BEZIER CURVE INTERPOLATION ON ROAD DESIGN

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BEZIER CURVE INTERPOLATION ON ROAD DESIGN

by

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

AASHTO	The American Association of State Highway and Transportation Officials
CAD	Computer-aided design
IPS	Institut Pengajian Siswazah
JKR	Jabatan Kerja Raya Malaysia
PLUS	Projek Lebuhraya Utara Selatan Berhad
USM	Universiti Sains Malaysia

LIST OF SYMBOLS

B_i ith Bezier curve

B_i^r rth derivative of ith Bezier curve

C^k parametric continuity of degree k

G^k geometric continuity of degree k

κ curvature

L_j jth interval length

min minimum

n degree

P_i ith control point obtained from parameterization method

P_i' ith control point obtained from satisfying continuity

Q_i ith data point

T affine transformation

t_j jth node

INTERPOLASI LENGKUNG BEZIER PADA REKABENTUK JALAN RAYA

ABSTRAK

Kajian ini bertumpu kepada pembinaan semula lengkung jalan raya dengan menggunakan lengkung Bezier yang dipadankan dengan peta. Kaedah kebiasaan dalam menghasilkan lengkung Bezier yang menggunakan titik kawalan boleh menjadi sangat rumit. Memandangkan lengkung Bezier tidak menginterpolasi titik kawalan tersebut, maka pereka perlu menganggar kedudukan titk kawalan supaya lengkung tersebut dapat dipadankan dengan elok. Bagi memudahkan proses tersebut, kami akan menghasikan lengkung Bezier dengan menggunakan kaedah pemparameteran yang mana maklumat titik data diperlukan dan bukannya seperti kaedah biasa yang memerlukan maklumat titik kawalan. Walaubagaimanapun, kaedah pemparameteran tidak berfungsi dengan baik pada darjah yang lebih tinggi kerana bentuk lengkungan mungkin akan terjejas. Salah satu kaedah untuk menyelesaikan masalah ini adalah dengan menggunakan cebisan lengkung Bezier dibentuk oleh beberapa lengkung Bezier darjah rendah yang telah diparameter. Kami mencadangkan satu kaedah untuk memenuhi ciri-ciri keselanjaran di sepanjang cebisan lengkung Bezier yang telah diparameter ini. Kaedah ini telah diimplementasikan kepada model dua dimensi dan model tiga dimensi. Dengan menggunakan kaedah ini, kami berjaya menghasilkan lengkung Bezier yang dapat menginterpolasi bilangan titik data yang banyak di samping dapat memenuhi sifat-sifat keselanjaran di sepanjang lengkung tersebut.

BEZIER CURVE INTERPOLATION ON ROAD DESIGN

ABSTRACT

This research focuses on reconstructing the road curve by using Bezier curve fitting on a map. The usual way of constructing the Bezier curve is by using control points which can be very tedious. Since Bezier curves do not interpolate the control points, designers need to estimates the position of control points so that the curve fits well. In order to ease up the process, we will construct the Bezier curve by using the parameterization method where the data point information is required instead of the usual way of using control points. However, this method does not work on Bezier curve of high degree as the curves tend to become perturbed. One way of solving the problem is by using piecewise Bezier curve made up of several parameterized Bezier curves of lower degree. We propose a method to satisfy the continuity properties along this piecewise parameterized Bezier curve. The method had been implemented on two-dimensional model and spatial model. By using this method, we manage to construct a Bezier curve that can interpolate high number of data points while satisfying the continuity properties along the curve.

CHAPTER 1

INTRODUCTION

1.1 Definition

Transportation can be defined as a process of moving people or things from one place to another (Transport — Wikipedia, The Free Encyclopedia, 2018). Meanwhile, road is a route on a land that connects two places that has been paved or otherwise improved to allow travel by motorized or non-motorized carriages. So, road transportation can be defined as a process of moving people or things from one place to another by using road. As the roads are no longer constructed solely for the use of cars, now it also used by pedestrian, cyclists, motorcyclists, farm machinery, trucks, and many more. Because of this, designing the transport facility require creative and flexible approaches along with the solutions for the road design to preserve the characteristics of the community since they are the majority users (Faghri et al., 2004).

There are many advantages of road transportation in comparison to other means of transportation. Firstly, the investment required is very less as the cost of construction, operating and maintenance of the road is less than that of the railway, port or airport. Secondly, it enables door-to-door delivery of goods as it provides a very effective means of cartage, loading, and unloading. Sometimes, it is the only way of transportation in rural areas which are not catered to by rail, water or air transport. Road vehicle is not tied to a particular route like the railways (Fenelon, 2017). Besides, for a short distance travel, most people prefer traveling by road as it is more flexible.

