METHODS FOR APPROXIMATING AND STABILIZING THE SOLUTION OF NONLINEAR RICCATI MATRIX DELAY DIFFERENTIAL EQUATION

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by

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LIST OF ABBREVIATIONS

ADM	Adomian Decomposition Method
AREs	Algebraic Riccati Equations
ARMEs	Algebraic Riccati Matrix Equations
DDEs	Delay Differential Equations
$\boldsymbol{\lambda}$**(s)**	General Lagrange Multiplier
HAM	Homotopy Analysis Method
ODEs	Ordinary Differential Equations
RDEs	Riccati Differential Equations
REs	Riccati Equations
RMDDEs	Riccati Matrix Delay Differential Equations
RMDEs	Riccati Matrix Differential Equations
RMEs	Riccati Matrix Equations
VIM	Variational Iteration Method

KAEDAH-KAEDAH UNTUK MENGANGGARKAN DAN MENSTABILKAN PENYELESAIAN PERSAMAAN PEMBEZAAN KELEWATAN MATRIKS RICCATI BUKAN LINEAR

ABSTRAK

Persamaan pembezaan matriks Riccati bukan linear mempunyai bentuk:

$$\dot{X}(t) + X(t)A + A^T X(t) - X(t)BX(t) + C = 0$$

di mana A, B dan C merupakan matriks $n \times n$ dengan $B = B^T$, $C = C^T$ dan $X(t) \in R^{n \times n}$. Persamaan pembezaan matriks Riccati bukan linear ini boleh juga dilihat sebagai suatu persamaan pembezaan kuadratik biasa. Persamaan di atas boleh digunakan secara umum untuk persamaan pembezaan kelewatan dengan hujah-hujah terbantut, di mana jangka kelewatan berlaku sebagai kelewatan masa malar dalam $X(t)$ tetapi bukan dalam $\dot{X}(t)$ (derivatif tersebut akan hilang dan persamaan itu akan menjadi persamaan matriks algebra Riccati selepas keadaan awal digunakan). Tesis ini mempunyai empat matlamat utama. Matlamat pertama adalah untuk mengkaji kaedah lelaran bervariasi dan kemudian, menggunakan teknik ini untuk menyelesaikan persamaan pembezaan matriks Riccati bukan linear dan persamaan pembezaan kelewatan matriks Riccati bukan linear. Pendekatan penyelesaian memerlukan, pada mulanya, derivasi kaedah lelaran bervariasi untuk menyelesaikan jenis persamaan sedemikian dan kemudian, membuktikan penumpuannya kepada penyelesaian yang tepat dalam dua kes dengan dan tanpa kelewatan. Matlamat kedua adalah untuk menggunakan kaedah penguraian Adomian untuk menyelesaikan persamaan pembezaan matriks Riccati bukan linear dalam dua kes dengan dan tanpa kelewatan, dan kemudian, untuk menyatakan dan membuktikan teorem penumpuan untuk kedua-dua kes. Matlamat ketiga adalah untuk melaksanakan kaedah

analisis homotopy untuk menyelesaikan persamaan pembezaan matriks Riccati bukan linear dan persamaan pembezaan kelewatan matriks Riccati bukan linear, dan kemudian, untuk menyatakan dan membuktikan teorem penumpuan untuk kedua-dua kes tersebut. Untuk menunjukkan kecekapan dan ketepatan kaedah-kaedah ini, satu kajian perbandingan telah dijalankan di antara mereka, dan ralat sisa mutlak telah diperolehi. Matlamat keempat, yang boleh dianggap sebagai yang paling penting di antara empat matlamat tersebut, adalah untuk menyelesaikan dan menstabilkan sistem dinamik bukan linear, persamaan pembezaan matriks Riccati bukan linear 2×2 dan persamaan pembezaan kelewatan matriks Riccati bukan linear 2×2 dengan menggunakan dan mengubah suai kaedah 'backstepping' yang dikenali dan berkesan untuk menstabilkan masalah kawalan sempadan.

METHODS FOR APPROXIMATING AND STABILIZING THE SOLUTION OF NONLINEAR RICCATI MATRIX DELAY DIFFERENTIAL EQUATION

ABSTRACT

The nonlinear Riccati matrix differential equation has the form:

