PRESERVING POSITIVITY AND MONOTONICITY OF REAL DATA USING BÉZIER-BALL FUNCTION AND RADIAL BASIS FUNCTION

AFIDA BINTI AHMAD

UNIVERSITI SAINS MALAYSIA

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PRESERVING POSITIVITY AND MONOTONICITY OF REAL DATA USING BÉZIER-BALL FUNCTION AND RADIAL BASIS FUNCTION

by

AFIDA BINTI AHMAD

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LIST OF ABBREVIATIONS

USM	Universiti Sains Malaysia
RBF	Radial Basis Function
MQ	Multiquadric
CAGD	Computer Aided Geometric Design
RMS	Root Mean Square
CSRBF	Compactly Supported Radial Basis Function

LIST OF SYMBOLS

ε	shape parameter
ϕ	radial function
λ	expansion coefficents for RBF
λ	unknown in Bézier-Ball sufficient condition
δ	tolerance
ω	rational Ball weight
Δ	changes in f
θ	local variable
d_i	first derivative at each ordinate
σ	parameter for Bézier-Ball in preserving positivity
ς	parameter for Bézier-Ball in preserving positivity
α	derivative of Bézier-Ball curve at 0
β	derivative of Bézier-Ball curve at 1
γ	parameter for Bézier-Ball in preserving monotonicity
ψ	a function
φ	changes in f_y
ν	parameter for Bi-Cubic Bézier-Ball
η	parameter for Bi-Cubic Bézier-Ball
κ	parameter for Bi-Cubic Bézier-Ball

τ	parameter for Bi-Cubic Bézier-Ball
$\mathscr{L}_{\mathscr{D}} f(x)$	MQ quasi interpolation

MENGEKALKAN KEPOSITIFAN DAN KEEKANADAAN DATA SEBENAR MENGGUNAKAN FUNGSI BÉZIER-BALL DAN FUNGSI ASAS JEJARI

ABSTRAK

Dalam tesis ini, sebuah fungsi rasional kubik Bézier-Ball dengan empat parameter yang menggunakan titik kawalan dan pekali daripada fungsi Ball telah dicadangkan untuk mengekalkan kepositifan dan keekanadaan data sebenar. Sebuah fungsi bikubik Bézier-Ball yang mengandungi lapan parameter digunakan untuk mengekalkan kepositifan sesebuah permukaan yang dihasilkan daripada data sebenar dan daripada beberapa fungsi yang diketahui. Interpolasi menggunakan fungsi jejari asas (RBF) juga diperkenalkan sebagai alternatif dalam mengekalkan kepositifan sesebuah data. Dua jenis RBF iaitu fungsi Multiquadrik (MQ) dan fungsi 'Gaussian' yang mengandungi satu parameter telah digunakan. Sempadan parameter (had bawah dan atas) yang mengekalkan kepositifan data juga dicadangkan. Perbandingan antara kaedah interpolasi yang dicadangkan dengan kaedah yang sedia ada dalam kajian telah dibuat dengan menggunakan kaedah ralat punca min kuasa dua (RMS). Didapati bahawa interpolasi dengan fungsi MQ dan fungsi rasional kubik Bézier-Ball adalah setara dengan interpolasi lain yang sedia ada dalam mengekalkan kepositifan sesebuah lengkung dan permukaan. Kedua-dua fungsi ini juga berjaya mengekalkan kepositifan sesebuah lengkung dan permukaan. Fungsi rasional kubik Bézier-Ball juga mampu mengekalkan keekanadaan sesebuah lengkung, tetapi tidak bagi MQ-quasi memandangkan perbezaan RMS yang besar oleh MQ-quasi. Fungsi Gaussian menunjukkan keupayaan untuk mengekalkan kepositifan sesebuah lengkung dan permukaan, namun menghasilkan lengkung yang berayun dan tidak rata.

PRESERVING POSITIVITY AND MONOTONICITY OF REAL DATA USING BÉZIER-BALL FUNCTION AND RADIAL BASIS FUNCTION

ABSTRACT

In this thesis, a rational cubic Bézier-Ball function which refers to a rational cubic Bézier function expressed in terms of Ball control points and weights are used to preserve positivity and monotonicity of real data sets. Four shape parameters are proposed to preserve the characteristics of the data. A rational Bi-Cubic Bézier-Ball function is introduced to preserve the positivity of surface generated from real data set and from known functions. Eight shape parameters proposed can be modified to preserve the positivity of the surface. Interpolating 2D and 3D real data using radial basis function (RBF) is proposed as an alternative method to preserve the positivity of the data. Two types of RBF which are Multiquadric (MQ) function and Gaussian function, which contains a shape parameter are used. The boundaries (lower and upper limit) of the shape parameter which preserves the positivity of real data are proposed. Comparisons are made using the root-mean-square (RMS) error between the proposed interpolation methods with existing works in literature. It was found that MQ function and rational cubic Bézier-Ball is comparable with existing literature in preserving positivity for both curves and surfaces. For preserving monotonicity, the rational cubic Bézier-Ball is comparable but the MQ quasi-interpolation introduced can only linearly interpolate the curve and the RMS values are big. Gaussian function is able to preserve positivity of curves and surfaces but with unwanted oscillations which result to unsmooth curves.

