MODIFIED SUMUDU TRANSFORM ANALYTICAL APPROXIMATE METHODS FOR SOLVING BOUNDARY VALUE PROBLEMS

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by

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LIST OF ABBREVIATIONS

ADM	Adomian decomposition method
BVPs	Boundary value problems
CF	Cubic function
DE	Differential equation
FE	Fractional equation
FDE	Fractional differential equation
FPDE	Fractional partial differential equation
HAM	Homotopy analysis method
HPM	Homotopy perturbation method
IVPs	Initial value problems
LF	Linear function
MSTHAM	Modified Sumudu transform homotopy analysis method
ST	Sumudu transform
MSTHPM	Modified Sumudu transform homotopy perturbation method
MSTVIM	Modified Sumudu transform variational iteration method
ODEs	Ordinary differential equations
PDE	Partial differential equation
QF	Quadratic function

RKF45	Runge-Kutta-Fehlberg 45
SRE	Square residual error
STHAM	Sumudu transform homotopy analysis method
STHPM	Sumudu transform homotopy perturbation method
STVIM	Sumudu transform variational iteration method
STM	Sumudu transformation method
VIM	Variational iteration method

LIST OF SYMBOLS

α	Unknown parameter
β	Constant
γ	Constant
Γ	Domain
δ	Convolution property
ε	Parameter
η	Transform parameter
λ	General Lagrange multiplier
λ_1	Lagrange multiplier
λ_2	Lagrange multiplier
τ	Constant
ξ	Variable
Φ	Continuous function
ϕ	Continuous function
χ	Parameter
$\hbar$	Auxiliary convergence parameter (HAM)
$\mathscr{H}$	Homotopy
$H(t)$	Auxiliary function (HAM)

L	Linear operator
N	Nonlinear operator
$\mathscr{N}$	Nonlinear operator (HAM)
p	Embedding parameter (HPM)
q	Embedding parameter (HAM)
$\mathscr{R}$	Linear operator
$\mathbb{S}$	Sumudu Transform
$\mathbb{T}$	Topological space
$\mathbb{Y}$	Topological space
$\mathscr{Z}$	Trial function

KAEDAH HAMPIRAN ANALITIKAL JELMAAN SUMUDU TERUBAHSUAI BAGI PENYELESAIAN MASALAH NILAI SEMPADAN

ABSTRAK

Dalam kajian ini, penekanan diberikan kepada kaedah hampiran analitik. Kaedah-kaedah ini termasuk gabungan jelmaan Sumudu dengan kaedah homotopi usikan, iaitu kaedah usikan homotopi jelmaan Sumudu, gabungan jelmaan Sumudu dengan kaedah ubahan lelaran iaitu kaedah ubahan lelaran jelmaan Sumudu dan akhirnya, gabungan jelmaan Sumudu dengan kaedah analisis homotopi, iaitu kaedah analisis homotopi jelmaan Sumudu. Walaupun kaedah-kaedah standard ini telah berjaya digunakan dalam menyelesaikan pelbagai jenis persamaan pembezaan, ia masih mengalami kelemahan dalam pemilihan tekaan awal. Di samping itu, ia memerlukan bilangan lelaran yang tak terhingga yang memberi kesan negatif kepada ketepatan dan penumpuan penyelesaian. Objektif utama tesis ini adalah untuk mengubah suai, menggunakan dan menganalisis kaedah-kaedah ini untuk mengatasi kesukaran dan kelemahan serta mencari penyelesaian hampiran analitik bagi beberapa kes persamaan pembezaan biasa linear dan tak linear. Kes-kes ini termasuk masalah nilai sempadan dua-titik peringkat kedua, singular serta sistem persamaan bagi masalah nilai sempadan dua-titik peringkat kedua. Bagi kaedah-kaedah yang dicadangkan, fungsi cubaan digunakan sebagai penghampiran awal untuk menyediakan penyelesaian hampiran yang lebih tepat bagi masalah yang dipertimbangkan. Di samping itu, bagi kaedah ubahan lelaran jelmaan Sumudu, suatu algoritma baru telah dicadangkan untuk menyelesaikan pelbagai jenis masalah nilai sempadan dua-titik peringkat kedua yang linear dan tak linear. Dalam algoritma ini, teorem konvolusi telah digunakan untuk mencari suatu pekali Lagrange optimum. Kaedah-kaedah yang dicadangkan memberikan penyelesaian dalam suatu

siri penumpuan yang pantas, yang mana dalam kebanyakan kes, membawa kepada penyelesaian bentuk tertutup. Kaedah-kaedah ini digunakan untuk suatu kelas masalah nilai sempadan yang luas, yang mana keputusan yang diperolehi dibandingkan dengan kaedah-kaedah standard dan antara satu sama lain. Keputusan yang diperoleh mengesahkan keupayaan dan kecekapan kaedah-kaedah terubahsuai ini dalam menyediakan penyelesaian hampiran yang mempunyai ketepatan yang baik, dengan cara yang lebih mudah dan ringkas daripada kaedah-kaedah standard.