$$\dot{X}(t) + X(t)A + A^T X(t) - X(t)BX(t) + C = 0$$

where A, B and C are $n \times n$ matrices such that $B = B^T$, $C = C^T$ and $X(t) \in R^{n \times n}$. This nonlinear Riccati matrix differential equation may also be viewed as a quadratic ordinary differential equation. The above equation may be generalized for delay differential equations with retarded arguments, in which the delay term occurs as a constant time delay in $X(t)$ but not in $\dot{X}(t)$ (the derivative will disappear and the equation will become algebraic Riccati matrix equation after the initial condition is used). In this thesis we study the variational iteration method and use it to solve nonlinear Riccati matrix differential equation and nonlinear Riccati matrix delay differential equations. The solution approach requires, initially, the derivation of the variational iteration method for solving such types of equations and then proof of its convergence to the exact solution in two cases with and without delay. The Adomian decomposition method is then applied for solving nonlinear Riccati matrix differential equation in two cases with and without delay. The convergence theorems are stated and proved for both cases. The homotopy analysis method for solving nonlinear Riccati matrix differential equation and nonlinear Riccati matrix delay differential equations are then studied. To show the efficiency and accuracy of these methods, a comparative study between them is conducted, and the absolute residual error is obtained. The solution and stability of a nonlinear dynamical system, nonlinear 2×2 Riccati matrix differential equation and nonlinear 2×2 Riccati matrix delay differential

equation by using and modifying a well-known and effective backstepping method is then undertaken.

CHAPTER 1
INTRODUCTION

1.1 Background

Functional differential equations have a wide range of applications in science and engineering. The simplest and perhaps most natural type of functional differential equation is a "delay differential equation", that is, a type of differential equations where the time derivatives at the current time depend on the solution, and possibly its derivatives, at previous times instead of a simple initial condition, and an initial history function φ must be specified as the initial condition. Then the equation can be expressed as delay differential equations (DDEs), which are also known as difference-differential equations and were initially introduced in the 18th century by Laplace and Condorcet (Gorecki et al., 1989). However, the rapid development of the theory and the applications of equations did not come until after the Second World War, and continue to current days. A number of applications, in which the delayed argument occurs in the derivative of the state variables and in the state variables itself, exist. Models can be formulated with linear or nonlinear, which are called mixed or advanced DDEs (Sun, 2005).

Delay differential equations arise when the rate of change in a time dependent process in a mathematical model is not only determined by its present state but also at certain past state known as its history. Introduction of delays in models enriches the dynamics of such models and allow a precise description of certain real-life phenomena. Delay differential equations are frequently encountered in various areas and are important in many applications, such as mixing of liquids, population growth and automatic control systems (Driver 1977); mechanics, physics, engineering, economics, biology and technology (Asl and Ulsoy, 2003); signal processing, digital images and control systems (Fridman et al. 2000); lasers and traffic models (Davis, 2003), metal cutting, epidemiology, neuroscience and population dynamics (Kuang, 1993); and chemical kinetics (Epstein and Luo, 1991). Particularly, DDEs are fundamental when ordinary differential equations (ODEs) based models fails. Unlike ODEs where the initial conditions are specified at the initial point, DDEs require the history of the system over the delayed interval and are given as initial conditions. Consequently, delay systems are slightly complex. Thus, DDEs are difficult to analyze analytically, therefore numerical and approximate methods are necessary.

Many different methods, such as variational iteration method (VIM) by He (1997), Adomain decomposition method (ADM) by Evans and Raslan (2005) and homotopy analysis method (HAM), have recently been introduced to solve DDEs. These methods are powerful and efficient and give approximations of higher accuracy and closed-form solutions, if existing (Wawaz, 2007). Variational iteration method is an approximate analytical method (since the solutions are polynomials), which is a modification of the general Lagrange's multiplier method (Inokuti et al., 1978). It has been shown to solve effectively, easily and accurately a wide class of nonlinear problems with approximations that converge rapidly to accurate solutions (He, 1997), (He, 1998), (He, 1999), (He, 2000), (He, 2003), (He et al. 2004), (He, 2006), (He and Wu, 2006), (He, 2007). It was introduced and developed by Chinese mathematician He (1999) in He (2006), He (1998), He (2000), He (2008) and He (2007). It has been used by many authors, including Abassy (2010), Momani and Abuasad (2006), and Wazwaz (2009) and found to be reliable and efficient for various scientific applications. This method provides a solution in the form of rapidly convergent successive approximations that may yield the exact solution, if existing (Wazwaz, 2009). The application of VIM to differential equations usually involves three steps, that is, obtaining the correction functional, identifying the Lagrange multiplier and determining a good initial approximation.

