, for player i, the subset $D_i \subseteq A_i$ of dominated actions defined as follows:

$$D_i = \left\{a_i \in A_i, \text{ s.t., } U_i^{pes}\left(a_i, \pi_{-i}\right) \leq U_i^{pes}(\pi)\right\}. \tag{2.12}$$

3. Compute the new mixed strategy of the improvement procedure Improve(G, π, i) such:
   - if $\forall a_i \notin D_i, \pi_i\left(a_i\right) < 1$. Then, π_i cannot be improved unilaterally because the possibility distribution should be normalized. Therefore:

     $$\text{Improve}(G, \pi, i) \leftarrow \pi;$$

   - if $\exists a_i \notin D_i$ such $\pi_i\left(a_i\right) = 1$, player i can move from π_i to a more specific mixed strategy π_i' without losing the normalization. π_i' is defined as follows:
     - $\pi_i'\left(a_i\right) \leftarrow \pi_i\left(a_i\right), \forall a_i \notin D_i$;
     - $\pi_i'\left(a_i\right) \leftarrow n(U_i^{pes}(\pi))^{-}, \forall a_i \in D_i$ where $n\left(U_i^{pes}(\pi)\right)^{-}$ is the degree in Δ just below and $n(U_i^{pes}(\pi)) = 1 - U_i^{pes}(\pi)$.

     Therefore:

     $$\text{Improve}(G, \pi, i) \leftarrow (\pi_i'.\pi_{-i}).$$

Algorithm 2.1 details the Improve function:

We note that the complexity of Improve(G, π, i) is dominated by that of the computation of $\left\{\{U_i^{pes}\left(a_i, \pi_{-i}\right)\}_{i \in N, a_i \in A_i}\right\}$. These can be computed in polynomial time. Therefore, Improve(G, π, i) can itself be performed in polynomial time.

(Ben Amor et al., 2017) exploits the improvement procedure to propose a polynomial-time algorithm to find a *least specific* mixed Nash equilibrium in an ordinal game outlined in Algorithm 2.2.

Algorithm 2.1: IMPROVE.

Data: $G = \langle N, A, \mu \rangle$, π, i
Result: $\pi' = (\pi'_1, \ldots, \pi'_n)$
$D_i \leftarrow \varnothing$;
forall $a_i \in A_i$ **do**
  if $U_i^{pes}(a_i, \pi_{-i}) \leq U_i^{pes}(\pi)$ **then** $D_i \leftarrow D_i \cup \{a_i\}$;
end
if $\forall a_i \notin D_i,\ \pi_i(a_i) < 1$ **then** $\pi' \leftarrow \pi$;
else
  forall $a_i \in A_i$ **do**
    if $a_i \in D_i$ **then** $\pi'_i(a_i) \leftarrow n\big(U_i^{pes}(\pi)\big)^-$;
    else $\pi'_i(a_i) \leftarrow \pi_i(a_i)$;
  end
end
return π'

Algorithm 2.2: Finding a possibilistic mixed Nash equilibrium in an ordinal game.

Data: $G = \langle N, A, \mu \rangle$
Result: $\pi^* = (\pi^*_1, \ldots, \pi^*_n)$, a ΠME
$\pi^0 \leftarrow (\pi^0_1, \ldots, \pi^0_n)$ /* $\pi^0_i(a_i) = 1, \forall i \in N, \forall a_i \in A_i,$ */
$t \leftarrow 0$
repeat
  $\pi^{loc} \leftarrow \pi^t$
  forall $i \in N$ **do**
    $\pi^{loc} \leftarrow$ IMPROVE (G, π^{loc}, i)
  end
  $\pi^{t+1} \leftarrow \pi^{loc}$
  $t \leftarrow t + 1$
until $\pi^t = \pi^{t-1}$
$\pi^* \leftarrow \pi^t$
return π^*

Example 2.6 (Cont. Example 2.2)**.** *Consider the ordinal game depicted in Table 2.6:*

		N: X	N: Y
E	X	*0.5 , 0.5*	*0.25 , 1*
	Y	*0.25 , 0.75*	*0.5 , 0*

Details of Algorithm 2.2 are as follows:
Let us start with $\pi^0_E = \pi^0_N = [1; 1]$, *that is,* $\pi^0_E(X) = \pi^0_N(X) = \pi^0_E(Y) = \pi^0_N(Y) = 1$: *Uncertainty is maximal. Now,*

$$
\begin{aligned}
U_E^{pes}\left(X, \pi^0_{-E}\right) &= \min_{a_N \in \{X,Y\}} \max\left(1 - \left(\pi^0_N\left(a_N\right)\right), \mu_E\left(X, a_N\right)\right) \\
&= \min\left(\mu_E(X,X), \mu_E(X,Y)\right) = 0.25 \\
U_E^{pes}\left(Y, \pi^0_{-E}\right) &= \min\left(\mu_E(Y,X), \mu_E(Y,Y)\right) = 0.25 \\
U_N^{pes}\left(X, \pi^0_{-N}\right) &= \min\left(\mu_N(X,X), \mu_N(Y,X)\right) = 0.5 \\
U_N^{pes}\left(Y, \pi^0_{-N}\right) &= \min\left(\mu_N(X,Y), \mu_N(Y,Y)\right) = 0
\end{aligned}
$$

Furthermore, $U_E^{pes}(\pi^0) = 1$*,* $U_N^{pes}(\pi^0) = 0, D_N\{Y\}$ *and* $A_E^* = \{X, Y\}$ *. Since* $A_E \backslash A_E^* = \varnothing$*, thus* $\textsc{Improve}(G, \pi^0, i) = \pi^0$*. Since* $A_N \backslash D_N = \{X\}$ *we get* $\textsc{Improve}(G, \pi^0, i) = \left[\pi^0_N(X); n\left(U_N^{pes}(\pi^0)\right)^-\right] = [1; 0.75]$*. Another round of improvement does not give anymore changes. So,* $\pi^* = (\pi^*_E, \pi^*_N)$*, where* $\pi^*_E = [1;1]$ *and* $\pi^*_N = [1; 0.75]$*, forms a possibilistic mixed equilibrium of the ordinal game.*

2.6 Succinct Ordinal Games

In order to model ordinal games where players utilities depend on a subset of players, (Azzabi et al., 2020) proposed a compact representation of ordinal games called: "*Min-based polymatrix games*", "*ordinal graphical games*" and "*Min-based hypergraphical games*". They have shown that the global utility of a player is captured by a *min* operator.

Consider a coordination game (as described is Example 2.1) with multiple players. This kind of game is typically based on a graph and the satisfaction of an agent is the minimum, over all her neighbors, of the satisfaction she gets in local games with a single neighbor. Of course, agents may have more gradual preferences, e.g., because they prefer some providers to other ones.

In coordination games, the satisfaction of an agent may depend on the number of neighbors choosing the same provider as this agent (Simon and Wojtczak, 2017), e.g., where the satisfaction of an agent depends on the number of neighbors who choose the same provider than herself. Unfortunately, since a min operation cannot be captured by a sum (min is idempotent, the sum is not), sum-based polymatrix games cannot capture some problems such coordination games, as soon as more than two players are involved. In this context, min-based polymatrix games have been recently proposed by (Azzabi et al., 2020).

Definition 2.4 (Min-based polymatrix game)**.** *A min-based polymatrix game is a polymatrix game* $G = \langle N, E, A, \mu \rangle$ *where the utility of player* $i \in N$ *for the joint action* $a \in A$

is:

$$\mu_i(a) = \min_{j \in N, \{i,j\} \in E} \mu_{i,j}(a_i.a_j). \tag{2.13}$$

Example 2.7 (Cont. Example 2.2)**.** *If we consider a min-based polymatrix game with three player where a central player, say player* 2*, is related to the two other ones, while player* 1 *and* 3 *are related to the central player only (see the graph of Figure 2.1). Each local 2-player game is a coordination game as depicted in Table 2.1. The utilities of each local SNF game are depicted in Table 2.7.*

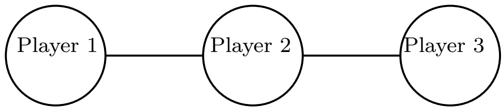

Figure 2.1: A graph of neighborhood between three players.

		Player 2	
		x	y
Player 1	x	$\alpha_{1,x}$, $\alpha_{2,x}$	$\beta_{1,x}$, $\beta_{2,x}$
	y	$\beta_{1,x}$, $\beta_{2,x}$	$\alpha_{1,y}$, $\alpha_{2,y}$

(a) Local SNF game between player 1 and player 2.

		Player 2	
		x	y
Player 3	x	$\alpha_{3,x}$, $\alpha_{2,x}$	$\beta_{3,x}$, $\beta_{2,x}$
	y	$\beta_{3,x}$, $\beta_{2,x}$	$\alpha_{3,y}$, $\alpha_{2,y}$

(b) Local SNF game between player 2 and player 3.

Table 2.7: Local SNF games.

It can be checked that $(x.x.x)$ *and* $(y.y.y)$ *are the only two PNE when* $\beta_{i,x}$ *and* $\beta_{i,y}$ *are low. Suppose now that player* 1 *really dislikes action* x*, i.e.,* $\beta_{1,x} > \alpha_{1,x}$*. Then* $(x.x.x)$ *is not a PNE anymore (player* 1 *would prefer to move to* y*).*

(Azzabi et al., 2020) has also defined the qualitative counterpart of graphical games and hypergraphical games so-called "*ordinal graphical games*" and "*ordinal hypergraphical games*" respectively.

The global utility of player i in an ordinal hypergraphical game is defined as the minimum overall local utilities. Formally:

Definition 2.5 (Min-based hypergraphical game)**.** *Min-based hypergraphical game is a game* $G = \langle N, E, A, \mu \rangle$ *where the utility of any player* $i \in N$ *for the joint action* $a \in A$ *is:*

$$\mu_i(a) = \min_{e \in E, i \in e} \mu_i^e(a_e). \tag{2.14}$$

Computing pure and mixed Nash equilibria in ordinal graph-based games has also been investigated in (Azzabi et al., 2020). They show that, for graph-based games, determining whether a PNE exists is an NP-hard problem They have, also, proposed a polynomial-time algorithm to compute possibilistic mixed equilibria for graph-based games.

2.7 Conclusion

Ordinal games are defined to model situations where players do not have an adequate utility function to evaluate their outcomes. However, they allow players to rank their preferences among their different outcomes. Ordinal games can be seen as the qualitative counterpart of cardinal games.

This chapter presented the basic concepts of ordinal games. Then, it detailed the notion of mixed Nash equilibrium which is based on the possibility theory.

All previous games assume that all players have complete information about the game being played: the players, their actions and their utilities. However, in some situations, players may lack some information about the game. These situations are modeled by "*games with incomplete information*" which will be presented in the next chapter.

Chapter 3

Games with Incomplete Information

3.1 Introduction

All frameworks described in the previous chapters, assume that the utilities of players depend only on the joint actions and amount to a representation of every player's utilities for every possible outcome. Game-theoretic analyses typically assume that all players have full knowledge of other ones, especially of the utilities for each outcome.

However, in real life, like security (Jain et al., 2008, Liu et al., 2006, Mohi et al., 2009) players may lack some information about some important aspects of the game that they are playing. As mentioned by (Harsanyi, 1967a), they may lack full information about their own utility functions or actions available to other players or even to themselves, or the utilities of other players.

To capture such situations, *Bayesian games* have been proposed by (Harsanyi, 1967a). They relax this assumption by allowing agents to have different "types", representing different beliefs about the game being played, and to have uncertainty about the types of the other players. The type of a player summarizes all the relevant information about that player. Furthermore, in Bayesian games, it is assumed that every player eventually knows her own type and has a probability distribution over the joint types, i.e., her type and the types of the other players. This probability distribution is common to all players.

In the remainder of this chapter, Section 3.2 presents the historical context of incomplete information games and their standard normal form representation. Section 3.3 defines Bayesian games. Sections 3.4 and 3.5 present the notions of strategies and Nash equilibria in games with incomplete information. Then, Section 3.6 shows the transformation of Bayesian games into an equivalent SNF game with complete information and the transformation to a 2-player Bayesian game into an equivalent polymatrix game.

Section 3.7 illustrates the different classes of games with incomplete information.

3.2 Games with Incomplete Information

The first paper about game theory is due to (Von Neumann and Morgenstern, 1944), it defines *extensive form games* (see Section 1.2.2). These latter are complex and difficult to analyze. That is why, in the same paper, authors argued that an extensive form game can be transformed into an equivalent normal form game. In other words, the multistage game in extensive form can be reduced into a one-stage normal form game (where all players play simultaneously and independently). In a normal form game, all players have to prepare their actions before playing the game. After that, no player gets any additional information.

In (Von Neumann and Morgenstern, 1944), the authors use the term of *incomplete information* to refer to a game in which a part of the normal form is unspecified. They note that the uncertainty of the players about the parameters of the game can be modeled by an extensive form game with imperfect information. In order to analyze these games, (Luce and Adams, 1956, Luce and Raiffa, 1957) propose generalized normal form game that does not make the assumption that every player knows every opponent's utility function. In (Luce and Raiffa, 1957), the authors proposed n-player normal form games which contain n^2 utility functions. Each utility function presents the belief of a player i about the utility function of another player j. However, it is difficult to analyze this generalized normal form game: *(i)* it does not consider the uncertainty about the actions of the players and *(ii)* it does not address the question of what player k may believe about player j's beliefs about player i's beliefs, etc.

In 1962, Harsanyi discussed a more general case of uncertainty in a game. He recognized problems of modeling players' *hierarchical beliefs* about the beliefs of other players (Harsanyi, 1962). For more explanation, let us take an example of a 2-player game. Player 1's (resp. player 2's) strategy will depend on what she expects to be player 2's (resp. player 1's) utility function. This expectation about player 2's (resp. player 1's) utility function may be called player 1's (resp. player 2's) "*first order beliefs*". However, the chosen action of player 1 (resp. player 2) will also depend on what she expects to be player 2's (resp. player 1's) first-order belief about her utility function. This is called player 1's (resp. player 2's) "*second order belief*". The latter can be considered as an expectation concerning the first-order belief and so on ad infinity.

In his paper, Harsanyi shows that the players' beliefs can be represented by *conditional probability distributions*[1]. However, this is difficult to analyze. That is why, Harsanyi

[1]Player 1's first-order belief will have the nature of a conditional probability distribution over all

proposes the construction of some complete information games theoretically equivalent to the original incomplete information game. The idea is to generate an incomplete information game in terms of *one* unique conditional probability distribution (derived from the sequence of a probability distribution over alternative order beliefs using the Bayes rule). These games are called "*games with incomplete information*" (Harsanyi, 1967b).

In games with incomplete information, the lack of some parameters can be declined in three cases:

1. the players may not know the physical outcome function of the game, i.e, the outcome produced by each combination of actions;

2. the players may not know their own or some other opponents' utility functions;

3. the players may not know their own or some other players' possible actions, i.e., the set of all strategies available to various players.

All other types of games with incomplete information can be reduced to this three basic cases[2].

In an incomplete information game, each player *(i)* knows the set of her possible actions, *(ii)* has some knowledge about her opponents, and *(iii)* receives some information that could affect her beliefs about what her opponents know. To encapsulate all this information, Harsanyi proposed the notion of "*type*". Let Θ_i be the possible pieces of information which can be received by player i: Θ_i is called the set of "types" of player i. The question is then, for each player, to determine an action for each of her types. Thus, in games with incomplete information the set of states of the world is omitted and only the types are considered. Θ_i is the local state space for player i and $\Theta = \Theta_1 \times \cdots \times \Theta_n$ is the effective global state space. The idea of Harsanyi when defining types was that a player's local state can encapsulate all the information to which the player has access: it contains the status of the external world that the player has observed but can also contain her introspective mental states. See (Brandenburger, 1993, J. Aumann and Brandenburger, 1995, Battigalli and Bonanno, 1999, Brandenburger, 2008, Dekel and Siniscalchi, 2015) for the links between belief states and types, and more generally for further developments about epistemic game theory. This kind of interpretation also complies with the semantics of epistemic logic (Fagin et al., 1996).

In Harsanyi's approach, every player has a set of possible types that represent the different information that this player may have. Before playing the game, everyone

alternative utility functions that player 2 may have and vice versa. On the other hand, player 1's second-order belief will be a conditional probability distribution over all alternative first-order belief probability distribution that player 2 may choose, etc.

[2]For some examples of reduction see (Harsanyi, 1967b) Section 2.

receives some private information to have her types but not the types of the other players.

This information is given by a "*conditional probability distribution*" over the joint types of the other players (Harsanyi, 1967a).

Formally an incomplete information game is defined as:

Definition 3.1 (Standard Normal Form of Incomplete Information Game)**.** *An incomplete information game is a tuple* $G = \langle N, A, \Theta, P^*, \mu \rangle$ *where:*

- $N = \{1, \ldots, n\}$ *is a finite set of* n *players;*
- $A = \times_{i \in N} A_i$*, where* A_i *is a finite set of actions available to player* $i \in N$*;*
- $\Theta = \times_{i \in N} \Theta_i$*, where* Θ_i *is the set of types of player* $i \in N$*.* Θ *gathers all the configurations of types, i.e., vectors* $\theta = (\theta_1, \ldots, \theta_n)$*.* θ_i *denotes the type of player* i *in* θ *and* $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$*. Hence* $\forall\ \theta, \theta' \in \Theta$*,* $\theta'_i.\theta_{-i}$ *belongs to* Θ*. Likewise* $\Theta_{-i} = \times_{j \neq i} \Theta_j$*;*
- $P^* = \{(P^*_i)_{i \in N}\}$ *where* $P^*_i = P^*_i(\theta_{-i} | \theta_i)$ *is the conditional probability distribution over the set* Θ_{-i}*;*
- $\mu = \{(\mu_i)_{i \in N}\}$ *is a set of utility functions.* $\mu_i(a, \theta)$ *captures the utility of player* $i \in N$ *for the joint action* $a \in A$ *and the joint type* $\theta \in \Theta$*.*

3.3 Bayesian Games

In 1967, Harsanyi (Harsanyi, 1967b) introduced an alternative model of games with incomplete information, named *"Bayesian game"*. It differs only in that the probability specifies a *joint probability distribution* over the set of joint types rather than n conditional probability distributions over the set of the joint types of the other players. From the "Bayesian game", using the Bayes' rule, it is possible to have an equivalent incomplete information game with the same set of players, actions, types, and utilities. Formally, a Bayesian game is defined as follows:

Definition 3.2 (Bayesian Game)**.** *A Bayesian game is a tuple* $G = \langle N, A, \Theta, P, \mu \rangle$ *where:*

- $N = \{1, \ldots, n\}$ *is a finite set of* n *players;*
- $A = \times_{i \in N} A_i$*, where* A_i *is a finite set of actions available to player* $i \in N$*;*
- $\Theta = \times_{i \in N} \Theta_i$*, where* Θ_i *is the set of types of player* $i \in N$*;*
- P*:* $\Theta \to [0, 1]$ *is a joint probability distribution over the combinations of types* Θ*;*

		Lisa				
		M			Av	
		C	B		C	B
Bob	C	1 , 2	0 , 0	C	1 , 0	0 , 1
	B	0 , 0	2 , 1	B	0 , 2	2 , 0
		$p(M) = 0.5$			$p(Av) = 0.5$	

Table 3.1: A Battle of the sexes game with 2 types combinations.

- $\mu = \{(\mu_i)_{i \in N}\}$ *is a set of utility functions.* $\mu_i(a, \theta)$ *captures the utility of player* i *for the joint action* a *and the joint type* θ*.*

Conversely, we can derive an equivalent Bayesian game from an incomplete information game if and only if the players' beliefs in the original game are common prior, i.e., the conditional probability distribution is known by all players (Harsanyi, 1967a). These two games differ only on the probabilities: in the games with incomplete information, the probability functions specify conditional probabilities whereas in the Bayesian game there exists a joint probability distribution over the joint types.

Note that, in a *"Bayesian game"*, each player's utility depends on the joint actions chosen by all players and on joint types. In addition to that, all players are assumed to know the joint probability distribution over the joint types. However, the type of player i is eventually known only by herself, i.e., each player knows only her type but she does not know the types of her opponents and, of course, she has some information about the opponents but does not know exactly how much information the opponent will have about her.

A Bayesian game can be equivalently defined as a set of $|\Theta|$ normal form games with the same set of players N and the same set of actions A. More precisely, for each $\theta \in \Theta$, there is a normal form game $G^\theta = \langle N, A, \mu^\theta \rangle$ where $\forall i \in N$:

$$\mu_i^\theta(.) = \mu_i(., \theta) \text{ and } p(G^\theta) = p(\theta). \tag{3.1}$$

Example 3.1 (Battle of the sexes game)**.** *Bob and Lisa wish to go out. Their main concern is to go out together. However, Bob prefers to go to the beach (B) and Lisa prefers to go to the cinema (C). This game is often referred to as the "Battle of the Sexes"; for the standard story behind it see (Luce and Raiffa, 1989). Bob does not know whether Lisa wishes to meet (M) or wishes to avoid (Av) him. Therefore, this is a Bayesian game where Lisa has two types, i.e.,* $\Theta_{Lisa} = \{M, Av\}$*. Suppose that these two types have probability 0.5, i.e.,* $p(M) = p(Av) = 0.5$*. There are two combinations of types thus two possible games (see Table 3.1) the probability degrees of which are:* $p(G^M) = p(M) = 0.5$ *and* $p(G^{Av}) = p(Av) = 0.5$.

3.4 Strategies in Bayesian Games

As in classical games, players can play a pure or mixed strategy. In this section, we will define these two strategies:

In a game with incomplete information, the action of player i depends only on the information $\theta_i \in \Theta_i$ that she receives. A joint pure strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ is thus a tuple of functions σ_i that map each possible information (each "type" $\theta_i \in \Theta_i$) to an action $a_i \in A_i$. Formally, a pure strategy is defined as follows:

Definition 3.3 (Pure Strategy in an Incomplete Information Game)**.** *A pure strategy is a vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ of functions $\sigma_i : \Theta_i \to A_i$.*

$\sigma_i(\theta_i)$ specifies the action that player i will execute when receiving the private information θ_i. Given a strategy σ and a configuration of the players types $\theta \in \Theta = \Theta_1 \times \cdots \times \Theta_n$, $\sigma(\theta) = (\sigma_1(\theta_1), \ldots, \sigma_n(\theta_n))$ denotes the joint action a (the element of A) prescribed by strategy σ when θ occurs. Let Σ_i denotes the set of all functions from Θ_i to A_i and $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ the set of all joint strategies.

As in classical games, a mixed strategy in an incomplete information game is defined as a probability distribution over pure strategies. It consists of randomizing over the set of available pure strategies according to some probability distribution. As before, let s_i denote a mixed strategy of player i and S_i be the set of all the mixed strategies of player i. Furthermore, we denote $s_i(a_i|\theta_i)$ the probability that player i of type θ_i plays action a_i under mixed strategy s_i.

In SNF games with complete information, the utility of a player for a pure strategy is derived directly from the utility table. In a Bayesian game, the utility of a player is computed using the expected utility even for pure and mixed strategies. More precisely, Harsanyi (Harsanyi, 1967a) proposed three meaningful notions of expected utilities, detailed and named by (Myerson, 2004), as follows:

1. ***Ex-Ante expected utility:*** the player does not know anybody's type, i.e., no player knows her type nor the types of her opponents. Formally, the Ex-Ante utility of player i is evaluated as follows:

   Definition 3.4 (Ex-Ante Expected Utility)**.** *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game.*

   - *the Ex-Ante expected utility of player $i \in N$ for the pure strategy $\sigma \in \Sigma$ is:*

$$EU_i^{Ex-Ante}(\sigma) = \sum_{\theta \in \Theta} p(\theta)\mu_i(\sigma(\theta), \theta). \tag{3.2}$$

- *the Ex-Ante expected utility of player $i \in N$ for the mixed strategy $s \in S$ is:*

$$EU_i^{Ex\text{-}Ante}(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \Big(\prod_{j \in N} s_j(a_j|\theta_j) \Big) \mu_i(a, \theta). \tag{3.3}$$

For Harsanyi, computing the Ex-Ante utility of a player i before her type is learned does not have a decision-theoretic interest in the game. In other words, the Ex-Ante utility is meaningless. Instead, he proposed to consider each player's conditional utility given her type. This conditional utility is called "*Ex-Interim*" utility.

2. ***Ex-Interim expected utility:*** considers the setting in which a player knows eventually her own type but not the types of the other players. Formally, the Ex-Interim utility of player i for type θ_i is evaluated as follows:

Definition 3.5 (Ex-Interim Expected Utility)**.** *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game.*

- *the Ex-Interim expected utility of player $i \in N$ of type $\theta_i \in \Theta_i$ for the pure strategy $\sigma \in \Sigma$ is:*

$$EU_i^{Ex\text{-}Interim}(\sigma, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \mu_i(\sigma(\theta_i.\theta_{-i}), \theta_i.\theta_{-i}). \tag{3.4}$$

- *the Ex-Interim expected utility of player $i \in N$ of type $\theta_i \in \Theta_i$ for the mixed strategy $s \in S$ is:*

$$EU_i^{Ex\text{-}Interim}(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \Big(\prod_{j \in N} s_j(a_j|\theta_j) \Big) \mu_i(a, \theta_i.\theta_{-i}). \tag{3.5}$$

where $p(\theta_{-i}|\theta_i)$ is computed using the Bayes rule.

3. ***Ex-Post expected utility:*** considers that each player knows all the types of her opponents, i.e., the joint type θ is known to all players. Formally, the Ex-Post expected utility of player i for the joint type θ is evaluated as follows:

Definition 3.6 (Ex-Post Expected Utility)**.** *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game.*

- *the Ex-Post expected utility of player $i \in N$ for the pure strategy $\sigma \in \Sigma$ given all players' actual types $\theta \in \Theta$ is:*

$$EU_i^{Ex\text{-}Post}(\sigma, \theta) = \mu_i(\sigma(\theta), \theta). \tag{3.6}$$

- *the Ex-Post expected utility of player $i \in N$ for the mixed strategy $s \in S$ given all players' actual types $\theta \in \Theta$ is:*

$$EU_i^{Ex\text{-}Post}(s,\theta) = \sum_{a \in A} \Big(\prod_{j \in N} s_j(a_j|\theta_j) \Big) \mu_i(a,\theta). \tag{3.7}$$

Computing Ex-Post expected utility assumes that every player knows every other players types. In other words, every player knows all the information of the game. Thus, it's not interesting to study Ex-Post expected utility in Bayesian games since the game will be a complete information game.

Example 3.2 (Cont. of Example 3.1). *Let s be a mixed strategy where:*

$$\begin{array}{ll} s_{Lisa}(C|M) = 0.6, & s_{Lisa}(B|M) = 0.4, \\ s_{Lisa}(C|Av) = 0.4, & s_{Lisa}(B|Av) = 0.6, \\ s_{Bob}(B) = 0.8, & s_{Bob}(C) = 0.2. \end{array}$$

The Ex-Interim expected utility of Lisa for type M is equal to:

$$\begin{aligned} EU_{Lisa}^{Ex-Interim}(s,M) = & \Big(p(M) \times \Big(\big(s_{Bob}(C) \times s_{Lisa}(C|M) \times \mu_{Lisa}(C.C,M)\big) + \\ & \big(s_{Bob}(C) \times s_{Lisa}(B|M) \times \mu_{Lisa}(C.B,M)\big) + \\ & \big(s_{Bob}(B) \times s_{Lisa}(C|M) \times \mu_{Lisa}(B.C,M)\big) + \\ & \big(s_{Bob}(B) \times s_{Lisa}(B|M) \times \mu_{Lisa}(B.B,M)\big)\Big)\Big) \\ EU_{Lisa}^{Ex-Interim}(s,M) = & \Big(0.5 \times \Big(\big(0.2 \times 0.6 \times 2\big) + \big(0.2 \times 0.4 \times 0s, M\big) + \\ & \big(0.8 \times 0.6 \times 0\big) + \big(0.8 \times 0.4 \times 1\big)\Big)\Big) \\ EU_{Lisa}^{Ex-Interim}(s,M) = & \ 0.28. \end{aligned}$$

The *Ex-Ante* expected utility of player i depends on her *Ex-interim* expected utility. Formally:

Proposition 3.1. *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game.*

- *the Ex-Ante expected utility of player $i \in N$ for the pure strategy $\sigma \in \Sigma$ is:*

$$EU_i^{Ex-Ante}(\sigma) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i^{Ex-Interim}(\sigma, \theta_i). \tag{3.8}$$

- *the Ex-Ante expected utility of player $i \in N$ for the mixed strategy $s \in s$ is:*

$$EU_i^{Ex\text{-}Ante}(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i^{Ex\text{-}Interim}(s, \theta_i). \tag{3.9}$$

Note that, when we write $EU_i^{Ex\text{-}Ante}(\sigma)$ as $\sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i^{Ex\text{-}Interim}(\sigma, \theta_i)$ (Equation (3.8)) we can observe that $EU_i^{Ex\text{-}Interim}(\sigma_i.\sigma_{-i}, \theta_i)$ does not depend on strategies that player i would play if her type were not θ_i. Thus, we are, in fact, performing independent maximization of player i's Ex-Interim expected utilities conditioned on each type that she could have. In other words, if all actions are the best after the type is received, it is preferable to establish a conditional plan, in advance, to know what to do if the latter is received simply if each player maximizes her $EU_i^{Ex\text{-}Interim}(\sigma, \theta_i)$, then $EU_i^{Ex\text{-}Ante}(\sigma)$ will be greater. Thus, it is more interesting to study $EU_i^{Ex\text{-}Interim}(\sigma, \theta_i)$ rather than $EU_i^{Ex\text{-}Ante}(\sigma)$.

In the following of this thesis, we will focus on studying $EU_i^{Ex\text{-}Interim}(\sigma, \theta_i)$.

3.5 Nash Equilibrium in Bayesian Games

3.5.1 Pure Nash Equilibrium

To ensure that a given player can best respond to other players, it is necessary to know what action each player will adopt for each of her possible types. A *best response* for player i is computed *knowing the provisional strategies of the other players*, i.e., knowing σ_{-i}. This is the action a_i which maximizes $EU_i^{Ex\text{-}Interim}(\sigma, \theta_i)$ (again, when considering a joint strategy σ, the uncertainty of player i only bears on the joint type).