CHAPTER 1

INTRODUCTION

Curves generation from discrete data are well-known in the area of Computer Aided Geometric Design. Nevertheless, sampled data is often unable to represent the underlying information of the data and this is where interpolation comes in handy. Interpolation provides graphical representation of information in a more effective way. In this thesis, we consider the problem of curve and surface interpolations that preserves the inherited properties of real data. In general, interpolation of real data involved a set of n data points $\{x_i, f_i\}_{i=1}^{n}$ where f_i are dependent on x_i, and a set of basis functions passing through these points. For real data, the properties of the sampled data are often known and it is important that the interpolants preserve the shape features such as positivity, monotonicity and convexity. The shapes that are of interest in this study are positive and monotone.

Preserving positivity involved physical quantity that cannot be negative, for example, stability of radioactive substance, population statistics, resistance of an electric circuit, gas discharge during chemical reactions and rainfall measurements (Hussein and Sarfraz, 2008; Hussein et al., 2010; Karim and Pang, 2014). The level of uric acid in gout patients, economic forecasting, data generated from stress and strain of a material and graphical display of Newton's law of cooling are some examples comprising data with monotonic feature (Abbas et al., 2012). To make these data meaningful, it is vital to preserve the shape of the data.

There are various methods suitable for preserving positivity and monotonicity. Spline interpolation schemes are common methods used in interpolation of real data sets. In this thesis, we have used a converted rational Bézier to Ball (refer to as Bézier-Ball in the entire thesis) function, rational Bi-Cubic Bézier-Ball function with shape parameters and two types of Radial Basis functions to preserve the shapes of curves and surfaces.

Fritsch and Butland (1984) first introduced the use of piecewise cubic polynomials to preserve monotonicity. However, this type of polynomials produced more oscillations (Asim and Brodlie, 2003). An alternative to this is using piecewise rational splines. Many researches were done in spline formulation using rational Bézier curves to solve positivity and monotonicity shape-preserving interpolations problems for scalar curves (Delbourgo and Gregory, 1985; Delgado and Pena, 2006; Sarfraz, 2007; Jaafar et al., 2014; Zakaria et al., 2016).

De Casteljau's algorithm is a recursive method used to evaluate polynomials in Bernstein form or Bézier curve. Since the algorithm is recursive, the rendering algorithm can be time consuming as it keeps breaking the curve into sub-curves.

An alternative basis function other than Bézier curve is Ball function. The generalised Ball polynomials were named differently according to Wang-Ball and Said-Ball polynomials.

The cubic Said-Ball curve defined in Dejdumrong et al. (2001) as:

$$S(x) = \sum_{i=0}^{3} V_i S_i^3(x)\,, \quad 0 \leq x \leq 1, \tag{1.1}$$

where $\{V_i\}_0^3$ are the control points and $\left\{S_i^3(x)\right\}_0^3$ are the Said-Ball basis function which can be defined as:

$$\begin{aligned}
S_0^3(x) &= 1-x,\\
S_1^3(x) &= 2x(1-x)^2,\\
S_2^3(x) &= 2x^2(1-x),\\
S_3^3(x) &= x.
\end{aligned} \tag{1.2}$$

The cubic Wang-Ball curve defined in Dejdumrong et al. (2001) comprises:

$$W(x) = \sum_{i=0}^{3} p_i A_i^3(x), \quad 0 \le x \le 1, \tag{1.3}$$

where $\{p_i\}_0^3$ are the control points and $\left\{A_i^3(x)\right\}_0^3$ are the Wang-Ball basis function which can be defined by:

$$\begin{aligned}
A_0^3(x) &= (1-x)^2,\\
A_1^3(x) &= 2x(1-x),\\
A_2^3(x) &= 2x^2(1-x),\\
A_3^3(x) &= x^2.
\end{aligned} \tag{1.4}$$

Said basis function in Equation (1.2) is different from Wang basis function given in Equation (1.4). Delgado and Pena (2006) compared both these representations and concluded that Said-Ball basis is better-conditioned (lower condition number than Wang-Ball basis) and has better shape-preserving properties since Wang-Ball basis is not strictly monotonicity preserving when the degree is 2 and 3. Dejdumrong (2007) verified that Said-Ball polynomials preserved the shape of Said-Ball curve while Wang-Ball polynomials did not preserve the shape of Wang-Ball curve.

For the above reasons, Said-Ball will be used in this thesis. However, the setback of Ball function is that it cannot exactly represent conics which are usually used in aircraft and machine design. The advantage of using Ball basis function is that it can evaluate a polynomial more efficiently than using Bernstein basis (Hu et al., 1996). Hu et al. (1996) produced a recursive algorithm for Said-Ball and Wang-Ball representation to evaluate any point on a Bézier curve. They came out with the conclusion that to get a point on a Bézier curve, the algorithms using Ball curves are more efficient in comparison to de Casteljau algorithm. Since a Bézier curve can be presented in terms of Ball function (and vice versa), we have used the relationship between these two functions to preserve the properties of a data.