Adomian decomposition method is a powerful approximate analytic technique for strongly nonlinear problems. This technique was introduced and developed by George Adomian, which can be applied successfully to different types of equation and is proven to be powerful and effective and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and linear and nonlinear integral equations. This technique has been successfully proven by many authors, such as Rao (2010) and Gbadamosi et al. (2012). This method involves expressing ODEs, linear or nonlinear, in an operator form, then applying the inverse operator to both sides of the equation written in an operator form and decomposing the unknown function of this equation into a sum of an infinite number of components defined by the decomposition series. It provides an efficient computational procedure for equations and the approximate solution by constructing a series that converges to the exact solution. An important point can be make here is that this method attacks the problem, homogeneous or inhomogeneous, in a straightforward manner without the need for any transformation formulas and does not involve any linearization of assumptions (Wazwaz, 2005); the zeroth component is identified by the terms that arise from integrating the inhomogeneous term and the initial or boundary conditions. The successive terms are determined in a recursive manner (Wazwaz, 2009). The VIM has been shown by many authors to be more powerful than other techniques, such as ADM. One of the advantages of VIM over ADM is that it has no specific requirements for nonlinear operators, whereas ADM suffers from the cumbersome work needed for the derivation of Adomian polynomials for nonlinear terms. Another advantage of VIM over ADM is that it can be used directly with no requirement or restrictive assumptions for nonlinear terms, whereas computational algorithms are used for ADM to handle nonlinear terms (Wazwaz, 2009). He's VIM provides several successive approximations by using the iteration of the correction functional. By contrast, the ADM provides the components of the exact solution, where these components should follow the summation. Moreover, VIM requires the evaluation of the Lagrange multiplier, whereas ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic calculations. Unlike the successive approximations obtained by VIM, ADM provides the solution in successive components that will be added to obtain the series solution. Furthermore, VIM reduces the volume of calculations by not requiring the Adomian polynomials; hence VIM is direct and straightforward. By contrast, ADM requires the use of Adomian polynomials for nonlinear terms, and this requirement leads to additional work. In VIM, the initial condition can be selected freely with some unknown parameters, unlike in ADM. The VIM is strongly and simply capable of solving a large class of linear or nonlinear equations without the tangible restriction of sensitivity to the degree of the nonlinear term (AL-Bar, 2017), homogeneous and nonhomogeneous (Wazwaz, 2007). For nonlinear equations that arise frequently to express nonlinear phenomena, He's variational iteration method facilitates the computational work and provides the solution rapidly compared with ADM (Wazwaz, 2007).

The homotopy analysis method (HAM) is an analytic approximation method for highly nonlinear equations in science, finance and engineering. This method was first proposed by Liao (1992) and was later developed in Liao (1995), Liao (2003), Liao (2004) and Liao (2009); it is a general analytic approach for acquiring series solutions of various types of nonlinear equations, including algebraic equations, ODEs, partial differential equations, differential-integral equations and differential-difference equation. This method is based on homotopy, which is a basic concept in topology. The validity of HAM is independent of whether small/large physical parameters exist in the considered. This method has been successfully applied to various nonlinear problems in science and engineering (Hayat and Sajid, 2007), (Abbasbandy, 2008) and (Liao, 2009). Without performance computer and symbolic computation software, such as Mathematica, solving high order deformation equation rapidly to obtain approximations at high order is impossible. Without performance computer and symbolic computation software, it is also impossible to select a proper value of convergence-control parameter $\hbar$ by analyzing the high order approximation. The solution obtained using HAM depends on the selection of linear operator $\boldsymbol{L}$, auxiliary function $H(t)$, initial approximation $u_0(t)$ and the value of the auxiliary parameter $\hbar$, which is nonzero and allows control over the convergence of the series. We can adjust the region wherein the series is convergent and the rate at which the series converges by varying the auxiliary parameter $\hbar$. Having selected a linear operator, an auxiliary function and an initial approximation, we can solve the deformation equations and develop a solution series. The solution we obtain in this way will still contain the auxiliary parameter $\hbar$, and this solution should be valid for a range of values of $\hbar$. In order to determine the optimum value of $\hbar$, we can plot the so-called $\hbar$-curves of the solution. The $\hbar$-curves will be essentially horizontal over the range of $\hbar$ for which the solution converges. As long as we can select $\hbar$ in this horizontal region, the solution must converge to the exact solution. These curves are obtained by plotting the partial sums $u_m(x,t)$ and/or their first few derivatives evaluated at a specific value of t against parameter $\hbar$. More importantly this method exhibits many advantages, such as providing a direct scheme for solving the problem, that is, without the need for linearization or any transformation (Awawdeh et al. 2009). This method also provides us a simple way to adjust and control the convergence of solution series by using an auxiliary parameter $\hbar$; and it is proved us that well-known ADM is a special case of HAM when $\hbar = -1$ (Cang et al., 2009). Another advantage is that HAM provides with great freedom to select a proper base function that approximates a nonlinear problem (Liao, 2003) and allows us to fine turn the region and rate of convergence of a solution by allowing the auxiliary parameter $\hbar$ to vary. Furthermore, an important property of this method is that any initial approximation that satisfies the initial condition can be used.