Definition 3.7 (Best Response for a pure strategy in a Bayesian game)**.** *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game. The best response of player $i \in N$ of type $\theta_i \in \Theta_i$ to σ_{-i} is:*

$$BR_i(\sigma_{-i}, \theta_i) = \underset{a_i \in A_i}{\arg\max}\ EU_i^{Ex\text{-}Interim}(a_i, \sigma_{-i}, \theta_i). \tag{3.10}$$

where:

$$EU_i^{Ex\text{-}Interim}(a_i, \sigma_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \mu_i(a_i.\sigma_{-i}(\theta_{-i}), \theta_i.\theta_{-i}). \tag{3.11}$$

A pure Nash Equilibrium is a joint strategy σ from which no player i will deviate unilaterally knowing σ_{-i}. Formally:

Definition 3.8 (Pure Nash Equilibrium in a Bayesian Game)**.**
Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game. The joint pure strategy σ is a pure Nash equilibrium iff $\forall i \in N,\ \forall \theta_i \in \Theta_i,\ \forall a_i^{'} \in A_i$:

$$EU_i^{Ex-Interim}(\sigma_i(\theta_i), \sigma_{-i}, \theta_i) \geq EU_i^{Ex-Interim}(a_i', \sigma_{-i}, \theta_i)$$

This definition generalizes Definition 1.9, which is recovered when $|\Theta| = 1$ (only one possible type per player). As for a classical normal form game, a pure Nash equilibrium may not exist for a Bayesian game.

(Conitzer and Sandholm, 2002) show that checking the existence of a pure Nash equilibrium in a Bayesian game is an NP-complete problem. Hardness holds even for symmetric, two-player games, while membership holds even if only one among $P(.)$ and the various $\mu_i(.)$ are given explicitly as a table.

Example 3.3. *(Cont. Example 3.1) Consider the pure strategy σ^* where Bob chooses to go to the beach and Lisa chooses to go to the beach if she wants to meet Bob and to go to the cinema if she wants to avoid him:*

$$\sigma^*_{Bob} = B,\ \sigma^*_{Lisa}(M) = B \text{ and } \sigma^*_{Lisa}(Av) = C.$$

The Ex-Interim expected utility of Lisa for type M is: $EU_{Lisa}^{Ex-Interim}(\sigma^*, M) = 0.5$ and for type Av is $EU_{Lisa}^{Ex-Interim}(\sigma^*, Av) = 1$.

Similarly, the Ex-Interim expected utility of Bob is: $EU_{Bob}^{Ex-Interim}(\sigma^*) = 1$.

It can be checked that σ^* is a pure Nash equilibrium (using Definition 3.8). The Ex-Interim expected utility of Bob playing C (resp. B) is equal to 0.5 (resp. 1). So, he will prefer to play B rather than C. The Ex-Interim expected utility of Lisa for type M playing C (resp. B) is equal to 0 (resp. 0.5). So, she will prefer to play B rather than C. The Ex-Interim expected utility of Lisa for type Av playing B (resp. C) is equal to 0 (resp. 1). So, she will prefer to play C rather than B.

3.5.2 Bayes-Nash Equilibrium

A *best response* for player i of type θ_i is computed knowing the provisional mixed strategies of the other players, i.e., knowing s_{-i}. More precisely, it is a mixed strategy s_i that maximises the *Ex-Interim* expected utility $EU_i^{Ex-Interim}(s, \theta_i)$.

Definition 3.9 (Best Response for a mixed strategy in a Bayesian game)**.** *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game. The best response of player $i \in N$ of type $\theta_i \in \Theta_i$ for the mixed strategy $s_{-i} \in S_{-i}$ is:*

$$BR_i(s_{-i}) = \underset{s_i \in S_i}{\operatorname{argmax}}\ EU_i^{Ex-Interim}(s_i.s_{-i}, \theta_i). \tag{3.12}$$

A Bayes-Nash equilibrium in a Bayesian game is a mixed strategy $s^* = (s_1^*, ..., s_n^*)$

where no player $i \in N$ of type $\theta_i \in \Theta_i$ can improve her Ex-Interim expected utility by changing her mixed strategy $s_i \in S_i$. Formally:

Definition 3.10 (Bayes-Nash equilibrium in a Bayesian game). *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game. The joint mixed strategy s^* is a Bayes-Nash equilibrium iff $\forall i \in N, \forall \theta_i \in \Theta_i, \forall s'_i \in S_i$:*

$$EU_i^{Ex\text{-}Interim}(s^*, \theta_i) \geq EU_i^{Ex\text{-}Interim}(s'_i.s^*_{-i}, \theta_i). \tag{3.13}$$

3.6 Bayesian Games Transformations

3.6.1 Transforming a Bayesian Game into a Standard Normal Form Game

Harsanyi proposed a transformation of Bayesian game into a normal form game with complete information (Harsanyi, 1967a). The idea is that the action of player i in the transformed game is a distinct mapping from Θ_i to A_i and her utility for the pure strategy a is then equal to the Ex-Ante expected utility of player i for the pure strategy σ^a where $a_i = \sigma_i^a$. Formally, the normal form transformed game is defined as follows:

Definition 3.11 (standard Normal form representation of a Bayesian game). *Let $G = \langle N, A, \Theta, P, \mu \rangle$ be a Bayesian game. $\tilde{G} = \langle \tilde{N}, \tilde{A}, \tilde{u} \rangle$ is its standard normal form representation where:*

- $\tilde{N} = N$;
- $\tilde{A}_i = \times_{\theta_i \in \Theta_i} A_i$;
- $\tilde{u}_i(\tilde{a}) = EU_i^{Ex\text{-}Ante}(\sigma^a)\ \forall \tilde{a} \in \tilde{A}, i \in \tilde{N}$, *where σ^a is the strategy of the original game defined by: $\sigma_i^a = \tilde{a}_i$.*

The pure Nash equilibria (resp. Bayes-Nash equilibria) of the Bayesian game are in bijection with the pure Nash equilibria (resp. mixed Nash equilibria) of its transformed SNF game. This fact allows us to find a pure or mixed equilibrium of a Bayesian game by transforming it into its equivalent normal form game.

As to the complexity of the transformation of a Bayesian game to a normal form game, consider that the games are extensively represented, by tables. The transformed game contains n utility functions of size $\prod_{i=1,n} |A_i|^{|\Theta_i|}$. If we write for simplification purpose that $|A_i| = d$ and $|\Theta_i| = t\ \forall i \in N$ (same number of types and the same number of actions for all players), this means that a game containing n utility functions of a size $(d^t)^n$ is transformed into a game containing n utility functions of size $(d^t)^n$.

Example 3.4 (Cont. Example 3.1)**.** *The transformed game $\tilde{G}$ of the Bayesian game detailed is Table 3.1 has two players: $\tilde{N} = \{Bob, Lisa\}$ such that $\tilde{A}_{Bob} = \{C, B\}$ and $\tilde{A}_{Lisa} = \{CC, CB, BC, BB\}$ and the utilities are detailed in Table 3.2.*

		Lisa			
		CC	CB	BC	BB
Bob	C	1 , 1	0.5 , 1.5	0.5 , 0	0 , 0.5
	B	0 , 1	1 , 0	1 , 1.5	2 , 0.5

Table 3.2: The SNF of Battle of the sexes Bayesian game depicted in Table 3.1.

The joint action $a^* = (B.BC)$ of $\tilde{G}$ corresponds to σ in G where $\sigma_{Bob} = B$, $\sigma_{Lisa}(M) = B$ and $\sigma_{Lisa}(Av) = C$. It can be checked that the strategy a^* is a pure Nash equilibrium of $\tilde{G}$: The utility of Bob playing B (resp. C) is equal to 1 (resp. 0.5). Then, he prefers to play B than C. The utility of Lisa playing BC is greater than her utility playing CC, CB, or BB.

3.6.2 Transforming a two-player Bayesian Game into a Polymatrix Game

In 1974, Howson and Rosenthal have shown that any Bayesian game with two players can be transformed into an equivalent polymatrix game (see Section 1.6.1 for more details about polymatrix games). The idea is to consider as many players as the number of pairs (i, θ_i), i.e., the number of players is equal to $|\Theta_1| + |\Theta_2|$. Each player (i, θ_i) has A_i as a set of available actions. For each joint strategy $a \in A$, the utility of player (i, θ_i) in the game $\{(i, \theta_i), (j, \theta_j)\} \in E$ in the polymatrix game is equal to the utility of the joint action $a \in A$, to player i of type θ_i where j is of type θ_j (Howson et al., 1974).

Definition 3.12 (Polymatrix representation of a 2-player Bayesian Game)**.** *Let $G = \langle N = \{1, 2\}, A, \Theta, P, \mu\rangle$ be a 2-player Bayesian game. $\tilde{\tilde{G}} = \langle \tilde{\tilde{N}}, \tilde{\tilde{E}}, \tilde{\tilde{A}}, \tilde{\tilde{u}}\rangle$ is the polymatrix game where:*

- $\tilde{\tilde{N}} = \{(i, \theta_i), \forall i \in \{1, 2\}, \forall \theta_i \in \Theta_i\}$*;*
- $\tilde{\tilde{E}} = \{\{(i, \theta_i), (j, \theta_j)\}, i \neq j\}$*;*
- $\tilde{\tilde{A}}_{(i,\theta_i)} = A_i, \ \forall (i, \theta_i) \in \tilde{\tilde{N}}$*;*
- $\tilde{\tilde{u}}_{(i,\theta_i),(j,\theta_j)}(\tilde{\tilde{a}}) = p(\theta_j|\theta_i) \times \mu_i(a, \theta_i.\theta_j), \ \forall \tilde{\tilde{a}} \in \tilde{\tilde{A}}, \forall (i, \theta_i) \in \tilde{\tilde{N}}$*.*

(Howson et al., 1974) has proved that the equilibria in two-player Bayesian games are equivalent to the equilibria in its equivalent polymatrix game.

As for the complexity of the transformation of a 2-player Bayesian game into a polymatrix game, consider that the games are extensively represented, by tables. The transformed polymatrix game contains $|\Theta_1| + |\Theta_2|$ utility functions of size $|A_1| \times |A_2|$. If we write for simplification purpose that $|A_i| = d$ and $|\Theta_i| = t \ \forall i \in \{1, 2\}$ (same number of types and the same number of actions for both players). This means that a Bayesian game containing 2 utility functions of a size $(d \cdot t)^2$ is transformed into a game containing t^2 local games. Each containing 2 utility functions of size d^2. Note that the computation of $\tilde{\tilde{u}}_{(i,\theta_i),(j,\theta_j)}$ takes constant time, so the overall time (and space) complexity is $O(d^2 \cdot t^2)$.

Example 3.5 (Cont. Example 3.1)**.** *The equivalent polymatrix game $\tilde{\tilde{G}}$ of the two-player battle of the sexes game depicted in Table 3.1 contains three players: $\tilde{\tilde{N}} = \{(Bob), (Lisa, M), (Lisa, Av)\}$, all having the same set of actions $\{C, B\}$.*

$\tilde{\tilde{G}}$ contains 2 local games (2 edges) and the utilities of each player in the equivalent game are detailed in Figure 3.1.

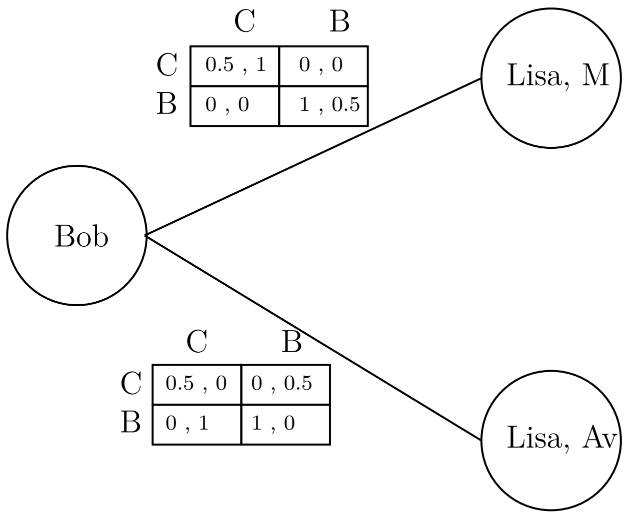

Figure 3.1: The polymatrix game of equivalent Bayesian game of Example 3.1.

The joint action $a^\sigma = (B.B.C)$ of $\tilde{\tilde{G}}$ corresponds to σ^* in G. Using Equation (1.10), the utilities of a^σ in $\tilde{\tilde{G}}$ are:

- $\mu_{Bob}(a^\sigma) = \mu_{Bob,(Lisa,M)} + \mu_{Bob,(Lisa,Av)} = 1$*;*
- $\mu_{Lisa,M}(a^\sigma) = \mu_{(Lisa,M),Bob} = 0.5$*;*
- $\mu_{Lisa,Av}(a^\sigma) = \mu_{(Lisa,Av),Bob} = 1$*.*

It can be checked that a^σ is a PNE in the equivalent polymatrix game.

3.7 Classes of Games with Incomplete Information

The well-know classes of complete information games have been extended to model games with incomplete information such that: repeated games, Boolean games and fuzzy games. In the following, we present the essential related work:

Repeated Games with Incomplete Information

Repeated games with incomplete information are first studied by (Aumann and Maschler, 1967), then, by (Zamir, 1992, Kohlberg, 1975, Mertens and Zamir, 1971) and (Zamir, 1971). They are defined as repeated games with complete information (defined in Section 1.5.2): a Bayesian game is played many times by the same set of players, actions and types. At each stage game, each player knows her previous played actions as well as the past information she has received, i.e., her past types.

Boolean Games with Incomplete information

Boolean games with incomplete information have been studied in different ways. In 2011, Grant et al. incorporated uncertainty in the Boolean games (see Section 1.6.4 for more details about Boolean games) by introducing a set of environment variables outside the control of any agent (Grant et al., 2011). Each agent has some beliefs about the value of the environment variables.

(De Clercq et al., 2014) studied Boolean games with incomplete information where agents can be uncertain about other agents' goals. Every player has her own beliefs about the goal of their opponents.

Ordinal Boolean Games with Incomplete Information

In (De Clercq et al., 2014), authors used possibilistic logic to model uncertainty, in Boolean games, i.e., to encode graded beliefs about other players' goals. They modeled this uncertainty by associating with each player a possibility distribution over the universe of all possible games. The same authors have extended this framework and proposed the use of generalized possibilistic logic to define Boolean games with incomplete information (De Clercq et al., 2015). They used possibilistic logic to compactly describe players' preferences and generalized possibilistic logic to describe incomplete knowledge about another player's preferences.

(De Clercq et al., 2018) proposed an evaluation of Nash equilibrium in terms of their possibility and necessity degrees to be an Nash equilibrium for the true (ill-known) ordinal game.

In possibilistic Boolean games (De Clercq et al., 2018), the knowledge of each player i can be captured by a distinct possibility distribution π_i (the knowledge is not common) and the players do not receive any private information before playing the game. Authors in (De Clercq et al., 2018) propose to compute the possibility and the necessity of a given joint action being a PNE in the usual sense (Definition 1.9): every player computes these indices according to her knowledge. Authors then consider the problem from the external point of view of an observer who proceeds to a fusion of these distributions and deduces a unique $\pi = \min_{i \in N} \pi_i$ over types of players.

3.8 Conclusion

This chapter presented games with incomplete information. These games model the situations where at least one player lacks some information about the game, such that, the utilities of her opponents or the states of nature. Then, it focused on Bayesian games, their strategies, and equilibria. Finally, it presented the different classes of games with incomplete information: repeated games with incomplete information, Boolean games with incomplete information, and ordinal boolean games with incomplete information.

We have also presented the work of (De Clercq et al., 2014) on ordinal games with incomplete information.

To the best of our knowledge, all works dedicated to the study of ordinal games with complete information are limited to normal form games (Xu, 2000, Ouenniche et al., 2016, Cruz and Simaan, 2000) and their succinct form representations (Azzabi et al., 2020). Ordinal games with incomplete information are limited to Boolean games (De Clercq et al., 2014,0). There are no work that present the semantic level of ordinal games with incomplete information as the qualitative counterpart of Bayesian games. In this thesis, we stay at the semantic level and work out the idea of "*possibilistic games with incomplete information*" (Π-games). This will be presented in the next chapter.

Part II

Contributions

Chapter 4

Possibilistic Games with Incomplete Information: Π-games

4.1 Introduction

In game theory, the cardinal notion is needed in two cases at least: (i) when the game is repeated (and outcomes are "collected" and assumed to be additive), and (ii) when the outcomes depend on a probabilistic event, e.g., in the prisoner dilemma if the verdict does not only depend on the confession of the prisoners, but also on the result of the trial.

To capture such incomplete information situations, *Bayesian games* have been proposed by Harsanyi (Harsanyi, 1967a). The latter assume that the utility degrees are additive in essence and the knowledge of the players can be quantified in a probabilistic way. This kind of approach does not apply to ordinal games, where the utility degrees do not capture more than a ranking, nor to situations of decision under qualitative uncertainty (Harsanyi, 1967a).

In this chapter, following the seminal work of (De Clercq et al., 2018) on possibilistic Boolean games, we propose to use possibility theory to model qualitative uncertainty in ordinal games. Then, we extend the notion of secure strategy, pure Nash equilibrium and (possibilistic) mixed Nash equilibrium.

This chapter is organized as follows: Section 4.2 proposes a possibilistic model for ordinal games with incomplete information called "*possibilistic games with incomplete information*" (Π-games). Section 4.3 generalizes the notion of secure strategy, pure and mixed Nash equilibrium to this framework. Section 4.4 shows how a possibilistic game with incomplete information can be transformed into an equivalent ordinal normal form game with complete information and Section 4.5 presents complexity results about Π-games. All proofs are omitted in the end of this chapter.

The main results of this chapter are published in (Ben Amor et al., 2018) and (Ben Amor et al., 2019a).

4.2 Possibilistic Games with Incomplete Information: Π-games

We propose in the following a model for (ordinal) games under possibilistic information where the knowledge of the players is modeled by a qualitative theory: the possibility theory. This framework is called "*possibilistic game with incomplete information*" (Π-game).

The framework of (De Clercq et al., 2018) assumes that the knowledge of each player i is captured by a possibility distribution π_i. The fusion of these distributions leads to a unique $\pi = \min_{i \in N} \pi_i$ over the joint types of players. In our framework, unlike (De Clercq et al., 2018), we do not develop a complex language. We assume all players have a unique possibility distribution π, i.e., all players have a common knowledge.

We follow Harsanyi's games based on types (see Definition 3.2) to define a possibilistic game with incomplete information (Π-game). A Π-game has a finite set of players N, each player i has a finite set of actions A_i and types Θ_i. The utility μ_i of a player i in a Π-game depends on all chosen actions of all players and on a joint type. The knowledge of players is modeled by a possibility distribution π over the joint types. A possibilistic game with incomplete information is defined as follows:

Definition 4.1 (Possibilistic Game with Incomplete Information)**.** *A possibilistic game with incomplete information (Π-game) is a tuple $G = \langle N, A, \Theta, \pi, \mu \rangle$ where:*

- $N = \{1, ..., n\}$ *is a finite set of n players;*
- $A = \times_{i \in N} A_i$ *where A_i is the set of actions of player $i \in N$;*
- $\Theta = \times_{i \in N} \Theta_i$*, where Θ_i is the set of types of player $i \in N$. Θ gathers all the possible combinations of types;*
- π*: $\Theta \to \Delta$ is a joint normalized possibility distribution over the combinations of types;*
- $\mu = \{(\mu_i)_{i \in N}\}$ *where $\mu_i : A \times \Theta \to \Delta$ is the utility function of player $i \in N$. Typically the ordered scale $\Delta = [0, 1]$, but any ordered scale may be used.*

Possibility distribution π captures the common knowledge of the players. The information that the players have about the real world corresponds to a $\theta \in \Theta$ but this information is not common: player i does not know θ, but only θ_i (θ_i is the private knowledge of player i).

$\pi(.|\theta_i)$ captures the knowledge that player i has when learning θ_i. On the other hand, utility $\mu_i(a, \theta)$ (utility of the joint action a for player i when learning θ) will be obtained once all players have played their actions and revealed their types - that is why μ_i depends on the whole θ and not only on θ_i.

As in Bayesian games, a strategy of a player i in a Π-game is a function σ_i that maps each possible information (each "type" $\theta_i \in \Theta_i$) to an action in $a_i \in A_i$. Similarly, a joint strategy $\sigma = (\sigma_1, \dots, \sigma_n)$ is a tuple of functions σ_i (see Definition 3.3).

Definition 4.2. *Let $G = \langle N, A, \Theta, \pi, \mu \rangle$ be a Π-game. G can be equivalently defined as a set of $|\Theta|$ normal form games with the same set of players N and the same set of actions A. More precisely, for each $\theta \in \Theta$, there is a normal form game $G^\theta = \langle N, A, \{\mu^\theta\} \rangle$ where $\forall i \in N$:*

$$\mu_i^\theta(.) = \mu_i(., \theta) \text{ and } \pi(G^\theta) = \max_{\theta' \in \Theta,\ s.t.,\ G^{\theta'} = G^\theta} \pi(\theta'). \quad (4.1)$$

Example 4.1 (Coordination Π-game)**.** *Let us consider the coordination game (described in Example 2.1) with incentives. We suppose that the preferences of the agents may also depend on an external event, e.g., an incentive that some of them receive - typically, an offer from some provider. Of course, the belief of a player about what offers her neighbors receive depends on what the player receives herself: normally, if I receive an offer, so do my neighbors; and if I do not receive anything, they do not either. But I may receive something while my neighbors do not.*

For the sake of brevity, we assume that incentives concern only action x: each player has two types r_i ("i receives an incentive for x") and $\overline{r}_i$ ("i does not receive an incentive for x"), so $\Theta = \{r_1, \overline{r}_1\} \times \dots \times \{r_n, \overline{r}_n\}$.

In our example $\forall i \in N$, $A_i = \{x, y\}$ and $\Theta_i = \{r_i, \overline{r}_i\}$. The satisfaction of player $i \in N$ is equal to $\beta_{i,x}$ (resp. $\beta_{i,y}$) if she prefers x (resp. y) and does not coordinate with her neighbors. If she coordinates and receives an incentive her satisfaction when playing x is increased to $\delta > \max\{\alpha_{i,x}, \alpha_{i,y}\}$ and remains to $\alpha_{i,x}$ if she coordinates and does not receive an incentive. Hence the following utility functions:

- *if $\exists j \in Neigh(i)$, s.t., $a_i \neq a_j$ then:*
  - $\mu_i(x.a_{-i}, \theta) = \beta_{i,x}$, $\forall \theta$;
  - $\mu_i(y.a_{-i}, \theta) = \beta_{i,y}$, $\forall \theta$;
- *if $\forall j \in Neigh(i)$, $a_i = a_j$ then:*

– $\mu_i(x.a_{-i}, \overline{r}_i.\theta_{-i}) = \alpha_{i,x},\ \forall\theta_{-i}$;

– $\mu_i(x.a_{-i}, r_i.\theta_{-i}) = \delta,\ \forall\theta_{-i}$;

– $\mu_i(y.a_{-i}, \theta) = \alpha_{i,y},\ \forall\theta$.

There are two "normal" states in $\Theta = \{r_1, \overline{r}_1\} \times \dots \times \{r_n, \overline{r}_n\}$*: everybody receives an incentive for* x *(state* $r = (r_1, r_2, \dots, r_n)$*) and nobody receives anything (state* $\overline{r} = (\overline{r}_1, \overline{r}_2, \dots \overline{r}_n)$*). Cases were only some players have an incentive are of course possible. This knowledge is captured by a joint possibility distribution on* Θ*:* $\pi(r) = \pi(\overline{r}) = 1$ *and* $\pi(\theta) = \gamma$ *(with* $0 \leq \gamma \leq 1$ *for other combinations of types* $\theta \in \Theta \backslash \{r, \overline{r}\}$*).*

In the following, we will present a coordination game with incentives for two players case. There are four combinations of types $\{(r_1.r_2), (r_1.\overline{r}_2), (\overline{r}_1.r_2), (\overline{r}_1.\overline{r}_2)\}$*, and thus four possible games the possibility degrees of which are:*

$$\pi(G^{r_1.r_2}) = \pi(r_1.r_2) = 1, \quad \pi(G^{r_1.\overline{r}_2}) = \pi(r_1.\overline{r}_2) = \gamma,$$
$$\pi(G^{\overline{r}_1.r_2}) = \pi(\overline{r}_1.r_2) = \gamma, \quad \pi(G^{\overline{r}_1.\overline{r}_2}) = \pi(\overline{r}_1.\overline{r}_2) = 1.$$

The utility degrees are in $\Delta = \{0, \gamma, \beta_{i,x}, \beta_{i,y}, \alpha_{i,x}, \alpha_{i,y}, \delta, 1\}$ *where:*

- $0 \leq \gamma \leq \beta_{i,x} \leq \beta_{i,y} \leq \alpha_{i,x} \leq \alpha_{i,y} \leq \delta \leq 1$ *if player* i *prefers* y *to* x*;*
- $0 \leq \gamma \leq \beta_{i,y} \leq \beta_{i,x} \leq \alpha_{i,y} \leq \alpha_{i,x} \leq \delta \leq 1$ *if player* i *prefers* x *to* y*.*

Table 4.2 details the utility functions for the two players case.

		Player 2			
		r_2		$\overline{r}_2$	
Player 1		x	y	x	y
r_1	x	δ , δ	$\beta_{1,x}$, $\beta_{2,y}$	δ , $\alpha_{2,x}$	$\beta_{1,x}$, $\beta_{2,y}$
	y	$\beta_{1,y}$, $\beta_{2,x}$	$\alpha_{1,y}$, $\alpha_{2,y}$	$\beta_{1,y}$, $\beta_{2,x}$	$\alpha_{1,y}$, $\alpha_{2,y}$
		$\pi(r_1.r_2) = 1$		$\pi(r_1.\overline{r}_2) = \gamma$	
$\overline{r}_1$	x	$\alpha_{1,x}$, δ	$\beta_{1,x}$, $\beta_{2,y}$	$\alpha_{1,x}$, $\alpha_{2,x}$	$\beta_{1,x}$, $\beta_{2,y}$
	y	$\beta_{1,y}$, $\beta_{2,x}$	$\alpha_{1,y}$, $\alpha_{2,y}$	$\beta_{1,y}$, $\beta_{2,x}$	$\alpha_{1,y}$, $\alpha_{2,y}$
		$\pi(\overline{r}_1.r_2) = \gamma$		$\pi(\overline{r}_1.\overline{r}_2) = 1$	

Table 4.1: A coordination Π-game between two players (with two types per player) where $\Delta = \{0, \gamma, \beta_{i,x}, \beta_{i,y}, \alpha_{i,x}, \alpha_{i,y}, \delta, 1\}$.

In the following, we translate the equivalent coordination Π*-game depicted in Table 4.1 into an equivalent coordination* Π*-game where utilities are represented in numerical ordered scale* $\Delta = \{0, 0.1, 0.2, 0.3, 0.7, 0.8, 0.9, 1\}$*. The utility functions for the two players are in Table 4.2.*

Player 1		Player 2: r_2		Player 2: $\overline{r}_2$	
		x	y	x	y
r_1	x	0.9 , 0.9	0.3 , 0.3	0.9 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(r_1.r_2) = 1$		$\pi(r_1.\overline{r}_2) = 0.1$	
$\overline{r}_1$	x	0.8 , 0.9	0.3 , 0.3	0.8 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(\overline{r}_1.r_2) = 0.1$		$\pi(\overline{r}_1.\overline{r}_2) = 1$	

Table 4.2: A coordination Π-game between two players (with two types per player) where $\Delta = \{0, 0.1, 0.2, 0.3, 0.7, 0.8, 0.9, 1\}$.

If player i receives an incentive, she conditions her knowledge and we get: $\pi(r_{-i}|r_i) = 1$ and $\pi(\theta_{-i}|r_i) = 0.1$ if $\theta_{-i} \neq r_{-i}$, i.e., she rather believes that her neighbors also receive an incentive. Symmetrically, when she does not receive an incentive, we get: $\pi(\overline{r}_{-i}|\overline{r}_i) = 1$ and $\pi(\theta_{-i}|\overline{r}_i) = 0.1$ if $\theta_{-i} \neq \overline{r}_{-i}$.

Clearly, Π-games properly generalize classical games (with complete information). Indeed:

Proposition 4.1. *Any classical normal form game $G = \langle N, A, \mu \rangle$ is a* Π*-game with $|\Theta_i| = 1, \ \forall i \in N$.*

Π-games can be related to the framework proposed by (De Clercq et al., 2018) as a semantic for possibilistic Boolean games. Authors in (De Clercq et al., 2018) assume that no player receives any private information before playing the game. This is an *ex-ante* situation. Adapting these notions to Π-games, where the prior knowledge is common, we can compute the *ex-ante* possibility and *ex-ante* necessity that the joint action a is a PNE:

$$\Pi(a \text{ is a PNE}) = \max_{\theta, a \text{ is PNE for } G^\theta} \pi(\theta). \tag{4.2}$$

$$N(a \text{ is a PNE}) = 1 - \max_{\theta, a \text{ is not a PNE for } G^\theta} \pi(\theta). \tag{4.3}$$

Contrarily to the framework of (De Clercq et al., 2018), ours can handle the knowledge that each player i has *when receiving some private information θ_i*, i.e., in our framework every player knows, before playing the game, her own type θ_i but not the types θ_{-i} of the other players. Then, the posterior necessity (resp. possibility) that a is a PNE can be different from one agent to another. The *ex-interim* possibility and *ex-interim* necessity that a joint action a be a PNE can be different from one player to another:

$$\Pi_i(a \text{ is a PNE}|\theta_i) = \max_{\theta_{-i}, a \text{ is a PNE for } G^{\theta_i.\theta_{-i}}} \pi(\theta_{-i}|\theta_i). \tag{4.4}$$

$$N_i(a \text{ is a PNE}|\theta_i) = 1 - \max_{\theta_{-i}, a \text{ is not a PNE for } G^{\theta_i.\theta_{-i}}} \pi(\theta_{-i}|\theta_i). \tag{4.5}$$

4.3 Solution Concepts in Π-games

4.3.1 Secure Strategy in Π-games

The notion of *secure strategy* is relevant for a player, say player i, who has some information (θ_i) about the game but does not know anything about the strategies of the other players. This player can then evaluate the degree of security of her own strategy: it is the minimal level of satisfaction the player may receive if executing it. In the absence of other information, a very cautious attitude is to maximize this degree of security. Taking the qualitative knowledge into account, we get:

Definition 4.3 (Level of security, Secure strategy). *Let $G = \langle N, A, \Theta, \pi, \mu \rangle$ be a Π-game. The level of security of player $i \in N$ of type $\theta_i \in \Theta_i$ for the action $a_i \in A_i$ is:*

$$u_i^{secure}(a_i, \theta_i) = \min_{\theta_{-i} \in \Theta_{-i}} \max\Big(1 - \pi(\theta_{-i}|\theta_i), \min_{a_{-i} \in A_{-i}} \mu_i(a_i.a_{-i}, \theta_i.\theta_{-i})\Big). \tag{4.6}$$

σ_i is a secure strategy for i iff $\forall i \in N$, $\forall \theta_i \in \Theta_i$, $\forall a'_i \in A_i$,

$$u_i^{secure}(\sigma_i(\theta_i), \theta_i) \geq u_i^{secure}(a'_i, \theta_i). \tag{4.7}$$

Example 4.2 (Cont. Example 4.1). *Using Equation (4.6), we have:*

- *the level of security of player 1 for action x when she receives an incentive ($\theta_1 = r_1$) is:*

$$\begin{aligned} u_1^{secure}(x, r_1) &= \min\Big(\max\Big(1 - \pi(r_2|r_1), \min\big(\mu_1(x.x, r_1.r_2), \mu_1(x.y, r_1.r_2)\big)\Big), \\ &\qquad \max\Big(1 - \pi(\overline{r}_2|r_1), \min\big(\mu_1(x.x, r_1.\overline{r}_2), \mu_1(x.y, r_1.\overline{r}_2)\big)\Big)\Big) \\ u_1^{secure}(x, r_1) &= 0.3. \end{aligned}$$

- *the level of security of player 1 for action y when she receives an incentive ($\theta_1 =$*

r_1) *is:*

$$\begin{aligned}
u_1^{secure}(y, r_1) &= \min\Bigg(\max\Big(1 - \pi(r_2|r_1), \min\big(\mu_1(y.x, r_1.r_2), \mu_1(y.y, r_1.r_2)\big)\Big), \\
&\qquad \max\Big(1 - \pi(\overline{r}_2|r_1), \min\big(\mu_1(y.x, r_1.\overline{r}_2), \mu_1(y.y, r_1.\overline{r}_2)\big)\Big)\Bigg) \\
u_1^{secure}(y, r_1) &= 0.2.
\end{aligned}$$

Using Equation (4.6), we have $u_1^{secure}(x, \overline{r}_1) = 0.3$ *and* $u_1^{secure}(y, \overline{r}_1) = 0.2$.