However, in spite of various merits, road transportation has major limitations. For instance, there is a higher chance of accidents compared to other means of transportation. Moreover, it has its own issues such as traffic jam especially during peak hour or festival periods. The speed in road transportation is also limited due to the condition of the road at a certain area that involves sharp corner or steep elevation. Due to all of this drawback, some people reconsider another choice of transportation to save a lot of time.

1.2 Problem Statement

By reconstructing curve on a road map, we can extract a lot of information from the formulation of the curve such as curvature value and design speed at any point on the road. There had been researches in road design field that use Bezier curve to reconstruct road map. It shows that Bezier curve has potential to be utilized in these fields. Moreover, the parameterization method that has been introduced later managed to solve problem regarding Bezier curve that unable to interpolate its control points. Parameterized Bezier curve will interpolate data points given by generating suitable sets of control points.

However, parameterized Bezier curve interpolation is not suitable to interpolate high number of data points as the resulting curve may become perturbed near the end-points. In this thesis, we propose a method to interpolate high number of data points by using multiple parameterized Bezier curve of suitable degree that are connected as piecewise curve to ensure the entire curve is well-constructed without any distortion near the end points.

1.3 Aims and Objective of Studies

In this research, we would like to look further on improving the reconstructing of the road map by using Bezier curve. There are two objectives fold in this research which are;

- to develop a piecewise parametrized Bezier curve with G^1 and G^2 continuity to interpolate data points.
- to generate curves representing road map with an accuracy related to visual observation and continuity on the planar and spatial models.

1.4 Structure of Thesis

This thesis consists of 6 chapters. In Chapter 1, we get a brief view about important terms in this research which are road transportation and Bezier curve interpolation. We also review the aim and objective of this research. Besides that, we go through with some past researches that had been done by other researchers all around the world. Chapter 2 contains discussion that had been done by previous researchers regarding the title of this thesis.

Chapter 3 focuses more on the background of the Bezier curve. In other words, we will get to know about defining the formula, control polygon, degree and the properties of Bezier curve. Then, piecewise Bezier curve will be introduced and how continuity aspects play an important role in maintaining the smoothness of it. Meanwhile, in Chapter 4, we will discuss parameterization methods. There are many parameterization methods that had been introduced to overcome the weakness of normal Bezier curve but this research focuses only on the uniform, chordal and centripetal parameterization methods. We also will propose a method to generate piecewise parameterized Bezier curve while maintaining the continuity properties at the joining points.

Chapter 5 is about applying the proposed method to fit the road map. We also extend the fitting for the spatial model by using three-dimensional coordinate as the data points. We will compare the result from the curve fitting process with the actual road.

Lastly, we will conclude our research in Chapter 6. We also have some suggestions for the future researches to be conducted on this topic.

CHAPTER 2

LITERATURE REVIEW

2.1 Road Transportation

Jabatan Kerja Raya (1986) has published a guideline on the geometric design of roads in Malaysia. This guideline was applicable to all new construction and improvement of roads for vehicular traffic managed by Jabatan Kerja Raya. Chapter 4 of the guideline discussed on the element of road design that needs to be considered which are;

- Sight distance: length of the road ahead visible to drivers. This criterion is important because a sight distance of sufficient length can enable the driver to control the speed of vehicles so as to avoid striking an unexpected obstacle.
- Horizontal alignment: it is necessary to establish a proper relationship between the design speed and curvature and the joint relations with superelevation and side friction.
- Vertical alignment: the vertical profile of road affects the performance of the road by providing a gradual change from one tangent grade to another.

Jabatan Kerja Raya (1986) has also stated that the notable function of road is to provide aid in transportation and this can be further divided into two sub-functions which are mobility and accessibility. Mobility is the ability and level of ease of moving goods and services while accessibility is the quality of travel in providing access to various land uses. However, these two sub-functions are in trade-off. To enhance one, the other may be limited. Generally, roads in Malaysia can be categorized into two groups by area which are rural and urban. The design standards for roads are further classified into seven groups (R6, R5, R4, R3, R2, R1, and R1a) for rural areas and into seven groups (U6, U5, U4, U3, U2, U1, and U1a) for urban areas. These are in descending order of hierarchy. Standard R6/U6 is for expressways which functions to provide long-distance travel and require higher mobility by increasing the design speed on that particular road. Jabatan Kerja Raya (1986) has defined design speed as the maximum safe speed that can be maintained over a specified section of the road when conditions are so favorable that the design features of the road govern. The American Association of State Highway and Transportation Officials (AASHTO) had elaborated more on design speed. It is interpreted as a selected speed used to determine the various geometric design features of a roadway (American Association of State Highway and Transportation Officials, 2011). The selected design speed should be high enough so that an applicable regulatory speed limit will not be more than it. It is expected that the speed at which drivers are operating comfortably will be close to the assigned speed limit. Meanwhile, standard R1a/U1a is for roads that serve local traffic. So, they need to focus more on accessibility compared to mobility by providing the intra-community continuity services.