Riccati differential equation (RDE) is named after an Italian nobleman, Count Jacopo Francesco Riccati (1676-1754). According to the aforementioned advantages of VIM, ADM and HAM, many authors such as Batiha et al. (2007) applied VIM to solve classical and difficult nonlinear differential equations, such as RDEs; Bulut and Evans (2002) applied ADM to solve RDEs and Das et al. (2016) solved RDE by HAM. The book of Reid (1972) contains the fundamental theories of Riccati algebraic equation with applications to random processes and optimal control, in addition to important engineering science applications that are currently considered classical, such as stochastic realization theory, robust stabilization and network synthesis. The newer applications include areas such as financial mathematics (Anderson and Moore, 1999), (Lasiecka and Triggiani, 1991).

Riccati differential equation is a class of nonlinear differential equations of much importance and plays a significant role in many fields of applied sciences. A more general form of this equation is the nonlinear Riccati matrix differential equation (RMDE). This equation, in one form or another, has an important role and appears in most optimal control problems, multivariable and large scale systems, scattering theory and estimation (Jamshidi, 1980) and (Reid, 1972).

The solution of RMDE is difficult to obtain from two points of view. One is that it is nonlinear, and the other is that it is in matrix form. Most general methods for solving an RMDE with a terminal boundary condition are obtained on transforming this equation into an equivalent linear differential Hamiltonian system (Jodar and Navarro, 1992). By using this approach, the solution of RMDE is obtained by partitioning the transition matrix of the associated Hamiltonian system, (Reid, 1965) and (Razzaghi, 1997). Another class of methods is based on transforming the RMDE into a linear matrix differential equation and then solving it analytically or computationally (Razzaghi, 1978), (Razzaghi, 1979) and in optimal control (Nazarzadeh et al., 1998).

1.2 Motivation

Delay differential equations arise from the inherent time-delays in system components or from the deliberate introduction of time-delays into systems for control purposes. Such time-delays occur often in systems in engineering, biology, chemistry, physics and ecology (Niculescu, 2001). Furthermore, DDEs models can be more effective and accurate compared to ODE based models, when it is necessary to capture oscillatory dynamics with specific periods and amplitudes (Kuang, 1993). Also, Riccati scalar or matrix differential equations are applied to random processes and optimal control in physics and engineering. Moreover, important engineering science applications that are presently considered classical, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, include such areas in financial mathematics (Reid, 1972), (Anderson and Moore,1999), (Lasiecka and Triggiani, 1991). These kinds of differential equations are a class of nonlinear differential equations of considerable importance, for several reasons, an RDE comprises a highly significant class of nonlinear ODEs. Firstly, this equation is closely related to ordinary linear homogeneous differential equations of the second order. Secondly, the solution of RDE possesses a very particular structure in which the general solution is a fractional linear function of the constant limits of integration. Thirdly, the solution of RDE is involved in the reduction of *nth*-order linear homogeneous ODEs (Aminikhah, 2013); (see theorem (3.1)). Due to this importance, we solve such types of equations with the other more general form of nonlinear Riccati matrix differential equation and modify them to nonlinear Riccati matrix delay differential equations (RMDDEs) in connection with the method of steps for solving DDEs to find the approximate analytical solution by VIM, ADM and HAM. These approximations converges rapidly to an accurate solution (He, 1999), (He, 2006) and (Wazwaz, 2009). Finally, nonlinear RMDEs play an important role and appear in most optimal control system design problems, and in order to make the system of nonlinear RMDE and system of nonlinear RMDDE asymptotically stable, we use and propose a modified backstepping method to solve and stabilize such types of equations.

1.3 Research Objectives

The research objectives are as follows:

1- To solve nonlinear RMDE and nonlinear RMDDE by using VIM, by deriving the sequence of approximate solutions and then proving that this sequence of approximate solutions converges to the exact solution in two cases, nonlinear RMDE and nonlinear RMDDEs according to location of delay terms in linear and nonlinear parts. For this objective, a comparative study between the exact and approximate solutions with RK4 is conducted; also the efficiency of the obtained results is validated using residual error.

2- To apply ADM for solving nonlinear RMDE, a comparison with the exact solution RK4, and VIM is conducted ,as well as, solving nonlinear RMDDEs in connection with the method of steps for solving DDEs in which the accuracy of the results are evaluated using residual error, as well as, comparison with the VIM. Then to state and prove the convergence theorems for ADM in two cases, nonlinear RMDE and nonlinear RMDDEs according to location of delay terms in linear and nonlinear parts.

3- To implement HAM for solving nonlinear RMDE, a comparison with the exact solution of VIM, RK4 and ADM is conducted and solving nonlinear RMDDEs in connection with the method of steps for solving DDEs and a comparison with the ADM and VIM in addition to residual error is performed. Then to state and prove the convergence theorems for HAM in two cases, nonlinear RMDE and nonlinear RMDDEs based on location of delay terms in linear and nonlinear parts.