Since, $u_1^{secure}(x, r_1) \geq u_1^{secure}(y, r_1)$ *and* $u_1^{secure}(x, \overline{r}_1) \geq u_1^{secure}(y, \overline{r}_1)$. *Thus, the secure strategy of player* 1 *is to play* x *in any case, i.e., whether she receives the incentive or not.*

Similarly, the secure strategy of player 2 *is to play* y *in any case, i.e., whether she receives the incentive or not.*

To summarize, in our coordination Π*-game, the secure strategy of player* i *is her preferred action or not her she receives the incentive or not.*

4.3.2 Pure Nash Equilibrium in Π-games: PNE

The concepts proposed in (De Clercq et al., 2018) are based on the definition of strategies as profiles of actions as in the case in a classical normal form game. This definition does not take the knowledge about the types of players into account: it assumes that the full θ is not known by all the players, and thus each player computes her *ex-ante* utility. In other words, the definition of the above *ex-ante* necessity and *ex-ante* possibility measures (Equations (4.2) and (4.3)) only suits games that are presently incomplete.

In our framework, we adopt an *ex-interim* approach. We assume that player i only knows $\pi(.|\theta_i)$ when receiving θ_i then decides what to play. As in any incomplete information game, strategy σ specifies an action of each player i for each θ_i. σ is a PNE when no player i has interest to deviate from her own action choice $\sigma_i(\theta_i)$ for θ_i, given $\pi(.|\theta_i)$ and σ. Using possibilistic qualitative decision theory, the utility of $\sigma(\theta_i)$ is evaluated by its possibilistic pessimistic utility (we assume that the player is cautious).

Definition 4.4 (Utility of an action, Utility of a pure strategy)**.**
Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ *be a* Π*-game. The utility of player* $i \in N$ *of type* $\theta_i \in \Theta_i$ *for an*

action $a_i \in A_i$ in the context of $\sigma_{-i} \in \Sigma_{-i}$ is:

$$U_i^{pes}(a_i, \sigma_{-i}, \theta_i) = \min_{\theta_{-i} \in \Theta_{-i}} \max\Big(1 - \pi(\theta_{-i}|\theta_i), \mu_i\big(a_i.\sigma_{-i}(\theta_{-i}), \theta_i.\theta_{-i}\big)\Big). \tag{4.8}$$

The utility of the pure strategy $\sigma \in \Sigma$ to player $i \in N$ of type $\theta_i \in \Theta_i$ is:

$$U_i^{pes}(\sigma, \theta_i) = U_i^{pes}\big(\sigma_i(\theta_i), \sigma_{-i}, \theta_i\big). \tag{4.9}$$

Note that $U_i^{pes}(\sigma_i(\theta_i), \sigma_{-i}, \theta_i)$ is independent of the choices of player i when her type is different from θ_i.

The level of security of player i for type θ_i can be computed as the minimum over all her pessimistic utilities for all possible other players joint actions $a_{-i} \in A_{-i}$. The following proposition holds:

Proposition 4.2. *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. The level of security of player i of type $\theta_i \in \Theta_i$ is:*

$$u_i^{secure}(a_i, \theta_i) \quad = \quad \min_{a_{-i} \in A_{-i}} U_i^{pes}(a_i, \sigma_{-i}(a_{-i}), \theta_i). \tag{4.10}$$

A *best response* for player i of type θ_i is computed *knowing the provisional strategies of the other players*, i.e., knowing σ_{-i}. This is the action a_i which maximizes $U_i^{pes}(a_i, \sigma_{-i}, \theta_i)$. Again, when considering a joint pure strategy σ, the uncertainty of player i only bears on the types of the other players. What they plan to do depending on their type is known, prescribed by σ.

Definition 4.5 (Best Response in a Π-game)**.** *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. The best response of player $i \in N$ of type $\theta_i \in \Theta_i$ to σ_{-i} is:*

$$BR_i(\sigma_{-i}, \theta_i) = \operatorname*{argmax}_{a_i \in A_i} U_i^{pes}(a_i, \sigma_{-i}, \theta_i). \tag{4.11}$$

A pure Nash Equilibrium in a Π-game (PNE) is a joint strategy from which no player i will deviate unilaterally knowing σ_{-i}, i.e., every player plays her best response to the other players possible strategies. In the possibilistic context, we set the following definition:

Definition 4.6 (Pure Nash Equilibrium in a Π-game)**.**
Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. σ is a pure Nash equilibrium (PNE) iff: $\forall i \in N, \forall \theta_i \in \Theta_i, \forall a_i' \in A_i$,

$$U_i^{pes}(\sigma_i(\theta_i), \sigma_{-i}, \theta_i) \geq U_i^{pes}(a_i', \sigma_{-i}, \theta_i). \tag{4.12}$$

This definition is consistent with the original definition (Definition 1.9), which is recovered when $|\Theta| = 1$ (only one possible type per player). As a consequence, a pure equilibrium may not exist for a Π-game (a classical normal form game does not always admit a PNE).

Example 4.3 (Cont. Example 4.1)**.** *Consider the joint strategy where the two players play x if they receive an incentive and play y otherwise:*

$$\sigma_1^*(r_1) = x, \quad \sigma_1^*(\overline{r}_1) = y, \quad \sigma_2^*(r_2) = x, \quad \sigma_2^*(\overline{r}_2) = y.$$

- *the pessimistic utility of player 1 if she receives an incentive is:*

$$\begin{aligned} U_1^{pes}(\sigma^*, r_1) &= \min\Bigg(\max\Big(1 - \pi(r_2|r_1), \mu_1\big(x.x, r_1.r_2\big)\Big), \\ &\qquad \max\Big(1 - \pi(\overline{r}_2|r_1), \mu_1\big(x.y, r_1.\overline{r}_2\big)\Big)\Bigg) \\ U_1^{pes}(\sigma^*, r_1) &= 0.9. \end{aligned}$$

- *the pessimistic utility of player 1 if she does not receive an incentive is:*

$$\begin{aligned} U_1^{pes}(\sigma^*, \overline{r}_1) &= \min\Bigg(\max\Big(1 - \pi(r_2|\overline{r}_1), \mu_1\big(y.x, \overline{r}_1.r_2\big)\Big), \\ &\qquad \max\Big(1 - \pi(\overline{r}_2|\overline{r}_1), \mu_1\big(y.y, \overline{r}_1.\overline{r}_2\big)\Big)\Bigg) \\ U_1^{pes}(\sigma^*, \overline{r}_1) &= 0.7. \end{aligned}$$

We have, $U_1^{pes}(x, \sigma_2^*, r_1) = 0.9$ and $U_1^{pes}(y, \sigma_2^*, r_1) = 0.2$.
Thus, $U_1^{pes}(y, \sigma_2^*, r_1) \leq U_1^{pes}(x, \sigma_2^*, r_1)$. Hence, the best response to σ_2^* for player 1 when she receives an incentive is x.

Similarly, we have $U_1^{pes}(y, \sigma_2^*, \overline{r}_1) = 0.7$ and $U_1^{pes}(x, \sigma_2^*, \overline{r}_1) = 0.3$.
Thus, $U_1^{pes}(y, \sigma_2^*, \overline{r}_1) \geq U_1^{pes}(x, \sigma_2^*, \overline{r}_1)$. Hence, the best response to σ_2^* for player 1 when she does not receive an incentive is y.

We have seen above that the best response, to strategy σ_2^*, of player 1 when she receives an incentive is x and when she does not receive an incentive is y. Similarly, player 2 has not incentive to move from x to y if she receives for x an incentive since her utility decreases from 0.9 to 0.3. If she does not receive an incentive, her utility playing y is greater than her utility playing x (i.e., $U_2^{pes}(y, \sigma_1^*, \overline{r}_2) \geq U_2^{pes}(x, \sigma_1^*, \overline{r}_2)$). Thus, σ^* is a PNE.

4.3.3 Mixed Nash Equilibrium in Π-games: Π-MNE

4.3.3.1 Possibilistic Mixed Strategy

We define a possibilistic mixed joint strategy in a Π-game as a tuple $\upsilon = (\upsilon_1, ..., \upsilon_n)$ of $\upsilon_i : \Theta_i \mapsto \Psi_i$ that maps each type of player i to a mixed possibilistic strategy (a normalized possibility distribution over actions in A_i). Formally:

Definition 4.7 (Possibilistic mixed strategy in a Π-game)**.** *A mixed joint strategy is a vector $\upsilon = (\upsilon_1, \ldots, \upsilon_n)$ of functions $\upsilon_i : \Theta_i \to \Psi_i$ where $\Psi_i = \{\pi : A_i \to \Delta\}$ is the set of the normalized possibility distributions over A_i.*

As in ordinal normal form games, in a Π-game, the possibilistic mixed strategy $\upsilon_i(\theta_i)$ of player i of type θ_i has a dual interpretation in terms of preference and likelihood (see (Ben Amor et al., 2017) and Section 2.5): for player i of type θ_i, $\upsilon_i(\theta_i)$ ranks the actions according to her preferences. Then, for all the other players, $\upsilon_i(\theta_i)$ measures the likelihood that player i of type θ_i plays each action a_i.

Example 4.4 (Cont. Example 4.1)**.** *Let us consider a joint mixed strategy $\upsilon = (\upsilon_1.\upsilon_2)$ of the coordination* Π*-game such that:*

- *$\upsilon_1(r_1)$ and $\upsilon_1(\overline{r}_1)$ are two possibility distributions over $A_1 = \{x, y\}$:*
  - *$\upsilon_1(r_1)$: when she receives an incentive, player 1 prefers to play x, i.e., she plays x with a possibility* 1*, and plays y with a possibility* 0.3*, i.e., $\upsilon_1(r_1)(y) = 0.3$;*
  - *$\upsilon_1(\overline{r}_1)$: when she does not receive an incentive, player* 1 *prefers to play x, i.e., she plays x with a possibility 1, and plays y with a possibility* 0.3*, i.e., $\upsilon_1(\overline{r}_1)(y) = 0.3$;*
- *$\upsilon_2(r_2)$ and $\upsilon_2(\overline{r}_2)$ are two possibility distributions over $A_2 = \{x, y\}$:*
  - *$\upsilon_2(r_2)$: when she receives an incentive, player* 2 *prefers to play y, i.e., she plays y with a possibility 1, and plays x with a possibility* 0.8*, i.e., $\upsilon_2(r_2)(y) = 0.8$;*
  - *$\upsilon_2(\overline{r}_2)$: when she does not receive an incentive, player* 2 *prefers to play y, i.e., she plays y with a possibility 1, and plays x with a possibility* 0.8*, i.e., $\upsilon_2(\overline{r}_2)(y) = 0.8$.*

For the sake of readability, let $\upsilon_i(a_i|\theta_i) = \upsilon_i(\theta_i)(a_i)$ denote the possibility that player i plays action a_i when her type is θ_i. According to (Ben Amor et al., 2017), when the configuration of types is θ, the joint possibility distribution over the profiles of actions is defined as the minimum of the individual players possibility distributions over individual actions:

$$\upsilon(a|\theta) = \min_{i \in N} \ \upsilon_i(a_i|\theta_i). \tag{4.13}$$

Let $\pi_\upsilon(a, \theta_{-i}|\theta_i)$ be the possibility distribution on $A \times \Theta_{-i}$ defined by υ, given the type θ_i and prior knowledge $\pi : \Theta \to \Delta$:

$$\pi_\upsilon(a, \theta_{-i}|\theta_i) = \min(\pi(\theta_{-i}|\theta_i), \upsilon(a|\theta_i.\theta_{-i})). \tag{4.14}$$

Let us now study the evaluation of strategies. In order to stay in accordance with the previous assumptions of cautiousness and ordinality, and for the purpose of homogeneity, strategies will be evaluated using the pessimistic possibilistic utility.

Definition 4.8 (Utility of a mixed strategy)**.** *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. The pessimistic utility for player $i \in N$ of type $\theta_i \in \Theta_i$ of the mixed strategy υ is:*

$$U_i^{pes}(\upsilon, \theta_i) = \min_{\theta_{-i}\in\Theta_{-i}, a\in A} \max\big(1 - \pi_\upsilon(a, \theta_{-i}|\theta_i), \mu_i(a, \theta)\big). \tag{4.15}$$

Proposition 4.3. *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. The pessimistic utility of player $i \in N$ of type θ_i of the mixed strategy υ is:*

$$U_i^{pes}(\upsilon, \theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1 - \pi(\theta_{-i}|\theta_i), \min_{a\in A} \max\big(1 - \upsilon(a|\theta_i.\theta_{-i}), \mu_i(a, \theta)\big)\Big). \tag{4.16}$$

The pessimistic utility of player i of type θ_i of the mixed strategy υ can be written as:

Proposition 4.4. *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. The pessimistic utility for player $i \in N$ of type $\theta_i \in \Theta_i$ of the mixed strategy υ is:*

$$U_i^{pes}(\upsilon, \theta_i) = \min_{a_i\in A_i} \max\big(1 - \upsilon_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)\big) \tag{4.17}$$

where:

$$U_i^{pes}(a_i, \upsilon_{-i}, \theta_i) = \min_{\substack{a_{-i}\in A_{-i}\\ \theta_{-i}\in\Theta_{-i},}} \max\Big(1 - \pi(\theta_{-i}|\theta_i), 1 - \upsilon_{-i}(a_{-i}|\theta_{-i}), \mu_i(a_i.a_{-i}, \theta_i.\theta_{-i})\Big). \tag{4.18}$$

Example 4.5 (Cont. Example 4.1)**.** *Consider the strategy υ defined in Example 4.4. It holds that:*

- *the pessimistic utility of player* 1 *when she plays x and receives an incentive for* *the mixed strategy υ is equal to:*

$$\begin{aligned}
U_1^{pes}(x, \upsilon_2, r_1) &= \min\Bigg(\max\Big(1 - \pi(r_2|r_1), 1 - \upsilon_2(x|r_2), \mu_1\big(x.x, r_1.r_2\big)\Big), \\
&\qquad \max\Big(1 - \pi(r_2|r_1), 1 - \upsilon_2(y|r_2), \mu_1\big(x.y, r_1.r_2\big)\Big), \\
&\qquad \max\Big(1 - \pi(\overline{r}_2|r_1), 1 - \upsilon_2(x|\overline{r}_2), \mu_1\big(x.x, r_1.\overline{r}_2\big)\Big), \\
&\qquad \max\Big(1 - \pi(\overline{r}_2|r_1), 1 - \upsilon_2(y|\overline{r}_2), \mu_1\big(x.y, r_1.\overline{r}_2\big)\Big)\Bigg) \\
U_1^{pes}(x, \upsilon_2, r_1) &= 0.3.
\end{aligned}$$

- *the pessimistic utility of player* 1 *when she plays y and receives an incentive for the mixed strategy υ is:* $U_1^{pes}(y, \upsilon_2, r_1) = 0.2$.

Thus, the pessimistic utility of player 1 *when she receives an incentive for the mixed strategy υ is:*

$$\begin{aligned}
U_1^{pes}(\upsilon, r_1) &= \min\Bigg(\max\Big(1 - \upsilon_1(x|r_1), U_1^{pes}(x, \upsilon_2, r_1)\Big), \\
&\qquad \max\Big(1 - \upsilon_1(y|r_1), U_1^{pes}(y, \upsilon_2, r_1)\Big)\Bigg) \\
U_1^{pes}(\upsilon, r_1) &= 0.3.
\end{aligned}$$

Similarly, we get:

- $U_1^{pes}(x, \upsilon_2, \overline{r}_1) = 0.3$ *and* $U_1^{pes}(x, \upsilon_2, \overline{r}_1) = 0.2$. *Thus,* $U_1^{pes}(\upsilon, \overline{r}_1) = 0.3$*;*
- $U_2^{pes}(x, \upsilon_1, r_2) = 0.7$ *and* $U_2^{pes}(y, \upsilon_1, r_2) = 0.3$. *Thus,* $U_2^{pes}(\upsilon, r_2) = 0.3$*;*
- $U_2^{pes}(x, \upsilon_1, \overline{r}_2) = 0.7$ *and* $U_2^{pes}(y, \upsilon_1, \overline{r}_2) = 0.3$. *Thus,* $U_2^{pes}(\upsilon, \overline{r}_2) = 0.3$.

4.3.3.2 Possibilistic Mixed Nash Equilibrium

A possibilistic mixed Nash equilibrium in a Π-game is a mixed strategy $\upsilon^* = (\upsilon_1^*, ..., \upsilon_n^*)$ where no player i of type θ_i can improve her pessimistic utility by changing her mixed strategy υ_i. Formally:

Definition 4.9 (Possibilistic mixed Nash equilibrium in a Π-game)**.**
*Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. υ^* is a possibilistic mixed Nash equilibrium (*Π*-MNE) in a* Π*-game iff:* $\forall i \in N, \forall \theta_i \in \Theta_i, \forall \upsilon_i'$,

$$U_i^{pes}(\upsilon^*, \theta_i) \geq U_i^{pes}(\upsilon_i'.\upsilon_{-i}^*, \theta_i). \tag{4.19}$$

Example 4.6 (Cont. Example 4.1)**.** *Consider the joint mixed strategy* $\upsilon^* = (\upsilon_1^*.\upsilon_2^*)$ *such that:*

$$\upsilon_1^*(x|r_1) = 1, \quad \upsilon_1^*(y|r_1) = 0.7, \quad \upsilon_1^*(x|\overline{r}_1) = 1, \quad \upsilon_1^*(y|\overline{r}_1) = 0.7,$$
$$\upsilon_2^*(x|r_2) = 1, \quad \upsilon_2^*(y|r_2) = 1, \quad \upsilon_2^*(x|\overline{r}_2) = 1, \quad \upsilon_2^*(y|\overline{r}_2) = 1.$$

It can be checked that υ^* *is a* Π*-MNE of the coordination* Π*-game. In fact:*

- $\forall \upsilon_1'(r_1) \neq \upsilon_1^*(r_1)$, $U_1^{pes}(\upsilon_1'(r_1).\upsilon_2^*, r_1) \leq U_1^{pes}(\upsilon_1^*(r_1).\upsilon_2^*, r_1) = 0.3$;
- $\forall \upsilon_1'(\overline{r}_1) = \upsilon_1^*(\overline{r}_1)$, $U_1^{pes}(\upsilon_1'(\overline{r}_1).\upsilon_2^*, \overline{r}_1) \leq U_1^{pes}(\upsilon_1^*(\overline{r}_1).\upsilon_2^*, \overline{r}_1) = 0.3$;
- $\forall \upsilon_2'(r_2) \neq \upsilon_2^*(r_2)$, $U_2^{pes}(\upsilon_2'(r_2).\upsilon_1^*, r_2) \leq U_2^{pes}(\upsilon_2^*(r_2).\upsilon_1^*, r_2) = 0.3$;
- $\forall \upsilon_2'(\overline{r}_2) \neq \upsilon_2^*(\overline{r}_2)$, $U_2^{pes}(\upsilon_2'(\overline{r}_2).\upsilon_1^*, \overline{r}_2) \leq U_2^{pes}(\upsilon_2^*(\overline{r}_2).\upsilon_1^*, \overline{r}_2) = 0.3$.

4.4 Transforming a Π-game into an Equivalent Ordinal Normal Form Game

In this section, we show that any incomplete information game can be transformed into an equivalent normal form game with *complete information*, the secure strategy, pure and mixed Nash equilibria of which are in bijection with the secure strategy, pure and mixed Nash equilibria of the Π-game. This representation result is a qualitative counterpart of Harsanyi's transformation of Bayesian games into normal form games under complete information (Harsanyi, 1967a). We follow the idea of (Howson et al., 1974) and we consider as many players as the number of pairs (i, θ_i), each player (i, θ_i) having A_i as a set of available actions. A joint strategy in this (classical) normal form game then corresponds to a strategy σ of the Π-game.

Definition 4.10 (Complete normal form representation of a Π-game)**.**
Let $G = \langle N, A, \Theta, \pi, \mu \rangle$ *be a* Π*-game. The complete normal form representation (C-SNF-representation) of* G *is the complete information game* $\tilde{G} = \langle \tilde{N}, \tilde{A}, \tilde{\mu} \rangle$*, where:*

- $\tilde{N} = \{(i, \theta_i), i \in N, \theta_i \in \Theta_i\}$;
- $\tilde{A}_{(i,\theta_i)} = A_i, \ \forall (i, \theta_i) \in \tilde{N}$;
- $\tilde{\mu}_{(i,\theta_i)}(\tilde{a}) = U_i^{pes}(\sigma, \theta_i), \ \forall \tilde{a} \in \tilde{A}, \forall (i, \theta_i) \in \tilde{N}$, *where* $\sigma_i(\theta_i) = \tilde{a}_{(i,\theta_i)}, \forall (i, \theta_i) \in \tilde{N}$.

Thus, a pure strategy in the Π game associates an action to each type θ_i of each player i. In the C-SNF-representation of the Π-game, a pure strategy associates an action to each player (i, θ_i).

Definition 4.11 (Complete normal form representation of a pure strategy)**.**
Let $G = \langle N, A, \Theta, \pi, \mu \rangle$ *be a* Π*-game and* σ *be a pure strategy in* G*. The C-SNF-*

representation of σ is the profile of actions $\tilde{a}$ such that:

$$\forall (i,\theta_i) \in \tilde{N}, \tilde{a}_{(i,\theta_i)} = \sigma_i(\theta_i). \tag{4.20}$$

It is easy to see that $\tilde{a}_{(i,\theta_i)}$ is a profile of actions of $\tilde{G}$ and that the strategies of G and $\tilde{G}$ are in bijection. It follows that:

Proposition 4.5. *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. Action a_i is a secure strategy for player i of type θ_i iff $\tilde{a}_{(i,\theta_i)} = a_i$ is a secure strategy for player (i,θ_i) in $\tilde{G}$.*

Proposition 4.6. *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. The pure strategy σ is a PNE for G iff $\tilde{a}$ is a PNE in $\tilde{G}$.*

Finally, every mixed strategy in the Π-game has an equivalent mixed strategy in its C-SNF-representation and the mixed equilibria of the two games are in bijection.

Definition 4.12 (Complete normal form representation of a mixed strategy)**.**
Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game and υ be a mixed strategy in G. The C-SNF-representation of υ is the mixed strategy $\tilde{\pi}$ such that:

$$\forall (i,\theta_i) \in \tilde{N}, \tilde{\pi}_{(i,\theta_i)}(\tilde{a}_{(i,\theta_i)}) = \upsilon_i(a_i|\theta_i). \tag{4.21}$$

It is easy to show that:

$$\tilde{\pi}(\tilde{a}) = \upsilon(a|\theta). \tag{4.22}$$

As a consequence of Definition 4.12, we get:

Proposition 4.7. *Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. The mixed strategy υ is a Π-MNE for G iff $\tilde{\pi}$ is a ΠME in $\tilde{G}$.*

Example 4.7 (Cont. Example 4.1)**.** *The C-SNF-representation of the coordination Π-game $\tilde{G}$ has 4 players:*
$\tilde{N} = \{(1,r_1),(1,\overline{r}_1),(2,r_2),(2,\overline{r}_2)\}$ *such that* $\tilde{A}_{(1,r_1)} = \tilde{A}_{(1,\overline{r}_1)} = \tilde{A}_{(2,r_2)} = \tilde{A}_{(2,\overline{r}_2)} = \{x,y\}$.

Using the definition of secure strategy in SNF game (Definition 1.4), we have:

- *the secure strategy of players $(1,r_1)$ and $(1,\overline{r}_1)$ is x;*
- *the secure strategy of players $(2,r_2)$ and $(2,\overline{r}_2)$ is y.*

The joint pure strategy σ^* such that: $\sigma_1^*(r_1) = x$, $\sigma_1^*(\overline{r}_1) = y$, $\sigma_2^*(r_2) = x$, $\sigma_2^*(\overline{r}_2) = y$ corresponds to the joint action $\tilde{a}^* = (x.y.x.y)$ in $\tilde{G}$. The utilities of $\tilde{a}^*$ in $\tilde{G}$ are:

$$\tilde{\mu}_{(1,r_1)}(\tilde{a}^*) = U_1^{pes}(\sigma^*, r_1) = 0.9, \quad \tilde{\mu}_{(1,\overline{r}_1)}(\tilde{a}^*) = U_1^{pes}(\sigma^*, \overline{r}_1) = 0.7,$$
$$\tilde{\mu}_{(2,r_2)}(\tilde{a}^*) = U_2^{pes}(\sigma^*, r_2) = 0.9, \quad \tilde{\mu}_{(2,\overline{r}_2)}(\tilde{a}^*) = U_2^{pes}(\sigma^*, \overline{r}_2) = 0.7.$$

Using the definition of a PNE in a SNF game (Definition 1.9), it can be check that $\tilde{a}^*$ is a PNE in $\tilde{G}$:

- *for player $(1, r_1)$, her utility playing x (resp. y) is equal to 0.9 (resp. 0.2): she prefers to play x than y;*

- *for player $(1, \overline{r}_1)$, her utility playing y (resp. x) is equal to 0.7 (resp. 0.3): she prefers to play y than x;*

- *for player $(2, r_2)$, her utility playing x (resp. y) is equal to 0.9 (resp. 0.3): she prefers to play x than y;*

- *for player $(2, \overline{r}_2)$, her utility playing y (resp. x) is equal to 0.7 (resp. 0.3): she prefers to play y than x.*

The joint mixed strategy υ^* corresponds to the joint mixed strategy $\tilde{\pi}^*$ such that:

$$\begin{array}{llll} \tilde{\pi}^*_{1,r_1}(x) = 1, & \tilde{\pi}^*_{1,r_1}(y) = 0.7, & \tilde{\pi}^*_{1,\overline{r}_1}(x) = 1, & \tilde{\pi}^*_{1,\overline{r}_1}(y) = 0.7, \\ \tilde{\pi}^*_{2,r_2}(x) = 1, & \tilde{\pi}^*_{2,r_2}(y) = 1, & \tilde{\pi}^*_{2,\overline{r}_2}(x) = 1, & \tilde{\pi}^*_{2,\overline{r}_2}(y) = 1. \end{array}$$

Using the definition of ΠME in a SNF game (Definition 2.2), it can be checked that the joint mixed strategy $\tilde{\pi}^*$ is a ΠME.

Regarding the complexity of the transformation, consider that the original game is extensively represented by tables (the game is in standard normal form). The transformed game contains $\tilde{n} = \sum_{i=1,n} |\Theta_i|$ utility functions of size $\prod_{(i,\theta_i)\in\tilde{N}} |\tilde{A}_{(i,\theta_i)}| = \prod_{i=1,n} \prod_{\theta_i\in\Theta_i} |A_i| = \prod_{i=1,n} |A_i|^{|\Theta_i|}$.

If we write, for simplification purpose, $|A_i| = d$ and $|\Theta_i| = t$, $\forall i \in N$ (i.e, the same number of actions and the same number of types for all players), thus a game containing n utility functions of size $(d \cdot t)^n$ is transformed into a game containing $\tilde{n} = n \cdot t$ utility functions of size $d^{\tilde{n}} = (d^n)^t$, since the number of players $\tilde{n}$ in the transformed game is equal to $(n \cdot t)$.

Except when the number of types is very small, the transformation does not provide a convenient way to solve the game. Proposition 4.7 is, as in the Bayesian case, more a representation result than a solving tool.

4.5 Complexity Results

4.5.1 Complexity Results on Secure Strategy

Recall that in a Π-game, the level of security of a pure action a_i for a given player $i \in N$ of type $\theta_i \in \Theta_i$ is defined in Equation (4.6) as:

$$u_i^{secure}(a_i, \theta_i) = \min_{\theta_{-i} \in \Theta_{-i}} \max\Big(1 - \pi(\theta_{-i}|\theta_i), \min_{a_{-i} \in A_{-i}} \mu_i(a_i.a_{-i}, \theta)\Big).$$

Thus, the optimal secure strategy for player i of type θ_i is simply obtained by computing:

$$\forall i \in N, \forall \theta_i \in \Theta_i, a_i^* = \arg\max_{a_i \in A_i} u_i^{secure}(a_i, \theta_i). \tag{4.23}$$

We conclude that the optimal (pure) secure strategy problem always has a solution and we will see that can be found in polynomial time since $u_i^{secure}(a_i, \theta_i)$ can be computed in polynomial time in the size of the Π-game (see Algorithm 5.1 in Chapter 5).

4.5.2 Complexity Results on PNE

In Section 4.4, we have shown that a Π-game can be transformed into an equivalent normal form game with complete information. Since, it is not guaranteed to get a PNE in a normal form game. Thus, it is also not guaranteed that a Π-game admits a PNE.

In the following, we study the complexity of the problem of deciding whatever Π-game admits a PNE. Let us first define the **Π-PNE** problem:

Definition 4.13 (**Π-PNE** problem)**.** *Let $G = \langle N, A, \Theta, \pi, \mu \rangle$ be a* Π*-game. The* Π***-PNE*** *problem consists in determining a PNE in the* Π*-game G if it exists.*

Deciding the existence of a PNE in a Π-game is a difficult problem as stated by the following proposition:

Proposition 4.8. Π-***PNE*** *is NP-Complete, even in symmetric*[1] *2-player games where π corresponds to total ignorance, i.e., $\forall \theta_1 \in \Theta_1$ and $\forall \theta_2 \in \Theta_2, \pi(\theta_1, \theta_2) = 1$.*

4.5.3 Complexity results on Π-MNE

In Section 4.4, we have shown that a Π-game can be transformed into an equivalent normal form game with complete information. Since, in (Ben Amor et al., 2017), authors have shown that an ordinal normal form game admits at least one mixed

[1]A game is symmetric if all players gave the same set of actions, and the utilities to playing a given action depends only on the actions being played, not on who plays them.