1.1 Research Background

Many researches were done in using Ball and Bézier as the basis function when interpolating real data as well as to preserve the shape of the data. Converting Bézier curves to Ball curves and vice versa is sometimes useful depending on the purpose of the design. In this thesis, a rational cubic Bézier curve that used the control points and weights of a rational cubic Ball as in Tien et al. (1999) are proposed. The function is referred to as **Bézier-Ball** function through out this thesis.

To preserve the nature of the data, the method used by these functions is by differentiating the function and then varying its shape parameters. To generate a curve, several segments are joined together which give more freedom for the user to control the shape of a curve. However, using this method, changing the position of a single data point will change the shape of the interpolating curve locally. An alternative function that preserves the shape of the data without derivation, applicable for scattered data and fit the data as a whole is investigated in this thesis.

Radial basis function (RBF) methods are an emerging field in shape preserving area. Though RBF has been widely used in fields such as engineering, business studies, biology and mathematics (Kansa, 1990; Sharan et al., 1997; Sarra and Kansa, 2009). In the field of mathematics, RBF is commonly used in surface reconstruction, numerical solution of partial differential equations, scattered data interpolation and neural network modelling. In this thesis, interpolations of small number of real data points are presented.

There are many types of RBFs available and they can be categorised into three main categories which are compactly supported and finitely smooth such as Wendland's function; infinitely smooth and containing a free parameter such as multiquadrics (MQ), Gaussians and Inverse Multiquadrics; and piecewise smooth and parameter-free such as thin-plate splines (Gneiting, 2002; Wendland, 1995; Wu, 1995). Some classical types of RBF are given in Table 1.1.

Table 1.1: Commonly used type of radial basis functions ($r = ||x - x_i||$).

Radial Basis Function	$\phi(r)$
Gaussian	$e^{-(\varepsilon r)^2}$
Multiquadric	$\sqrt{1+(r\varepsilon)^2}$
Inverse Multiquadric	$\frac{1}{\sqrt{1+(r\varepsilon)^2}}$
Thin Plate Spline	$r^2 ln(r)$

As in Table 1.1, not all basis functions contain a free shape parameter. The radial functions $\phi(r)$ used in this thesis are multiquadric and Gaussian as both these functions contain a parameter that can be varied to preserve the shape of data.

Although numerous researches had been done in choosing the best shape parameter that produced the best interpolation, (Rippa, 1999; Fornberg and Zuev, 2007; Scott and Sturgill, 2009; Wei et al., 2009; Wu et al., 2010; Wu et al., 2010; Xiang et al., 2012; Ranjbar, 2015; Biazar and Hosami, 2016) to our knowledge, there is no clear research done on RBFs that can preserve positivity and monotonicity of **real data** for curves and surfaces.

1.2 Problem Statement

Given a set of data points (real data or data generated using function), construct a function that can preserve the properties (i.e. positive or monotone) of the data as well as producing smooth curves/surfaces.

1.3 Research Objectives

The objectives of the thesis are

1. To provide sufficient conditions for the rational cubic Bézier-Ball curve with four shape parameters in order to preserve positivity and monotonicity of real data,
2. To provide sufficient conditions for the rational Bi-Cubic Bézier-Ball surface with four shape parameters in order to preserve positivity of real data,

3. To propose a boundary condition on the shape parameter of Multiquadric and Gaussian functions that can ensure smooth interpolants while preserving positivity and monotonicity of curves of real data,

4. To propose a boundary condition on the shape parameter of Multiquadric and Gaussian functions that can ensure smooth interpolants while preserving positivity of surfaces of real data and functional data.

1.4 Scope of Thesis

In this thesis, the Bézier-Ball function is used to preserve positivity and monotonicity of positive and monotone data. For a 2D data set $\{(x_i, f_i)$ where $i = 1, 2, ...n\}$, for simplicity, we assume that $x_1 < x_2 < ... < x_n$ and the data set is positive if

$$f_1 > 0, f_2 > 0, ..., f_n > 0. \tag{1.5}$$

The data is monotonically increasing if

$$f_i \leq f_{i+1}. \tag{1.6}$$

For a 3D data set $\{(x_i, y_j, F_{i,j})$ where $i = 1, 2, ...m, j = 1, 2, ..., n\}$, assuming $x_1 < x_2 < ... < x_m$ and $y_1 < y_2 < ... < y_n$, the data is said to be positive if

$$F_{i,j} > 0, \tag{1.7}$$

and monotonically increasing if

$$F_{i+1,j} > F_{i,j} \text{ and } F_{i,j+1} > F_{i,j}. \tag{1.8}$$

1.5 Motivation to study Radial Basis Function

Bézier-Ball interpolates the data by joining several segments while RBF interpolates all data points at once. Hence, for RBF, the order in which the data points are arranged is not important. For example, MQ interpolant is given by

$$s(x) = \sum_{i=1}^{n} \lambda_i \sqrt{1 + (||x - x_i||)^2 \varepsilon^2}, \tag{1.9}$$

where n is the number of data points, λ_i is the expansion coefficient, ε is the shape parameter and x_i are the centres where $i = 1, 2, ..., n$. A set of positive data from Sarfraz (2007) is tabulated in Table 1.2. Table 1.3 shows a set of modified data from Table 1.2.