4- To propose and modify the well known backstepping technique to solve and study the stability of nonlinear RMDE and nonlinear RMDDE associated with the method of steps for solving DDEs.

1.4 Methodology

Firstly, we derive the sequence of approximate solutions of RMDE by using VIM and then prove that this sequence of approximate solutions converges to the exact solution. Then the VIM is implemented to find the approximate analytical solution for nonlinear 2×2 RMDE for each component. In this method, the initial condition may be selected freely. We perform a comparative study with the exact solution. Secondly, the nonlinear RMDE is modified to four types of nonlinear RMDDEs in connection with the method of steps. Similarly, we derive the sequence of approximate solutions of these equations using the VIM for each time step and then prove its convergence to the exact solution for two cases based on location of the delay terms in linear and nonlinear parts. Then VIM is implemented to find the approximate analytical solution for nonlinear 2×2 RMDDEs for the four types based on the location of delay terms in connection with the method of steps for solving DDEs. The important point can be make here is that we have used the term of delay in first and second examples in the linear parts while in the third and fourth examples in the nonlinear parts and more importantly is that the term of delay disappears after applying the method of steps for all types and systems that are transformed into a special type of RMDE, which can be solved similarly for the solving system of the first order ODEs.

Adomian decomposition method is implemented to find the approximate analytical solutions for nonlinear 2×2 RMDE and for the four types nonlinear 2×2 RMDDEs in connection with the method of steps for solving DDEs. The convergence theorems in two cases for nonlinear RMDE and nonlinear RMDDEs based on the location of delay terms in linear and nonlinear parts is stated and proven. Firstly, we apply the ADM for nonlinear 2×2 RMDE for each component and compare it with the exact solution and VIM. Secondly, for nonlinear 2×2 RMDDEs for the four types based on the location of delay term in connection with the method of steps for solving DDEs, and the important point can be make here is that we have used the term of delay in the first and second examples in the linear parts while in the third and fourth examples in the nonlinear parts. The term of delay disappears after applying the method of steps for all types and systems transformed to a special type of RMDE, which can be solved similarly for the solving system of the first order ODEs. A comparative study is conducted with VIM and absolute residual error.

Homotopy analysis method is implemented to determine the approximate analytical solutions for nonlinear 2×2 RMDE and nonlinear 2×2 RMDDEs for the four types in connection with the method of steps. The convergence theorems of the approximate solution of nonlinear RMDE and nonlinear RMDDEs based on the location of the delay terms in linear and nonlinear parts is stated and proven. Firstly, for the nonlinear 2×2 RMDE, we implement HAM for each component and compare it with the exact solutions of ADM and VIM. Secondly, for the nonlinear 2×2 RMDDEs for the four types based on the location of delay term in connection with the method of steps for solving DDEs, the important point can be make here is that we have used the term of delay in the first and second examples in the linear parts and in the third and fourth examples in the nonlinear parts. The term of delay disappears after applying the method of steps for all types and systems transformed to a special type of RMDE, which can be solved similarly for the solving system of the first order ODEs. A comparative study is conducted with ADM, VIM and absolute residual error.

Finally, we apply the modified backstepping method to solve and stabilize the dynamical, nonlinear 2×2 RMDE and nonlinear 2×2 RMDDE systems by evaluating *n*-number of Lyapunov functions that depend on system dimension. These functions stabilize the system in *n*-steps by evaluating control functions $u_1, u_2, ..., u_n$, which make those Laypunov functions stabilize system and then evaluate the solution. All results of numerical examples are obtained by Mathematica 10 and Mathcad 14.

1.5 Organization of Thesis

This section presents the organization of thesis as follows:

Chapter 1 contains the introduction of this thesis, along with the literatures of the study. The motivation, objectives and methodology of the study are also introduced.

Chapter 2 presents the literature review for DDEs. Beginning with history and development of methods with application for solving DDEs, nonlinear RMDEs and nonlinear RDEs using different methods in various fields of sciences are provided.

Chapter 3 explains some basic concepts related to our study. A brief description and basic ideas for some approximate analytical methods, such as VIM, ADM and HAM, are presented. Finally, some theorems related to nonlinear RDE and nonlinear RMDE are proven.

Chapter 4 describes the application of VIM for solving nonlinear RMDE with and without time delay in connection with the method of steps for solving DDEs. The convergence theorems for nonlinear RMDE and nonlinear RMDDEs by VIM are stated and proven. This method is investigated to solve several different examples. A comparative is conducted, in which the results are tabulated and analyzed, as well as, the residue error is calculated.