Jabatan Kerja Raya (1986) has allocated different design speed for each design standard of the road as shown in Table 2.1 and Table 2.2. For example, the maximum design speed limit at expressways in rural areas is 110 km/h, but this is further reduced to 100 km/h at rolling terrain and 90 km/h at mountainous terrain. The reduction in allocated design speed is applicable to other design standards as well.

Table 2.1: Design standard for roads in rural area with its maximum design speed limit (Jabatan Kerja Raya, 1986).

Standard	Max design speed limit (km/h)	Minimum lane width (m)	Application
JKR R6	110	3.5	Expressways
JKR R5	100	3.5	Primary roads and highways
JKR R4	90	3.25	Main / secondary roads
JKR R3	80	3.0	Secondary roads
JKR R2	60	2.75	Minor roads
JKR R1	40	5.0*	Single-lane minor roads (country lane)
JKR R1a	40	4.5*	Single lane roads (roads to restricted areas such as quarries)

* - Total width of 2-way road

Table 2.2: Design standard for roads in urban area with its maximum design speed limit (Jabatan Kerja Raya, 1986).

Standard	Max design speed limit (km/h)	Minimum lane width (m)	Application
JKR U6	90	3.5	Expressways
JKR U5	80	3.5	Arterial roads and partial access municipal highways
JKR U4	70	3.25	Arterial / collector roads
JKR U3	60	3.0	Collector roads / local streets
JKR U2	50	2.75	Local streets
JKR U1	40	5.0*	Single-lane street (in towns)
JKR U1a	40	4.5*	Single-lane street (as in low-cost housing areas)

* - Total width of 2-way road

There are several type of research regarding the road design and its impact on the road safety. For example, Srinivasan (1982) noted that an isolated narrow curve right after a road with straight alignment is more dangerous than a succession of curves of the same radius. In addition, O'Cinneide (1998) has stated that horizontal curves have higher accident rates than straight sections of similar length and traffic composition and this differences become apparent at radii less than about 1000m. Moreover, the increase in accident rates become particularly significant at radii below 200m. Lastly, Jiang et al. (2016) have presented a new methodology to summarize the relationship between design speed and operating speed. One of their main findings is that there exists a discrepancy between design speed and operating speed, which may cause safety problems. It is very important to consider all the opinions when choosing the best curve if we want to reconstruct a road so that it will become safer for road user.

Another aspect that related to road transportation is the path planning problems. In recent years, path planning has been discussed a lot as one of the main requirement in autonomous driving. The history of the autonomous car begins with the introduction of world's first radio controlled car, Linriccan Wonder in 1926 (Bimbraw, 2015). Since then, the research on autonomous driving has been growing rapidly. An efficient autonomous vehicle should be able to search for the shortest path while at the same time capable of maintaining the curvature continuity of the path chosen (Han et al., 2010). One of the challenges in path planning algorithms is to avoid producing feasible path which can cause the vehicle to fail in tracking path due to sharp turns. There are many kinds of path planner that has been discussed by researchers and Bezier curve is shown to be a have advantage due to its simplicity and curvature continuity properties (Han et al., 2010).

2.2 Curve Interpolation

In the mathematical field of numerical analysis, curve interpolation is known as a process of constructing a curve that can satisfy known non-noisy data points. In another word, the curve constructed will pass through each data point. For noisy data, there is another method that is quite similar to interpolation which is the curve approximation. Curve approximation is about constructing a smooth function that just approximately fits the data. Mortenson (1997) describes that under an approximating-fitting scheme, a curve must pass reasonably close to the data points but is not required to pass through them. Noisy data points usually can be found in the statistical field. Curve fitting for noisy data is required so that a pattern can be observed which can lead to good estimation to be made. There are numerous types of interpolation method that have been introduced such as Bezier curve interpolation, B-spline curve interpolation, and Lagrange curve interpolation.

2.3 Bezier Curve Interpolation

Bezier curve is a parametric curve frequently used in the computer-aided geometric design. Note that the spelling of “Bezier” and “Bézier” are used interchangeably in many research papers. We will use the “Bezier” style of writing throughout this thesis.