Nash equilibrium. Thus, a Π-game always admits at least one possibilistic mixed Nash equilibrium.

Adapting the algorithm proposed in (Ben Amor et al., 2017) in Π-games, a Π-MNE can be computed in polynomial time in the size of the Π-game. In the following chapter, a polynomial time algorithm is proposed to compute a Π-MNE in a Π-game. Especially, least-specific ones.

4.6 Conclusion

In this chapter, we have defined possibilistic games as a new representation framework for (ordinal) games under possibilistic incomplete information. We have extended the standard solution concepts: secure strategy and Nash equilibrium (pure and mixed). Then, we have proposed a transformation of a possibilistic game with incomplete information into an equivalent ordinal normal form game with complete information. We have shown that the secure strategies, pure and mixed Nash equilibrium in the original Π-game are in bijection of the secure strategies, pure and mixed Nash equilibrium in the equivalent complete standard normal form game. In the end, we have studied the complexity of solving a Π-game.

In the next chapter, we will show how to solve possibilistic games with incomplete information. First, we will propose a polynomial time algorithm to compute a secure strategy. The, we will propose a MILP formulation to find one pure Nash equilibrium if it exists. Finally, a polynomial time algorithm will be proposed to compute possibilistic mixed Nash equilibrium.

Proofs

Proof of Proposition 4.1.
Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game where $\forall i \in N$, $|\Theta_i| = 1$. Then it exists a unique type combination $\theta \in \Theta$, i.e., $|\Theta| = 1 \Rightarrow$ there is one possible game G^θ such that $\pi(G^\theta) = 1$ since π is normalized. Thus G is equal to classical normal form game, i.e., $G = G^\theta = \langle N, A, \{\{\mu_i^\theta\}_{i\in N}\}\rangle$. □

Proof of Proposition 4.2.

$$\begin{aligned}
u_i^{secure}(a_i,\theta_i) &= \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \min_{a_{-i}\in A_{-i}} \mu_i(a_i.a_{-i},\theta_i.\theta_{-i})\Big)\\
&= \min_{a_{-i}\in A_{-i}} \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \mu_i(a_i.a_{-i},\theta_i.\theta_{-i})\Big)\\
u_i^{secure}(a_i,\theta_i) &= \min_{a_{-i}\in A_{-i}} U_i^{pes}(a_i,\sigma_{-i}(a_{-i}),\theta_i).
\end{aligned}$$

□

Proof of Proposition 4.3.

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}, a\in A} \max\big(1-\pi_\upsilon(a,\theta_{-i}|\theta_i), \mu_i(a,\theta)\big)$

Based on Equation (4.14):

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}, a\in A} \max\big(1-\min(\pi(\theta_{-i}|\theta_i), \upsilon(a|\theta_i.\theta_{-i})), \mu_i(a,\theta)\big)$

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}, a\in A} \max\big(1-\pi(\theta_{-i}|\theta_i), 1-\upsilon(a|\theta_i.\theta_{-i}), \mu_i(a,\theta)\big)$

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \min_{a\in A}\max\big(1-\upsilon(a|\theta_i.\theta_{-i}), \mu_i(a,\theta)\big)\Big)$ □

Proof of Proposition 4.4.

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \min_{a\in A}\max\big(1-\upsilon(a|\theta), \mu_i(a,\theta)\big)\Big)$

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \min_{a\in A}\max\big(1-\min(\upsilon_i(a_i|\theta_i), \upsilon_{-i}(a_{-i}|\theta_{-i})), \mu_i(a,\theta)\big)\Big)$

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \min_{a\in A}\max\big(1-\upsilon_i(a_i|\theta_i), 1-\upsilon_{-i}(a_{-i}|\theta_{-i}), \mu_i(a,\theta)\big)\Big)$

$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}, a\in A} \max\Big(1-\pi(\theta_{-i}|\theta_i), 1-\upsilon_i(a_i|\theta_i), 1-\upsilon_{-i}(a_{-i}|\theta_{-i}), \mu_i(a,\theta)\Big)$

$U_i^{pes}(\upsilon,\theta_i) = \min_{a_i\in A_i}\Big(1-\upsilon_i(a_i|\theta_i), \min_{\substack{a_{-i}\in A_{-i}\\ \theta_{-i}\in\Theta_{-i},}} \max\big(1-\pi(\theta_{-i}|\theta_i), 1-\upsilon_{-i}(a_{-i}|\theta_{-i}), \mu_i(a_i.a_{-i},\theta)\big)\Big)$

$U_i^{pes}(\upsilon,\theta_i) = \min_{a_i\in A_i}\max(1-\upsilon_i(a_i|\theta_i), U_i^{pes}(a_i,\upsilon_{-i},\theta_i))$ □

Proof of Proposition 4.5.

In order to prove that the secure strategy of player i of type θ_i is equal to the secure strategy of player (i,θ_i), it is enough to show that:

$$u_i^{secure}(a_i,\theta_i) = \mu_{(i,\theta_i)}^{secure}(\tilde{a}_{(i,\theta_i)}).$$

By Definition 4.3: $u_i^{secure}(a_i,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \min_{a_{-i}\in A_{-i}} \mu_i(a_i.a_{-i},\theta)\Big)$

$u_i^{secure}(a_i,\theta_i) = \mathit{min}_{a_{-i}\in A_{-i}} \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \mu_i(\sigma_i(\theta_i).a_{-i},\theta)\Big)$

By Definition 4.4, we have: $u_i^{secure}(a_i,\theta_i) = \min_{a_{-i}\in A_{-i}} U_i^{pes}(\sigma,\theta_i)$

Note that, by Definition 4.10 we have $\tilde{\mu}_{(i,\theta_i)}(\tilde{a}) = U_i^{pes}(\sigma,\theta_i)$

Then, $u_i^{secure}(a_i,\theta_i) = \mu_{(i,\theta_i)}^{secure}(\tilde{a}_{(i,\theta_i)})$.

Thus, proposition 4.5 holds. □

Proof of Proposition 4.6.

$\Rightarrow$ Assume that a is a PNE in $\tilde{G}$. Then, , $\forall i \in N, \theta_i \in \Theta_i, a'_i \in A_i$

$$\tilde{\mu}_{(i,\theta_i)}(a'_i.a_{-i}) \leq \tilde{\mu}_{(i,\theta_i)}(a).$$

But, since $\tilde{\mu}_{(i,\theta_i)}(a) =_{def} U_i^{pes}(a^\sigma,\theta_i)$, we get $U_i^{pes}(a'_i, a^\sigma_{-i},\theta_i) \leq U_i^{pes}(a^\sigma,\theta_i)$.

Thus, a^σ is a PNE of G.

$\Leftarrow$ Now, let σ be a PNE of G, define $\tilde{G}$ and joint action a: $a_{(i,\theta_i)} = \sigma_i(\theta_i), \forall(i,\theta_i)$. Then, again, $\tilde{\mu}_{(i,\theta_i)}(a) = U_i^{pes}(\sigma,\theta_i)$. And since σ is a PNE in G, we get $U_i^{pes}(a'_i,\sigma_{-i},\theta_i) \leq U_i^{pes}(\sigma,\theta_i), \forall i,\theta_i,a'_i$ and, $\forall i,\theta_i,a'_i$

$$\tilde{\mu}_{(i,\theta_i)}(a'_{(i,\theta_i)}.a_{-(i,\theta_i)}) \leq \tilde{\mu}_{(i,\theta_i)}(a).$$

Thus, a is a PNE of $\tilde{G}$. □

Proof of Proposition 4.7.
First, note that the transformation $\upsilon \to \tilde{\pi}$ (Definition 4.12) between the sets of mixed strategies in games G and $\tilde{G}$ is bijective. Thus, in order to prove that mixed equilibria are the same in both games, it is enough to show that

$$\forall (i,\theta_i), U_i^{pes}(\upsilon,\theta_i) = \mu_{i,\theta_i}^{pes}(\tilde{\pi}). \tag{4.24}$$

To do so, first note that:
$U_i^{pes}(\upsilon,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i},a\in A} \max(1-\pi_\upsilon(a,\theta_{-i}|\theta_i),\mu_i(a,\theta))$, by Definition 4.8,
$\pi_\upsilon(a,\theta_{-i}|\theta_i) = \min(\pi(\theta_{-i}|\theta_i),\upsilon(a|\theta))$, by Equation (4.14),
and $\tilde{\pi}(\tilde{a}) = \upsilon(a|\theta)$, by Definition 4.12.

Thus, $U_i^{pes}(\upsilon,\theta_i) = \min_{a\in A} \max(1-\tilde{\pi}(\tilde{a}), \min_{\theta_{-i}\in\Theta_{-i}} \max(1-\pi(\theta_{-i}|\theta_i),\mu_i(a,\theta)))$.

Note that, by Definition 4.4, $U_i^{pes}(\sigma,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max(1-\pi(\theta_{-i}|\theta_i),\mu_i(\sigma(\theta),\theta))$, where σ is the unique pure strategy in G, defined from any pure strategy a in $\tilde{G}$.

Then,

$$\begin{aligned}
U_i^{pes}(\upsilon,\theta_i) &= \min_{a\in A} \max(1-\tilde{\pi}(\tilde{a}),U_i^{pes}(\sigma,\theta_i)),\\
U_i^{pes}(\upsilon,\theta_i) &= \min_{a\in A} \max(1-\tilde{\pi}(\tilde{a}),\tilde{\mu}_{(i,\theta_i)}(a)), \text{ by Definition 4.10},\\
U_i^{pes}(\upsilon,\theta_i) &= \mu_{i,\theta_i}^{pes}(\tilde{\pi}), \text{ by Equation (2.7).}
\end{aligned}$$

Thus, Proposition 4.7 holds. □

Proof of Proposition 4.8.
Membership. We prove the membership in NP for the more general case of N unbounded. In this case, the size of the input is exponential in the number of players n. The PNE can be solved by guessing a strategy σ, i.e., guessing an action for each pair player/type, then checking whether σ is a PNE or not. More precisely:

- for each player $i \in N$ and for each type $\theta_i \in \Theta_i$: compute $U_i^{pes}(\sigma,\theta_i)$;
- for each action $a_i \in A_i$: compute $U_i^{pes}(a_i,\sigma_{-i},\theta_i)$.

Then we should check if player i has incentive to deviate from $\sigma_i(\theta_i)$, i.e., we should compare $U_i^{pes}(\sigma,\theta_i)$ and $U_i^{pes}(a_i,\sigma_{-i},\theta_i)$. Under the assumption that π is represented by a table of

$|\Theta|$ lines, the complexity of computing $U_i^{pes}(\sigma, \theta_i)$ is in $O(|\Theta_{-i}|)$, from Definition 4.4. Thus the whole complexity is polynomial $O(n \cdot |\Theta_i| \cdot |\Theta_{-i}| \cdot |A_{\max}|) = O(n \cdot |\Theta| \cdot |A_{\max}|)$ where $|A_{\max}| = \max(|A_1|, ..., |A_n|)$. Algorithm 4.1 details this process:

Algorithm 4.1: CHECK_EQUILIBRIUM.

Data: $G = \langle N, A, \Theta, \pi, \mu \rangle$, σ
Result: IsPNE (Boolean)
IsPNE $\leftarrow$ *true*;
forall i *in* N **do**
  forall $\theta_i \in \Theta_i$ **do**
    forall $a_i \in A_i$ **do**
      if $U_i^{pes}(a_i, \sigma_{-i}, \theta_i) > U_i^{pes}(\sigma, \theta_i)$ **then** IsPNE$\leftarrow$ false;
    end
  end
end
return IsPNE

This easy to check that σ is a PNE using Algorithm 4.1 in $O(n \cdot |\Theta| \cdot |A_{\max}|)$.

Hardness: The hardness proof uses a reduction from the SET-COVER problem. It is inspired from (Conitzer and Sandholm, 2002):

Definition 4.14. *(**SET-COVER** (SC) problem)*
Given a set $S = \{s_1, ..., s_n\}$, subsets $\{S_1, S_2, ..., S_m\}$ of S with $\cup_{1 \le i \le m} S_i = S$ and an integer $K \le m$. We are asked whether there exists a subset of $\{S_1, S_2, ..., S_m\}$ of size K whose union equals S, i.e., $S_{c_1}, ..., S_{c_K}$ such that $\cup_{1 \le i \le K} S_{c_i} = S$.

We reduce an arbitrary $SC = \langle S, \{S_1, S_2, ..., S_m\}, K \rangle$ instance to a PNE instance: Let Π-game $G_{SC} = \langle N, A, \Theta, \pi, \mu \rangle$ where:

- $N = \{1, 2\}$;
- $A = A_1 \times A_2$ where $A_1 = A_2 = \{S_1, ..., S_m, s_1, ..., s_n\}$;
- $\Theta = \Theta_1 \times \Theta_2$ where $\Theta_1 = \Theta_2 = \{t_1, ..., t_K\}$ (both players belong to one of K types);
- $\forall \theta_1 \in \Theta_1, \theta_2 \in \Theta_2, \pi(\theta_1, \theta_2) = 1$ (π reflects total ignorance);
- We assume utility functions that do not depend on a specific type $\theta \in \Theta$. They are as follows:
  (i) $\mu_1(S_i.S_j) = \mu_2(S_j.S_i) = 0.25 \ \forall S_i, S_j$;
  (ii) $\mu_1(S_i.s_j) = \mu_2(s_j.S_i) = 0.25 + \frac{j}{4n+1} \ \forall S_i, \forall s_j \notin S_i$;
  (iii) $\mu_1(S_i.s_j) = \mu_2(s_j.S_i) = 0.5 \ \forall S_i, \forall s_j \in S_i$;
  (iv) $\mu_1(s_i.s_j) = \mu_2(s_j.s_i) = 0 \ \forall s_i, s_j$;
  (v) $\mu_1(s_j.S_i) = \mu_2(S_i.s_j) = 0.75 \ \forall S_i, \forall s_j \notin S_i$;

(vi) $\mu_1(s_j.S_i) = \mu_2(S_j.s_i) = 0 \ \forall S_i, \forall s_j \in S_i$.

Note that a SET-COVER instance SC can be represented in space $O(m \cdot n \cdot \log(n))$. The size of the Π-game G_{SC} is the size required to represent π, μ_1 and μ_2. Assuming that π is represented as a table (which is obviously not the most concise way to represent it), $|\pi| = O(K^2) = O(n^2)$ and $|\mu_1| = |\mu_2| = O((m+n)^2)$. The latter size may become $O((m+n)^2 K^2)$ if we store $\mu_i(\sigma, \theta)$ even though utilities are independent of θ. Thus, G_{SC} requires space polynomial in that of SC to be represented. And since every $\pi(\theta)$ and $\mu_i(\sigma, \theta)$ requires constant time to be computed, the transformation is polynomial (time).

Note that, in Π-game with 2 players and π corresponding to total ignorance, the utility of the 2 players are computed as follows:

$$U_1^{pes}(\sigma, \theta_1) = \min_{\theta_2 \in \Theta_2} \max\left(1 - \pi(\theta_2|\theta_1), \mu_1(\sigma_1(\theta_1).\sigma_2(\theta_2))\right).$$

Since $\pi(\theta_2|\theta_1) = 1$, thus:

$$U_1^{pes}(\sigma, \theta_1) = \underset{\theta_2 \in \Theta_2}{min} \ \mu_1(\sigma_1(\theta_1).\sigma_2(\theta_2)).$$

In the same way, $U_2^{pes}(\sigma, \theta_2) = \underset{\theta_1 \in \Theta_1}{min} \ \mu_2(\sigma_1(\theta_1).\sigma_2(\theta_2))$.

Now we show that G admits a PNE $\Leftrightarrow$ SC admits a SET-COVER.

SC admits a SET-COVER $\Rightarrow$ G admits a ΠPNE

First suppose there exist $S_{c_1}, ..., S_{c_K}$ such that $\cup_{1 \le i \le K} S_{c_i} = S$. Suppose both players $i = \{1, 2\}$ play $S_{c_{\theta_i}}$ when their type is θ_i, i.e., $\forall \theta_1 \in \Theta_1$, $\sigma_1(\theta_1) = S_{c_{\theta_1}}$ and $\forall \theta_2 \in \Theta_2$, $\sigma_2(\theta_2) = S_{c_{\theta_2}}$. We claim that $(\sigma_1.\sigma_2)$ is a PNE.

Player 1 (resp. 2) supposes that player 2 (resp. 1) employs this strategy. Then, note that for any s_j, there is at least one S_{c_i} such that $s_j \in S_{c_i}$, since SC admits a SET-COVER $\{S_{c_1}, \ldots, S_{c_K}\}$. So, if player 1 of type θ_1, for example, changes her strategy by replacing $S_{c_{\theta_1}}$ with some s_j, this will decrease her utility from 0.25 (i) to 0 (vi), since s_j is covered by some $S_{c_{\theta_2}}$ played by player 2 of type θ_2. Of course, the same holds for the other player, so that no player has the interest to deviate from the SET-COVER play. It follows that playing any of the S_j is optimal. So there is a PNE.

G admits a PNE $\Rightarrow$ SC admits a SET-COVER

Suppose that G admits a PNE $\sigma^* = (\sigma_1^*.\sigma_2^*)$. We are going to show by contradiction that $\{\sigma_1^*(\theta_1)\}_{\theta_1 \in \Theta_1}$ and $\{\sigma_2^*(\theta_2)\}_{\theta_2 \in \Theta_2}$ form Set covers of SC.

1. Assume player 1 plays some $\sigma_1(\theta_1) \in S$ for some $\theta_1 \in \Theta_1$. We show that we have $\sigma_2(\theta_2) \subseteq S, \forall \theta_2 \in \Theta_2$. Indeed, if any $\sigma_2(\theta_2) \in S$ for some $\theta_2 \in \Theta_2$, then $U_2^{pes}(\sigma, \theta_2) = \mu_2(\sigma_1(\theta_1), \sigma_2(\theta_2)) = 0$ while $U_2^{pes}(\sigma', \theta_2) = 0.25$ if $\sigma_2(\theta_2) \in S$ is replaced with any $\sigma_2'(\theta_2) = S_c$.

2. Now, forget about Player 1 and assume that $\sigma_2(\theta_2) \subseteq S, \forall \theta_2 \in \Theta_2$ and that $S \setminus \cup_{\theta_2} \sigma_2(\theta_2)$ is non-empty. Then, obviously, $\sigma_1(\theta_1) \in S \setminus \cup_{\theta_2} \sigma_2(\theta_2), \forall \theta_1 \in \Theta_1$, since this provides utility 0.75 to player 1 of any type θ_1. However, let s_{j^*} be the state with minimum index for which there exists θ_1 s.t. $\sigma_1(\theta_1) = s_{j^*}$. s_{j^*} is not "covered" by any $\sigma_2(\theta_2)$ (so, $U_2^{pes}(\sigma, \theta_2) = 0.25 + \frac{j^*}{4n+1}$). However, considering (ii), Player 2 of any type θ_2 will be better off trading $\sigma_2(\theta_2)$ for some S_c such that $s_{j^*} \in S_c$, since this will increase the the smallest index of uncovered states.

   Thus, we have a contradiction, and $\cup_{\theta_2} \sigma_2(\theta_2) = S$.

3. The final step is the following: In step 1 we proved that, when G admits a PNE, if for some $\theta_1 \in \Theta_1, \sigma_1(\theta_1) \in S$, then $\sigma_2(\theta_2) \subseteq S, \forall \theta_2 \in \Theta_2$. Then, in step 2 we proved that if $\sigma_2(\theta_2) \subseteq S, \forall \theta_2 \in \Theta_2$, then $\cup_{\theta_2} \sigma_2(\theta_2) = S$. However, it may happen that $\sigma_1(\theta_1) \subseteq S, \forall \theta_1 \in \Theta_1$. But in this case, symmetrically to step 2, we can show that $\cup_{\theta_1} \sigma_1(\theta_1) = S$. In this case too, we have proved that there exists a set cover.

$\square$

Chapter 5

Solving Possibilistic Games with Incomplete Information

5.1 Introduction

In Chapter 4, we defined possibilistic games with incomplete information (Π-games). Then we gave the definition of a secure strategy, pure Nash equilibrium (PNE) and mixed Nash equilibrium (Π-MNE). In this chapter, we aim to solve these games by computing a secure strategy, a PNE and a Π-MNE.

This chapter is organized as follows: Section 5.2 proposes a polynomial-time algorithm to compute a secure strategy in a given Π-game. Section 5.3 proposes a Mixed Integer Linear Programming encoding to find a PNE if it exists. And Section 5.4 proposes a polynomial-time algorithm for building Π-MNE in Π-games. Finally, an experimental study, reported in Section 5.5, is proposed to confirm the feasibility of these approaches. All proofs are omitted in the end of this chapter.

5.2 A Polynomial Time Algorithm for Building Secure Strategies in Π-games

In this part, we propose an algorithm to compute a secure strategy $\sigma_i^*(\theta_i)$ of player i for type θ_i in a given Π-game $G = \langle N, A, \Theta, \pi, \mu \rangle$. $\sigma_i^*(\theta_i)$ is the result of the SECURE STRATEGY function. This latter takes as input the Π-game $G = \langle N, A, \Theta, \pi, \mu \rangle$, the player i and her type θ_i, then computes the optimal secure strategy $a_i^* = \sigma_i^*(\theta_i)$ using Equation (5.1).

$$\begin{aligned}\forall i \in N, \forall \theta_i \in \Theta_i, a_i^* &= \arg\max_{a_i \in A_i} u_i^{secure}(a_i, \theta_i) \\ &= \arg\max_{a_i \in A_i} \min_{\theta_{-i} \in \Theta_{-i}} \max\Big(1 - \pi(\theta_{-i} | \theta_i), \min_{a_{-i} \in A_{-i}} \mu_i(a_i.a_{-i}, \theta_i.\theta_{-i})\Big)\end{aligned} \tag{5.1}$$

Algorithm 5.1 details the SECURE STRATEGY(G, i, θ_i) function which implements Equation (5.1).

Algorithm 5.1: SECURE STRATEGY.

Data: $G = \langle N, A, \Theta, \pi, \mu \rangle$, i, θ_i
Result: a_i^*: optimal secure strategy for player i of type θ_i

$maximum \leftarrow 0$
forall $a_i \in A_i$ **do**
  $\bar{u}_i^{secure}(a_i, \theta_i) \leftarrow 1$
  forall $\theta_{-i} \in \Theta_{-i}$ **do**
    $\bar{\mu}_i(a_i, \theta_i.\theta_{-i}) \leftarrow 1$
    forall $a_{-i} \in A_{-i}$ **do**
      if $\mu_i(a_i.a_{-i}, \theta_i.\theta_{-i}) \leq \bar{\mu}_i(a_i, \theta_i.\theta_{-i})$ **then**
        $\bar{\mu}_i(a_i, \theta_i.\theta_{-i}) \leftarrow \mu_i(a_i.a_{-i}, \theta_i.\theta_{-i})$
      end
    end
    if $1 - \pi(\theta_i.\theta_{-i}) \geq \bar{\mu}_i(a_i, \theta_i.\theta_{-i})$ **then** $m \leftarrow 1 - \pi(\theta_i.\theta_{-i})$
    else $m \leftarrow \bar{\mu}_i(a_i, \theta_i.\theta_{-i})$
    if $m \leq \bar{u}_i^{secure}(a_i, \theta_i)$ **then** $\bar{u}_i^{secure}(a_i, \theta_i) \leftarrow m$
  end
  if $maximum \leq \bar{u}_i^{secure}(a_i, \theta_i)$ **then**
    $maximum \leftarrow \bar{u}_i^{secure}(a_i, \theta_i)$
    $a_i^* \leftarrow a_i$
  end
end
return a_i^*

Proposition 5.1 (Complexity of Algorithm 5.1)**.** *The* SECURE STRATEGY(G, i, θ_i) *function takes a polynomial time in the size of the* Π*-game* $G = \langle N, A, \Theta, \pi, \mu \rangle$*, i.e.,* SECURE STRATEGY(G, i, θ_i) *can be performed in polynomial time in the size of the standard normal form* Π*-game. The whole complexity of Algorithm 5.1 is:* $O(|A| \times |\Theta_{-i}|)$.

5.3 Finding a Pure Nash Equilibrium in Π-games: a MILP Formulation

Taking advantage of the efficiency of modern solvers, we propose a Mixed Integer Linear Programming (MILP) formulation for finding, if it exists, a PNE in Π-games (we follow in this the line opened by (Ceppi et al., 2009) for solving Bayesian games).

- the main decision variables are Boolean variables encoding the strategy searched for: each σ_{i,a_i,θ_i} is a Boolean variable indicating whether action a_i is prescribed for type θ_i of player i:

$$\forall i \in N, \forall a_i \in A_i, \forall \theta_i \in \Theta_i : \sigma_{i,a_i,\theta_i} \in \{0,1\};$$

- the utilities are encoded by continuous variables: U_{i,a_i,θ_i} is the utility (according to σ_{-i}) of player i if action a_i is chosen for type θ_i (i.e., if $\sigma_i(\theta_i) = a_i$):

$$\forall i \in N, \forall a_i \in A_i, \forall \theta_i \in \Theta_i : U_{i,a_i,\theta_i} \in [0,1].$$

We will also use the following Boolean variables to constrain the U_{i,a_i,θ_i} to be equal to the $\min_{\theta_{-i}\in\Theta_{-i}}$ of $\max\big(1-\pi(\theta_{-i}|\theta_i), \mu_i(a,\theta)\big)$ (and not only lower than the *min*):

$$\forall i \in N, \forall a_i \in A_i, \forall \theta \in \Theta : M_{i,a_i,\theta} \in \{0,1\}.$$

Hence, the MILP will contain the following constraints:

- $\forall i \in N, \forall \theta_i \in \Theta_i$,

$$\sum_{a_i\in A_i} \sigma_{i,a_i,\theta_i} = 1. \tag{5.2}$$

- $\forall i \in N, \forall a_i, a_i' \in A_i$, s.t, $a_i \neq a_i', \forall \theta_i \in \Theta_i$,

$$U_{i,a_i,\theta_i} - U_{i,a_i',\theta_i} \geq \sigma_{i,a_i,\theta_i} - 1. \tag{5.3}$$

- $\forall i \in N, \forall a \in A, \forall \theta \in \Theta$,

$$U_{i,a_i,\theta_i} \leq \max\big(1-\pi(\theta_{-i}|\theta_i), \mu_i(a,\theta)\big) + \sum_{j\in N,\ j\neq i} (1-\sigma_{j,a_j,\theta_j}) \tag{5.4}$$

- $\forall i \in N, \forall a \in A, \forall \theta \in \Theta$,

$$U_{i,a_i,\theta_i} + M_{i,a_i,\theta} + \sum_{j\in N,\ j\neq i} (1-\sigma_{j,a_j,\theta_j}) \geq \max(1-\pi(\theta_{-i}|\theta_i), \mu_i(a,\theta)) \tag{5.5}$$

- $\forall i \in N, \forall a_i \in A_i, \forall \theta_i \in \Theta_i$,

$$\sum_{\theta_{-i}\in\Theta_{-i}} (1-M_{i,a_i,\theta_i.\theta_{-i}}) = 1. \tag{5.6}$$

- constraints (5.2) ensure that the strategy σ searched for specifies exactly one action per type, for each player i.
- constraints (5.3) require that σ is a PNE: when $\sigma_{i,a_i,\theta_i} = 1$, this constraint requires

$U_{i,a_i,\theta_i} \geq U_{i,a'_i,\theta_i}$, i.e., that player i has no incentive to deviate from a_i. When actions a_i is not chosen for θ_i, ($\sigma_{i,a_i,\theta_i} = 0$) the constraint is always satisfied ($U_{i,a_i,\theta_i} - U_{i,a'_i,\theta_i}$ is always greater than -1).

- constraints (5.4) implement Definition (4.4). They ensure that the utility of player i playing $\sigma(\theta_i) = a_i$ is lower than all, i.e., the minimum over the θ_{-i} of the $\max(1-\pi(\theta_{-i}|\theta_i), \mu_i(a_i.\sigma_{-i}(\theta_{-i}), \theta))$. Indeed, for any profile of action a that does not correspond to what is prescribed by σ, $\Sigma_{j\neq i}(1-\sigma_{j,a_j,\theta_j}) \geq 1$ and the constraint is always satisfied ($U_{i,a_i,\theta_i} \leq 1$). If a_{-i} is chosen for θ_{-i}, then $\sigma_{j,a_j,\theta_j} = 1\ \forall j \neq i$ and $\Sigma_{j\neq i}(1-\sigma_{j,a_j,\theta_j}) = 0$: the constraint becomes $U_{i,a_i,\theta_i} \leq \max(1-\pi(\theta_{-i}|\theta_i), \mu_i(a,\theta))$.

- constraints (5.5) and (5.6) ensure that U_{i,a_i,θ_i} is equal to the *min* over θ_{-i} of $\max(1-\pi(\theta_{-i}|\theta_i), \mu_i(a,\theta))$: If a_{-i} does not correspond to σ_{-i}, $\Sigma_{j\neq i}(1-\sigma_{j,a_j,\theta_j})$ is at least equal to 1 and the constraints (5.5) are always satisfied. Otherwise, (a_{-i} correspond to σ_{-i}) the *sum* is equal to 0 and does not annihilate the constraint. The *min* is reached if $U_{i,a_i,\theta_i} = \max(1-\pi(\theta_{-i}|\theta_i), \mu_i(a,\theta))$. Whenever $M_{i,a_i,\theta} = 1$, Equation (5.5) holds, and Equation (5.6) ensures that (5.5) is an equality for one θ_{-i} (minimizing $\max(1-\pi(\theta_{-i}|\theta_i), \mu_i(a,\theta))$).

The above formulation is linear (the max operator which appears in constraints (5.4) and (5.5) deals with constants only). Furthermore, it does not lead to a combinatorial explosion of the required space. Recall that the size of the original problem is $n \cdot |\Theta| \cdot |A| + |\Theta|$. Let us denote d (resp. t) the number of actions (resp. types) of each player. The MILP formulation contains:

- $O(n \cdot t \cdot d)$ continuous variables U_{i,a_i,θ_i};
- $O(n \cdot t \cdot d)$ Boolean variables σ_{i,a_i,θ_i};
- $O(n \cdot d \cdot t^n)$ Boolean variables $M_{i,a_i,\theta}$;
- $O(n \cdot t)$ constraints (5.2), each involving $O(a)$ variables;
- $O(n \cdot t \cdot d \cdot (d-1))$ constraints (5.3), each involving 3 variables;
- $O(n \cdot t^n \cdot d^n)$ constraints (5.4) each involving $O(n)$ variables;
- $O(n \cdot t^n \cdot d^n)$ constraints (5.5) each involving $O(n+1)$ variables;
- $O(n \cdot d \cdot t)$ constraints (5.6) each involving $O(\frac{t^n}{d})$ variables.

The size of the MILP encoding is thus in $O(n^2 \cdot |\Theta| \cdot |A|)$ (polynomial in the size of the original size of the problem (i.e., $n \cdot |\Theta| \cdot |A| + |\Theta|$)).