MODIFIED SUMUDU TRANSFORM ANALYTICAL APPROXIMATE METHODS FOR SOLVING BOUNDARY VALUE PROBLEMS

ABSTRACT

In this study, emphasis is placed on analytical approximate methods. These methods include the combination of the Sumudu transform (ST) with the homotopy perturbation method (HPM), namely the Sumudu transform homotopy perturbation method (STHPM), the combination of the ST with the variational iteration method (VIM), namely the Sumudu transform variational iteration method (STVIM) and finally, the combination of the ST with the homotopy analysis method (HAM), namely the Sumudu transform homotopy analysis method (STHAM). Although these standard methods have been successfully used in solving various types of differential equations, they still suffer from the weakness in choosing the initial guess. In addition, they require an infinite number of iterations which negatively affect the accuracy and convergence of the solutions. The main objective of this thesis is to modify, apply and analyze these methods to handle the difficulties and drawbacks and find the analytical approximate solutions for some cases of linear and nonlinear ordinary differential equations (ODEs). These cases include second-order two-point boundary value problems (BVPs), singular and systems of second-order two-point BVPs. For the proposed methods, the trial function was employed as an initial approximation to provide more accurate approximate solutions for the considered problems. In addition, for the STVIM method, a new algorithm has been proposed to solve various kinds of linear and nonlinear second-order two-point BVPs. In this algorithm, the convolution theorem has been used to find an optimal Lagrange multiplier. The proposed methods provide the solution in a rapid convergent series, which leads to a closed form of the

solution in the majority of the cases. These methods were applied to a wide class of BVPs, in which the obtained results were compared with those obtained from the standard methods and with each other. The obtained results verified the capability and efficiency of these modified methods in providing approximate solutions with good accuracy, in an easier and simpler way than the standard methods.

CHAPTER 1

INTRODUCTION

1.1 Research Introduction

In the field of mathematics studies, a differential equation (DE) is an equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables. If a DE contains only ordinary derivatives of one or more unknown functions with respect to a single independent variable, it is said to be an ODE. An equation involving partial derivatives of one or more unknown functions of two or more independent variables is called a partial differential equation (PDE). Many phenomena in the engineering and sciences fields can be modeled using linear and nonlinear ODEs with associated supplementary conditions. If the ODE is of second-order and the supplementary conditions are given at two different points, then second-order two-point BVPs result. Such problems often occur in engineering and sciences and many field of study.

Accordingly, ODEs can be classified according to whether the equations are linear or nonlinear. When the dependent variables and all their derivatives only appear in the first degree and are not multiplied together, the DE is linear, otherwise, it is nonlinear (Zill, 2016). A further classification of DEs can be carried out according to the highest ordered derivative, which appears in the equation. Therefore, any DE needs supplementary conditions that correspond to the highest order derivative to solve it. For example, solving a problem that is described by a DE of second order requires two supplementary conditions to obtain a unique solution. If these conditions are given at one starting point, then we have an initial value problem (IVP), and if these conditions are given at two points then we have a two-point BVP.

In general, the exact analytical solution of second order two-point BVPs is usually not available, especially for nonlinear equations because of their complexity. Thus numerical and analytical techniques were used to obtain the approximate solution for such problems. Although numerical approximate methods are applicable to a wide range of practical cases, analytical approximate methods provide highly accurate solutions and subsequently, increase our insights into the natural behavior of complex systems. One of the important advantages of analytical approximate methods involves the ability to provide an analytical representation of the solution that provides better solution information over time intervals. On the other hand, the numerical methods provide solutions in numerical and discretized form, which makes it somewhat complicated in achieving a continuous representation. The focus of this thesis is to study and develop analytical methods for the solution of second-order two-point BVPs as well as systems of BVPs.

1.2 Two-Point Boundary Value Problems (BVPs)

In this thesis, the focus will be on second-order two-point BVPs of the following form:

$$u''(t) = f(t,u,u'), \quad t \in [a,b],$$

with the following boundary conditions:

- Dirichlet: $u(a) = \alpha, \quad u(b) = \beta,$

- Neumann: $u'(a) = \alpha, \quad u'(b) = \beta$,

- Mixed: $u(a) + u'(a) = \alpha, \quad u(b) + u'(b) = \beta$,

where f is a linear or a nonlinear continuous function on the set $A = \{|(t, u, u')|,\ a \leq t \leq b,\ u \in R\}$ and a, b, α, and β are real numbers.

1.3 Motivation

The main motivation of this study is to develop efficient approximate techniques that provide solutions to BVPs. In this regard and in most cases, these types of problems do not have exact analytical solutions and, therefore, several methods for the analytical approximate solutions were used in solving the equations, including the homotopy perturbation method (HPM) (Chun and Sakthivel, 2010; He et al., 2008), variational iteration method (VIM) (Khuri and Wazwaz, 2013; Lu, 2007; Mo and Wang, 2009), and homotopy analysis method (HAM) (Hassan and El-Tawil, 2011; Liao and Tan, 2007). Furthermore, many authors improved these methods that are capable of handling linear, as well as nonlinear boundary value problems, these methods include the works of Niu and Wang (2010), Ghorbani et al. (2011), Shivanian and Abbasbandy (2014), Abbasbandy and Shivanian (2010) and Khuri and Sayfy (2017). Also, these methods have been combined with ST to remove its drawbacks, such as, STHPM (Singh and Devendra, 2011), STVIM (Abedl-Rady et al., 2014) and SSTHAM (Rathore et al., 2012).