Bezier curve was named after Dr. Pierre Bezier, who invented it in the early 1960’s with Paul de Casteljau as a result of continuous researches on improving curve fitting process (Sederberg, 1997). During that time, he worked as an engineer in the Renault car company. He intended to develop a curve formulation for use in shape design that can be used by designers who have no background in mathematics. Since then, there are researchers who proposed new interpolating curve based on the blending function of Bezier curve. For example, Catmull and Rom (1974) have introduced the Catmull-Rom Curve that manages to interpolate all of its control points. Another interpolating curve arises in this field is proposed by Timmer (1980) which is the Timmer's Parametric Cubic. The curve mimics its control polygon more tightly compared to Bezier curve. Abbas et al. (2014) had shown that Timmer basis functions managed to approximate a circular arc up to 2π (but not including 2π) without resorting to negative weights. As a comparison, the turning angle for rational cubic Bezier curve without negative weight is not more than $4\pi/3$.

Bezier curve interpolation is popular among designer because of the simplicity in the mathematical descriptions. It is proven to be useful in graphics animation and they are widely available in various CAD systems, graphics, and animation packages. They are easy to use in higher dimensions (3D and up). If the user wants to modify the curve, they just need to alter the location of the control points for that curve. In vector graphics, Bezier curves are used to model smooth curve that can be scaled indefinitely. It can also be used in the time domain such as in specifying the velocity over time for an object moving from point A to point B along the curve generated. This can be done by finding the time derivative of the function that defines the curve. Bezier curve also can be stitched together as a piecewise Bezier curve to represent any shape desired.

One way of achieving Bezier curve interpolation is by making each data points as end points of the Bezier curve and put control points in between each data point (number of control points depends on degree desired). Another way to achieve this is by using the parameterization method. Parameterization method does not deal with the blending function but instead, it deals with the way the control point is defined with respect to the node. Parameterization method has been implemented on other parametric curve such as B-spline and Catmull-Rom curve to create curve that can interpolate data. Ahlberg et al. (1967) had proposed the chordal parameterization in which the length of line segment between each node is taken into consideration for parameterization. After that, Lee (1989) had proposed the centripetal parameterization method. He claimed that centripetal parameterization can produce a better curve compared to chordal parameterization. Saux and Daniel (2003) stated that even though many solutions have been proposed for choosing the parameter value, no solution can be considered as the best choice in any situation. Yuksel et al. (2011) showed that for cubic Catmull-Rom curves, centripetal parameterization is the only parameterization in this family that guarantees that the curves do not form cusps or self-intersections within curve segments. So, we can say that centripetal parameterization does a great choice in preventing technical error although it does not guarantee a result that best fit with designer's expectation.

Many type of researches had been done in discussing the usage of the Bezier curve. Rusdi and Yahya (2015) had used the quartic Bezier curve in the reconstruction of Arabic fonts while Roslan and Yahya (2015) has used cubic Bezier curve to generate Japanese fonts with the aid of differential evolution.

In the field of road design, Misro et al. (2015) had used the curvature information of the cubic Bezier curve in approximating the maximum speed. To estimate design speed from curvature value, Misro et al. (2015) fit a road map in Balik Pulau, Malaysia by using Piecewise Cubic Bézier Curve. Point continuity had been considered in joining the piecewise curves. From the mapping, they had calculated the maximum speed for driving by using the curvature information extracted at a certain location along the road. They believe that the estimation of speed is acceptable even though the Bezier curve does not interpolate exactly on the curve of interest. As mentioned in Section 2.1, Bezier curve is shown to be useful in path planning problems. It can help in producing a smooth pre-planned path. Recent research by Khan et al. (2017) had used the Bezier curve approximation for smoothing the square spiral coverage path. It leads to a fast coverage path and less energy consumption required to cover the path. Another recent research by Latip and Omar (2017) shows that Bezier curves have the capability of making the planned path feasible. They also managed to implement the Bezier curves in a path planning algorithm for an autonomous vehicle with kinematic constraints.

We have discussed on the interpolation of the Bezier curve by using the parameterization method. Next, we decide to improve the parameterization method for Bezier curve interpolation in term of the technicality without changing much the way of choosing the parameter value. In facts, the proposed method can be used with all parameterization method discussed previously.

CHAPTER 3

BEZIER CURVE

Bezier curve is one of the well-known interpolating curve used in the design. Historically, other curve forms such as Timmer Curve and Ball Curve evolved independently at several different industrial sites, each faced with the common problem of making free-form curve accessible to designers with no mathematical background. An artist can quickly master the process of designing shapes using Bezier curves by moving the control points and most drawing system like Adobe Illustrator use Bezier curves. This chapter attempts to show the mechanics in the Bernstein polynomials as the blending functions for Bezier curve that made it is superb compared to other interpolating curves.