Example 5.1 (Cont. Example 4.1)**.** *Solving the MILP formulation for the coordination* Π*-game detailed in Table 5.1.*

Player 1		Player 2			
		r_2		$\overline{r}_2$	
		x	y	x	y
r_1	x	0.9 , 0.9	0.3 , 0.3	0.9 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(r_1.r_2) = 1$		$\pi(r_1.\overline{r}_2) = 0.1$	
		x	y	x	y
$\overline{r}_1$	x	0.8 , 0.9	0.3 , 0.3	0.8 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(\overline{r}_1.r_2) = 0.1$		$\pi(\overline{r}_1.\overline{r}_2) = 1$	

Table 5.1: A coordination Π-game between two players (with two types per player) where $\Delta = \{0, 0.1, 0.2, 0.3, 0.7, 0.8, 0.9, 1\}$.

Let us give some constraints of the MILP formulation:

- $\sigma_{1,x,r_1} + \sigma_{1,y,r_1} = 1$ *ensures that player 1 plays x or y when she receives the incentive;*
- $U_{1,x,r_1} - U_{1,y,r_1} \geq \sigma_{1,x,r_1} - 1$ *ensures that if player 1 chooses x then $U_1^{pes}(x, \sigma_2, r_1) \geq U_1^{pes}(y, \sigma_2, r_1)$.*

The result is as follows:
$\sigma_{1,x,r_1} = 1$, $\sigma_{1,y,r_1} = 0$, $\sigma_{1,x,\overline{r}_1} = 0$, $\sigma_{1,y,\overline{r}_1} = 1$, $\sigma_{2,x,r_2} = 1$, $\sigma_{2,y,r_2} = 0$, $\sigma_{2,x,\overline{r}_2} = 0$, $\sigma_{2,y,\overline{r}_2} = 1$.
This means that player 1 plays x when she receives the incentive ($\sigma_{1,x,r_1} = 1$) and plays y when she does not receive the incentive $\sigma_{1,y,\overline{r}_1} = 1$. Similarly for player 2, she plays x when she receives the incentive ($\sigma_{2,x,r_2} = 1$) and plays y when she does not receive the incentive $\sigma_{2,y,\overline{r}_2} = 1$.

5.4 A Polynomial Time Algorithm for Building Possibilistic Mixed Nash Equilibria in Π-games

A possibilistic mixed Nash equilibrium (Π-MNE) can be seen as the result of a negotiation where players negotiate to improve their pessimistic utilities until no player has the interest to negotiate anymore. Following the idea of (Ben Amor et al., 2017), we propose a polynomial time algorithm to compute one Π-MNE. This algorithm presents the negotiation process where, at each stage, each player i for type θ_i tries to maximize her pessimistic utility $U_i^{pes}(\upsilon, \theta_i)$ by moving her mixed strategy $\upsilon_i(\theta_i)$ to $\upsilon_i'(\theta_i)$. Indeed, we will see that if a player moves to a more specific mixed strategy $\upsilon_i'(\theta_i) \leq \upsilon_i(\theta_i)$, her pessimistic utility increases.

Let us first recall the definition of specificity relation over possibility distributions.

Definition 5.1 (Specificity relation)**.** *Given the mixed strategies υ and υ', we say that υ' is at least as specific as υ iff all the distributions $\upsilon'_i(\theta_i)$ are all more specific than the distributions $\upsilon_i(\theta_i)$, $\forall i \in N$:*

$$\forall i \in N, \forall \theta_i \in \theta_i, \upsilon' \leq \upsilon \Leftrightarrow \upsilon'_i(\theta_i) \leq \upsilon_i(\theta_i) \tag{5.7}$$

where $\upsilon'_i(\theta_i) \leq \upsilon_i(\theta_i) \Leftrightarrow \upsilon'_i(a_i|\theta_i) \leq \upsilon_i(a_i\theta_i), \forall a_i \in A_i$.

The following proposition shows that given a possibilistic joint strategy υ, a player i of type θ_i never loses utility, in terms of her pessimistic utility, if the joint strategy υ' of all players is made more specific, i.e., if other players precise their intentions.

Proposition 5.2. *Let $G = \langle N, A, \Theta, \pi, \mu \rangle$ be a Π-game, υ be a mixed strategy and υ' be a more specific mixed strategy ($\upsilon' \leq \upsilon$). Then:*

$$\forall i \in N, \forall \theta_i \in \Theta_i, U_i^{pes}(\upsilon, \theta_i) \leq U_i^{pes}(\upsilon', \theta_i). \tag{5.8}$$

Given a Π-game, if player i updates her possibilistic mixed strategy υ_i to a more specific υ'_i her utility will not decrease if all other players stick to their strategies. However, the more specific mixed strategy υ'_i may allow other players to change their possibilistic mixed strategies and increase their pessimistic utilities. Therefore, the result of the negotiation process leads to a least-specific Π-MNE.

Formally, a least-specific possibilistic mixed Nash equilibrium in a Π-game is defined as:

Definition 5.2 (Least-specific Possibilistic mixed Nash Equilibrium)**.**
Let $G = \langle N, A, \Theta, \pi, \mu \rangle$ be a Π-game. $\upsilon^* = (\upsilon_1^*, \ldots, \upsilon_n^*)$ is a least specific possibilistic mixed equilibrium for G iff:

1. *υ^* is a possibilistic mixed equilibrium for G;*
2. *there exist no υ', s.t., $\upsilon^* \prec \upsilon'$ and υ' is a possibilistic mixed equilibrium, where $\upsilon^* \prec \upsilon'$ if and only if $\upsilon^* \leq \upsilon'$ and there exists (i, θ_i, a_i) such that $\upsilon^*(a_i|\theta_i) < \upsilon'(a_i|\theta_i)$.*

In order to find a least-specific possibilistic mixed equilibrium (which may not be unique, since $\leq$ is a partial order), we adapt the algorithm proposed by (Ben Amor et al., 2017). The idea is that each player i improves her mixed strategy $\upsilon_i(\theta_i)$, for each type θ_i, to a more specific one $\upsilon'_i(\theta_i)$ in order to increase her pessimistic utility. Algorithm 5.2 outlines the Finding a least-specific Π-MNE function:

Where the Improve function takes as input: *(i)* a Π-game $G = \langle N, A, \Theta, \pi, \mu \rangle$, *(ii)* a mixed strategy $\upsilon = (\upsilon_1, \ldots, \upsilon_n)$, *(iii)* the player i and *(iv)* her type θ_i then returns

Algorithm 5.2: Finding possibilistic mixed Nash equilibrium in Π-game.

Data: $G = \langle N, A, \Theta, \pi, \mu \rangle$
Result: $\upsilon^* = (\upsilon_1^*, \dots, \upsilon_n^*)$, a Π-MNE
$\upsilon^0 \leftarrow (\upsilon_1^0, \dots, \upsilon_n^0)$ /* $\upsilon_i^0(\theta_i, a_i) = 1, \forall i \in N, \forall a_i \in A_i, \forall \theta_i \in \Theta_i$ */
$t \leftarrow 0$
repeat
  $\upsilon^{loc} \leftarrow \upsilon^t$
  forall $i \in N$ **do**
    forall $\theta_i \in \Theta_i$ **do** $\upsilon^{loc} \leftarrow$ Improve $(G, \upsilon^{loc}, i, \theta_i)$
  end
  $\upsilon^{t+1} \leftarrow \upsilon^{loc}$
  $t \leftarrow t + 1$
until $\upsilon^t = \upsilon^{t-1}$
$\upsilon^* \leftarrow \upsilon^t$
return υ^*

a more specific joint mixed strategy. Algorithm 5.3 details the Improve function process.

Algorithm 5.3: Improve.

Data: $G = \langle N, A, \Theta, \pi, \mu \rangle$, υ, i, θ_i
Result: $\upsilon' = (\upsilon'_1, \dots, \upsilon'_n)$
$A^*_{i,\theta_i} \leftarrow \varnothing$
forall $a_i \in A_i$ **do**
  if $U_i^{pes}(a_i, \upsilon_{-i}, \theta_i) \le U_i^{pes}(\upsilon, \theta_i)$ **then** $A^*_{i,\theta_i} \leftarrow A^*_{i,\theta_i} \cup \{a_i\}$
end
if $\forall a_i \in A_i \setminus A^*_{i,\theta_i}$, $\upsilon_i(a_i|\theta_i) < 1$ **then** $\upsilon' \leftarrow \upsilon$
else
  forall $a_i \in A_i$ **do**
    if $a_i \in A^*_{i,\theta_i}$ **then** $\upsilon'_i(a_i|\theta_i) \leftarrow n(U_i^{pes}(\upsilon, \theta_i))^-$
    else $\upsilon'_i(a_i|\theta_i) \leftarrow \upsilon_i(a_i|\theta_i)$
  end
end
return υ'

We note that $n(U_i^{pes}(\upsilon, \theta_i))^-$ is the degree in the ordinal scale Δ just below and $n(U_i^{pes}(\upsilon, \theta_i)) = 1 - U_i^{pes}(\upsilon, \theta_i)$. As an example, suppose that $\Delta = \{0, 0.25, 0.5, 0.75, 1\}$ and $U_i^{pes}(\upsilon, \theta_i) = 0.25$. Thus, $n(U_i^{pes}(\upsilon, \theta_i)) = 1 - U_i^{pes}(\upsilon, \theta_i) = 0.75$ and the degree just below 0.75 in Δ is 0.5. Therefore, $n(U_i^{pes}(\upsilon, \theta_i))^- = 0.5$.

Proposition 5.3 (Complexity of Algorithm 5.3)**.** Improve *function takes time polynomial in the size the the* Π*-game G. The whole complexity of the* Improve *function is:* $O(|A_{\max}| \times |A| \times |\Theta_{-i}|)$ *where* $|A_{\max}| = \max(|A_1|, ..., |A_i|, ..., |A_n|)$.

Algorithm 5.2 calls $N \times |\Theta_i|$ times Algorithm 5.3. Thus, based on proposition 5.3:

Corollary 5.1 (Complexity of Algorithm 5.2)**.** The whole complexity is: $O(|N| \times |\Theta_{\max}| \times |\Delta| \times |A_{\max}| \times |A| \times |N|) = O(|N|^2 \times |\Theta_{\max}| \times |\Delta| \times |A|)$ where $|\Theta_{\max}| = \max(|\Theta_1|, ..., |\Theta_i|, ..., |\Theta_n|)$.

Algorithm 5.2 is polynomial. In the following, we prove that this algorithm converges towards a least-specific possibilistic mixed Nash equilibrium.

Proposition 5.4 (Convergence)**.** *Algorithm 5.2 converges in a finite number of steps and convergence occurs in the size of the* Π*-game* G*.*

Based on Proposition 5.4, the following proposition holds:

Proposition 5.5 (Soundness)**.** *Let* $G = \langle N, A, \Theta, \pi, \mu \rangle$ *be a* Π*-game. If Algorithm 5.2 has converged towards* υ^**, then* υ^* *is a possibilistic mixed equilibrium of* G*.*

Example 5.2 (Cont. Example 4.1)**.** *Consider the* Π*-game detailed in Table 5.2,*

		Player 2			
		r_2		$\overline{r}_2$	
Player 1		x	y	x	y
r_1	x	0.9 , 0.9	0.3 , 0.3	0.9 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(r_1.r_2) = 1$		$\pi(r_1.\overline{r}_2) = 0.1$	
		x	y	x	y
$\overline{r}_1$	x	0.8 , 0.9	0.3 , 0.3	0.8 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(\overline{r}_1.r_2) = 0.1$		$\pi(\overline{r}_1.\overline{r}_2) = 1$	

Table 5.2: A coordination Π-game between two players (with two types per player) where $\Delta = \{0, 0.1, 0.2, 0.3, 0.7, 0.8, 0.9, 1\}$.

Let us start with:

$$\upsilon_1^0(x|r_1) = 1, \quad \upsilon_1^0(y|r_1) = 1, \quad \upsilon_1^0(x|\overline{r}_1) = 1, \quad \upsilon_1^0(y|\overline{r}_1) = 1,$$
$$\upsilon_2^0(x|r_2) = 1, \quad \upsilon_2^0(y|r_2) = 1, \quad \upsilon_2^0(x|\overline{r}_2) = 1, \quad \upsilon_2^0(y|\overline{r}_2) = 1.$$

We have:
$U_1^{pes}(x, \upsilon_2^0, r_1) = 0.3$, $U_1^{pes}(y, \upsilon_2^0, r_1) = 0.2$. *Thus,* $U_1^{pes}(\upsilon^0, r_1) = 0.2$ *and* $A^*_{1,r_1} = \{y\}$.
$U_1^{pes}(x, \upsilon_2^0, \overline{r}_1) = 0.3$, $U_1^{pes}(y, \upsilon_2^0, \overline{r}_1) = 0.2$. *Thus,* $U_1^{pes}(\upsilon^0, \overline{r}_1) = 0.2$ *and* $A^*_{1,\overline{r}_1} = \{y\}$.
Since, $A_{1,r_1} \backslash A^*_{1,r_1} = \{x\}$ *and* $A_{1,\overline{r}_1} \backslash A^*_{1,\overline{r}_1} = \{x\}$*, then player 1 can move to a more specific possibilistic mixed strategy* υ_1^1 *and decrease* $\upsilon_1^0(y|r_1)$ *and* $\upsilon_1^0(y|\overline{r}_1)$*. Thus we get:* $\upsilon_1^1(x|r_1) = 1$, $\upsilon_1^1(y|r_1) = 0.7$, $\upsilon_1^1(x|\overline{r}_1) = 1$ *and* $\upsilon_1^1(y|\overline{r}_1) = 0.7$.

Similarly, we have:
$U_2^{pes}(x, \upsilon_1^0, r_2) = 0.3$, $U_2^{pes}(y, \upsilon_1^0, r_2) = 0.3$. *Thus,* $U_2^{pes}(\upsilon^0, r_2) = 0.3.$ *and* $A^*_{2,r_2} = \{x, y\}$.

$U_2^{pes}(x, \upsilon_1^0, \overline{r}_2) = 0.3$, $U_2^{pes}(y, \upsilon_1^0, \overline{r}_2) = 0.3$. *Thus,* $U_2^{pes}(\upsilon^0, \overline{r}_2) = 0.3$ *and* $A^*_{2,\overline{r}} = \{x, y\}$. *Since,* $A_{2,r_2} \backslash A^*_{2,r_2} = \varnothing$ *and* $A_{2,\overline{r}} \backslash A^*_{2,\overline{r}} = \varnothing$*, then player 2 cannot change her possibilistic mixed strategy since she will more to a non normalized mixed strategy.*

Thus, at the end of the round 1 we get:

$$\upsilon_1^1(x|r_1) = 1, \quad \upsilon_1^1(y|r_1) = 0.7, \quad \upsilon_1^1(x|\overline{r}_1) = 1, \quad \upsilon_1^1(y|\overline{r}_1) = 0.7,$$
$$\upsilon_2^1(x|r_2) = 1, \quad \upsilon_2^1(y|r_2) = 1, \quad \upsilon_2^1(x|\overline{r}_2) = 1, \quad \upsilon_2^1(y|\overline{r}_2) = 1.$$

Another round of improvement does not give any more changes. Therefore, the possibilistic mixed Nash equilibrium υ^* *such that:*

$$\upsilon_1^*(x|r_1) = 1, \quad \upsilon_1^*(y|r_1) = 0.7, \quad \upsilon_1^*(x|\overline{r}_1) = 1, \quad \upsilon_1^*(y|\overline{r}_1) = 0.7,$$
$$\upsilon_2^*(x|r_2) = 1, \quad \upsilon_2^*(y|r_2) = 1, \quad \upsilon_2^*(x|\overline{r}_2) = 1, \quad \upsilon_2^*(y|\overline{r}_2) = 1.$$

5.5 Experimental Study

The goal of experiments is to show the efficiency and feasibility of our proposed algorithms and MILP formulation to solve possibilistic games with incomplete information. To do that, we have to test our approaches for different Π-games. We do not develop random games. We adapt GAMUT game generators (Nudelman et al., 2004) to generate Π-games. GAMUT basically produces exclusively normal form games with complete information. It contains instances of games from different game classes.

The following of this section details our Π-game generator. Then details the experimental protocol and finally presents the experimental results.

5.5.1 A Π-game Generator

We introduce a novel generator for possibilistic games with incomplete information based on GAMUT (Nudelman et al., 2004). Based on Definition 4.2, where every Π-game can be equivalently defined as a set of $|\Theta|$ normal form games with the same set of players N and actions A.

We follow the approach of (Ceppi et al., 2009) for the generation of Bayesian games. More precisely, given the number of players n, the number of actions d and the number of types t per player, the idea is to generate, using GAMUT, for each combination of types $\theta \in \Theta$ a normal form game G^θ and derive the possibilities over the combinations of types.

We assume that the number of actions and types for all players are equal, i.e., $\forall i, j \in N, |A_i| = |A_j| = d$ and $\forall i, j \in N, |\Theta_i| = |\Theta_j| = t$.

To generate a Π-game, $G = \langle N, A, \Theta, \pi, \mu \rangle$, we need as inputs: *(1)* the class and the name of the game, *(2)* the number n of players, *(3)* the number of degrees in Δ, *(4)* the number of types $|\Theta_i|$ for each player i and *(5)* the number of actions $|A_i|$ for each player i.

Then, we ask GAMUT to generate $|\Theta|$ normal form games of the class given in input, the range of utility of which is Δ and we generate a normalized distribution $\pi : \Theta \mapsto \Delta$ (a randomly selected θ receives degree 1; the degrees of the other elements of Θ are selected in Δ following a normalized distribution). Finally, the utility $\mu_i(a, \theta)$ is simply the utility of the joint action a for player i in the normal form game $G^\theta = \langle N, A, \{\{\mu_i^\theta\}_{i\in N}\}\rangle$ (see Definition 4.2).

Inputs of the Generator

The class of the game: first, to generate a Π-game, we have to choose the class of the game. GAMUT offers the possibility to generate different game classes. In our generator, we select only appropriate games, i.e., possible to be presented in the normal form, and we classify these games into six classes:

1. **class 1:** 2×2 games: normal games with 2 players and 2 actions per player: Battle Of The Sexes, Chicken, Hawk And Dove, Matching Pennies and Prisoners Dilemma.
2. **class 2:** 2×3 games: normal games with 2 players and 3 actions per player: Rock Paper Scissors and Shapley's Game.
3. **class 3:** $2 \times m$ games: normal games with 2 players and $m \geq 2$ actions per player: Grab The Dollar, Random Zero Sum.
4. **class 4:** $n \times 2$ games: normal games with $n \geq 2$ players and 2 actions per player: N Player Chicken and Random Compound Game.
5. **class 5:** $n \times n$ games: normal games with $n \geq 2$ players and n actions per player: Collaboration Game and Coordination Game.
6. **class 6:** $n \times m$ games: normal games with $n \geq 2$ players and $m \geq 2$ actions per player: Co variant Game, Dispersion Game, Majority Voting, Minimum Effort Game, Random Game, Travelers Dilemma.

Number of players: by default, in GAMUT, for games in classes 1, 2, and 3, the number of players is equal to two. For the remaining games (classes 4, 5 and 6), the user should specify the number of players n. In this case, we set the GAMUT parameter *-players* to the specified number of players n.

Number of actions: by default, in GAMUT, for games in classes 1 and 4, the number of actions per player is equal to two. For games in class 2, the number of actions per player is equal to three. However, for the other games in classes 3 and 6, the user should specify the number of actions d per player. In this case, we set the GAMUT parameter *-actions* to the specified number of actions d per player.

Let us notice that in games of class 5, the number of actions d is by default equal to the number of players in the game, i.e., $d = n$.

Ordinal Scale: In a Π-game, the utilities should be in an ordinal scale Δ. We ask the user to give the maximum value of the scale Δ. By default, the minimum value is equal to 0. For that, we set the following GAMUT parameters:

- ***-min_payoff:*** minimum utility when normalization is used (generally is equal to 0);
- ***-max_payoff:*** maximum utility when normalization is used (equal to $|\Delta|$);
- ***-int_payoffs:*** all utilities should be converted to integers rather than output as doubles;
- ***-int_mult:*** multiplier used before rounding when converting double to integer utilities. Defaults to 10,000. (In our generator, we set it to 1).

Number of Types: For all player $i \in N$, the user should specify the number of types t per player.

Possibilities of joint Types Combinations: In a Π-game, each $\theta \in \Theta$ has a possibility $\pi(\theta)$. We generate $\pi(\theta)$ randomly and we ensure that the possibility distribution is normalized, i.e., $\exists \theta \in \Theta$, s.t., $\pi(\theta) = 1$.

Utilities of Players: In this step, we run GAMUT $|\Theta|$ times to generate all players' utilities for the different combinations of types. The utility of player i for the joint type θ and joint action a is equal to the utility of that player for the joint action a in the normal form game G^{θ}.

5.5.2 Experimental Protocol

In our experiments, we vary the number of players from 2 to 10, the number of types from 2 to 10 and the number of actions from 2 to 10. We fix $\Delta = \{0, 0.25, 0.5, 0.75, 1\}$. For each combination of parameters, we have generated 100 instances and measured the time necessary to get:

(i) a secure strategy;

(ii) a pure Nash equilibrium (or a negative result);

(iii) a possibilistic mixed Nash equilibrium.

We present, in the following, results for 6 game classes: *covariant game*, *dispersion game*, *majority voting game*, *minimum effort game*, *random game* and *travelers dilemma game*. In our evaluation, we bounded the execution time for a single game to 10 minutes as in (Sandholm et al., 2005, Porter et al., 2008).

All experiments were conducted on an Intel Xeon E5540 processor and 64GB RAM workstation. We use CPLEX (CPLEX, 2009) as a MILP solver and Java 8 as a programming language. The implementations of the transformation of the Π-game as a normal form game (C-SNF) and MILP solver are available online `https://www.irit.fr/%7EHelene.Fargier/PossibilisticGames.html`.

5.5.3 Experimental Results

5.5.3.1 Results on Secure Strategy

The following part of our experimental study consisted of implementing Algorithm 5.1 and computing the time necessary to find a secure strategy of a Π-game. We varied the number of actions and types per player while fixing the number of players to 2 (see Figures 5.1 and 5.2). Then, we fixed the number of actions (resp., types) per player while varying the number of players (resp., actions) and we computed the average execution time to find the secure strategies for different types (resp., players). The results are given in Figure 5.3 (resp., Figure 5.4).

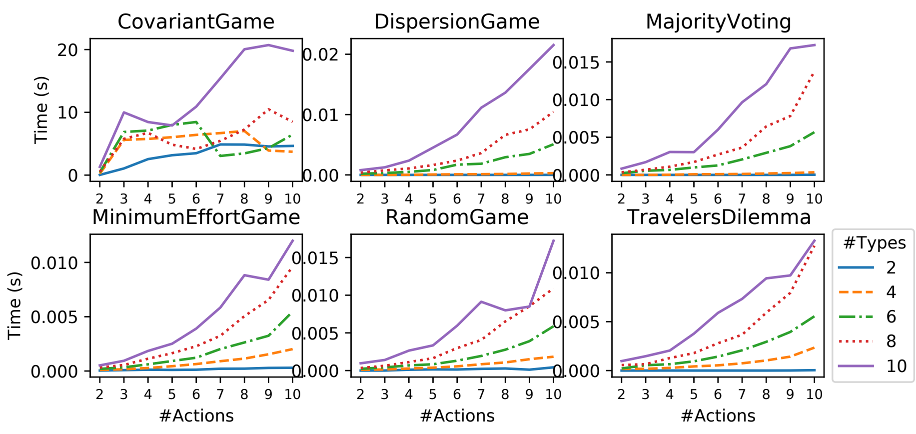

Figure 5.1: Average execution time (s) to find one secure strategy in a Π-game, fixing $n = 2$, varying $|A_i|$ from 2 to 10 for $|\Theta_i|$ from 2 to 10, $|\Delta| = 5$.

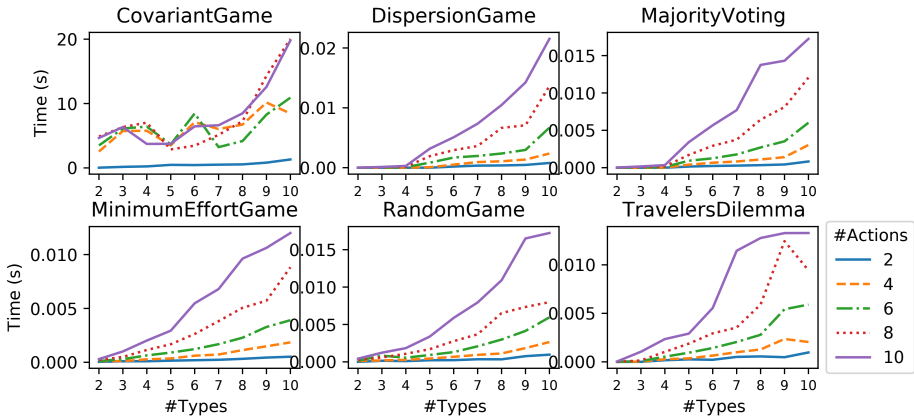

Figure 5.2: Average execution time (s) to find one secure strategy in a Π-game, fixing $n = 2$, varying $|\Theta_i|$ from 2 to 10 for $|A_i|$ from 2 to 10, $|\Delta| = 5$.

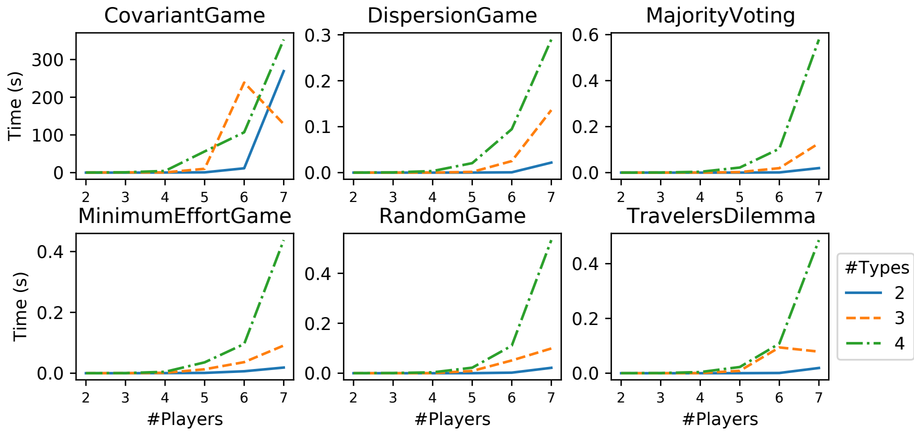

Figure 5.3: Average execution time (s) to find one secure strategy in a Π-game, fixing $|A_i| = 2$, varying n from 2 to 7 for $|\Theta_i|$ from 2 to 4, $|\Delta| = 5$.

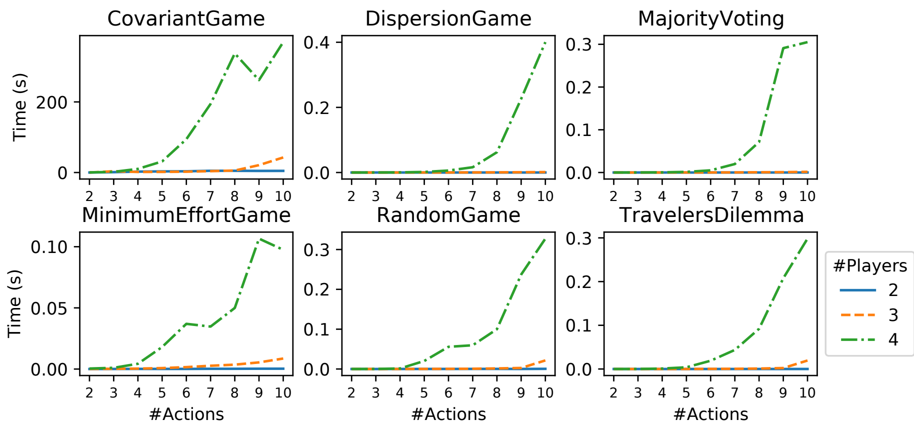

Figure 5.4: Average execution time (s) to find one secure strategy in a Π-game, fixing $|\Theta_i| = 2$, varying $|A_i|$ from 2 to 10 for n from 2 to 4, $|\Delta| = 5$.

Figure 5.5 (resp, Figure 5.6) presents the average execution time needed to find a secure strategy fixing the number of actions (resp, types) per player to 2 and varying the number of types (resp., player) for different players (resp, actions).

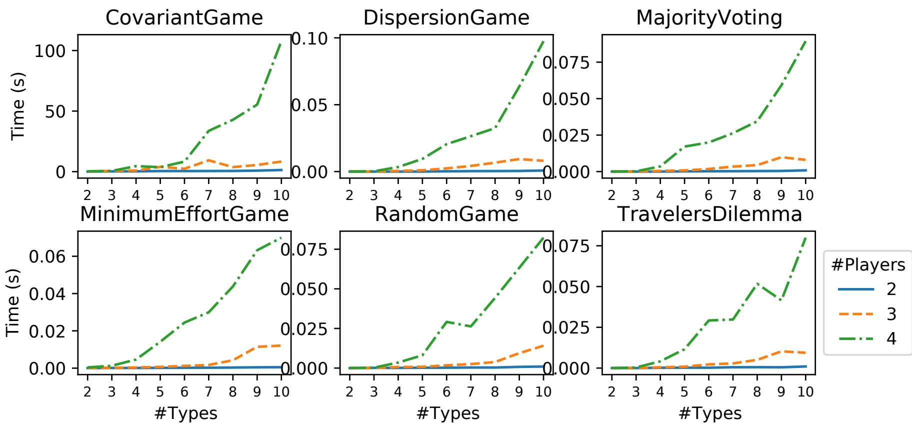

Figure 5.5: Average execution time (s) to find one secure strategy in a Π-game, fixing $|A_i| = 2$, varying $|\Theta_i|$ from 2 to 10 for n from 2 to 4, $|\Delta| = 5$.

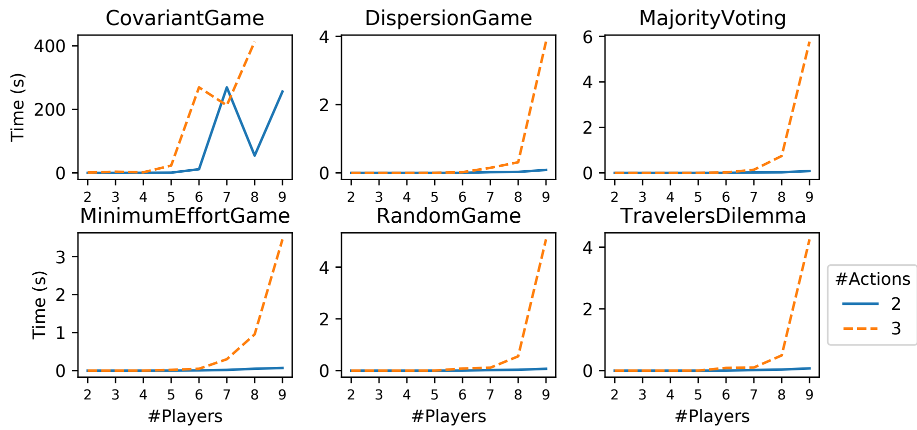

Figure 5.6: Average execution time (s) to find one secure strategy in a Π-game, fixing $|\Theta_i| = 2$, varying n from 2 to 9 for $|A_i|$ from 2 to 3, $|\Delta| = 5$.