Although these analytical approximate methods have been widely used in solving various types of BVPs, several drawbacks of these methods were recurrently reported by many authors. For example, a suitable choice of the initial guess satisfying the boundary conditions is necessary. In addition, an infinite number of iterations is required to obtain the approximate solutions, where at each step, an integration is needed to obtain the results. Also, the general Lagrange multiplier used in the STVIM are restricted. These drawbacks are presented and discussed further in detail in Chapters 3, 4 and 5. Therefore, developing new techniques basing on the existing methods to overcome these drawbacks and reduce the computational work and make computations easier are necessary. Also, motivated by Kilicman and Gadain (2009) approach, the convolution theorem will be employed to find the optimal Lagrange multiplier. This represents the motivation of the present study.

1.4 Problem Statement

The exact analytical solution of second-order two-point BVPs is usually is not available, especially for nonlinear equations because of their complexity. Therefore, several analytical approximate methods such as HPM, VIM, HAM, STHPM, STVIM and STHAM were widely used to provide analytical approximate solutions for this type of differential equations. However, these methods still suffer from the weakness in the choice of the so-called initial guess; in addition, they require an infinite number of iterations which negatively affect the accuracy and convergence of the solutions. Hence, this study aims to develop new techniques which will reduce the volume of calculations introduced by the standard methods. Also, it can remove the task of having to randomly choose the initial guess by setting a specific rule so that the solution algorithms give more powerful.

1.5 Research Objectives

The objectives of this study are as follows:

- To formulate a new modification based on the ST with both methods HPM and HAM, namely the modified Sumudu transform homotopy perturbation method (MSTHPM) and the modified Sumudu transform homotopy analysis method (MSTHAM), respectively, using power series as an initial approximation to solve linear and nonlinear second-order two-point BVPs.

- To develop a new algorithm based on the ST and the VIM which is called the modified Sumudu transform variational iteration method (MSTVIM), using the convolution theory to obtain the optimal general Lagrange multiplier and employing the power series as an initial approximation to solve linear and nonlinear second-order two-point BVPs.

- To apply MSTHPM, MSTVIM and MSTHAM to solve linear and nonlinear singular second-order two-point BVPs as well as systems of this type.

- To investigate the efficiency and the accuracy of MSTHPM, MSTVIM and MSTHAM by comparing with known exact solutions and the existing STHPM, STVIM and STHAM methods.

1.6 Methodology

The methodology of this study is provided and discussed in this section. The focus will be on the STHPM, STVIM and STHAM. The general structure of these methods will be studied. Subsequently, these methods will be constructed and formulated to solve linear and nonlinear second-order two-point BVPs as well as singular and systems of BVPs. This step will provide a basis for the research to follow. New modifications of the STHPM, STVIM, and STHAM will be proposed and applied to solve the linear and the nonlinear second-order BVPs, as well as singular and systems of the BVPs. Numerical experiments will be carried out to illustrate the efficiency of these modifications. The obtained results using the three methods and their modifications will be presented and analyzed in addition to comparisons with exact solutions or known results wherever possible. All the numerical examples in this study will be investigated using Mathematica 11.

1.7 Basic Concepts and Techniques

This section consists of a discussion of the fundamental concepts and techniques which will be used throughout this thesis.

1.7.1 Power Series

In mathematics, a power series (in one variable) is an infinite series of the form (Sánchez-Reyes and Chacón, 2003):

$$\sum_{n=0}^{\infty} a_n (t-c)^n,$$

where a_n represents the coefficient of the n^{th} and c is a constant. a_n is independent of t and may be expressed as a function of n (e.g., $a_n = \frac{1}{n!}$). Power series are useful in analysis since they arise as Taylor series of infinitely differentiable functions. In many situations c (the center of the series) is equal to zero, for instance when considering a Maclaurin series. In such cases, the power series takes the simpler form:

$$\sum_{n=0}^{\infty} a_n \, t^n.$$

1.7.2 Sumudu Transform (ST)

Watugala (1993) introduced a new integral transform, named the ST and further applied it to the solution of ODE in control engineering problems. The ST is defined by the following formula (Eltayeb and Kilicman, 2010):

$$F(\eta) = \mathbb{S}(f(t)) = \frac{1}{\eta} \int_0^{\infty} e^{-\frac{t}{\eta}} f(t) dt,$$

for any function $f(t)$, and $-\tau_1 \leq \eta \leq \tau_2$.

We state the general properties of the ST in the next theorems which are very useful in the study of the DEs.

Theorem 1.1 (Belgacem and Karaballi, 2006)

The ST amplifies the coefficients of the power series function,

$$f(t) = \sum_{n=0}^{\infty} a_n t^n,$$

by sending it to the power series function,

$$F(\eta) = \sum_{n=0}^{\infty} n! a_n \eta^n.$$

So, the linear function $f(t) = c_0 + c_1 t$ transforms to itself, $F(\eta) = c_0 + c_1\eta = f(\eta)$.