3.1 Bezier curve

The general Bezier curve of degree n is defined by $n+1$ control points $P_0, P_1, \dots,$ P_n and given by

$$B(t) = \sum_{i=0}^{n} P_i b_{i,n}(t), \quad 0 \leq t \leq 1 \tag{3.1}$$

where $b_{i,n}(t)$ are the Bernstein polynomials or Bernstein basis functions of degree n defined by

$$b_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}(t)^i \tag{3.2}$$

and $\binom{n}{i}$ is the Binomial coefficient defined as

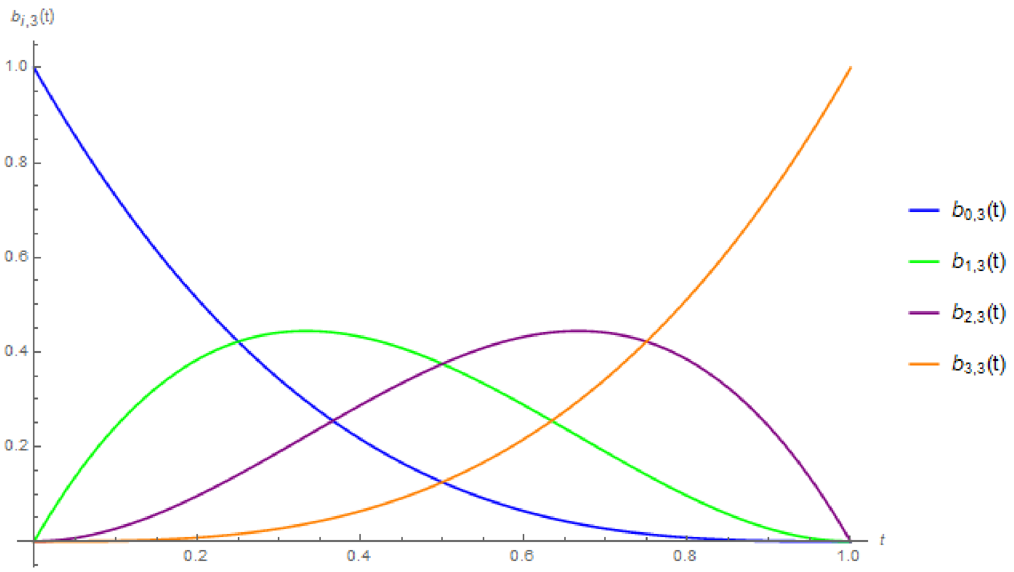

Figure 3.1: Bernstein polynomials of cubic Bezier curve.

$$\binom{n}{i} = \frac{n!}{(n-i)!i!}. \tag{3.3}$$

The Bernstein polynomials of degree n can be illustrated by applying Equation (3.2) for $i = 0, 1, \ldots, n$. For example, the Bernstein polynomials of degree 3 are illustrated in Figure 3.1. It is constructed from four Bernstein polynomials with t ranging from 0 to 1 as follows:

$$\begin{aligned} b_{0,3}(t) &= (1-t)^3, \\ b_{1,3}(t) &= 3(1-t)^2 t, \\ b_{2,3}(t) &= 3(1-t)t^2, \\ b_{3,3}(t) &= t^3. \end{aligned} \tag{3.4}$$

For every value of t in the interval $[0, 1]$, the total value of the four Bernstein polynomials is equal to 1 or in other words,