Table 1.2: A positive data set from Sarfraz (2007).

i	**1**	**2**	**3**	**4**	**5**
x_i	0	2	3	9	11
f_i	0.5	1.5	7	9	13

Table 1.3: Modified data from Table 1.2.

i	**1**	**2**	**3**	**4**	**5**
x_i	0	2	**9**	**3**	11
f_i	0.5	1.5	**9**	**7**	13

Figure 1.1 shows that the order of the data does not make a difference when MQ interpolation method is used. However, in Figure 1.2, cubic Ball produced a totally different curve when using data with different arrangement.

Further, to preserve the shape of the data, when using Bézier-Ball spline, the function will be differentiated and some conditions were imposed in which the shape parameters in the function must comply. For RBF, the shape parameter is used directly to preserve the shaped data. The optimal shape parameter will be discussed later.

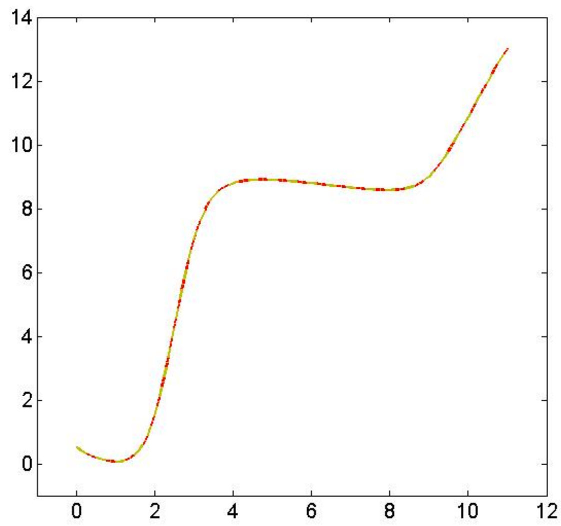

Figure 1.1: The red curve is MQ interpolant using data from Table 1.2; The dashed-line refers to MQ interpolant produced using data from Table 1.3.

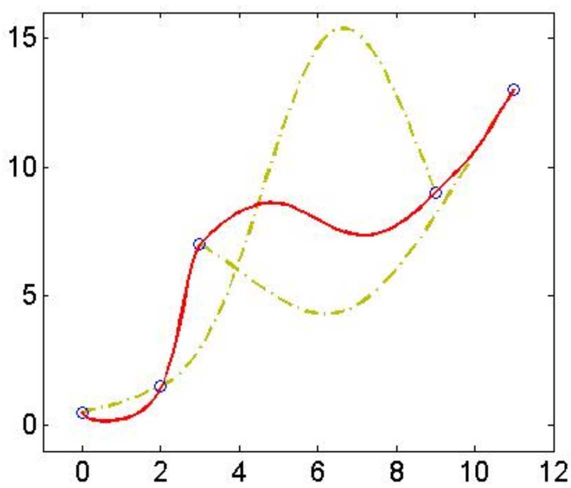

Figure 1.2: The red curve is Ball interpolant using data from Table 1.2; The dashed-line refers to Ball interpolants produced using data from Table 1.3.

1.6 Significance of Findings

This thesis provides the sufficient conditions required to preserve positivity and monotonicity between a Bézier curve expressed in terms of Ball control points. Furthermore, the research contributes an alternative function for shape preserving. There has been no research done in using RBFs in preserving the shape of real data for curves and surfaces.

1.7 Thesis Outline

This thesis is organised as follows. There are 7 chapters in this thesis. Chapter 2 provides reviews and development related to shape preserving which mainly involved rational cubic Ball function, rational cubic Bézier function and RBF.

Chapter 3 discusses the positivity and monotonicity preserving method using rational cubic Bézier-Ball basis function. Four shape parameters are proposed in order to preserve the shape of the curves. Some examples using real data are also included.

Chapter 4 includes preserving positivity of surfaces using rational Bi-Cubic Bézier-Ball function. Three examples are also included which include one real data and two positive data generated by two positive functions.

Chapter 5 introduces the basic concepts on RBFs, particularly MQ and Gaussian function. Then, the proposed interpolation method which involves varying the shape parameter values in order to preserve positivity and monotonicity of the data are presented. In preserving positivity, a max-min test using derivatives is included in order to prove that the interpolating curves are positive. For monotonicity, MQ quasi interpolation is used as the basis. Numerical examples are also included in this chapter.

Chapter 6 introduces the extended version of RBF introduced in Chapter 5 to produce positive surface. This chapter also includes the method used to set the boundaries to the shape parameters of MQ and Gaussian functions. Analysis to determine the positivity of the surface is also included which uses the Second Derivative Tests. The final results are concluded in Chapter 7.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter includes studies on positivity and monotonicity shape preserving for curves and surfaces using Ball and Bézier functions. In this thesis, cubic spline is used in the analysis. In general, cubic splines are more favourable since linear splines have discontinuous first derivatives, quadratic splines have discontinuous second derivatives while quartic or higher-order splines produced instabilities due to the use of higher order polynomials. Cubic Bézier and cubic Ball have long been used in preserving positivity. In the thesis, we introduced a converted cubic Bézier to cubic Ball form of equation and the researches done on this are also included. The development of two types of RBF which are Multiquadric and Gaussian in preserving positivity is presented for curves and surfaces.