Theorem 1.2 (Belgacem et al., 2003)

If $c_1 \geq 0$, $c_2 \geq 0$ and $c \geq 0$ are any constants, and $f_1(t)$, $f_2(t)$ and $f(t)$ are any functions having the ST $F_1(\eta)$, $F_2(\eta)$ and $F(\eta)$, respectively, then

$$\begin{aligned} i.\ \mathbb{S}(c_1 f_1(t) + c_2 f_2(t)) &= c_1\mathbb{S}(f_1(t)) + c_2\mathbb{S}(f_2(t)) \\ &= c_1 F_1(\eta) + c_2 F_2(\eta). \\ ii.\ \mathbb{S}(f(ct)) &= F(c\eta). \\ iii.\ \mathbb{S}\left(t\frac{df(t)}{dt}\right) &= \eta\frac{dF(\eta)}{d\eta}. \end{aligned}$$

The next theorem deals with the affect of the differentiation of the function $f(t)$ on the ST $F(\eta)$.

Theorem 1.3 (Asiru, 2002)

If $F(\eta)$ is the ST of $f(t)$, then the ST of differentiation of the function $f(t)$ for n times is

$$\begin{aligned} i.\ \mathbb{S}(f'(t)) &= \frac{F(\eta) - f(0)}{\eta}, \\ ii.\ \mathbb{S}(f''(t)) &= \frac{1}{\eta^2}F(\eta) - \frac{1}{\eta^2}f(0) - \frac{1}{\eta}f'(0), \\ iii.\ \mathbb{S}(f^{(n)}(t)) &= \frac{1}{\eta^n}F(\eta) - \frac{1}{\eta^n}\sum_{k=0}^{n-1}\eta^k f^{(k)}(0). \end{aligned}$$

where $f^{(0)}(0) = f(0)$, $f^{(k)}(0)$, $k = 1, 2, 3, \ldots, n-1$ are the k^{th} derivatives of the function $f(t)$ evaluated at $t = 0$.

Theorem 1.4 (Belgacem and Karaballi, 2006)

If $\mathbb{S}(f(t)) = F(\eta)$, then:

$$i.\ \mathbb{S}(tf(t)) = \eta^2 \frac{d}{d\eta}F(\eta) + \eta F(\eta).$$

$$ii.\ \mathbb{S}(t^2 f(t)) = \eta^4 \frac{d^2}{d\eta^2}F(\eta) + 4\eta^3 \frac{d}{d\eta}F(\eta) + 2\eta^2 F(\eta).$$

$$iii.\ \mathbb{S}(t^n f(t)) = \eta^n \sum_{k=0}^{n} a_k^n \eta^k F_k(\eta).$$

$$iv.\ \mathbb{S}(t^{n+1} f(t)) = \eta^{n+1} \sum_{k=0}^{n+1} a_k^{n+1} \eta^k F_k(\eta),$$

where $a_0^n = n!$, $a_n^n = 1$, $a_1^n = n!n$, $a_{n-1}^n = n^2$, and for $k = 2, 3, ..., n-2$,

$$a_k^n = a_{k-1}^{n-1} + (n+k)a_k^{n-1}.$$

The next theorem very useful in study of differential equations having non constant coefficient.

Theorem 1.5 (Eltayeb and Kilicman, 2010)

If Sumudu transform of the function $f(t)$ given by $\mathbb{S}(f(t)) = F(\eta)$, then

$$i.\ \mathbb{S}(tf'(t)) = \eta^2 \frac{d}{d\eta}\left(\frac{F(\eta) - f(0)}{\eta}\right) + \eta\left(\frac{F(\eta) - f(0)}{\eta}\right).$$

$$ii.\ \mathbb{S}(tf''(t)) = \eta^2 \frac{d}{d\eta}\left(\frac{F(\eta) - f(0) - f'(0)}{\eta^2}\right) + \eta\left(\frac{F(\eta) - f(0) - f'(0)}{\eta^2}\right).$$

$$iii.\ \mathbb{S}(t^2 f''(t)) = \eta^2 \frac{d^2}{d\eta^2}\left(\frac{F(\eta) - f(0) - f'(0)}{\eta^2}\right) + 4\eta^3 \frac{d}{d\eta}\left(\frac{F(\eta) - f(0) - f'(0)}{\eta^2}\right) + 2\eta^2\left(\frac{F(\eta) - f(0) - f'(0)}{\eta^2}\right).$$

Theorem 1.6 (Eltayeb et al., 2010)