The implementation of ADM for solving nonlinear RMDE and nonlinear RMDDEs in connection with the method of steps to find the approximate analytical solution is presented in Chapter 5. The convergence theorems for nonlinear RMDE and nonlinear RMDDEs by ADM are stated and proven. This method is investigated to solve different numerical examples, in which the results are tabulated and analyzed, and the residual error is obtained. A comparative study with VIM is conducted.

Homotopy analysis method is implemented to solve nonlinear RMDE and nonlinear RMDDEs in connection with the method of steps to find the approximate analytical solutions in Chapter 6. The convergence theorems for nonlinear RMDE and nonlinear RMDDEs by HAM are stated and proven. Several examples are considered, in which the results are tabulated and analyzed, and the residual error is obtained. A comparative study with VIM and ADM is conducted.

A new technique for solving and studying the asymptotic stability of nonlinear RMDE and nonlinear RMDDE by applying the modified backstepping method is discussed in Chapter 7. This approach provides a recursive method for stabilizing the system origin, and numerical examples are considered.

Finally, conclusion and suggestions for future work are presented in Chapter 8.

CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

This chapter provides a review of the literature and previous work of several researchers related to the present study. This chapter comprises four sections. Section 2.1 presents the introduction. The literature review of delay differential equations (DDEs) is presented in section 2.2. Riccati matrix differential equations (RMDEs) are reviewed in section 2.3. A summary of the issues is provided in section 2.4.

2.2 Semi Analytic Methods

Evans and Raslan (2005) solved DDEs using Adomian decomposition method (ADM) to find the approximate analytical solutions. They applied this by using initial value problems. Numerical examples were discussed, and numerical results showed that this method was quite efficient and practically suitable.

Taiwo and Odetunde (2010) studied the iterative decomposition method to determine a numerical solution for DDEs. They showed that the approximate solution of DDEs rapidly converged as an infinite series to the exact solution. Several examples were selected to demonstrate the validity and applicability of this method. Numerical results showed that this method was efficient, accurate and practically suitable.

Mohyud-Din and Yildirim (2010) applied the variational iteration method (VIM) to solve DDEs by using He’s polynomials. Illustrative examples for the proposed combination method were provided, and numerical results showed that this method was efficient and practically suitable.

Also, Chen and Wang (2010) solved neutral functional differential equations with proportional delays using VIM, in which they presented illustrative examples to show the efficiency of this method. They also compared the performance of the method with that of a particular Runge–Kutta method and others by considering neutral functional-differential equations with proportional delay.

Avaji et al. (2012) solved linear and nonlinear Volterra integral equations with time delay using VIM. They identified the general Lagrange multiplier optimally via variational theory, and the initial approximations can be freely selected in a form with unknown constants. The proposed method was highly accurate and rapidly converged to the exact solution.

Rangkuti and Noorani (2012) investigated the exact solution of DDEs by VIM with Taylor series expansion. They constructed the correction functional by using the general Lagrange multiplier for VIM. The terms of delay were considered as restricted variations. Then the Taylor series expansion was used for ignoring the small terms in each iteration, which were obtained via VIM. Furthermore, the exact solution of DDEs was obtained in good agreement with those obtained by previous researchers. Also, this method provided more realistic series solutions that converged rapidly to the exact solution. Numerical examples were selected to show the efficiency and accuracy of the method.

Ogunfiditimi (2015) used ADM to obtain the approximate analytical solution of DDEs, presented several examples, and demonstrated the efficiency of the method.

2.3 Riccati Matrix Differential Equations

Davison and Maki (1973) utilized a fast computational method to solve a Riccati matrix differential equation (RMDE) with finite terminal time by using a linear system and an associated quadratic feedback controller with a minimizes form. An illustrative example for the proposed method was given.

Razzaghi (1978) obtained an analytical solution for RMDE in optimal control. The proposed method calculated the optimal control for linear systems subject to quadratic cost criteria. This method transformed the problem into examining matrix differential equations. This problem could also be programmed and would appear to offer a computational algorithm for the optimal control. Several examples were selected to demonstrate the validity and applicability of the proposed method.

Razzaghi (1979) also presented a computational solution for RMDE. This work was concerned with the solution of the finite time Riccati equation. The solution was presented in terms of the partition of the transition matrix. Matrix differential equations for the partition of the transition matrix were derived and solved using computational methods, that is, this method transformed the problem into examining matrix differential equations of the type, which were extensively studied in the field of control theory. Illustrative examples and computational algorithms were presented to demonstrate the proposed method.

Jodar and Navarro (1992) discussed a closed analytical solution for RMDE. The solution was presented in terms of exponential of matrices related to the problem under the consideration that it was associated with an equivalent linear differential Hamiltonian system.