2.2 Shape Preserving using Spline

There are many researches done in constructing positivity-preserving interpolants. The method started with the use of piecewise polynomials either to preserve positivity, monotonicity or convexity. Among these methods, there are several authors who introduced additional knot in each subinterval to preserve shaped data or no additional knot. Instead, the shape parameters are varied in order to satisfy the data properties. The schemes involve additional knots inserted on the subinterval, see Asim and Brodlie (2003); Butt and Brodlie (1993) or no additional knot but some constraints imposed on the shape parameters in the basis functions, see Sarfraz (2002); Sarfraz (2010); Hussain et al. (2011); Abbas et al. (2013); Jaafar et al. (2014); Jamil and Piah (2014); Tahat et al. (2014); Han (2015).

Butt and Brodlie (1993) used piecewise cubic Hermite polynomials to preserve positivity. They added extra knots to the interval where the positivity of curve were lost. Fritsch and Carlson (1980) and Fritsch and Butland (1984) used piecewise cubic polynomials to preserve monotonicity of the data. Conditions were derived for the free parameter in an interval and the algorithm was then presented.

The problem with polynomials interpolation is when we have $n+1$ data points, we require n degree polynomials to fit the data. Higher degree polynomials tend to be unstable and produce oscillations. An alternative to this, more researches were done using piecewise rational splines instead of polynomials. With splines, a curve with a simple function can be presented on each interval between data points.

An alternative to this is to use Bézier form of rational splines to preserve positivity and monotonicity of a data. In the area of computer-aided geometric design (CAGD), the curves are generated by using an efficient recursive algorithm known as de Casteljau Algorithm (Farin, 1993).

There are many different types of rational functions that uses Bernstein-Bézier formulation to preserve the shape of data. Lam (1990) utilized the piecewise quadratic Bézier spline proposed in Schumaker (1983). In Schumaker (1983), additional knots must be inserted to preserve the shape of the data. Algorithm that abled to adjust the slopes and knot locations to obtain desired interpolants is presented. In Lam (1990), the method was improved where instead of inserting additional knots, the first order derivatives are obtained. Suitable parameters were set in order to produce a positivity preserving curve.

Delbourgo and Gregory (1985) used rational cubic Bézier with quadratic denominator. Hussain et al. (2011) used cubic Hermite interpolation with two families of free parameters to attain positivity of the interpolants. Abbas et al. (2013) used piecewise rational cubic Bézier function in cubic/quadratic form to preserve the inherited positive shape. The authors proposed curve scheme where the scheme is applicable for data with derivative or without derivatives. LU decomposition was used to solve the value for derivatives. Sarfraz (2002) used rational cubic Bernstein-Bézier formulation (cubic/cubic form) to preserve the positivity of the data. Hussain and Sarfraz (2008) used rational cubic spline to preserve positivity of 1D data and then extended it to a rational Bi-Cubic form to visualize positive surfaces. Sarfraz (2010) then used the rational cubic Bézier function and extended it to a rational Bi-Cubic partially blended functions to visualize the inherited shape of positive data. Karim (2014) proposed a cubic spline Bézier interpolant with linear denominator to preserve positivity of the data containing two shape parameters. The final shape of the interpolating curve depends on one shape parameter manipulated by the other shape parameter.

Sarfraz (2000) presented rational cubic Bézier functions with two parameters to preserve monotonicity. In this method, the user had no freedom to choose the parameters since the parameters were computed automatically. Abbas et al. (2012) proposed a rational cubic Bézier function with three shape parameters to preserve monotonicity of the data. One shape parameter was used to preserve the shape of the data, while the other two were used to control the shape of the data.

In the literature reviews mentioned above, all utilized the used of cubic function and constraints were imposed on the shape parameters in order to preserve positivity.

There were also other types of piecewise rational splines other than cubic Bézier. In Han (2015), a piecewise rational interpolant with quartic numerator and quadratic denominator was presented and a shape parameter on each subinterval was determined in order to meet the shape-preserving properties of the interpolant mainly positivity and monotonicity. Can et al. (2013) used piecewise cubic Hermite to compare with piecewise cubic Bessel and piecewise cubic polynomial interpolation to preserve positivity. Cubic Bessel and cubic polynomial produced a smooth curve, however, when there was abrupt change in data, the interpolants produced oscillations. In Dube and Rana (2014), for preserving positivity, a rational quadratic trigonometric function with three shape parameters was developed. Although the function managed to preserve positivity, some of the interpolants using different data were not smooth. In Sarfraz et al. (2015), Sarfraz used nonrational trigonometric quadratic spline to preserve positivity, monotonicity and convexity of the data. The trigonometric developed involved three shape parameters where two of the shape parameters were used to maintain the shape of the data, and one of the parameters was left free for shape modification of the preserving curve.