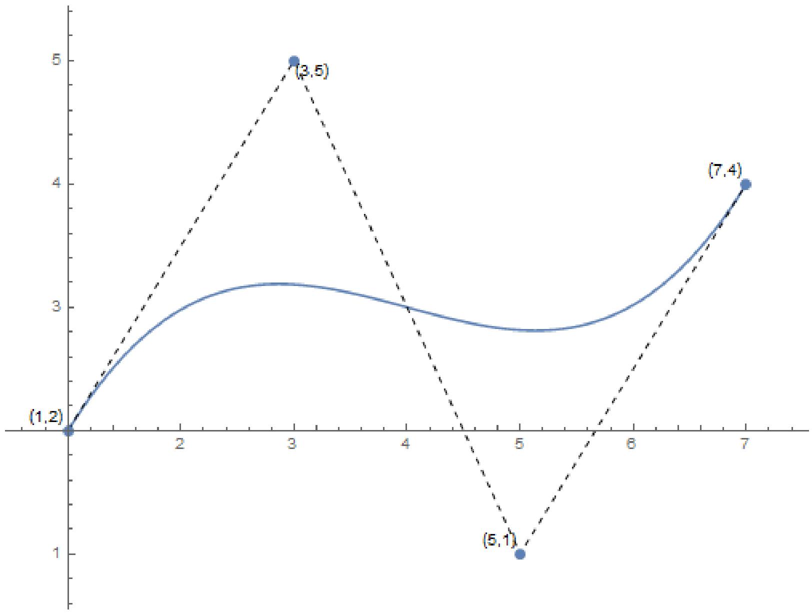

Figure 3.2: A cubic Bezier curve with $(1,2)$, $(3,5)$, $(5,1)$ and $(7,4)$ as its ordered control points. The dashed line represents the control polygon for the Bezier curve.

$$\sum_{i=0}^{n} b_{i,n}(t) = 1, \quad 0 \leq t \leq 1. \tag{3.5}$$

This satisfies the properties of partition of unity that will be discussed later on in Section 3.2

The polygon formed by joining the control points in the specified order is called as control polygon (refer Figures 3.2 and 3.3). Consider a planar Bezier curve of degree 4 (quintic Bezier curve) which has P_0, P_1, P_2, P_3 and P_4 as its ordered control points. Adjusting the position of any control point will always change the entire shape of the curve (Marsh, 2005). Some of the changes may be obvious and the other may be too little to be seen visually. By adjusting the first control point or the last control point of a Bezier curve, the starting point or the finishing point will be changed respectively. However, by changing the other control points, the starting point and the finishing point will remain unchanged, but it may change the starting and finishing directions as illustrated in Figure 3.4. If P_1 is adjusted to a new point, P_1' such that P_0, P_1, and P_1' are

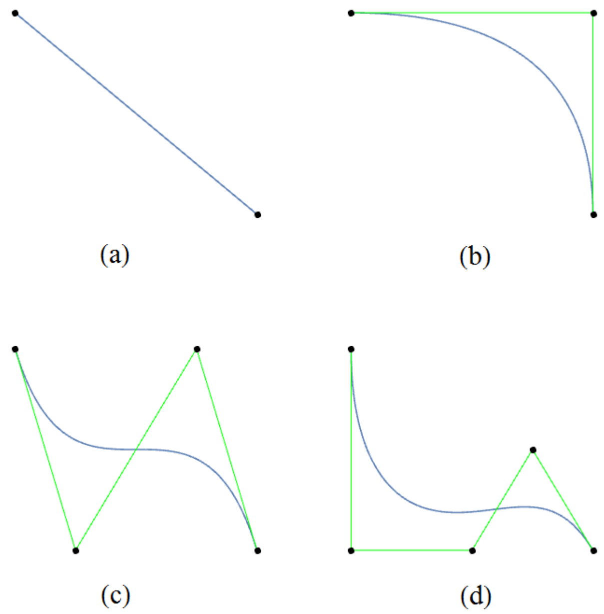

Figure 3.3: Bezier curves of (a) degree 1, (b) degree 2, (c) degree 3 and (d) degree 4. The control polygon for each of the curve is shown by the green line.

collinear, then the initial direction of the curve will not change. The finishing direction also remain unchanged if P_3 is adjusted to a new point, P_3' such that P_3, P_3' and P_4 are collinear. This behavior is important later when we want to construct a piecewise Bezier curve that satisfies continuity at its joining points.

Increasing the degree of a Bezier curve will give more freedom for the user to control the shape of the curve. This is because there are more control points to be manipulated. Moreover, the effect of changing a control point on the shape of the entire curve is decreasing with the higher degree.

3.2 Properties of Bezier Curve

A Bezier curve $B(t)$ of degree n that have control points of P_0, P_1, P_2, ..., P_n will satisfy the following properties (Marsh (2005)):

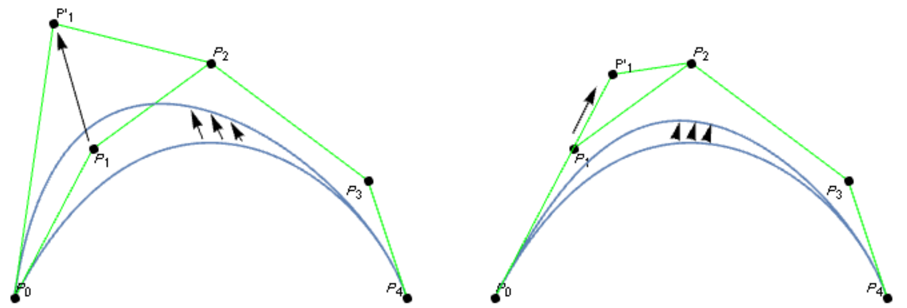

Figure 3.4: The effect of changing a control point from P_1 to P_1' such that initial direction (left) changed, and (right) unchanged.