Let $f(t)$ and $g(t)$ having Laplace transforms $F(s)$ and $G(s)$ respectively, and Sumudu transform $M(\eta)$ and $N(\eta)$, respectively. Then the Sumudu transform of the convolution of f and g

$$(f*g)(t) = \int_0^\infty f(t)g(t-\xi)d\xi,$$

is given by

$$\mathbb{S}((f*g)(t)) = \eta M(\eta)N(\eta).$$

1.8 Definition of Homotopy

A homotopy between two continuous functions $f(t)$ and $g(t)$ from a topological space $\mathbb{T}$ to a topological space $\mathbb{Y}$ is formally defined to be a continuous function $\mathscr{H}$: $\mathbb{T} \times [0,1] \to \mathbb{Y}$ from the product of the space $\mathbb{T}$ with the unit interval [0,1] to $\mathbb{Y}$ such that, if $t \in \mathbb{T}$ then (Liao, 2012)

$$\mathscr{H}(t,0) = f(t) \quad and \quad \mathscr{H}(t,1) = g(t).$$

1.9 Accuracy of Solution

For most ODE problems, the exact solutions are unknown. Therefore to check the accuracy of the approximate solution of these problems:

Firstly, we solve the problems by Runge-Kutta-Fehlberg Method (RKF45), then compare the numerical solution obtained by RKF45 with approximate solutions obtained by the analytical methods.

Secondly, we use the square residual error (SRE), which is a measure of how well the approximate solution $u(t)$ satisfies the original ODE. Consider a general nonlinear DE in the form

$$L(u(t)) + N(u(t)) = f(t), \tag{1.1}$$

with boundary conditions

$$\beta(u, \partial u/\partial t), \quad t \in \Gamma, \tag{1.2}$$

where L and N are a linear and nonlinear operators, respectively, $f(t)$ is a known analytical function, β is a boundary operator and Γ is the domain boundary for Ω .
The SRE is defined as

$$\int_a^b R^2(u(t))dt,$$

where a and b are the end points of the interest interval, and $R(u(t))$ is the residual error of Eq.(1.1) which is defined as the following:

$$R(u(t)) = L(u(t)) + N(u(t)) - f(t), \quad t \in [a,b]$$

and $u(t)$ is an approximate solution to Eq.(1.1). The SRE is in general terms a positive number, which is representative of the total error committed by using the approximate solution $u(t)$. The main reason to choose the SRE as an accuracy approach is that it is reliable and independent of numerical simulations. Finally SRE would be zero only for the case where $u(t)$ turns out to be the exact solution of the differential equation (Filobello-Nino et al., 2017).
On the other hand, if the exact solution u_{exact} of a problem is known, then we can directly find the absolute error by calculating $|u_{exact} - u(t)|$.

1.10 Thesis Outline

The thesis is organized into six chapters. Figure 1.1 presents the flow chart of the study. Chapter 2 reviews the previous studies that were recently conducted by many authors to find the approximate solutions of various kinds of differential equations. Chapter 3 investigates the analytical solutions of second-order two-point BVPs as well as singular BVPs by using STHPM and MSTHPM. A new algorithm is proposed, and some numerical examples are tested. A comparison of the results that were obtained by STHPM and MSTHPM with exact solutions is also provided. Moreover, the convergence of MSTHPM is discussed. In Chapter 4, the STVIM and MSTVIM are applied to solve various problems of second-order two-point BVPs. The convolution theorem has been used in the structure of STVIM algorithm which contributed to finding an optimal Lagrange multiplier. Comparison of results by these methods with exact solutions is also given. In Chapter 5, the STHAM and MSTHAM are introduced and applied to solve the problems that were solved in Chapters 3 and 4. Subsequently, a comparison of the obtained results by MSTHPM, MSTVIM and MSTHAM is carried out. In Chapter 6, systems of linear and nonlinear second-order two-point BVPs are solved using MSTHPM, MSTVIM and MSTHAM. Comparisons of the obtained results by MSTHPM, MSTVIM and MSTHAM are performed. Finally, Chapter 7 provides the main results of the study and recommendations are forwarded for further research.