The results show that we can find a secure strategy efficiently even for large games (more players, actions, or types per player). For example, in Figure 5.6 for a 9-player Π-game 2 types and 3 actions per player, finding a secure strategy, generally, does not take more than 5 seconds in all game classes. As in the previous experiment, the average execution time needed to find a secure strategy mainly depends on the number of players in the game, e.g., if the number of types per player is equal to 5 and the number of actions per players is equal to 2, the average execution time needed to find a secure strategy for a minimum effort game with 4 players is equal to 0.01 s. Nevertheless, for a minimum effort game with 5 players, the average execution time to find a secure strategy is equal to 0.1 s and equal to 0.4 s if the number of players is equal to 7.

5.5.3.2 Results on PNE

First, we start by implementing the transformation of the Π-game as a normal form game (C-SNF). This method, which is exponential in time and space, cannot be considered as a solving method, and this is supported by the experimental results.

Game class	Θ_i A_i	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	
Covariant Game	**2**	**.00**	**.02**	**.05**	**.10**	**.30**	**.70**	**1.44**	**5.72**	**6.94**	C-SNF
		.30	13.34	11.72	17.08	13.3	19.54	24.74	26.22	25.62	MILP
	3	**.01**	**.24**	**1.76**	13.70	42.40	69.96	38.76	103.72	160.66	C-SNF
		.02	14.48	10.26	**10.92**	**14.82**	**17.08**	**19.16**	**20.42**	**23.6**	MILP
	4	**.51**	**6.6**	128.48	558.84	753.86	889.18	1.1k	1.4k	1.7k	C-SNF
		1.02	8.42	**11.4**	**12**	**14.64**	**17.22**	**19.71**	**21.7**	**23.84**	MILP
Dispersion Game	**2**	**.00**	**.05**	**.10**	**.13**	**.31**	**.74**	**5.62**	**1.36**	**7.02**	C-SNF
		.30	.50	.60	1.00	5.38	1.44	6.78	2.34	7.72	MILP
	3	**.16**	**.32**	1.76	15.76	39.12	69.94	29.64	88.46	126.38	C-SNF
		.48	.64	**1.2**	**1.68**	**2.4**	**11.28**	**7.12**	**11.84**	**13.04**	MILP
	4	**.20**	6.9	135.28	600.74	478.22	818.52	1.1k	1.5k	1.8k	C-SNF
		3.24	**1.28**	**4.02**	**4.84**	**10.08**	**10.82**	**10.52**	**13.2**	**20.26**	MILP
Travelers Dilemma Game	**2**	**.01**	**.11**	**.13**	**.17**	**.20**	**.60**	**.96**	**1.52**	**6.94**	C-SNF
		.23	.32	.41	.90	.56	1.44	7.38	2.94	10.74	MILP
	3	**.05**	**.32**	1.84	10.22	42.38	70.1	30.38	60.18	173.82	C-SNF
		.32	.64	**.88**	**8.48**	**11.96**	**11.68**	**21.08**	**22.36**	**33.52**	MILP
	4	.09	3.84	50.62	546.46	1.5k	1.5k	1.6k	2.8k	3.4k	C-SNF
		.03	**2.36**	**5.34**	**9.92**	**14.82**	**15.14**	**19.42**	**21.32**	**23.00**	MILP

Table 5.3: Average memory usage of MILP & C-SNF (MB), $n = 2$.

Game class	Θ_i A_i	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	
Covariant Game	**2**	**.5**	**.4**	**.8**	**1.9**	**4.3**	**10.3**	**35.7**	**61.3**	**144.1**	C-SNF
		12.8	850.2	633.3	1.6k	2k	2.2k	2.4k	1.9k	2.1k	MILP
	3	**.3**	**.34**	**43**	**261.8**	**947.2**	2.4k	5.9k	14.7k	22.6k	C-SNF
		21.8	1.1k	1.5k	2.1k	1.8k	**1.3k**	**1.8k**	**1.7k**	**1.6k**	MILP
	4	**1.9**	**139.9**	2.4k	3.9k	85.6k	318.5k	-	-	-	C-SNF
		23.9	853.6	**1.3k**	**885.8**	**1.7k**	**1.3k**	**1.6k**	**867.8**	**1.1k**	MILP
Dispersion Game	**2**	**.4**	**.3**	**.7**	**1.8**	**4.6**	**8.8**	**28.3**	**64.8**	116.8	C-SNF
		11.8	44.5	53.5	81.8	78.3	80.3	115.5	96	**110.2**	MILP
	3	**.3**	**.0**	**35.8**	220.9	813	2.3k	5.7k	14.3k	27.4k	C-SNF
		62.5	65.1	46.6	**63.1**	**50.1**	**60.1**	**63.5**	**97.6**	**109.2**	MILP
	4	**1.9**	116.1	1.3k	9.7k	41.5k	113.2k	295.6k	478.1k	-	C-SNF
		87.5	**58.7**	**34.3**	**35.3**	**54.3**	**65.0**	**88.0**	**98.9**	**145.6**	MILP
Travelers Dilemma Game	**2**	**.2**	**.2**	**.6**	**1.5**	**3.7**	**8.4**	**18.9**	**40.5**	**76**	C-SNF
		21.2	59.3	78.4	89.1	76.6	95.8	103.2	93.9	182.8	MILP
	3	**.3**	**2.7**	**20.3**	**104.2**	**365.4**	932.9	2.3k	6.3k	13.1k	C-SNF
		51.0	71.8	59.5	643.4	578.8	**217.8**	**689.2**	**1k**	**1.6k**	MILP
	4	**1.5**	47.4	535.7	3.9k	18.9k	77.4k	271.8k	-	-	C-SNF
		18.8	**26.6**	**257.4**	**525.9**	**794.4**	**647.3**	**1.3k**	**632.8**	**670.5**	MILP

Table 5.4: Average execution time of MILP & C-SNF (ms), $n = 2$.

Table 5.3 and Table 5.4 present, respectively, the average memory (in MB) required to decide whether the problem admits a PNE or not and the average of execution time (in milliseconds) needed to find one PNE (best results are in bold). We vary $|A_i|$ from 2 to 10 and $|\Theta_i|$ from 2 to 4 for player i while fixing $n = 2$. "-" mentions that the execution time exceeds 10 minutes. Table 5.3 confirms that C-SNF cannot scale up contrarily to MILP which requires less memory, e.g., C-SNF requires more than 3.4k MB when MILP needs just 23MB to solve *Travelers Dilemma* game with 2 players, 10 actions and 4 types per player. The non-scalability of C-SNF is also observable in Table 5.4.

Second, we extend our study to the model of MILP by varying the Π-game parameters. More precisely, we start by varying the number of actions (resp., types and players) from 2 to 10 and the number of types (resp., players and actions) from 2 to 10 while we fix the number of players (resp., actions, and types) to 2 – see Figure 5.7 (resp., Figure 5.8, and Figure 5.9).

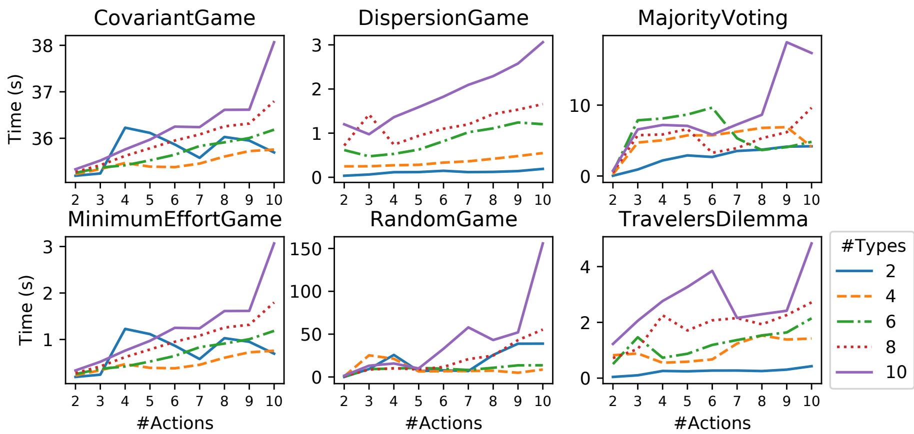

Figure 5.7: Average execution time (s) to find one PNE in a Π-game, fixing $n = 2$, varying $|A_i|$ from 2 to 10 for $|\Theta_i|$ from 2 to 10, $|\Delta| = 5$.

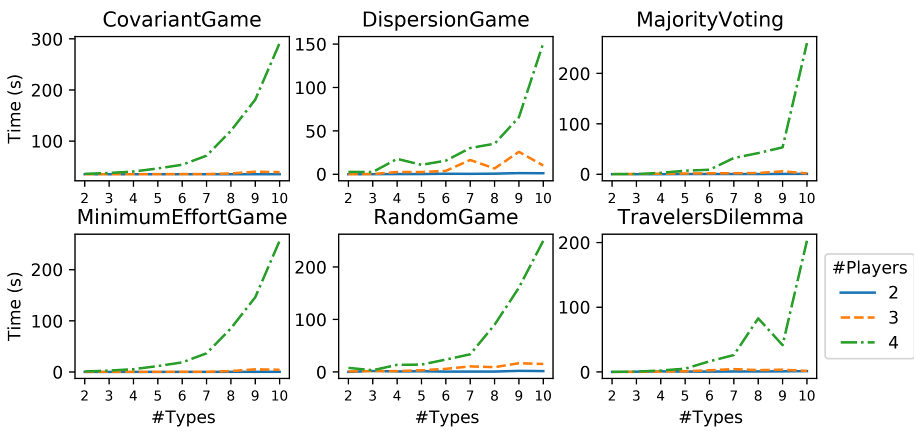

Figure 5.8: Average execution time (s) to find one PNE in a Π-game, fixing $|A_i| = 2$, varying $|\Theta_i|$ from 2 to 10 for n from 2 to 4, $|\Delta| = 5$.

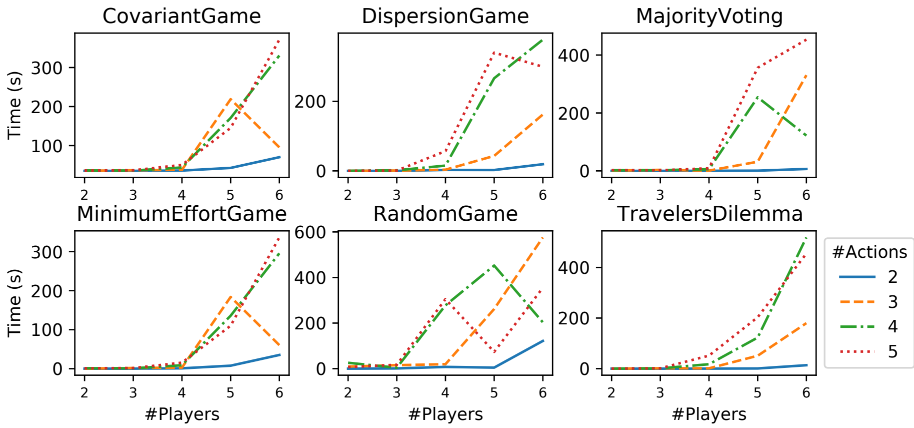

Figure 5.9: Average execution time (s) to find one PNE in a Π-game, fixing $|\Theta_i| = 2$, varying n from 2 to 6 for $|A_i|$ from 2 to 5, $|\Delta| = 5$.

Then, we vary the number of actions (resp., types and players) from 2 to 10 and the number of actions from 2 to 10 while we fix the number of playerto 2 – see Figure 5.10.

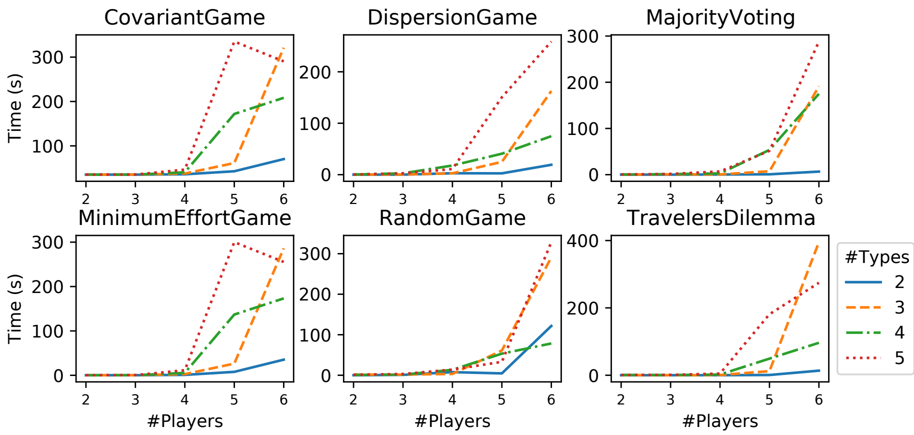

Figure 5.10: Average execution time (s) to find one PNE in a Π-game, fixing $|A_i| = 2$, varying n from 2 to 6 for $|\Theta_i|$ from 2 to 5, $|\Delta| = 5$.

Globally MILP results confirm the feasibility of the qualitative approach of incomplete information games advocated in this thesis. Results show that MILP has almost the same behavior in different games. We notice that if the number of actions or the number of types per player increases, the average execution time increases linearly, e.g., as shown in Figure 5.7, given a Travelers dilemma game with two players and 10 types per players, the average execution time needed to find a PNE where $\forall i \in N, |A_i| = 5$ is equal to 1 s, $\forall i \in N, |A_i| = 1.3$ is equal to 1.5 s, $\forall i \in N, |A_i| = 9$ is equal to 5.84 s.

However, the number of players highly impacts the execution time. Indeed, if the number of players increases, the average execution time increases exponentially, e.g., as shown in Figure 5.10, the average execution time needed to find a PNE in a Random game where every player has two actions ($|A_i| = 2$) and two types ($|\Theta_i| = 2$) is equal to

4.5 s if the number of players is equal to 5 ($n = 5$), 121 s if $n = 6$. These results are consistent with the theoretical complexity since adding a player directly increases $|\Theta|$ and $|A|$. Thus, the number of constraints of the MILP increases.

5.5.3.3 Results on Π-MNE

In this part, we use Algorithm 5.2 to find a Π-MNE in a Π-game. We start by fixing the number of players ($n = 2$) and varying the number of actions from 2 to 10 and the number of types from 2 to 10. The execution time needed to find the least specific Π-MNE is represented in Figures 5.11 and 5.12, respectively. We also fix the number of actions per player to 2 ($\forall i \in N$, $|A_i| = 2$) and we vary the number of players from 2 to 10 and the number of types from 2 to 10. Results are represented in Figures 5.13 and 5.14, respectively. Finally, we fix the number of types per player to 2 ($\forall i \in N$, $|\Theta_i| = 2$) and we vary for all players (resp., actions) the number of actions per player from 2 to 10 (resp., the number of players from 2 to 10). Results are represented in Figures 5.15 and 5.16, respectively.

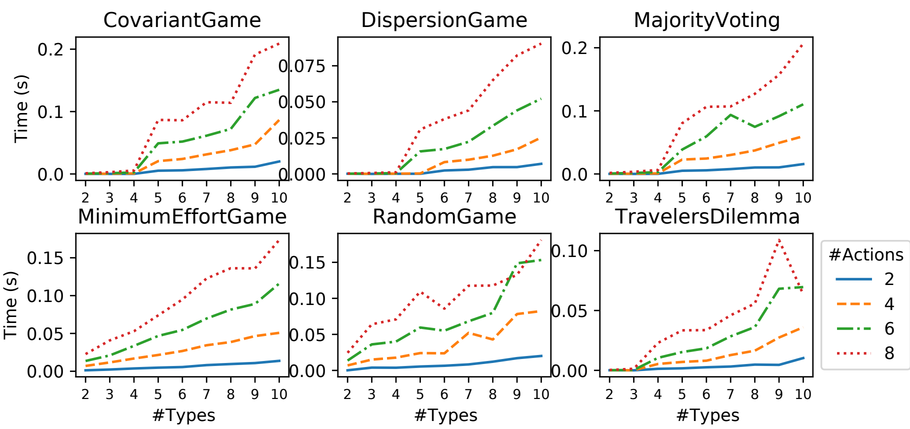

Figure 5.11: Average execution time (s) to find one Π-MNE in a Π-game, fixing $n = 2$, varying $|\Theta_i|$ from 2 to 10 for $|A_i|$ from 2 to 8, $|\Delta| = 5$.

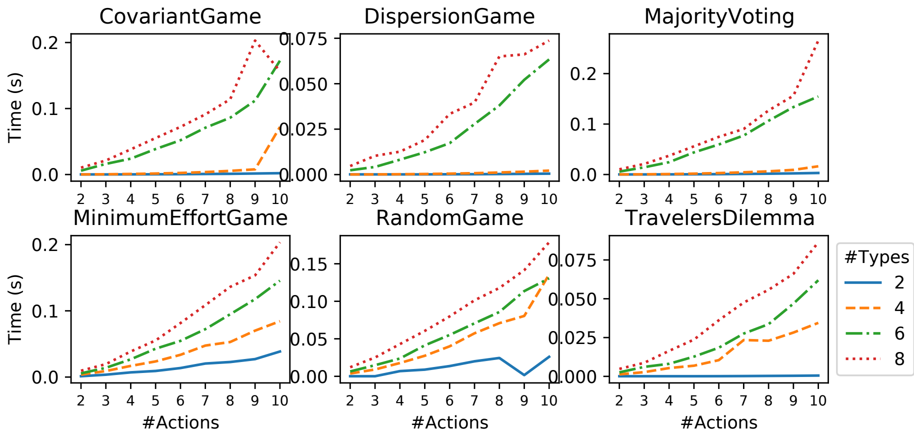

Figure 5.12: Average execution time (s) to find one Π-MNE in a Π-game, fixing $n = 2$, varying $|A_i|$ from 2 to 10 for $|\Theta_i|$ from 2 to 8, $|\Delta| = 5$.

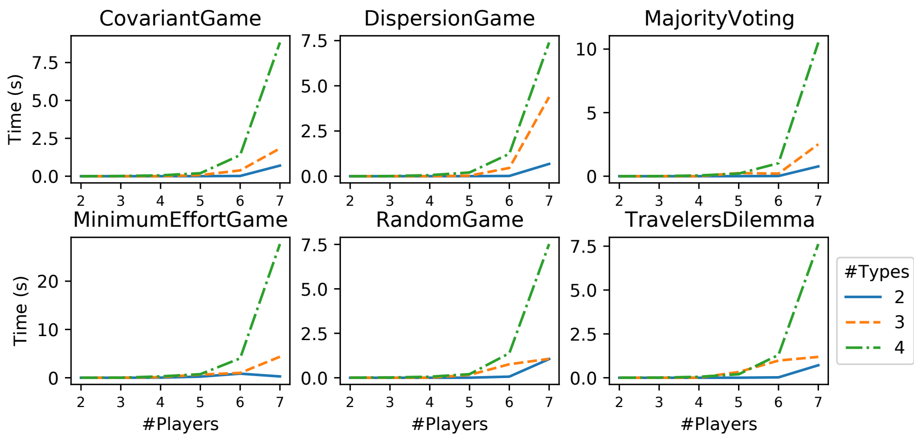

Figure 5.13: Average execution time (s) to find one Π-MNE in a Π-game, fixing $|A_i| = 2$, varying n from 2 to 7 for $|\Theta_i|$ from 2 to 10 from 2 to 4 , $|\Delta| = 5$.

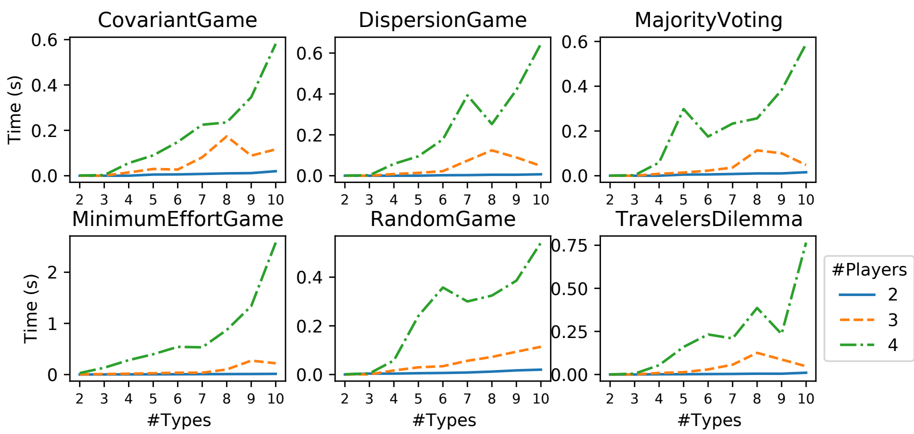

Figure 5.14: Average execution time (s) to find one Π-MNE in a Π-game, fixing $|A_i| = 2$, varying $|\Theta_i|$ from 2 to 10 for n from 2 to 4, $|\Delta| = 5$.

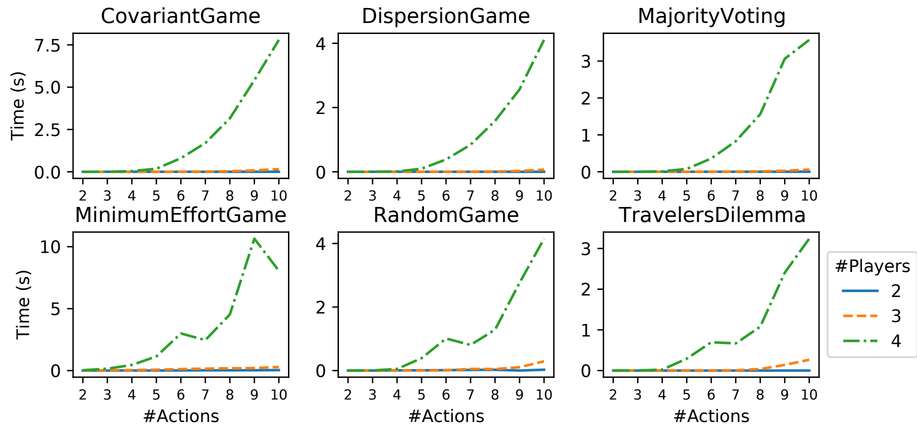

Figure 5.15: Average execution time (s) to find one Π-MNE in a Π-game, fixing $|\Theta_i| = 2$, varying $|A_i|$ from 2 to 10 for n from 2 to 4, $|\Delta| = 5$.

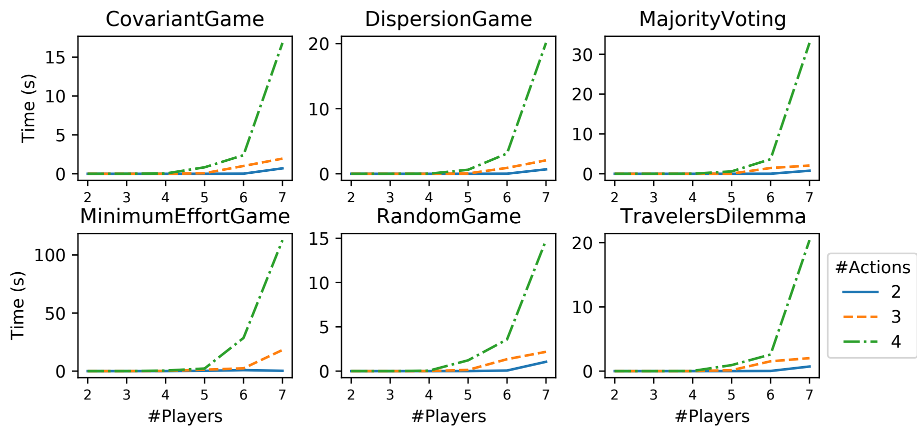

Figure 5.16: Average execution time (s) to find one Π-MNE in a Π-game, fixing $|\Theta_i| = 2$, varying n from 2 to 7 for $|A_i|$ from 2 to 4, $|\Delta| = 5$.

Globally, the results show that for the 6 games: covariant game, dispersion game, majority voting game, minimum effort game, random game and travelers dilemma game, Algorithm 5.2 can return the least specific Π-MNE in a reasonable time (less than 0.4 seconds when the number of players is equal to 2 and the number of actions and types are equal to 10). We find that the number of players in a Π-game is the "main" parameter that influences the average execution time needed to find a Π-MNE. As an example, in Figure 5.16, if the number of actions per player is equal to 4 and the number of types per player is equal to 2, the average execution time needed to find the least specific Π-MNE in a Covariant game with 5 players is equal to 0.8 s. However, when the number of players is equal to 6 (resp., 7) the average execution time is equal to 2.4 s (resp., 16.8 s).

The results empirically validate the theoretical part since Algorithm 5.2 requires at least $|N|^2 \times |\Theta_{\max}| \times |\Delta| \times |A| = 6^2 \times 2 \times 5 \times 3^6$ iterations if the number of players is equal to 6 whereas it requires $7^2 \times 2 \times 5 \times 3^7$ iterations if the number of players is equal to 7. Therefore, the execution time needed to find a Π-MNE in a 7-player Π-games is almost at least equal to $\frac{7^2 \times 2 \times 5 \times 3^7}{6^2 \times 2 \times 5 \times 3^6} \simeq 4.1$ of the time needed to find a Π-MNE in a 7-player Π-games.

5.6 Conclusion

This chapter focused on solving possibilistic games with incomplete information. It proposes polynomial times algorithms to build a secure strategy and a possibilistic mixed Nash equilibrium. In addition it proposes a MILP formulation to solve the problem of finding a PNE in a Π-game if it exists. The experimental study we led shows that, if the number of actions or types per player increases, the average execution time needed to find one PNE, Π-MNE or a secure strategy increases roughly linearly. However, if the number of players grows, the average execution time increases exponentially. This is observable for all game classes. All the experiments empirically confirm the theoretical results of this paper. Indeed, if the number of actions or types per player increases, the number of total strategies grows only polynomially. However, if the number of players increases, the number of strategies grows exponentially $(n \cdot (|\Theta| \cdot |A|)^n)$.

The normal form representation of an incomplete information game with n players, t types and d actions per player is very costly (n utility functions of size $t^n.d^n$ and a distribution over Θ, i.e., of size t^n) even when the problem involves local interactions only. In order to efficiently represent Π-games with local interactions, we propose in the next chapter, a less costly view of Π-games, namely *min-based polymatrix* Π*-games*. This framework allows, for instance, the compact representation of coordination games under uncertainty where the satisfaction of an agent is high if and only if her strategy is coherent with all of her neighbors, the game being possibly only incompletely known to the agents.

Proofs

Proof of proposition 5.1.
Under the assumption that π is represented by a table of $|\Theta|$ lines, the complexity of computing $\{\bar{\mu}_i(\sigma_i(\theta_i), \theta)\}$ is in $O(|A_{-i}|)$. Therefore complexity of computing $\{\bar{u}_i^{secure}(\sigma_i(\theta_i), \theta_i)\}$ is in $O(|\Theta_{-i}| \times |A_{-i}|)$. Thus, the whole complexity of Algorithm 5.1 is polynomial $O(|\Theta_{-i}| \times |A_{-i}| \times |A_i|)$=$O(|\Theta_{-i}| \times |A|)$. □

Proof of Proposition 5.2.

$$\begin{aligned}
\upsilon' \le \upsilon \quad &\Leftrightarrow \quad \upsilon'_i(\theta_i) \le \upsilon_i(\theta_i), \forall i \in N, \forall \theta_i \in \Theta_i,\\
&\Leftrightarrow \quad \upsilon'_i(a_i|\theta_i) \le \upsilon_i(a_i|\theta_i), \forall i \in N, \forall \theta_i \in \Theta_i, \forall a_i \in A_i,\\
&\Rightarrow \quad \min_{i\in N} \upsilon'_i(a_i|\theta_i) \le \min_{i\in N} \upsilon_i(a_i|\theta_i), \forall i \in N, \forall \theta_i \in \Theta_i, \forall a_i \in A_i,\\
&\Leftrightarrow \quad \upsilon'(a|\theta) \le \upsilon(a|\theta), \forall a \in A, \forall \theta \in \Theta,\\
&\Leftrightarrow \quad 1-\upsilon'(a|\theta) \ge 1-\upsilon(a|\theta), \forall a \in A, \forall \theta \in \Theta,\\
&\Rightarrow \quad \min_{a\in A} \max\big(1-\upsilon'(a|\theta), \mu_i(a,\theta)\big) \ge \min_{a\in A} \max\big(1-\upsilon(a|\theta), \mu_i(a,\theta)\big), \forall a \in A, \forall \theta \in \Theta,\\
&\Leftrightarrow \quad U_i^{pes}(\upsilon', \theta_i) \ge U_i^{pes}(\upsilon, \theta_i), \forall i \in N, \forall \theta_i \in \Theta_i \text{ Using Equation (4.16).}
\end{aligned}$$

□

Proof of proposition 5.3.
Under the assumption that π is represented by a table of $|\Theta|$ lines, the complexity of computing $\{U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)\}$ $\forall i \in N, \forall a_i \in A_i, \forall \theta_i \in \Theta_i$ is in $O(|A_{-i}| \times |\Theta_{-i}|)$ and the complexity of computing $U_i^{pes}(\upsilon, \theta_i)$ is in $O(|A_i| \times |A_{-i}| \times |\Theta_{-i}|) = O(|A| \times |\Theta_{-i}|)$.

Thus the whole complexity of Algorithm 5.3 is polynomial in the size of the Π-game: $O(|A_{\max}| \times |A| \times |\Theta_{-i}|)$ where $|A_{\max}| = \max(|A_1|, ..., |A_i|, ..., |A_n|)$. □

Proof of Proposition 5.4.
Given the definition of the Improve function (Algorithm 5.3), one can prove the following results:

Lemma 5.1 (Improvement function)**.** Let $G = \langle N, A, \Theta, \pi, \mu\rangle$ be a Π-game. Let υ be a joint mixed strategy and $(i \in N,\ \theta_i \in \Theta_i)$ be the player i of type θ_i. Then, the following facts hold:

1. $\textsc{Improve}(G, \upsilon, i, \theta_i)$ is a *normalized* joint mixed strategy.
2. $\textsc{Improve}(G, \upsilon, i, \theta_i) \le \upsilon$.
3. if $\textsc{Improve}(G, \upsilon, i, \theta_i) \prec \upsilon$, then $U_i^{pes}(\textsc{Improve}(G, \upsilon, i, \theta_i), \theta_i) > U_i^{pes}(\upsilon, \theta_i)$.
4. $\forall \upsilon''$ such that $\textsc{Improve}(G, \upsilon, i, \theta_i) \prec \upsilon'' \le \upsilon$, we have:
$$U_i^{pes}(\upsilon'', \theta_i) = U_i^{pes}(\upsilon, \theta_i) < U_i^{pes}(\textsc{Improve}(G, \upsilon, i, \theta_i), \theta_i)$$
5. if $\upsilon' = \textsc{Improve}(G, \upsilon, i, \theta_i)$, then, $\forall j \in N, \forall \theta_j \in \Theta_j$:
   - $U_j^{pes}(a_j, \upsilon'_{-j}, \theta_j) \ge U_j^{pes}(a_j, \upsilon_{-j}, \theta_j)$, $\forall a_j \in A_j$;
   - $U_j^{pes}(\upsilon', \theta_j) \ge U_j^{pes}(\upsilon, \theta_j)$.