Razzaghi (1997) presented a Schur method (a method used to find the solution of matrices according to whether a unitary matrix U exists, such that $U^H GU$ is triangular and has diagonal elements as the eigenvalues of G; an orthogonal matrix P also exists, such that $P^H GP$ is quasi- upper-triangular, where each diagonal element is either a 1×1 or 2×2 matrix that have complex conjugate eigenvalues) to obtain an analytic solution for RMDE by presenting the solution in terms of the multiple of two matrices. Several numerical examples were selected to illustrate the validity, applicability and feasibility of the method of solution.

Nazarzadeh et al. (1998) presented a solution for RMDE with a terminal boundary condition (boundary condition with final time), where this solution is given by using the solution of the algebraic form of the Riccati equation. This method solved RMDE with a terminal boundary condition for the linear quadratic optimal control problems then transformed the problem into examining matrix differential equations of the Liapunov type to get the solution. Numerical examples demonstrated the reliability and capability of the proposed method.

Bulut and Evans (2002) investigated the solutions of Riccati differential equations (RDEs) by ADM. The solutions obtained were in the form of a series with simply computable components. Several numerical examples were discussed to find the approximate analytic solution, and then a comparison was conducted with Rung –Kutta and Euler methods. Numerical results showed that the method was efficient, more accurate, easily and practically suitable.

Abbasbandy (2007) introduced a new application of VIM for one-dimensional quadratic RDE by using Adomian's polynomials. A comparison with the ADM and the exact solution were achieved. Finally, the results which obtained revealed that the proposed method was very effective and simple and could be applied to other nonlinear problems.

Batiha et al. (2007) applied VIM to solve a general one-dimensional RDEs by considering two equations; the first one with included a single variable coefficient, and the other was a special matrix form. A correction functional was constructed using VIM by a general Lagrange multiplier, which could be identified via variational theory and showed that VIM yielded an approximate solution in the form of a rapidly convergent series. Furthermore, a comparison with the exact solution and the fourth-order Runge-Kutta method was performed. The numerical results showed that VIM was very powerful, efficient and more accurate in finding the approximate analytical solution, as well as, numerical solutions for wide classes of linear and nonlinear differential equations.

Tan and Abbasbandy (2008) implemented a homotopy analysis method (HAM) for solving quadratic RDEs, selected numerical examples and conducted a comparative study with the ADM and the exact solution. The results indicated that the ADM was a special case of HAM when assuming the auxiliary parameter $\hbar = -1$ and $H(t) = 1$. The numerical results showed that HAM was very effective and simple.

Abazari (2009) applied the differential transform method to find the approximate analytical solution for the RMDE, in which the exact solution was obtained. Several examples were discussed. The numerical results showed that the method was efficient and practically suitable.

Geng et al. (2009) studied a piecewise VIM, which was a modified VIM (MVIM), to solve RDEs. The solutions, which were obtained for RDEs by using the classical VIM, presented good approximations only in the neighborhood of the initial position. However, using the MVIM was proven to obtain good approximations for a larger interval rather than a local vicinity of the initial position, that is, a highly accurate numerical solution was obtained. The advantage of the modified approach over the existing methods for solving this problem was also discussed. The solution obtained using the proposed method was efficient not only for a small values of the state variable but also for a large values.

Ghorbani and Momani (2010) presented an effective VIM algorithm for solving RDEs by proposing an easy-to-use piecewise-truncated VIM algorithm to overcome the shortcomings for the piecewise VIM, which provided a solution as a sequence of iteration that led to the calculation of unnecessary terms that are not needed and considerable time was consumed in repeated calculations for the series solutions. A comparison was conducted with the classical fourth-order Runge–Kutta method to verify that the new method was very effective and convenient for solving RDEs. The numerical results showed that the method was efficient, more accurate and practically suitable.

Rao (2010) solved general one-dimensional RDEs by ADM. Several numerical examples verified the validity and applicability of this method. A Comparison with the exact solution was performed. The numerical results showed that the proposed method was efficient, more accurate and practically suitable.

Geng (2010) introduced a modified VIM to solve RDEs. The RDE solutions which obtained using the traditional VIM presented good approximations only in the neighborhood of the initial position. The advantage of the modified VIM over the existing methods for solving this problem was proven, such that the solution of the considered equation using the proposed method, which was based on selecting an auxiliary parameter generally less than one to adjust the convergence region (to enlarge the convergence region of the sequence of successive approximations), was efficient not only for a small values of the state variable but also for a large values rather than a local vicinity of the initial position. Numerical examples were selected to demonstrate this method, and a comparison with the standard VIM method was conducted. The numerical results showed that the proposed method was more accurate, quite efficient and practically well suited.

Ntogramatzidis and Ferrante (2011) presented an exact solution of the RMDE by establishing closed formulae for the solution of this equation with a terminal condition that involved particular solutions of the associated algebraic Riccati equation. Numerical examples were selected to verify the validity and feasibility of the proposed method.