An alternative to Bézier form of equation is Ball function. Back in 1989, Said (1989) proposed basis functions that employed cubic polynomials. A recursive algorithm using de Casteljau algorithm for a curve of degree $2m+1$ was developed. From this paper, it was found that the basis functions proposed by Said has the same shape-preserving properties as the Bernstein polynomials (They are non-negative, sum to one and the interpolating curve contain inside the convex hull of the control polygon). It was also found that the algorithm generated by Said was more efficient than the de Casteljau algorithm for evaluating polynomials in Bézier form since the generalized Ball form is of lower degree curve resulting from coalescing interior control points.

There are a lot of researches done in preserving positivity and monotonicity of the data using Ball function. The researches done showed that the interpolants produced using Ball basis function managed to produce smooth and visually pleasing curves. Tahat et al. (2014) proposed the use of cubic Ball function to preserve positivity of the data. Four free parameters were utilized to preserve positivity of the curve. Two of the free parameters were used to preserve shape of the data and the other two shape parameters remain free to suit the designer's choices for a smoother curve. In Jamil and Piah (2014), sufficient conditions were derived and second derivative continuity is applied at each knot to modify the shape of the curve when desired. In Jaafar et al. (2014), no additional points were inserted in the presented scheme but instead, constraints were derived on shape parameter which guarantees the positivity of the data.

Jaafar et al. (2014) proposed rational cubic Ball function to preserve monotonicity of the data. The function consisted of one parameter used to preserve the shape of data and the other two were for the user to modify the shape of the curve produced. Tahat et al. (2016) proposed a rational cubic Ball function with four shape parameters. They managed to show that the method works on monotonicity increasing and monotonicity decreasing data.

Abbas et al. developed methods to preserve positivity of 3D positive data using a rational Bi-Cubic function in Abbas et al. (2012) and Abbas et al. (2014). In Abbas et al. (2012), 12 shape parameters were introduced where four parameters were used to preserve the positivity of data while another eight parameters can be used by the user to modify the shape of the curve and surfaces. In both papers, C^1 continuity are achieved. In Abbas et al. (2014), six shape parameters were introduced to conserve the positivity of 3D data. This paper includes shape control analysis involving the tension effect when the shape parameters were varied. Further, Abbas et al. computed the derivatives directly from data points hence this method works on data with or without derivatives.

Hussain and Sarfraz (2008) presented a C^1 piecewise rational Bi-Cubic function to preserve positivity of positive curves and surfaces. It requires no additional points to preserve the shape of the surface. It has two free parameters for the users to modify the shape of the curves and surfaces.

Tahat et al. (2016) proposed surface interpolation using rational Bi-Cubic Ball interpolation with four shape parameters. The derivatives are computed from the given data set. The managed to achieve C^1 interpolant with smooth interpolation.

There are various methods used to preserve the positivity of 3D surface data using Bézier or Ball Bi-Cubic function. With this, we are motivated to extend the use of a converted Bézier-Ball function to preserve the positivity of surface data.

Tien et al. (1999) proved that a Ball curve of n degree is also a Bézier curve of the same degree. When we have a Bézier curve of degree n and its control points $\{b_i\}_{i=0}^{n}$, control points of Ball curves $\{V_i\}_{i=0}^{n}$ in terms of b_i can be determined. Further explanation is given in Chapter 5. Dejdumrong (2007) verified that Bézier curves and Said-Ball curves preserve the shape of its control points while Wang-Ball curves did not preserve the shape of the control polygon. Dejdumrong (2007) has also verified that Bernstein polynomials preserve the shape of Bézier curve better than Said-Ball polynomials.

Dejdumrong et al. (2001) also presented an algorithm to present a point on a rational Bézier curve using rational Wang-Ball curve control points and weights. The Wang-Ball algorithm involved addition and multiplication while de Casteljau algorithm use recursive method to evaluate polynomials in Bernstein form. The method proposed was by converting the rational Bézier curve to rational Wang-Ball curve of the same degree, then using the algorithm proposed for the Wang-Ball curve to evaluate the point. From the paper, it was proven that to compute a point on a rational Bézier curve, the proposed algorithms is better for curve with degree five and above than using de Casteljau algorithm.

2.3 Shape Preserving using Radial Basis Function

Radial basis function (RBF) approximation method is commonly used in solving mathematical problems such as to solve differential equations on arbitrarily scattered data (Kansa, 1997; Hon and Mao, 1997; Sharan et al., 1997; Buhmann, 2003;

Wendland, 2004; Jenkinson et al., 2007; Parand and Hemami, 2016) and also modeling neural network to solve the problems of function learning (Chai,1996; Arkadan, 2007; Chang, 2008; Arana-Daniel et al., 2014). RBF interpolation method was first developed in 1968 by Roland Hardy in Hardy (1971) to construct a continuous function that can approximate the features of irregular surfaces. Hardy applied the use of Multiquadric (MQ) to interpolate data on a topographic surface. In Hardy (1971) also, the MQ equation was derived.