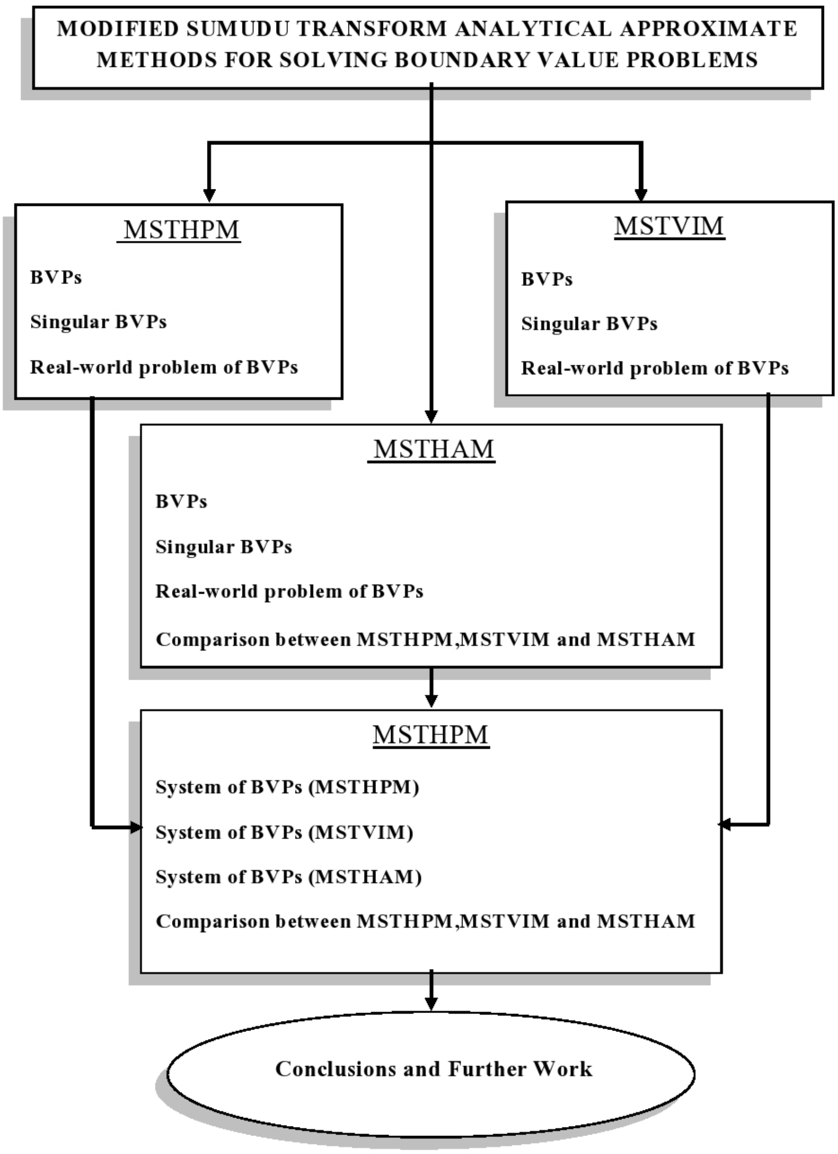

Figure 1.1: Flow chart of the thesis

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

For the last two decades or so, the field of DEs has received considerable attention from mathematicians and research scientists, where some promising analytical approximate and numerical methods were proposed and developed for solving various kinds of DEs. In this chapter, we review recent studies related to find analytical approximate solutions of DEs using the coupling of ST with many analytical approximate methods falling within the area of study interest. Some modifications of these methods and their advantages are reviewed in this chapter. A summary of issues and objectives will be discussed in the last section.

2.2 Sumudu Transformation Method (STM)

Ever since a long time ago, DEs have played an important role in all aspects of mathematics. In order to develop new technological processes, scientific computation is important and it helps in understanding and controlling our natural environment. Analysis of DEs helps in a profound understanding of mathematical problems. Various techniques may be used to solve DEs. In the literature, there are numerous integral transforms that are widely used in physics, astronomy as well as in engineering. The integral transform method is also an efficient method to solve differential equations. Watugala (1993) introduced a new transform named as ST. He applied this new transform to the solution of ODEs and control engineering problems.

The ST possesses many interesting properties such as the scale and unit-preserving properties, that make visualization easier and its application has been demonstrated in the solution of ODEs. The ST helps in solving complex problems in applied sciences and engineering mathematics without resorting to a new frequency domain. This is one of many strength points of this transform, especially with regards to applications in problems with physical dimensions. In fact, the ST which is itself linear, preserves linear functions, and hence in particular does not change units (Belgacem and Karaballi, 2006; Belgacem et al., 2003; Eltayeb and Kilicman, 2010; Kılıçman and Gadain, 2010).

The partial differential equations (PDEs) of the type Maxwell's equations were solved by Hussain and Belgacern (2007) using the ST method. The ST of Maxwell equations provides directly a solution in the time domain without the need for performing an inverse ST. The provided solution as well as its inverse ST, have the same characteristics. They provide equal information about the phenomenon of wave propagation. This property is referred to as the Sumudu reciprocity which is useful in engineering applications that involve solving DEs.

Kilicman and Gadain (2009) proposed the so-called double ST method to solve the linear second-order partial differential of the type wave equations in one dimension having a singularity at the initial conditions. The so-called double convolution theorem was used to solve this type of DEs. In addition, a comparison was made between the double Laplace transform and the double ST. The results showed that there was a high correlation between the two transforms, and that the proposed method was very effective and efficient.

Kiliçman and Eltayeb (2010) applied the ST method to solve the linear ODEs with constant and non-constant coefficients. The results confirmed that the proposed method is both efficient and reliable.

Eltayeb and Kilicman (2010) compared the Sumudu and Laplace transformations by applying both transforms to solve linear ODEs with constant and non-constant coefficients to investigate the differences as well as the similarities. The results showed that the solution is obtained by the Laplace transform in the complex domain, and it is obtained by the ST in the real domain.

The ST of the convolution was proposed and proved by Kiliçman et al. (2010) for matrices. It was used to solve the regular system of DEs. The obtained results proved that the integral transform is quite effective; it can solve the systems of DEs. However, in spite of the usefulness of ST, only a few investigations were found in the literature. In addition, ST is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as, STHPM, STVIM, and STHAM whose literature will be discussed in detail in the next sections.