Taiwo and Osilagun (2012) solved general RDEs by iterative decomposition algorithm to find an approximate analytic solution. Numerical examples were presented to illustrate the accuracy and efficiency of this method. A comparison with the exact solution and Runge–Kutta method was conducted. The numerical results showed that the proposed method was powerful and provided an accurate result with few terms.

Gbadamosi et al. (2012) applied ADM for solving RDE with variable and constant coefficients. Several illustrative examples were discussed, and a comparison of the results which obtained by using this method with the exact solution was conducted. The proposed method was proven more accurate, effective, reliable, powerful and practically well suited.

Allahviranloo and Behzadi (2012) solved a general RDE by using many iterative methods such as ADM, modified Adomian's decomposition method (MADM), VIM, modified variational iteration method (MVIM) and homotopy analysis method. Furthermore, several examples were discussed to demonstrate the capability and accuracy of these methods.

Goharee and Babolian (2014) applied a modified VIM to solve RDE. The modification was based on replacing the integrand involved in the corresponding correction functional by Taylor or Chebyshev expansions. Numerical examples were selected, and comparison with the standard VIM and exact solution was conducted. The numerical results showed that the proposed method was more accurate and very powerful in finding the approximate analytical solution.

Bildik and Deniz (2015) proposed a new technique for solving RDEs by using a modified ADM to improve the effectiveness of this method. The proposed technique was based on using Chebyshev polynomials instead of the Taylor polynomials to expand the source function. Several numerical examples were discussed. The results which obtained revealed that the proposed method was very effective and more accurate than the standard ADM.

Ghomanjani and Khorram (2015) presented an efficient method to study the approximate solution of a quadratic RDE by considering the Bezier curve method as an algorithm to find the approximate solution of the nonlinear RDE. Several examples were selected to demonstrate the capability, reliability, simplicity and efficiency of the proposed method.

Didgar et al. (2016) proposed an effective and simple method to obtain an approximate analytical solution for the RDE with high accuracy by using VIM. The proposed method was based on transforming the general RDE into a second order linear ODE and then applying VIM to solve the transformed equation.

Hamaresha et al. (2016) presented the approximate analytical solution for RDE of fractional order using optimal homotopy asymptotic method. The convergence rate and the region of the solution series was discussed via several auxiliary parameters over the homotopy analysis method that has only one auxiliary parameter which optimally determined. Numerical experiments were selected and a comparative study with the homptopy perturbation method is conducted. Numerical results showed that the proposed method was reliable, efficient and more accurate tool for solving such types equations.

Suresh and Piriadarshani (2016) obtained approximate analytical solution for various kinds of RDEs using differential transform method. Numerical examples are presented. Results which are obtained indicate that this technique is more effective, powerful and provides the solution in a rapidly convergent series with components that are computed elegantly and accurately.

Ismail et al. (2017) applied the VIM- Restrictive Padé to determine the approximate solution for RDEs of fractional order. Several examples are presented and a comparative study with standard VIM and VIM Padé is conducted. Numerical results emphized that the proposed method gives that Padé -VIM is more better than VIM but restrictive Padé VIM is the best result and less error from them.

Kashkari and Saleh (2017) proposed a variational homotopy perturbation method to solve nonlinear RMDEs and find the approximate analytical solution. Many examples are discussed and a comparative study with truncated Taylor series and rational approximation is performed. The results reveal that the proposed method is very effective and simple.

Osintcev and Sobolev (2017) explored the possibility of applying the method of order reduction of optimal estimation problem for singularly perturbed systems with low measurement noise for RMDEs. It is shown that RMDEs for the Kalman-Bucy filter has a periodic solution. They used a combination of geometric and asymptotic approaches for the regularization of RMDEs in some critical cases. Firstly, they found a formal solution for RMDEs as asymptotic expansions.

Pala and Ertas (2017) proposed an analytical method for solving general nonlinear RDEs by a new transformation which reduces RDEs of general type into a second order linear homogeneous differential equation which is readily solvable, therefore, there is no need for extra operation to reduce the transformed equation, the present method is further simple and can be preferred in teaching the solutions of RDEs and does not put any condition on the functions involved in the equation, the method is very general. Several examples are illustrated to explain the ability of the proposed method.

Masjed-Jamei and Shayegan (2017) implemented a numerical method for solving RDEs by adding a suitable real function on both sides of the quadratic RDE. A weighted type of Adams Bashforth rules is proposed for solving it, in which moments are used instead of the constant coefficients of Adams-Bashforth rules. Many examples are selected. Numerical results reveal that the proposed method is efficient and can be applied for other nonlinear problems.

Suresh and Piriadarshani (2017) applied differential transform method, He Laplace method and ADM for solving nonlinear RDEs to determine the approximate analytical solution. Numerical examples were selected and a comparative study