A radial function is a function that is radially symmetric around a set of centers. A radial basis function (RBF), $\phi(r)$ contains a function in which its function value depends on r (the distance of x and its neighbours). For a RBF interpolant, $s(x)$ is a linear combination of the following form

$$s(x) = \sum_{i=1}^{n} \lambda_i \, \phi(||x - x_i||), \; x \in \mathbb{R}, \tag{2.1}$$

where n is the number of points, x_i are called the *centres* of RBF, λ_i are the expansion coefficients or the weights of the RBF and $\phi(r)$ is some radial function and $||x - x_i||$ utilizes the use of Euclidean distances between a point and the centers of the radial basis functions (i.e. $||x - x_i|| = \sqrt{(x - x_i)^2}$).

In many cases, these centers are chosen from the data value even when they can be different from the data. Researches were done on cases where centers are different, mainly involving cases when the number of data points is huge (thousands of points that will result in a very huge matrix and the data might overfit).

In Kanungo (2002), for big data sets, clustering algorithms was introduced in order to find a set of centers which could accurately reflect the distribution of the data points. In this thesis, the number of data points are small. Therefore, all given data points are chosen to be the centres, meaning to say, if there are 11 data points, all 11 points will be the centres.

Further, in some cases, low order polynomials is added to the RBF interpolation (Wright, 2003). A RBF with polynomial terms has the form of

$$s(x) = \sum_{i=1}^{n} \lambda_i \phi(||x - x_i||) + \sum_{j=1}^{m} \gamma_j p_j(x),\ x \in \mathbb{R}^d, \tag{2.2}$$

where p_j is the basis function for the polynomials term, γ_j is the expansion coefficient, $j = 1, 2, ..., m$ and $\mathbb{R}^d$ is real number in any dimension. Polynomials are normally added in cases where the radial basis functions is not positive definite, which means that it has no unique solution (Schaback, 2007). Further, Bot compared the performance of a few types of RBFs in a test case. One of the function used was MQ with added polynomials. They have concluded that for this experiment, the added polynomial terms have slight impact to the experiment. Therefore, in this thesis, no polynomial terms were included.

Franke (1982) evaluated methods used for scattered data interpolations. Thin plate splines, B-splines, Hardy's MQs and Gaussian were amongst the methods used. The evaluations involved the sensitivity, accuracy, visual pleasantness and ease of implementation. In the evaluation, some methods are 'global' where the interpolant is dependent on all data points. Addition or deletion of any data point/s will cause a change in the whole interpolant (Hardy's MQ, Duchon's Thin plate Splines, Gaussian, B-Splines, Foley's Cubic spline). Some methods are 'local' which means addition or deletion of data point/s will only affect the nearby points and the interpolant of greater distance will not be affected.

Based on the evaluations, Franke concluded that global methods are more feasible for data up to 100-200 points. Global methods also produced a more visually pleasing interpolants as local methods produced poor behaviour near edge of data. In terms of sensitivity to parameters, for fixed data sets, both MQ and Gaussian are very sensitive and dependable on the shape parameter Franke (1982). The analysis done by Franke was solely based on extensive numerical experiments. There were no sufficient proof whether MQ method is nonsingular or uniquely solvable. Micchelli (1986) then proved that RBF was invertible, including Hardy's MQ form.

Throughout the thesis, we will represent the parameter as ε. Few researches have been done in determining the best ε value (Franke, 1979; Ralph and Thomas, 1991; Rippa, 1999; Scott and Sturgill, 2009; Wei et al., 2009; Xiang et al. 2012). The methods proposed to find for the optimal shape parameter in these researches are based on observations from numerical experiments and trial-and-error method.

Hardy (1971) suggested the use of $\varepsilon = 0.815d$ where d is the mean distance from each data point to the nearest neighbour. Franke (1979) suggested the use of $\varepsilon = 1.25D/\sqrt{n}$ where D is the diameter of the minimum circle enclosing all n data points.

Ralph and Thomas (1991) used six different sets of 100 data points and six different test functions. The root-mean-square (RMS) error between the MQ radial basis interpolant and the test function were computed using different ε values in order to determine the most optimal ε. Finally, $\varepsilon = 1/(1+120V)^2$ was chosen as the optimal parameter where V is the variance between the data points.

Rippa (1999) study the effect of number and distributions of data points on the value of ε using MQ, inverse MQ and Gaussian. An algorithm to select ε that minimised a cost function defined by the error between interpolants produced from two sets of data points and nine test functions. Then, by using RMS method, the best shape parameter that resulted the least error is used as the optimal shape parameter. Rippa then presented numerical results involving interpolation using MQ, inverse MQ and Gaussian. Different ε were proposed for different data sets obtained from 10 different known functions. The author concluded that their proposed minimised cost function can be used to determine the shape parameter.

There were a few researches on variable shape parameter where the shape parameter varies in every interval. This proposed shape parameter values are applicable for piecewise function. Biazar and Hosami (2016) proposed an interval $[\varepsilon_{min}, \varepsilon_{max}]$ to the variable shape parameter. The interval is based on the analysis of RMS error between an error function with known function where ε_{min} is the shape parameter with the minimum error and ε_{max} is the shape parameter that resulted in maximum error. The interval proposed is not the optimal interval that can be used in the whole domain since the interval varied according to the data. The numerical results showed that variable parameter within the interval is more accurate.