2.3 Sumudu Transform Homotopy Perturbation Method (STHPM)

The HPM was developed by He (1999a, 2000) by combining the homotopy in topology and classical perturbation techniques to solve many linear and nonlinear DEs because this method is proved to be very effective, simple, and convenient for both weakly and strongly nonlinear BVPs. In spite of the previous features of this method, an infinite number of iterations is required to obtain the accurate approximate solutions, where at each iteration step, an integration is needed to obtain the desired results. Consequently, it was necessary to develop new techniques based on the current method to overcome these defects and reduce the computational work, therefore, making computations easier are necessary. Hence, this method has been combined with other methods such as Laplace transform homotopy perturbation method (LTHPM) (Aminikhah, 2012; Khan and Wu, 2011; Tripathi and Mishra, 2016), variational homotopy perturbation method (VHPM) (Noor and Mohyud-Din, 2008), Elzaki transform homotopy perturbation method (EHPM) (Elzaki and Biazar, 2013), and Sumudu transform homotopy perturbation method (STHPM) (Singh and Devendra, 2011).

Singh and Devendra (2011) proposed the STHPM as a modification of HPM to find the analytical approximate solutions of nonlinear PDEs. The method is an elegant combination of the ST, the HPM and He's polynomials. The proposed method was applied to two examples of nonlinear PDEs with initial conditions. It is worth mentioning that the method is capable of reducing the volume of computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result.

Also, the proposed method was applied by Singh et al. (2013a) to solve nonlinear time-fractional gas dynamics equation with initial conditions. Further, the same problem is solved by the Adomian decomposition method (ADM). The results obtained by the two methods are in good agreement. Therefore, the STHPM has an advantage over the ADM which is, that it solves the nonlinear problems without using Adomian polynomials and hence this technique may be considered as an alternative and efficient method for finding approximate solutions of both linear and nonlinear fractional DEs.

Elbeleze et al. (2013) successfully applied the STHPM for getting the analytical solution of one type of partial fractional DEs, called the Black-Scholes option pricing equation. Two examples with initial conditions from the literature are presented. Further, the same equation is solved by the LTHPM. The results obtained by the two methods are in agreement. The STHPM is a very powerful and efficient method to find approximate solutions for this type of equations.

Kumar et al. (2013) employed STHPM to find the analytical approximate solutions for nonlinear nonhomogeneous fractional partial differential equations (FPDE) with initial conditions, called the Harry Dym equation. Furthermore, the same problem is solved by ADM. The results obtained by the two methods are in good agreement. The STHPM may be considered as a nice refinement in the existing numerical techniques and might find wide applications.

The STHPM was employed by Latifizadeh (2013) to solve partial differentials of the type heat and wave-like equations with initial conditions. The method gives more realistic series solutions that converge very rapidly in physical problems. The fact that the STHPM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Singh et al. (2013b) went deeply into using the STHPM to employ it in solving a system of nonlinear DEs governing the problem of two-dimensional and axisymmetric unsteady flows due to normally expanding or contracting parallel plates. The numerical solutions obtained by the proposed technique indicate that the approach is easy to implement and are computationally very attractive. The proposed method requires less computational work as compared to the other analytical methods.

Rathore et al. (2013) coupled the STHPM with Pade approximants to solve two-dimensional viscous flow with a shrinking sheet. The method is applied in a direct manner without any limitations. The results showed that the STHPM is a powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations. The STHPM could be a promising tool for solving more complex boundary equations.

Furthermore, Singh and Kumar (2014) used the STHPM to solve a certain type of PDEs called the magnetohydrodynamics (MHD) viscous flow due to a stretching sheet. An excellent agreement is achieved by comparing the obtained solution with the HPM and exact solution. The method is applied in a direct manner without the use of linearization, transformation, discretization, perturbation, or restrictive assumptions. The approach gave more practical solutions that converge very rapidly in physical problems. The numerical solutions obtained by the proposed method show that the approach is easy to implement and are computationally very attractive.

The STHPM has been used by Patra and Ray (2014) to evaluate ordinary fractional differential equations (FDEs) with boundary conditions. These equations represent the temperature distribution and effectiveness of convective radial fins with constant and temperature-dependent thermal conductivity. STHPM is a perturbation based iterative technique and it is an effective method for the solution of nonlinear FDEs. In each iteration, the method gave the solution directly as a polynomial expression and this is the main advantage of the method.

Karbalaie et al. (2014) used the STHPM to find the exact solution of nonlinear time-FPDEs with initial conditions. This method has been successfully applied to one- and two-dimensional FDEs and also for systems of more than two linear and nonlinear PDEs. The STHPM is shown to be an analytical method that runs by using the initial conditions only. Thus, it can be used to solve equations with fractional and integer order with respect to time. An important advantage of the new approach is its low computational cost.

Hamed et al. (2014) applied successfully STHPM for finding exact and approximate solutions for linear and nonlinear space-time fractional Schrödinger equation with initial conditions. The efficiency of this method was demonstrated by four numerical examples of a variety of linear and nonlinear equations. The results showed that the proposed method is reliable, effective, and easy to implement and produces accurate results. Thus, the method can be applied to solve other nonlinear FPDEs.

The STHPM method was employed by Singh et al. (2014a) to solve nonlinear FPDEs arising in spatial diffusion of biological populations in animals. The obtained results were compared with Sumudu decomposition method. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and accurate. These results reveal that the proposed method is computationally very attractive. It is worth mentioning that the proposed methods provide the solutions in terms of convergent series with easily computable components in a direct way without any limitations.