1. Convex Hull Property

   Given a set of points $X = \{x_0, x_1, ..., x_n\}$ the convex hull of X denoted by $CH\{X\}$, is defined to be the set of points

$$CH\{X\} = \left\{ a_0x_0 + ... + a_nx_n \middle| \sum_{i=0}^{n} a_i = 1, a_i \geq 0 \right\}. \tag{3.6}$$

   Marsh (2005) visualized the convex hull concept as stretching an elastic rubber band surrounding all the points in X. Once the rubber band is released, it may shrink around all or some of the points depend on the distribution of the point itself to form a polygon. The region bounded by the polygon is the convex hull of the set of points X. Several examples of convex hulls are illustrated in Figure 3.5. This property states that for all $t \in [0,1], B(t) \in CH\{P_0, P_1, P_2, ..., P_n\}$. Thus, the Bezier curve will not form or oscillate outside the convex hull. So, it is easier for the user to control the shape of the curve.

2. Endpoint Interpolation Property

   This property will ensure that the curve will interpolate the first and the last the

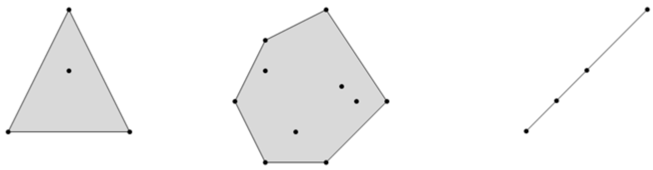

Figure 3.5: Several examples of convex hull.

control point since

$$B(0) = P_0 \text{ and } B(1) = P_n, \tag{3.7}$$

where P_0 is the first control point and P_n is the last control point of the Bezier curve.

3. Endpoint Tangent Property

The starting direction and the finishing direction for $B(t)$ is given by the following equation respectively:

$$B'(0) = n(P_1 - P_0) \text{ and } B'(1) = n(P_n - P_{n-1}). \tag{3.8}$$

As discussed in Section 3.1, this property will be used to satisfy the continuity in piecewise Bezier curve.

4. Variation Diminishing Property

For a planar Bezier curve $B(t)$, this property states that the number of intersections of a given line with $B(t)$ is less than or equal to the number of intersections of that line with the control polygon.

5. Invariance under Affine Transformations

An affine transformation is any transformation that preserves collinearity and the ratio of the distance between point. Preserving collinearity means that all points that lie on a line remain on a line after transformation. Meanwhile, preserving the ratio of the distance between point means that the midpoint of a line segment will remain the midpoint after transformation. Examples of affine transformation are such as translation, reflection, rotation, and enlargement. Let T be an affine transformation. Then

$$T\left(\sum_{i=0}^{n} P_i b_{i,n}(t)\right) = \sum_{i=0}^{n} T(P_i) b_{i,n}(t). \tag{3.9}$$

3.3 Derivatives of Bezier Curve

Many information regarding curve will require the calculation of derivative. For example, the tangent value, normal and curvature. From Equation (3.1), we can see that the control points are independent of the variable t. So, the derivatives of the Bezier curve can be obtained by computing the derivative of each Bernstein polynomials. The first derivative for a Bezier curve of degree n is

$$B'(t) = \sum_{i=0}^{n-1} P_i^{(1)} b_{i,n-1}(t), \tag{3.10}$$

where $P_i^{(1)} = n(P_{i+1} - P_i)$.

The rth derivative for a Bezier curve of degree n is

$$B^r(t) = \sum_{i=0}^{n-r} P_i^{(r)} b_{i,n-r}(t), \tag{3.11}$$

where $P_i^{(r)} = n(n-1)...(n-r+1)\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}P_{i+j}$.

3.4 Curvature of Bezier Curve

Curvature is the amount by which a geometric object deviates from being flat, or straight in the case of line (Misro et al., 2015). The curvature of a Bezier curve

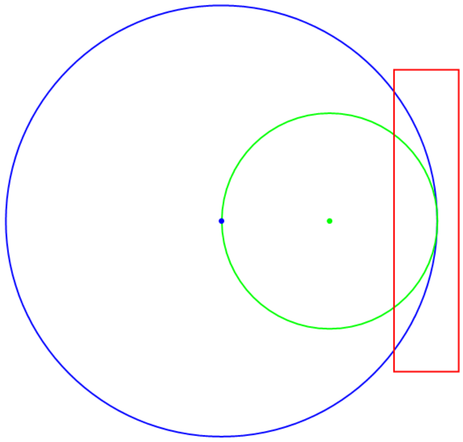

Figure 3.6: Comparing the curvature between two circles of different radius length.