Proof of Lemma 5.1.

1. IMPROVE$(G, \upsilon, i, \theta_i)$ only changes (potentially) $\upsilon_i(\theta_i)$. And when it does, we have taken the caution that the corresponding $\upsilon'_i(\theta_i)$ remains normalized.

2. the only potential change in υ' =IMPROVE(G, π, i, θ_i) occurs when:
   $\upsilon'_i(a_i|\theta_i) \leftarrow \min\big(\upsilon_i(a_i|\theta_i), n(U_i^{pes}(\upsilon,\theta_i))^-\big)$.
   Then: $\upsilon'_i(a_i|\theta_i) \le \upsilon_i(a_i|\theta_i)$.
   So: $\upsilon'_i \le \upsilon$ and since $\forall j \ne i, \upsilon'_j = \upsilon'_i$ we have $\Leftrightarrow \upsilon' \le \upsilon$.

3. let $\upsilon' = Improve(G, \upsilon, i, \theta_i)$. Since $\upsilon' \prec \upsilon$, we have:

   (i) $U_i^{Ex-Post}(\upsilon, \theta_i) \le 1$;

   (ii) $A^*_{i,\theta} \ne \varnothing$;

   (iii) $\forall a_i \in A^*_{i,\theta}, \upsilon'_i(a_i|\theta_i) = n(U_i^{Ex-Post}(\upsilon,\theta_i))^-$ then $1 - \upsilon'_i(a_i|\theta_i) > U_i^{Ex-Post}(\upsilon,\theta_i)$.

   Thus:

   - $\forall a_i \in A^*_{i,\theta}$: $\max(1 - \upsilon'_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)) \ge \max(1 - \upsilon_i(a_i|\theta_i), U_i^{pes}(\upsilon, \theta_i))$

     So: $\max(1 - \upsilon'_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)) > U_i^{pes}(\upsilon, \theta_i)$

   - $\forall a_i \in A_i \setminus A^*_{i,\theta_i}$: $U_i^{pes}(a_i, \upsilon_{-i}, \theta_i) \le U_i^{pes}(\upsilon, \theta_i)$

     So, $\max(1 - \upsilon'_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)) \ge U_i^{pes}(\upsilon, \theta_i)$

   In conclusion, $\forall a_i \in A_i$: $\max(1 - \upsilon'_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)) \ge U_i^{pes}(\upsilon, \theta_i)$.

   So, $U_i^{pes}(\upsilon'(\theta_i).\upsilon_{-i}, \theta_i) = \min_{a_i \in A_i} \max(1 - \upsilon'_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)) > U_i^{pes}(\upsilon, \theta_i)$.

   The "else" part is trivial since $\upsilon' \le \upsilon$ and $\upsilon' \nprec \upsilon$ then $\upsilon' = \upsilon$.

4. let $\upsilon' = Improve(G, \upsilon, i, \theta_i)$ then $\upsilon' \prec \upsilon'' \le \upsilon$ implies:

   (i) $\forall a_i \in A_i \setminus A^*_{i,\theta_i}$: $\upsilon''_i(a_i|\theta_i) = \upsilon'_i(a_i|\theta_i) = \upsilon_i(a_i|\theta_i)$;

   (ii) $\forall a_i \in A^*_{i,\theta}$: $\upsilon'_i(a_i|\theta_i) = n(U_i^{pes}(\upsilon,\theta_i))^- < \upsilon'_i(a_i|\theta_i)$;

   (iii) $\exists a_i^* \in A^*_{i,\theta_i}$ s.t, $\upsilon'_i(a_i^*|\theta_i) < \upsilon''_i(a_i^*|\theta_i) \le \upsilon_i(a_i^*|\theta_i)$.

   Thus, $1 - \upsilon'_i(a_i|\theta_i) < U_i^{pes}(\upsilon,\theta_i)^+$ and, $1 - \upsilon'_i(a_i|\theta_i) \le U_i^{pes}(\upsilon,\theta_i)$.

   Since, $\forall a_i \in A^*_{i,\theta_i}, U_i^{pes}(a_i, \upsilon_{-i}, \theta_i) \le U_i^{pes}(\upsilon, \theta_i)$

   Then, $\forall a_i \in A^*_{i,\theta_i}, \max(1 - \upsilon''_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)) \le U_i^{pes}(\upsilon, \theta_i)$

   $\min_{a_i \in A^*_{i,\theta_i}} \max(1 - \upsilon''_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i)) = U_i^{pes}(\upsilon, \theta_i)$

   Thus, $U_i^{pes}(\upsilon'', \theta_i) = U_i^{pes}(\upsilon, \theta_i)$ and since $U_i^{pes}(\upsilon', \theta_i) > U_i^{pes}(\upsilon, \theta_i)$

   Then, $U_i^{pes}(\upsilon'', \theta_i) = U_i^{pes}(\upsilon, \theta_i) < U_i^{pes}(\upsilon', \theta_i)$.

5. - we have:

$$U_j^{pes}(a_j, \upsilon'_{-j}, \theta_j) = \min_{\theta_{-j} \in \Theta_{-j}, a_{-j} \in A_{-j}} \max(1 - \pi(\theta_{-j}|\theta_j), 1 - \upsilon'_{-j}(a_{-j}|\theta_{-j}), \mu_i(a_i.a_{-j}, \theta_j.\theta_{-j}))$$

and since, $\upsilon' \prec \upsilon$ then, $\upsilon'_{-j} \prec \upsilon_{-j}$:

$$U_j^{pes}(a_j, \upsilon'_{-j}, \theta_j) \geq \min_{\theta_{-j} \in \Theta_{-j}, a_{-j} \in A_{-j}}$$
$$\max(1 - \pi(\theta_{-j}|\theta_j), 1 - \upsilon_{-j}(a_{-j}|\theta_{-j}), \mu_i(a_i.a_{-j}, \theta_j.\theta_{-j}))$$
$$U_j^{pes}(a_j, \upsilon'_{-j}, \theta_j) \geq U_j^{pes}(a_j, \upsilon'_{-j}, \theta_j)$$

- we have: $U_i^{pes}(\upsilon', \theta_i) = \min_{a_i \in A_i} \max(1 - \upsilon'_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon'_{-i}, \theta_i))$

  then: $U_i^{pes}(\upsilon', \theta_i) \geq \min_{a_i \in A^*_{i,\theta_i}} \max(1 - \upsilon'_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon'_{-i}, \theta_i))$

  since: $\forall a_i \in A^*_{i,\theta_i}$, $\upsilon'_i(a_i|\theta_i) < \upsilon_i(a_i|\theta_i)$

  thus: $U_i^{pes}(\upsilon', \theta_i) \geq \min_{a_i \in A^*_{i,\theta_i}} \max(1 - \upsilon_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i))$

  then: $U_i^{pes}(\upsilon', \theta_i) \geq \min_{a_i \in A_i} \max(1 - \upsilon_i(a_i|\theta_i), U_i^{pes}(a_i, \upsilon_{-i}, \theta_i))$
  $U_i^{pes}(\upsilon', \theta_i) \geq U_i^{pes}(\upsilon, \theta_i)$.

□

Properties 1, 2 and 3 show that Improve(G, π, i, θ_i) is at least as specific as υ and strictly improves the utility of player i of type θ_i or leaves υ unchanged. Property 4 shows that Improve(G, π, i, θ_i) is the least specific improvement of υ to player i of type θ_i, when it changes υ. Property 5 shows that Improve(G, π, i, θ_i) does not decrease the utilities of other players than i of type θ_i.

Let us note that the outer loop of Algorithm 5.2 requires $n \times |\Theta_{\max}|$ calls to the function Improve per iteration. Then, note that by Lemma 5.1, property 3, υ' =Improve$(G, \upsilon, i, \theta_i)$ is either more specific than υ, or equal to υ and that υ' can only differ from υ in its component $\upsilon'_i(\theta_i)$. Finally, note that the number of possible *strict* improvements of the mixed strategy $\upsilon_i(\theta_i) : A_i \to \Delta$ of a player i of type θ_i is upper bounded by $|\Delta| \times |A_i|$: for each improvement, one of the $|A_i|$ coordinates of the mixed strategy is decreased, and each coordinate belongs to Δ.

Therefore, Algorithm 5.2 converges after at most $n \times |\Theta_{\max}| \times |\Delta| \times |A_{\max}|$ calls to the Improve procedure, which itself takes time polynomial in the size of the expression of G.

Thus, the complexity of Algorithm 5.2 is $O(|N| \times |\Theta_{\max}| \times |\Delta| \times |A_{max}| \times |A_{\max}| \times |A| \times |N|) = O(|N|^2 \times |\Theta_{\max}| \times |\Delta| \times |A_{\max}|^2 \times |A|)$ where $|A_{\max}| = \max(|A_1|, ..., |A_i|, ..., |A_n|)$ and $|\Theta_{\max}| = \max(|\Theta_1|, ..., |\Theta_i|, ..., |\Theta_n|)$. □

Proof of proposition 5.5.
Since υ^* has been obtained after convergence of Algorithm 5.2, it verifies: $\upsilon^* =$ Improve$(G, \upsilon, i, \theta_i), \forall i \in N, \forall \theta_i \in \Theta_i$. This implies that $\forall i \in N, \forall \theta_i \in \Theta_i, \forall \upsilon_i(\theta_i)$,

$U_i^{pes}(\upsilon^*, \theta_i) \geq U_i^{pes}(\upsilon_i(\theta_i).\upsilon^*_{-i}, \theta_i)$, that is, υ^* is possibilistic mixed Nash equilibrium. □

Chapter 6

Ordinal Polymatrix Games with Incomplete Information

6.1 Introduction

In the present chapter, we study Π-games where the interactions between players are pairwise and the utility of a player depends on her neighborhood and not on all others players in the Π-game. We define the new framework of *min-based polymatrix* Π-*games*, which allows us to concisely specify Π-games with local interactions. This framework allows, for instance, the compact representation of coordination games under possibilistic uncertainty.

In the following, Section 6.2 defines the general representation framework that we propose: min-based polymatrix Π-games, and applies it to the coordination game example. Section 6.3 shows that any 2-player Π-game can be transformed into an equivalent *min-based polymatrix* game. This result is the qualitative counterpart of Howson and Rosenthals's theorem (Howson et al., 1974) linking 2-player Bayesian games to polymatrix games. Furthermore, as soon as a simple condition on the coherence of the players' knowledge about the world is satisfied, any polymatrix Π-game can be transformed in polynomial time into an equivalent min-based and *complete* information polymatrix game. Finally, Section 6.4 proposes a MILP formulation of the problem of deciding whether a polymatrix Π-game admits a PNE. Experimental results are reported in Section 6.5. All proofs are in the end of this chapter.

The main results of this chapter are published in (Ben Amor et al., 2020a) and (Ben Amor et al., 2020b).

6.2 Polymatrix Π-games

The normal form representation of an incomplete information game with n players, t types and d actions per player is very costly (n utility functions of size $t^n.d^n$ and distribution over Θ, i.e., of size t^n) even when the problem involves local interactions only. In example 4.1, when the type vector is fixed, the satisfaction of one player is the minimum of what this player gets in a series of two-player games like the one presented in Table 2.1. To efficiently represent such games, we now define polymatrix Π-games as min-based polymatrix games where each local game is a Π-game. Such a game can be much more compact than the equivalent SNF Π-game.

Definition 6.1 (Polymatrix Π-game)**.**
A polymatrix Π-game *is a tuple* $G = \langle N, E, A, \Theta, \mu, \pi\rangle$ *where:*

- $N = \{1, \dots, n\}$ *is the set of n players;*
- E *is a set of pairs of distinct players of* N*;*
- $A = \times_{i\in N} A_i$*, where* A_i *is the set of actions of player* i*;*
- $\Theta = \times_{i\in N}\Theta_i$*, where* Θ_i *is the set of types of player* i*;*
- $\mu = \{(\mu_{i,j}, \mu_{j,i}), \{i, j\} \in E\}$*, a set of pairs of utility functions on* $A_i \times A_j \times \Theta_i \times \Theta_j$ *taking their values in* Δ*;*
- $\pi = \{\pi_{i,j} : \theta_i \times \theta_j \mapsto \Delta, \{i, j\} \in E\}$ *a set of pairwise possibility distributions on the* $\Theta_i \times \Theta_j$ *product sets.*

In other terms, a polymatrix Π-game is a polymatrix game where each local game is a Π-game $\langle\{i, j\}, A_i \times A_j, \Theta_i \times \Theta_j, \pi_{i,j}, \{\mu_{i,j}, \mu_{j,i}\}\rangle$.

The condition of "common knowledge" is less natural in the present context of a series of local games than in SNF Π-games – here, we assume that the knowledge of each local Π-game is common to the two players involved in, but not to the full community of players. Each player is "myopic" and her knowledge is restricted to what she knows about her neighborhood. The knowledge of player i about the configurations of types of the global incomplete information game is:

$$\pi_i(\theta) = \min_{j\in N,\{i,j\}\in E} \pi_{i,j}(\theta_i.\theta_j). \tag{6.1}$$

We thus replace the condition of "common knowledge" by a condition of "coherent knowledge": there should be a π on Θ from which the $\pi_{i,j}$'s derive:

Assumption 6.1. $\exists \pi : \Theta \mapsto \Delta$ *such that:*

$$\forall i, j \in E,\ \ \pi_{i,j}(\theta_i.\theta_j) = \max_{\theta_{-\{i,j\}}} \pi(\theta_i.\theta_j.\theta_{-\{i,j\}}). \tag{6.2}$$

π is unknown, but one knows that $\pi_{i,j}(\theta_i.\theta_j) = \max_{\theta_{-\{i,j\}}} \pi(\theta_i.\theta_j.\theta_{-\{i,j\}})$.

Let us now study the global utility functions of the players. Each joint type $\theta \in \Theta$ defines a min-based polymatrix game. The global utility of player i for the joint action a when the configuration of types is θ is thus:

Definition 6.2 (Global utility in a polymatrix Π-game)**.** *Let $G = \langle N, E, A, \Theta, \mu, \pi\rangle$ be a polymatrix* Π*-game. The global utility of player i for the joint action a when the configuration of types is θ is:*

$$\mu_i(a, \theta) = \min_{j \in N, \{i,j\} \in E} \mu_{i,j}(a_i.a_j, \theta_i.\theta_j). \tag{6.3}$$

If one considers all the types and the associated distribution, the polymatrix Π-game (compactly) represents the SNF Π-game $\langle N, A, \Theta, \pi, \mu\rangle$. Then from the definition of the utility of a joint action/strategy for a player in a Π-game (Definition 4.4) we have:

Definition 6.3 (Utility of a strategy in a polymatrix Π-game)**.** *Let $G = \langle N, E, A, \Theta, \mu, \pi\rangle$ be a polymatrix* Π*-game. The pessimistic utility of player i of type θ_i for the joint strategy σ is:*

$$U_i^{pes}(\sigma, \theta_i) = \min_{\theta_{-i} \in \Theta_{-i}} \max\Big(1 - \pi(\theta_{-i}|\theta_i), \mu_i\big(\sigma(\theta_i.\theta_{-i}), \theta\big)\Big). \tag{6.4}$$

Definitions 6.1, 6.2 and 6.3 constitute, to the best of our knowledge, the first attempt to introduce a way to cope with uncertainty in polymatrix games and more generally in ordinal hypergraphical games.

Notice that in Definition 6.2, we compute, for each player and each type configuration, the utility of a player in the configuration and then compute the pessimistic utility of the player. We could have proceeded in the other way: compute the pessimistic utility in each local game and then aggregate the pessimistic utilities. The theory is fortunately sound: the two approaches coincide. We can indeed prove that:

Proposition 6.1. *The pessimistic utility of player i of type θ_i for the joint strategy σ is equal to:*

$$U_i^{pes}(\sigma, \theta_i) = \min_{j, \{i,j\} \in E} \min_{\theta_j \in \Theta_j} \max\Big(1 - \pi_{i,j}(\theta_j|\theta_i), \mu_{i,j}(\sigma(\theta_i.\theta_j), \theta_i.\theta_j)\Big) \tag{6.5}$$

As far as spatial complexity is concerned, it is easy to show that a polymatrix Π-game can be exponentially more compact than its standard normal form equivalent. Consider our running example:

Example 6.1 (Cont. Example 4.1)**.** *The SNF* Π*-game described in Table 4.2 is captured by the polymatrix* Π*-game* $G = \langle N, E, A, \Theta, \mu, \pi \rangle$ *(same players, same actions and same types) where* E *is the neighborhood relation of the original game and where the utility function* $\mu_{i,j}$ *of player* i *(w.r.t.* $j \in N$*,* $(i,j) \in E$*) is:*

- $\mu_{i,j}(x.y, \theta_i.\theta_j) = \beta_{i,x}$;
- $\mu_{i,j}(y.x, \theta_i.\theta_j) = \beta_{i,y}$;
- $\mu_{i,j}(x.x, r_i.\theta_j) = \delta$;
- $\mu_{i,j}(x.x, \overline{r}_i.\theta_j) = \alpha_{i,x}$;
- $\mu_{i,j}(y.y, \theta_i.\theta_j) = \alpha_{i,y}$.

This polymatrix game contains $2 \cdot |E|$ *possibility distributions of size* $2 \cdot 2$*, and* $2 \cdot |E|$ *utility functions of size* $2 \cdot 2 \cdot 2 \cdot 2$*, while the original SNF game involves one possibility distribution of size* 2^n *and* n *utility functions of size* $2^n \cdot 2^n$*, whatever the connectivity of the neighborhood graph.*

As far as time complexity is concerned, deciding whether a polymatrix Π-game admits a pure Nash equilibrium is an NP-hard problem because (i) any 2-player Π-game is a (degenerated) polymatrix game and (ii) deciding whether a 2-player Π-game admits a pure Nash equilibrium is an NP-complete problem. We show in the following section that the question is "only" NP-complete for polymatrix Π-games. In other terms, the possible gain in compactness does not increase complexity.

6.3 From Polymatrix Π-games to Min-based Polymatrix Games

In the following, we show that any polymatrix Π-game can be transformed into a min-based polymatrix game, the Nash equilibria of which are in bijection with the ones of the original game. To this extent, we first show that any 2-player Π-game can be transformed into an equivalent min-based polymatrix game. This result can be viewed as a qualitative counterpart of Howson and Rosenthals's theorem linking 2-player Bayesian games to polymatrix games (Howson et al., 1974).

6.3.1 Transforming a 2-player Π-game into a Min-based Polymatrix Game

In section 3.6.2, we have detailed the transformation of a 2-player Bayesian game into an equivalent polymatrix game. In this section, we follow (Howson et al., 1974) and propose a qualitative transformation of a 2-player Π-game into an equivalent min-based polymatrix game (Definition 2.4). The idea is to consider as many players as the number of pairs $\{(i,\theta_i)_{i\in N}\}$, i.e., the number of players is equal to $|\Theta_1| + |\Theta_2|$. In fact, each player (i,θ_i) has A_i as a set of available actions. For each joint strategy $a \in A$, the utility of player (i,θ_i) in the game $\{(i,\theta_i),(j,\theta_j)\} \in E$ in the polymatrix game is equal to the utility of the joint action $a \in A$, to player i of type θ_i where j is of type θ_j.

Definition 6.4 (Polymatrix representation of a 2-player Π-game)**.** *Given a 2-player* Π*-game* $G = \langle N = \{1,2\}, A, \Theta, \pi, \mu\rangle$*,* $\tilde{G} = \langle \tilde{N}, \tilde{E}, \tilde{A}, \tilde{\mu}\rangle$ *is the min-based polymatrix game where:*

- $\tilde{N} = \{(i,\theta_i), \forall i \in \{1,2\}, \forall \theta_i \in \Theta_i\}$*;*
- $\tilde{E} = \{\{(i,\theta_i),(j,\theta_j)\}, i \neq j\}$*;*
- $\tilde{A}_{(i,\theta_i)} = A_i,\ \forall (i,\theta_i) \in \tilde{N}$*;*
- $\tilde{\mu}_{(i,\theta_i),(j,\theta_j)}(a) = \max\Big(1 - \pi(\theta_j|\theta_i), \mu_i(a,\theta_i.\theta_j)\Big)$,
  $\forall a \in \tilde{A}, \forall i,j \in N,\ s.t.,\ i \neq j, \forall \theta_i \in \Theta_i, \forall \theta_j \in \Theta_j$*.*

Intuitively, each combination of types $(\theta_1.\theta_2)$ in G is mapped to an edge $\{(1,\theta_1),(2,\theta_2)\} \in E$.

Definition 6.5 (Transformation of a pure strategy)**.** *Let* $G = \langle N, A, \Theta, \pi, \mu\rangle$ *be a 2-player* Π*-game,* $\tilde{G} = \langle \tilde{N}, \tilde{E}, \tilde{A}, \tilde{\mu}\rangle$ *its min-based polymatrix representation and* σ *be a pure strategy in* G*. We define* a^σ *as the joint action in* $\tilde{A}$ *such that:*

$$a^\sigma_{(i,\theta_i)} = \sigma_i(\theta_i)$$

Based on Definition 2.4, we have:

$$\tilde{\mu}_{(i,\theta_i)}(a^\sigma) = \min_{\theta_j\in\Theta_j} \tilde{\mu}_{(i,\theta_i),(j,\theta_j)}(a^\sigma_{(i,\theta_i)}.a^\sigma_{(j,\theta_j)}). \tag{6.6}$$

We can then show that the utilities of σ in G and a^σ in $\tilde{G}$ are equal:

Proposition 6.2. *Let* $G = \langle N, A, \Theta, \pi, \mu\rangle$ *be a 2-player* Π*-game,* $\tilde{G} = \langle \tilde{N}, \tilde{E}, \tilde{A}, \tilde{\mu}\rangle$ *its polymatrix representation. It holds that, for any pure strategy* σ *of* G*:* $\forall i \in \{1,2\}, j \neq$

$i, (\theta_i.\theta_j) \in \Theta$:

$$U_i^{pes}(\sigma, \theta_i) = \tilde{\mu}_{(i,\theta_i)}(a^\sigma) = \min_{\theta_j \in \Theta_j} \tilde{\mu}_{(i,\theta_i),(j,\theta_j)}(a^\sigma_{(i,\theta_i)}.a^\sigma_{(j,\theta_j)}). \tag{6.7}$$

Finally, we can show that the PNE are the same in both games:

Proposition 6.3. *σ is a PNE in the 2-player Π-game $G = \langle N, A, \Theta, \pi, \mu \rangle$ iff a^σ is a PNE in its polymatrix representation $\tilde{G} = \langle \tilde{N}, \tilde{E}, \tilde{A}, \tilde{\mu} \rangle$.*

Example 6.2 (Cont. Example 4.1)**.** *Given the coordination Π-game detailed in Table 6.1.*

		Player 2			
		r_2		$\overline{r}_2$	
Player 1		x	y	x	y
r_1	x	0.9 , 0.9	0.3 , 0.3	0.9 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(r_1.r_2) = 1$		$\pi(r_1.\overline{r}_2) = 0.1$	
		x	y	x	y
$\overline{r}_1$	x	0.8 , 0.9	0.3 , 0.3	0.8 , 0.7	0.3 , 0.3
	y	0.2 , 0.2	0.7 , 0.8	0.2 , 0.2	0.7 , 0.8
		$\pi(\overline{r}_1.r_2) = 0.1$		$\pi(\overline{r}_1.\overline{r}_2) = 1$	

Table 6.1: A coordination Π-game between two players (with two types per player) where $\Delta = \{0, 0.1, 0.2, 0.3, 0.7, 0.8, 0.9, 1\}$.

The equivalent polymatrix game $\tilde{G}$ of the two-player coordination Π-game depicted in Table 6.1 contains four players:
$\tilde{N} = \{(1, r_1), (1, \overline{r}_1), (2, r_2), (2, \overline{r}_2)\}$, *all having the same set of actions $\{x, y\}$.*

Figure 6.1 presents the equivalent polymatrix game $\tilde{G}$. $\tilde{G}$ contains 4 local games (4 edges). The utilities of each player in the equivalent game are detailed in Table 6.2.

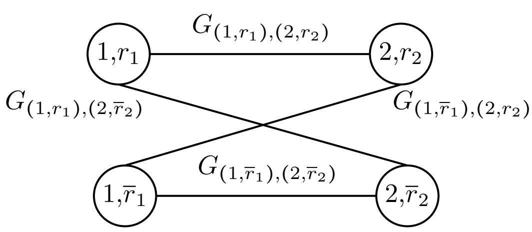

Figure 6.1: The polymatrix game equivalent of the 2-player Π-game depicted in Table 4.2.

The joint action $a^\sigma = (x.x.x.x)$ of $\tilde{G}$ corresponds to σ^x in G. The global utilities of a^σ in $\tilde{G}$ are:

$2, r_2$

		x	y
$1, r_1$	x	0.9, 0.9	0.3, 0.3
	y	0.2, 0.2	0.8, 0.8

(a) $G_{(1,r_1),(2,r_2)}$.

$2, \overline{r}_2$

		x	y
$1, r_1$	x	0.9, 0.9	0.9, 0.9
	y	0.9, 0.9	0.9, 0.9

(b) $G_{(1,r_1),(2,\overline{r}_2)}$.

$2, r_2$

		x	y
$1, \overline{r}_1$	x	0.9, 0.9	0.9, 0.9
	y	0.9, 0.9	0.9, 0.9

(c) $G_{(1,\overline{r}_1),(2,r_2)}$.

$2, \overline{r}_2$

		x	y
$1, \overline{r}_1$	x	0.7, 0.7	0.3, 0.3
	y	0.2, 0.2	0.8, 0.8

(d) $G_{(1,\overline{r}_1),(2,\overline{r}_2)}$.

Table 6.2: The utility degrees of the min-based polymatrix game of Figure 6.1.

$$
\begin{aligned}
&\mu_{(1,r_1)}(a^\sigma) = \min\big(0.9, \max(1-0.1, 0.9)\big) = 0.9,\\
&\mu_{(1,\overline{r}_1)}(a^\sigma) = \min\big(1-0.1, 0.7\big) = 0.7,\\
&\mu_{(2,r_2)}(a^\sigma) = \min\big(0.9, \max(1-0.1, 0.9)\big) = 0.9,\\
&\mu_{(2,\overline{r}_2)}(a^\sigma) = \min\big(\max(1-0.1, 0.7), 0.7\big) = 0.7.
\end{aligned}
$$

It can be checked that a^σ is a PNE in the equivalent min-based polymatrix game.

Notice that the nodes of the polymatrix game $\tilde{G}$ represent the types in $\Theta_1 \cup \Theta_2$, that the edges of $\tilde{G}$ correspond to types combinations θ in the Π-game G and that the graph of $\tilde{G}$ is bipartite. More generally, the transformation of G to $\tilde{G}$ is polynomial:

Proposition 6.4. *The transformation of a 2-player Π-game G into an equivalent polymatrix representation $\tilde{G}$ is in $O(d^2 \cdot t^2)$ where t (resp. d) is the maximal number of types (resp. actions) per player.*

Therefore, the transformation of G into $\tilde{G}$ is polynomial in time and space, contrarily to the transformation into an SNF game proposed in Chapter 4.

6.3.2 From Polymatrix Π-games to Min-based Polymatrix Games with Complete Information

When it comes to general polymatrix Π-games, we can show that polymatrix Π-games are not more expensive than classical (complete information) min-based polymatrix games. Indeed, recall that for any polymatrix Π-game, we have shown in Proposition 6.1 that:

$$
U_i^{pes}(\sigma, \theta_i) = \min_{\substack{j \in N, \{i,j\} \in E \\ \theta_j \in \Theta_j}} \max\Big(1 - \pi_{i,j}(\theta_j | \theta_i), \mu_{i,j}(\sigma_i(\theta_i).\sigma_j(\theta_j), \theta_i.\theta_j)\Big)
$$

We now reuse, for every pairs of players, the transformation of the previous Section, transforming a 2-player Π-game into an equivalent min-based polymatrix game:

Definition 6.6. *Given a polymatrix* Π*-game* $G = \langle N, E, A, \Theta, \mu, \pi \rangle$*, the min-based polymatrix game is* $\tilde{G} = \langle \tilde{N}, \tilde{E}, \tilde{A}, \tilde{U} \rangle$ *where:*

- $\tilde{N} = \{(i, \theta_i), \forall i \in N, \forall \theta_i \in \Theta_i\}$;
- $\tilde{E} = \{((i, \theta_i), (j, \theta_j)), i \neq j, \theta_i \in \Theta_i, \theta_j \in \Theta_j\}$;
- $\tilde{A}_{(i,\theta_i)} = A_i, \ \forall (i, \theta_i) \in \tilde{N}$;
- $\tilde{U}_{(i,\theta_i),(j,\theta_j)}(a_i.a_j) = \max\Big(1 - \pi_{i,j}(\theta_j|\theta_i), \mu_i(a_i.a_j, \theta_i.\theta_j)\Big)$,
  $\forall (i,j) \in E, \ \forall a_i.a_j \in A_i \times A_j, \ \forall \theta_i.\theta_j \in \Theta_i \times \Theta_j$.

Using this definition and Proposition 6.1, it follows that:

Proposition 6.5. *Let* G *be a* n*-player polymatrix* Π*-game,* $\tilde{G}$ *be the corresponding min-based polymatrix representation,* σ *be a pure strategy for* G *and* a^σ *its transformation according to Definition 6.5. It holds that* $\forall i \in N, \forall \theta_i \in \Theta_i$,

$$U_i^{pes}(\sigma, \theta_i) = \min_{\substack{j \in N, \{i,j\} \in E \\ \theta_j \in \Theta_j}} \tilde{U}_{(i,\theta_i),(j,\theta_j)}(a^\sigma_{(i,\theta_i)}.a^\sigma_{(j,\theta_j)}). \tag{6.8}$$

So, the utility of σ in G is equal to the utility of a^σ in $\tilde{G}$.

Since the set of pure strategies in G is bijectively related to the action set $\tilde{A}$ of $\tilde{G}$, the following proposition holds:

Proposition 6.6. σ *is a PNE in the* n*-player polymatrix* Π*-game* $G = \langle N, E, A, \Theta, \mu, \pi \rangle$ *iff* a^σ *is a PNE in the min-based polymatrix game* $\tilde{G} = \langle \tilde{N}, \tilde{E}, \tilde{A}, \tilde{U} \rangle$.