Singh et al. (2014b) computed an analytical approximate solution of the system of nonlinear DEs governing the problem of two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability. The numerical results clearly showed that the STHPM is capable of solving two-dimensional problems with successive rapid convergent approximations without any restrictive assumptions or transformations causing changes in the physical definition of the problem.

The PDEs of the type Jeffery-Hamel flow have been solved by Sushila et al. (2014) using STHPM. The results of the proposed method are in excellent agreement with the reproducing kernel Hilbert space method. The numerical solutions obtained by the proposed method indicate that the approach is effective for finding the solution of nonlinear PDEs. The method is straightforward, powerful and efficient technique in finding approximate solution for linear and nonlinear problems.

Yousif and Hamed (2014) applied STHPM to obtain exact analytical solutions of nonlinear non-homogenous time-FPDEs with initial conditions where the solutions were given in closed forms. Thus, this method is powerful, reliable and effective and easy to implement, and can be applied to solve many nonlinear problems in applied science.

The system of nonlinear PDEs with initial conditions, which is derived from the attractor for Keller-Segel was solved by Atangana (2015) using STHPM. The STHPM does not require linearization or the assumption of weak nonlinearity. The solutions are not generated in the form of a general solution, which is the case with the ADM. Moreover, Lagrange multipliers and correction functions are not required, which is the case with the VIM. The STHPM is more realistic compared with other methods used to simplify physical problems. If the exact solution of the PDE exists, the approximate solution rapidly converges to the exact solution using the STHPM.

Touchent and Belgacem (2015) presented STHPM to find the analytical approximate solution for the nonlinear systems of FPDEs with initial conditions. The results showed that the solutions obtained coincide with those of the ADM. However, the STHPM turns out to have a significant advantage over the ADM since it solves the nonlinear problems without the cumbersome need and use of Adomian polynomials.

Kumar et al. (2015) employed STHPM to find the analytical approximate solutions for the fractional multi-dimensional diffusion equations with the initial conditions which describes density dynamics in a material undergoing diffusion. The technique provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. Thus, it can be concluded that the STHPM is very powerful and efficient in finding analytical as well as numerical solutions for a wide class of FPDEs.

Dubey et al. (2015) presented STHPM for solving linear and nonlinear space-time fractional partial Fokker-Planck equations with initial conditions. It is easy to conclude that the solution continuously depends on the space-fractional derivatives and the approximate solutions obtained by using the ADM are the same as those obtained by STHPM. The numerical results showed that the method used is very simple and is straightforward to implement.

The nonlinear partial differential Schrödinger equations with initial conditions were solved by Koçak and Koç (2016) using the STHPM. The proposed method provided the solution in a rapid convergent series which may lead to the solution in a closed form. This method is very efficient, simple and can be applied to other linear and nonlinear problems.

Patra and Ray (2016) presented STHPM to find analytical approximate solutions for the FDEs. The proposed method is a perturbation based iterative technique and it was an effective method in the solution of nonlinear FDEs. In each iteration, the method gives directly the solution as a polynomial expression and this is the main advantage of the method.

The local fractional Tricomi equation with its applications in fractal transonic flow was solved and discussed by Singh et al. (2016) using the local fractional STHPM. The results showed that the proposed technique is very efficient and can be used to solve various kinds of local FDEs. Hence, the introduced method is a powerful tool for solving local fractional linear equations of physical importance.

Zhang et al. (2017) applied STHPM to solve nonlinear systems of time-space FDEs with initial conditions. The advantage of the STHPM is its capability in combining two powerful methods for obtaining exact and analytical approximate solutions for nonlinear systems. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation, or restrictive assumptions. The numerical results indicate that this method is effective and simple in constructing analytic or approximate solutions for fractional coupled systems.

Khader (2017) implemented STHPM to obtain the approximate solutions of the multi-dimensional nonlinear FPDEs of heat-like equations. The obtained approximate solution using the suggested method is in excellent agreement with the exact solution, and shows that these approaches can solve the problem effectively and illustrate the validity and the great potential of the proposed technique.

The fractional partial of Klein-Gordon equations was solved by Kumar et al. (2017) using STHPM. The proposed computational approach is very simple and easy to employ and computationally nice for solving local FDEs arising in various real world problems.

Choi et al. (2017) solved the time-fractional nonlinear nonhomogeneous PDEs with initial conditions by using STHPM. This method gives a series solutions which converge rapidly, and require less computational work and provide high accurate results for systems of nonlinear equations.

Kumar et al. (2018) presented the STHPM to find the analytical approximate solutions for fractional partial of fractal vehicular traffic flow equations. The solutions are presented in a closed form, which are very suitable for numerical computations. The result indicates that the suggested computational schemes are very simple and computationally sound for handling similar kinds of differential equations occurring in natural sciences.

The nonlinear local FPDEs arising in fractal media was solved by Prakash and Kaur (2018) using the STHPM. The numerical solution obtained by the proposed method is in closed form of the exact solution. The proposed numerical technique is