Scott and Sturgill (2009) then introduced three types of variable shape parameters for MQ, which are exponentially varying shape parameter, a linearly varying parameter and a random shape parameter stratergy. An exponentially varying shape parameter is given by

$$\varepsilon_j = \left[\varepsilon_{min}^2 \left(\frac{\varepsilon_{max}^2}{\varepsilon_{min}^2} \right)^{(j-1)/(N-1)} \right]^{1/2}, \quad j = 1,2,...,N. \tag{2.3}$$

where ε_{min} is the shape parameter with the minimum error and ε_{max} is the shape parameter that resulted in maximum error. A linearly varying parameter is given by

$$\varepsilon_j = \varepsilon_{min} + ((\varepsilon_{max} - \varepsilon_{min})/(N-1))j, \quad j = 0,1,...,N-1; \tag{2.4}$$

and a random shape strategy that uses generated random values ($rand(1,N)$) is given by

$$\varepsilon_j = \varepsilon_{min} + (\varepsilon_{max} - \varepsilon_{min}) \times rand(1,N). \tag{2.5}$$

Xiang et al. (2012) suggested the use of trigonometric variable shape parameter to interpolate one-dimensional interpolation with uniformly spaced nodes in the form of

$$\varepsilon_{min} + (\varepsilon_{max} - \varepsilon_{min}) \times \sin(j), \quad j = 1,...,N. \tag{2.6}$$

The author then compared (2.3) and (2.5) with (2.6) and concluded that MQ radial basis function worked best with trigonometric variable shape parameter than the other two.

Wei et al. (2009) proposed an algorithm to obtain ε by minimising RMS while controlling the runge phenomenon (see also Fornberg and Zuev (2007)). ε is chosen before the runge phenomenon occurs. The author presented numerical examples using MQ basis function in engineering application to approximate the mass, stress and torsion angle of an aeroplane wing. It was shown that MQ produced the best approximation with the right choice of ε. Inappropriate choice of ε resulted in interpolants with a lot of distortions.

Ranjbar (2015) introduced a new variable shape parameter called symmetric variable shape parameter (SVSP) for Gaussian basis function. The strategy refers to the use of different value of shape parameter at each center. Base on Equation (5.1), the new interpolation is given by the following

$$s(x) = \sum_{i=1}^{n} \lambda_i \, e^{-\varepsilon_i^2 (x-x_i)^2}, \; x \in \mathbb{R}. \tag{2.7}$$

Then, the errors between the exact and approximated solutions of the problems were calculated.

However, all these researches were done to obtain an optimal ε value for a visually pleasing curve or for a curve that produced the least error between the interpolant and the test functions. The proposed ε value or the proposed ε intervals are not suitable in shape preserving area. Few researches have been done in searching for the ideal ε value that preserves positivity.

Wu et al. (2010) first approximated positive data by minimising the error between the data values and the approximating function $F(x)$. The function is given by

$$F(x) = \sum_{i=1}^{N} \lambda_i \phi(||x - c_i||),$$

where $\phi(||x - c_i||)$ is a compactly supported radial basis function (CSRBP) with center

c_i and λ_i is the expansion coefficient. The method is by setting $\lambda_i > 0$. The final curve can only manage to approximate the data. Then, the paper continued with method used to interpolate the data. Since there is no free shape parameter involved, the author proposed a method of adding new data to the data set in such a way that the interpolated curve preserves positivity. For each data, an added data was chosen so that the radius of the added data is within its' proposed range. The author did not suggest in detail any number of optimal added data and the position of the added data is suitable to preserve positivity. The author then compared the result with constrained modified quadratic Shepard method (Brodlie et al. 2005). Even though Wu managed to preserve positivity, the curve produced is quite different from the one produced in (Brodlie et al. 2005).

Wu et al. (2010) used multiquadric quasi-interpolation operator given by $(\mathscr{L}_{\mathscr{D}}f)(x)$ to construct the interpolation function. They combined two RBFs which are MQ quasi, and also CSRBP. The CSRBP is used to define an error function $\varepsilon(x,y)$ that act as a minor modification to the MQ quasi interpolation. The final curve resulted in the following curve $y = (\mathscr{L}_{\mathscr{D}}f)(x) + \varepsilon(x,y)$. The data sets used were based on two known functions which were given by $y = x^3 + 2,\ x \in [0,1]$ and $y = \sqrt{1-4x^2},\ x \in [-1/2, 1/2]$. The authors had chosen 100 points as a testing set. Wu (2014) then presented an extension to his work in Wu et al. (2010) where he added scattered data to the analysis.

There were researches done to preserve monotonicity using MQ quasi-interpolation. Beatson and Powell (1992) came out with MQ approximation for scattered data. In this paper, the author had presented three kinds of MQ approximation. Wu then proposed the forth form of MQ quasi-interpolation which