6.3.3 Complexity

We can show that the size of the transformed min-based polymatrix game $\tilde{G}$ is in $O(|E| \cdot d^2 \cdot t^2)$ where t (resp. d) is the maximal number of types (resp. possible actions) per player: each local Π-game (we have $|E|$ local Π-games) is transformed using the transformation described in Section 6.3.1with a complexity in $O(d^2 \cdot t^2)$. Hence, the global computation cost is $O(|E| \cdot d^2 \cdot t^2)$ in time and space. Now, recall that deciding the existence of a PNE in a 2-player Π-game is an NP-hard problem. Because any Π-game can be transformed in polytime and space into an equivalent min-based polymatrix game, we first derive that:

Proposition 6.7. *Determining whether there exists a PNE in a min-based polymatrix game is NP-complete. The result holds even when the graph is bipartite.*

A second consequence is that deciding whether a polymatrix Π-game admits a PNE is NP-complete (but not harder). The problem is NP-hard because deciding whether a 2-player Π-game admits a PNE is NP-complete and belongs to NP since the previous transformation allows to solve it through a polytime reduction to a polymatrix min-based game.

Proposition 6.8. *Determining whether there exists a PNE in a polymatrix* Π*-game is NP-complete.*

Hence, the possible gain in compactness offered by polymatrix Π-games w.r.t. SNF Π-games comes with no increase in theoretical complexity.

6.4 Finding a Pure Nash Equilibrium in Min-based Polymatrix Games: a MILP Formulation

The basic computational problem is the search for an equilibrium in min-based polymatrix games since every polymatrix Π-game can be transformed into an equivalent min-based polymatrix game. Taking advantage of the efficiency of modern solvers, we propose a *Mixed Integer Linear Programming* (MILP) formulation of the problem.

- the main decision variables are boolean variables encoding the strategy searched for: each σ_{i,a_i} is a boolean variable indicating whether action a_i is prescribed for player i:

$$\forall i \in N, \forall a_i \in A_i, \sigma_{i,a_i} \in \{0,1\};$$

- utilities are encoded by continuous variables (we assume $\Delta = [0,1]$).
  - $U_{i,a_i,j}$ is a continuous variable indicating the utility of player i playing action a_i given the strategy of player j:

$$\forall \{i,j\} \in E, \forall a_i \in A_i, U_{i,a_i,j} \in [0,1];$$

  - U_{i,a_i} is a continuous variable indicating the utility of player i playing action a_i:

$$\forall i \in N, \forall a_i \in A_i, U_{i,a_i} \in [0,1].$$

We will also use the following boolean variables to constrain the U_{i,a_i} to be equal to the $\min_{j,\{i,j\}\in E} U_{i,a_i,j}$ (and not only lower than):

$$\forall i,j \in N, \text{ s.t., } \{i,j\} \in E, \forall a_i \in A_i, V_{i,a_i,j} \in \{0,1\}.$$

Hence, the MILP will contain the following constraints:

- $\forall i \in N$,
$$\sum_{a_i \in A_i} \sigma_{i,a_i} = 1. \tag{6.9}$$

- $\forall i \in N, \forall a_i, a_i' \in A_i$, s.t., $a_i \neq a_i'$,
$$U_{i,a_i} - U_{i,a_i'} \geq \sigma_{i,a_i} - 1. \tag{6.10}$$

- $\forall i, j \in N$, s.t., $\{i,j\} \in E, \forall a_i \in A_i$,
$$U_{i,a_i,j} = \sum_{a_j \in A_j} \mu_{(i,j)}(a_i.a_j) \times \sigma_{j,a_j}. \tag{6.11}$$

- $\forall i, j \in N$, s.t., $\{i,j\} \in E, \forall a_i \in A_i$,
$$U_{i,a_i} \leq U_{i,a_i,j}. \tag{6.12}$$

- $\forall i, j \in N$, s.t., $\{i,j\} \in E, \forall a_i \in A_i$,
$$U_{i,a_i} + V_{i,a_i,j} \geq U_{i,a_i,j}. \tag{6.13}$$

- $\forall i \in N, \forall a_i \in A_i$,
$$\sum_{j,\{i,j\} \in E} (1 - V_{i,a_i,j}) = 1. \tag{6.14}$$

- constraints (6.9) ensure that the strategy σ searched for specifies exactly one action a_i for each player i;

- constraints (6.10) require that the strategy built (the σ_{i,a_i} which are set to 1) is a PNE: when $\sigma_{i,a_i} = 1$, it writes $U_{i,a_i} \geq U_{i,a_i'}$, and thus requires that player i has no incentive to deviate from a_i. When action a_i is not chosen for player i, ($\sigma_{i,a_i} = 0$) the constraint is always satisfied ($U_{i,a_i} - U_{i,a_i'}$ is always greater than -1);

- constraints (6.12), (6.13) and (6.14) implement Equation (2.13). Constraints (6.12) ensure that the utility of player i playing a_i is lower than the minimum of utilities in local games played with players $j \in N$, i.e., the $\mu_{(i,j)}(a_i.a_j)$;

- constraints (6.13) and (6.14) ensure that U_{i,a_i} is equal to the above minimum. Whenever $V_{i,a_i,j} = 1$, Equation (6.13) holds, and Equation (6.14) ensures that (6.13) is an equality for a single j (minimizing $U_{i,a_i,j}$).

Let us denote d the maximal number of actions of any player in the polymatrix game and b the maximal number of local games in which a player can be involved. The MILP formulation contains:

- $O(n \cdot d \cdot b)$ continuous variables $U_{i,a_i,j}$;
- $O(n \cdot d)$ continuous variables U_{i,a_i};
- $O(n \cdot d)$ boolean variables σ_{i,a_i};
- $O(n \cdot d \cdot b)$ boolean variables $V_{i,a_i,j}$;
- $O(n)$ constraints (6.9), each involving $O(d)$ variables;
- $O(n \cdot d^2)$ constraints (6.10), each involving 3 variables;
- $O(n \cdot b \cdot d)$ constraints (6.11), each involving $O(d)$ variables;
- $O(n \cdot b \cdot d)$ constraints (6.12), each involving 2 variables;
- $O(n \cdot b \cdot d)$ constraints (6.13), each involving 3 variables;
- $O(n \cdot d)$ constraints (6.14), each involving $O(b)$ variables.

The MILP can then be easily encoded in a matrix of size $O(n^2 \cdot b \cdot d^2 \cdot (b + d))$.

6.5 Experimental Study

The goal of the first part of the experimental study is to compare the efficiency of the polymatrix encoding (PE) of 2-player Π-games to the one of the direct encoding (DE) (proposed in Chapter 5), and beyond, to prove that the resolution of polymatrix Π-games is not out of reach.

6.5.1 A Generator of a Polymatrix Π-game

To conduct our experimental study, we use the Π-game generator proposed in Section 5.5.1 to generate polymatrix Π-games. First, it takes as input the class of the game, the number of degrees in Δ, the number of players n, the number of types per player and if necessary, the number of actions per player. Then, for each player, we affect randomly the set of her neighbors. Finally, for each pair of players i and j such that $\{i, j\} \in E$, we generate a Π-game between i and j.

6.5.2 Experimental Protocol

In our experiments, we study the results of the class of coordination games available on GAMUT: *Battle Of The Sexes*, *Collaboration Game* and *Minimum Effort*

Game (Vaughan, 2004).

We start by studying the equivalent transformed min-based polymatrix games from 2-player Π-games. We varied, for each player, the number of types from 2 to 10. Then, according to the game's parameters, we varied just for minimum effort games the number of actions of each player (since in the battle of the sexes games both players have just 2 actions and collaboration games the number of actions of each player is equal to the number of players in the game). Then, we fixed the number of degrees Δ to 5, i.e., $\Delta = \{0, 0.25, 0.5, 0.75, 1\}$.

The second part of the experiments was dedicated to polymatrix Π-games. We generated coordination games with different numbers of players. We varied the number of players from 5 to 80. Then, we generated random interactions between players (ensuring that the interaction graph was connected). Then, for each edge, i.e., interaction, we generated a Π-game between 2 players using the Π-game generator proposed in (Ben Amor et al., 2019b). We varied the number of types from 2 to 9 and the number of actions (for minimum effort game) from 2 to 10. Then we computed the average execution time needed to find a PNE by transforming the original game into its equivalent min-based polymatrix game (using Definition 6.6) and solving the MILP of the latter. Notice that, the equivalent min-based polymatrix game contains $n \cdot t^2$ players where t is the maximal number of types per player.

Furthermore, for each combination of the parameters, we generated 100 different instances and we measured the average time necessary to get a PNE by solving the MILP proposed in Section 6.4 (we denote this approach DE, for direct encoding) and by transforming the Π-game into an equivalent min-based polymatrix game and solving the above MILP of the equivalent polymatrix game proposed in Definition 6.4 (we denote this approach PE, for polymatrix encoding). All experiments were conducted on an Intel Xeon E5540 processor and 64GB RAM workstation. We used CPLEX (CPLEX, 2009) as a MILP solver.

6.5.3 Experimental Results

6.5.3.1 Direct Encoding vs Polymatrix Encoding

Table 6.3 presents the average of execution times (in seconds) needed by direct encoding and polymatrix encoding to find one PNE, for 3 games classes. In the experiment reported in Table 6.3, we fixed the number of actions per player to 2, i.e., $|A_i| = 2$ and we varied the number of types $|\Theta_i|$ from 2 to 10.

Table 6.4 presents the average of execution times (in seconds), to find one PNE in a minimum effort game (we tested just for minimum effort game since the numbers of

$\|\Theta_i\|$	2	3	4	5	6	7	8	9	10	
battle of the sexes	0.12	**0.02**	0.15	0.03	0.22	**0.05**	0.22	**0.08**	0.26	DE
	0.02	**0.02**	**0.03**	**0.02**	**0.04**	0.06	**0.06**	0.10	**0.09**	PE
collaboration game	0.12	**0.02**	0.15	0.04	0.52	**0.75**	1.33	1.52	2.08	DE
	0.02	**0.02**	**0.03**	**0.03**	**0.35**	0.77	**1.12**	**1.50**	**1.89**	PE
minimum effort game	0.12	**0.02**	0.15	**0.02**	0.22	**0.04**	0.22	**0.07**	0.25	DE
	0.02	**0.02**	**0.03**	**0.02**	**0.04**	0.05	**0.05**	0.09	**0.08**	PE

Table 6.3: Average execution time (s) of direct encoding (DE) and polymatrix encoding (PE), $|A_i| = 2$.

$\|A_i\|$	2	3	4	5	6	7	8	9	10	
$\|\Theta_i\| = 4$	**0.07**	0.08	0.23	0.19	0.19	**0.19**	0.24	0.26	0.36	DE
	0.07	**0.07**	**0.18**	**0.15**	**0.18**	0.20	**0.23**	**0.23**	**0.33**	PE
$\|\Theta_i\| = 7$	**0.14**	0.16	0.18	0.28	0.33	**0.39**	**0.39**	0.53	**0.73**	DE
	0.14	**0.14**	**0.16**	**0.26**	**0.28**	**0.39**	0.40	**0.51**	0.78	PE
$\|\Theta_i\| = 10$	0.14	0.28	**0.31**	0.48	**0.51**	**0.67**	0.72	**1.11**	**3.28**	DE
	0.13	**0.26**	**0.31**	**0.44**	**0.51**	0.74	**0.70**	1.16	3.32	PE

Table 6.4: Average execution time (s) of direct encoding (DE) and polymatrix encoding (PE) for minimum effort game.

players and actions per player in the battle of the sexes (resp. collaboration game) game is equal to 2). We varied the number of types $|\Theta_i|$ from 2 to 10 and we varied the number of actions per player $|A_i|$ from 2 to 10. We present the results for 3 different numbers of types (4,7 and 10).

The results show that whether we vary the number of actions or the number of types, the execution time needed to find a PNE in a 2-player Π-game using direct encoding is very close to the execution time needed to find a PNE using polymatrix encoding.

6.5.3.2 Pure Nash Equilibrium in Ordinal Polymatrix Games with Incomplete Information

Figure 6.2 presents the average execution time needed to get one PNE in 3 different game classes: battle of the sexes, collaboration game and minimum effort game. We fixed the number of actions per player to 2.

Figure 6.3 (resp. 6.4) presents the average execution time needed to find one PNE in a minimum effort game fixing the number of types per player to 2 (resp. the number of players to 25).

Globally, MILP results confirm the feasibility of the qualitative approach of min-based polymatrix games. Furthermore, the results also show that the execution time needed to solve a polymatrix Π-game increases “reasonably” (less than 20 seconds for any

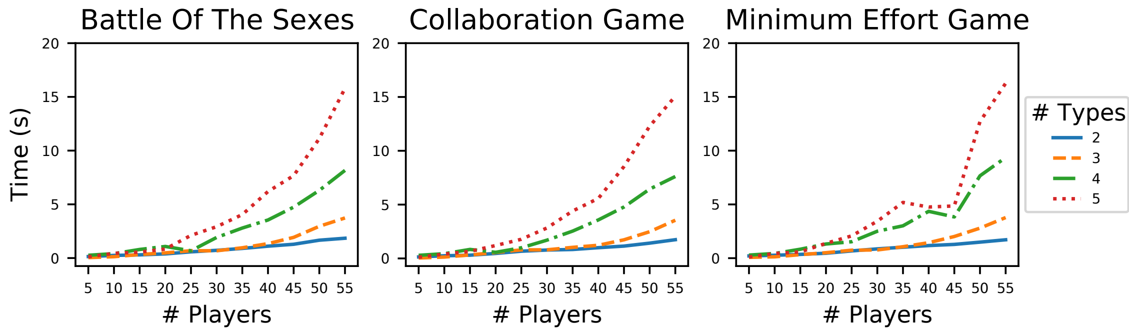

Figure 6.2: Average execution time (s) to find one PNE in a polymatrix Π-game, fixing $|A_i| = 2$, varying $|n|$ from 5 to 55 for $|\Theta_i|$ from 2 to 5 $|\Delta| = 5$.

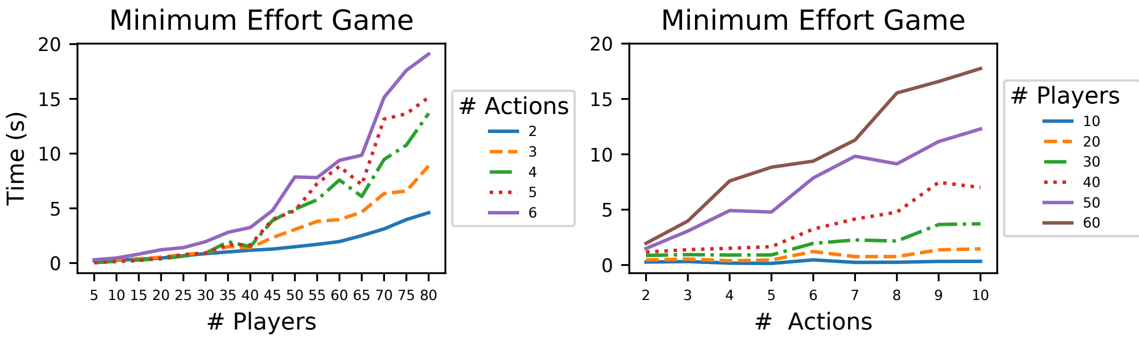

Figure 6.3: Average execution time (s) to find one PNE in a polymatrix Π-game, fixing $|\Theta_i| = 2$, varying $|n|$ from 5 to 80 and $|A_i|$ from 2 to 10, $|\Delta| = 5$.

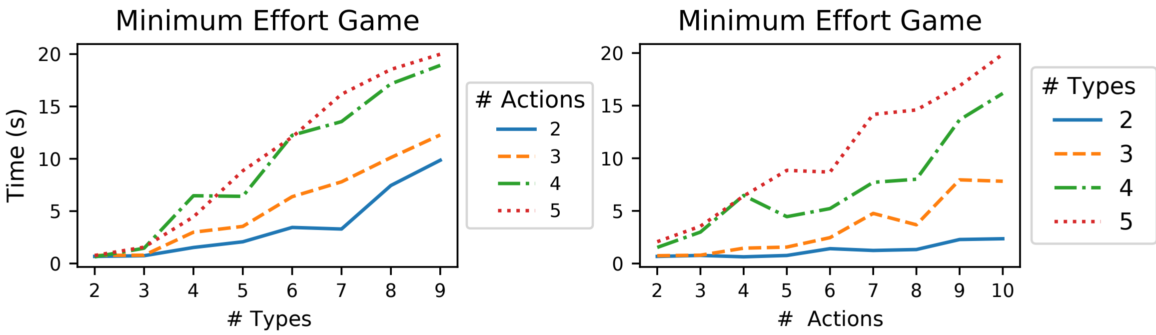

Figure 6.4: Average execution time (s) to find one PNE in a polymatrix Π-game, fixing $n = 25$, varying $|A_i|$ from 2 to 10 for $|\Theta_i|$ from 2 to 9, $|\Delta| = 5$.

configuration) when increasing the number of actions or types of players.

6.6 Conclusion

The main contributions of this chapter are threefold. First, we have defined a new framework for ordinal games, namely polymatrix Π-games, where local games are Π-games. Such games can be exponentially more compact than the equivalent SNF Π-game expression. Second, we have shown that any 2-player Π-game can be transformed into an equivalent min-based polymatrix game, proving a qualitative counterpart of Howson and Rosenthal's theorem linking 2-player Bayesian games to polymatrix games (Howson et al., 1974). Then we have shown that any polymatrix Π-game can itself be transformed in polytime into an equivalent min-based polymatrix game. As a consequence, the potential gain in succinctness comes with no increase in time complexity.

We have also studied the problem of deciding whether a min-based polymatrix Π-game admits a pure Nash equilibrium (the problem is NP-complete).

Finally, we have suggested solving the problem through a MILP formulation, taking advantage of the available state of the art solvers. This allowed us to prove the feasibility of our approach.

Proofs

Proof of Proposition 6.1.
Note that

$$U_i^{pes}(\sigma,\theta_i) = \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \mu_i(\sigma(\theta),\theta)\Big).$$

Now, from Definition 6.2, we have (slightly simplifying notations for readability):

$$\mu_i(\sigma(\theta),\theta) = \min_{j\in N,\{i,j\}\in E} \mu_{ij}(\sigma,\theta_i.\theta_j).$$

So, we have, for all $j \in N$ such that $\{i,j\} \in E$:

$$\begin{aligned}
U_i^{pes}(\sigma,\theta_i) &\le \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1-\pi(\theta_{-i}|\theta_i), \mu_{ij}(\sigma,\theta_i.\theta_j)\Big),\\
&\le \min_{\theta_j\in\Theta_j} \max\Big(\min_{\theta_{-ij}\in\Theta_{-ij}} (1-\pi(\theta_{-i}|\theta_i)), \mu_{ij}(\sigma,\theta_i.\theta_j)\Big),\\
&\le \min_{\theta_j\in\Theta_j} \max\Big(1-\max_{\theta_{-ij}\in\Theta_{-ij}} \pi(\theta_{-i}|\theta_i), \mu_{ij}(\sigma,\theta_i.\theta_j)\Big),\\
&\le \min_{\theta_j\in\Theta_j} \max\Big(1-\pi_{ij}(\theta_j|\theta_i), \mu_{ij}(\sigma,\theta_i.\theta_j)\Big).
\end{aligned}$$

Thus,

$$U_i^{pes}(\sigma, \theta_i) \leq \min_{j,\{i,j\}\in E} \min_{\theta_j \in \Theta_j} \max\Big(1 - \pi_{ij}(\theta_j|\theta_i), \mu_{ij}(\sigma, \theta_i.\theta_j)\Big).$$

Conversely, let j^* be such that $\mu_i(\sigma(\theta), \theta) = \mu_{ij^*}(\sigma, \theta_i.\theta_{j^*})$. We have,

$$\begin{aligned}
U_i^{pes}(\sigma, \theta_i) &= \min_{\theta_{-i}\in\Theta_{-i}} \max\Big(1 - \pi(\theta_{-i}|\theta_i), \mu_{ij^*}(\sigma, \theta_i.\theta_{j^*})\Big), \\
&= \min_{\theta_{j^*}\in\Theta_{j^*}} \max\Big(\min_{\theta_{-ij^*}\in\Theta_{-ij^*}} (1 - \pi(\theta_{-i}|\theta_i)), \mu_{ij^*}(\sigma, \theta_i.\theta_{j^*})\Big), \\
&= \min_{\theta_{j^*}\in\Theta_{j^*}} \max\Big(1 - \max_{\theta_{-ij^*}\in\Theta_{-ij^*}} \pi(\theta_{-i}|\theta_i), \mu_{ij^*}(\sigma, \theta_i.\theta_{j^*})\Big), \\
&= \min_{\theta_{j^*}\in\Theta_{j^*}} \max\Big(1 - \pi_{ij^*}(\theta_{j^*}|\theta_i), \mu_{ij^*}(\sigma, \theta_i.\theta_{j^*})\Big).
\end{aligned}$$

Thus,

$$U_i^{pes}(\sigma, \theta_i) \geq \min_{j\in N,\{i,j\}\in E} \min_{\theta_j \in \Theta_j} \max\Big(1 - \pi_{ij}(\theta_j|\theta_i), \mu_{ij}(\sigma, \theta_i.\theta_j)\Big).$$

□

Proof of Proposition 6.2.
Let us consider player 1 (the same proof holds for player 2, by symmetry). Using Definition 4.4, we get:

$$U_1^{pes}(\sigma, \theta_1) = \min_{\theta_2\in\Theta_2} \max\left(1 - \pi(\theta_2|\theta_1), \mu_1(\sigma(\theta), \theta)\right)$$

Now, in $\tilde{G}$, we have, by definition of $\sigma = a^\sigma$ and of the utility of a joint strategy in a min-based polymatrix game, and the fact that player (i, θ_i)'s utility is independent of the actions of players (i, θ_i') for $\theta_i' \neq \theta_i$:

$$\tilde{\mu}_{(1,\theta_1)}(a^\sigma) = \min_{\theta_2\in\Theta_2} \tilde{\mu}_{(1,\theta_1),(2,\theta_2)}(\sigma_1(\theta_1), \sigma_2(\theta_2)),$$

and, from the definition of $\tilde{\mu}_{(1,\theta_1),(2,\theta_2)}$ in Definition 6.4:

$$\begin{aligned}
\tilde{\mu}_{(1,\theta_1)}(a^\sigma) &= \min_{\theta_2\in\Theta_2} \max\Big(1 - \pi(\theta_2|\theta_1), \mu_1(\sigma(\theta), \theta)\Big) \\
&= U_1^{pes}(\sigma, \theta_1).
\end{aligned}$$

□

Proof of Proposition 6.3. We have proved in Proposition 6.2 that the utility of any pure strategy σ in G is equal to the utility of a^σ in $\tilde{G}$. In order to prove the equivalence of PNE in both games, it is enough to prove that the relation $\sigma \to a^\sigma$ forms a bijection between $\Sigma = \Sigma_1 \times \Sigma_2$ and $\tilde{A}$. To do so, simply note that (i) the $|\tilde{A}|$ and $|\Sigma|$ are equal (to $|\tilde{A}| = |\Sigma| = \prod_{i\in N} |A_i|^{|\Theta_i|}$), and (ii) the transformation is injective, i.e., if $\sigma_i(\theta_i)$ differs from $\sigma_i'(\theta_i)$ for any (i, θ_i), then the pure strategies a^σ and $a^{\sigma'}$ will differ in their components $a^\sigma_{(i,\theta_i)} \neq a^{\sigma'}_{(i,\theta_i)}$. As a result, the transformation is bijective, and since utilities of strategies are preserved, Nash equilibria are identical. □

Proof of Proposition 6.4. Let $G = \langle \{i,j\}, A_i \times A_j, \Theta_i \times \Theta_j, \pi_{i,j}, \{\mu_{i,j}, \mu_{j,i}\}\rangle$ be a 2-player Π-game and $\tilde{G} = \langle \tilde{N}, \tilde{E}, \tilde{A}, \tilde{\mu}\rangle$ its polymatrix representation. Note that the space and computation times of the transformation are dominated by the computation of the tables $\tilde{\mu}_{(i,\theta_i),(j,\theta_j)}(a)$. There are $2 \cdot |\Theta_1| \cdot |\Theta_2|$ such tables, each of size $|A_1| \cdot |A_2|$. The computation of $\tilde{\mu}_{(i,\theta_i),(j,\theta_j)}(a)$ for a given tuple $(\theta_i, \theta_j, a_i, a_j)$ is given in Definition 6.4:

$$\tilde{\mu}_{(i,\theta_i),(j,\theta_j)}(a) = \max\Big(1 - \pi(\theta_j|\theta_i), \mu_i(a, \theta_i.\theta_j)\Big).$$

This computation takes constant time, so the overall time (and space) complexity is $O(d^2 \cdot t^2)$. □

Proof of Proposition 6.5. The proof follows immediately from Proposition 6.1 and Definition 6.6:

Using Equation 6.5, we have

$U_i^{pes}(\sigma, \theta_i) = \min_{\substack{j \in N, \{i,j\} \in E \\ \theta_j \in \Theta_j}} \max\Big(1 - \pi_{i,j}(\theta_j|\theta_i), \mu_{i,j}(\sigma_i(\theta_i).\sigma_j(\theta_j), \theta_i.\theta_j)\Big).$

Based on Definition 6.6, $\tilde{U}_{(i,\theta_i),(j,\theta_j)}(a_i.a_j) = \max\Big(1 - \pi_{i,j}(\theta_j|\theta_i), \mu_i(a_i.a_j, \theta_i.\theta_j)\Big)$. Thus: $U_i^{pes}(\sigma, \theta_i) = \min_{\substack{j \in N, \{i,j\} \in E \\ \theta_j \in \Theta_j}} \tilde{U}_{(i,\theta_i),(j,\theta_j)}(a_i.a_j)$. □

Proof of Proposition 6.6. The proof is similar to that of Proposition 6.3 and is based on the bijection between the actions set in $\tilde{G}$ and the pure strategies set in G. □

Proof of Proposition 6.7.
Membership: We prove the membership in NP for the more general case of N unbounded. The PNE can be solved by guessing a strategy a, i.e., guessing an action for each player, then checking whether a is a PNE or not. More precisely:

- for each player i: compute $\mu_i(a)$ (Using equation (2.13));
- for each action $a_i \in A_i$: compute $\mu_i(a_i.a_{-i})$.

Then check if player i has incentive to deviate from a_i, i.e., we should compare $\mu_i(a)$ and $\mu_i(a_i.a_{-i})$.

The complexity of computing $\mu_i(a)$ is in $O(|E|)$, from Equation 2.13. Thus the whole complexity is polynomial $O(n \times |A_{\max}| \times |E|)$ where $|A_{\max}| = \max(|A_1|, ..., |A_n|)$.

Hardness: Any Π-game can be transformed in polytime and space into an equivalent min-based polymatrix game □

Conclusion

Possibilistic games with incomplete information (Π-games) constitute a suitable framework for the representation of ordinal decision problems under incomplete knowledge where the common knowledge of the players is ordinal and can be captured by a joint possibility distribution over the joint types.

In this thesis, we considered that all players share the same knowledge. For instance, in the coordination game, every player believes that if she receives an offer, so do her neighbors; and if she does not receive anything, her neighbors do not either. However, it may exist some players that have different beliefs: she may believe that she is the only one that receives an offer. Therefore, the study of possibilistic games with incomplete information under conditional qualitative knowledge becomes an active line of research. In this kind of game, every player has her own knowledge given by a conditional possibility distribution over the other players' joint types, i.e., each player i knows $\pi_i(\theta_{-i}|\theta_i)$, $\forall \theta_i \in \Theta_i$ and $\forall \theta_{-i} \in \Theta_{-i}$. We join the idea of (De Clercq et al., 2018) and we adopt an *ex-interim* approach.

Along this thesis, especially in Chapter 4, we studied three solution concepts: (i) secure strategy, (ii) pure Nash equilibrium, and (iii) possibilistic mixed Nash equilibrium. We plan to continue the picture by extending further solution concepts, e.g., dominance, and Pareto optimality.

The second part of this thesis introduced "*min-based polymatrix* Π-*games*", which allow us to concisely specify Π-games with local pairwise interactions. In this framework, the utility of a player depends on her neighborhood only. Such games can be exponentially more compact than the equivalent standard normal form Π-games. We have shown that each polymatrix Π-game can be transformed, in polynomial time, into an equivalent min-based polymatrix game. The transformation consists of transforming each local 2-player Π-game into an equivalent min-based polymatrix game. Therefore, the pure Nash equilibria in both games are in bijection. This transformation represents a qualitative counterpart of Howson and Rosenthals's theorem (Howson et al.,

1974) linking 2-player Bayesian games to polymatrix games.

Then, we have shown that the problem of deciding whether a min-based polymatrix game admits a pure Nash equilibrium is NP-complete and we have proposed a MILP formulation of this problem. We shall now develop direct algorithms for computing a pure Nash equilibrium of min-based polymatrix games with incomplete information (without transforming the polymatrix Π-game into an equivalent min-based polymatrix game) namely a possibilistic version of the constrained PNE algorithm (Simon and Wojtczak, 2017) or of the Valued Nash Propagation algorithm (Chapman et al., 2010). We plan also to study possibilistic mixed Nash equilibrium in polymatrix Π-games and to develop an adaptation of the NashProp algorithm (Ortiz and Kearns, 2002) to the possibilistic framework.

In Chapter 6, we focused on the study of Π-games with pairwise interactions where a player can be involved in several multiple players subgames. Thus, a straightforward extension of our work is a generalization to n-player hypergraphical Π-games. To this extend, we need to generalize the transformation of a 2-player Π-game into a min-based polymatrix game to a n-player Π-game into a min-based hypergraphical game.

The study of polymatrix games with an uncertainty on the edges represents an interesting line of research as advocated by (Deng et al., 2019). In the qualitative case, we shall define games where every player does not have complete information about the set of her neighborhood and her knowledge can be captured by a possibility distribution over the set of her possible neighbors. In other words, we get a degree of possibility of existence for each edge. This framework can be applied in coordination games where every player does not know with which neighbor she coordinates. The uncertainty about the existence of the edges will affect the pessimistic utilities.
This idea can also be applied to min-based hypergraphical games with uncertainty on the edges where every player has incomplete knowledge about the groups of players that she belongs to.

The results of our contributions presented in this thesis encourage us to explore further classes of possibilistic games. As long-term perspectives, we consider studying dynamic possibilistic games. For instance, extending competitive Markov decision processes (Filar and Vrieze, 1997), and partially observed stochastic games (Sorin, 2002). The competitive Markov decision processes extend both Markov Decision Processes (MDP) (Puterman, 1994) and cardinal games. Partially observed stochastic games extend both partially observed Markov decision processes (Cassandra et al., 1994) and cardinal games. Since possibilistic MDP have been introduced by (Sabbadin et al., 1998, Sabbadin, 2001), it is possible to define possibilistic competitive Markov decision processes and to develop a possibilistic extension of partially observed games.

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