at certain value t is the measurement of how sharply it curves at that instant. For a circle, the curvature of the circumference is inversely proportional to the radius length. Suppose there is a circle with radius length r. Then the curvature value of the circle denoted as κ is

$$\kappa = \frac{1}{r}. \tag{3.12}$$

Increasing the radius length will lower the curvature value of the circle as illustrated in Figure 3.6. We can see that in the red box, it appears that the smaller circle is more curved compared to larger circle. Circle is a special case of curve since it has the same value of curvature everywhere. For other curves including Bezier curve, the curvature value may change throughout it.

Marsh (2005) states that the curvature of a planar curve $C(t) = (x(t), y(t))$ is

$$\kappa(t) = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{3}{2}}}, \tag{3.13}$$

where $\dot{x}(t) = \frac{dx}{dt}$, $\dot{y}(t) = \frac{dy}{dt}$, $\ddot{x}(t) = \frac{d^2x}{dt^2}$ and $\ddot{y}(t) = \frac{d^2y}{dt^2}$.

By applying Equation (3.13) on Bezier curve, we will get

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}. \tag{3.14}$$

According to Gobithasan et al. (2013), the curvature at a certain point is given a positive sign or negative sign if the circle of curvature can be fitted on the left side or right side of the curve respectively. Equation (3.14) can only be used on planar Bezier curve. For a Bezier space curve in three-dimensions or in other words, spatial Bezier curve, the curvature value at the certain value of t is given by

$$\kappa(t) = \frac{\sqrt{(x'y'' - x''y')^2 + (y'z'' - y''z')^2 + (x''z' - x'z'')^2}}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}}. \tag{3.15}$$

Curvature is normally a scalar quantity, but one may also define a curvature vector that takes into account the direction of the bend in addition to its magnitude. The curvature vector for a point p on a curve points to the center of a circle best approximating the curve near p (Kay and Gray, 2005).

3.5 Piecewise Bezier Curve

It is rather tedious to design something with one single Bezier curve of high degree especially if the curve needs to pass through a few data points. To widen the range of shapes without using a curve of very high degree, a number of simpler Bezier curves can be joined end to end to form a single curve called piecewise Bezier curve as shown in Figure 3.7. In order to ensure that the entire curve is smooth especially at the joining point of each curve segment, we need to satisfy certain continuity properties. Next, we will discuss about these continuity properties and how they will help in producing a more reliable curve.

Continuity aspect is very important in piecewise Bezier curve to retain smoothness and to prevent the formation of the sharp corner at each joining point in between curve segments. Since Bezier curve is a polynomial function which is continuous (or C^∞) everywhere, then a piecewise Bezier curve is also continuous everywhere except at the parameter values corresponding to the joins of the individual curves (Marsh, 2005).

For example, we can see that curve (a) in Figure 3.7 is smooth everywhere except at the joint point. This is because there is a difference in the tangent direction between curve B_1 and B_2. Curve (b) in Figure 3.7 is the result after changes have been made on control point $P_{1,3}$ to satisfy the G^1 continuity properties. It appears to be smoother even at the joint point.

The formulation for continuity aspect at joint point in piecewise Bezier curve is as explained by DeRose and Barsky (1988). A piecewise Bezier curve consisting of two Bezier curve segments of degree u and v defined as $B_\alpha(t)$ (with control points $P_{\alpha,0}, P_{\alpha,1}, P_{\alpha,2}, \cdots, P_{\alpha,u}$) and $B_\beta(t)$ (with control points $P_{\beta,0}, P_{\beta,1}, P_{\beta,2}, \cdots, P_{\beta,v}$) respectively are said to be G^0 continuous or to have 0th order geometric continuity when the two adjacent curves share common endpoint or in other word, the two curves are connected end-to-end. Mathematically, G^0 continuity is achieved if

$$B_\alpha(1) = B_\beta(0) \tag{3.16}$$

which gives

$$P_{\beta,0} = P_{\alpha,u}. \tag{3.17}$$

Note that G^0 continuity also implies C^0 or 0th order parametric continuity and vice versa.

Next, we will introduce G^1 continuity or unit tangent continuity. It implies that the two curves shared the same tangent direction at joining point. G^1 continuity can be achieved if G^0 continuity is satisfied and

$$B_{\alpha}^{(1)}(1) = \mu_1 B_{\beta}^{(1)}(0); \mu_1 > 0 \tag{3.18}$$

which gives

$$P_{\beta,1} = \frac{-P_{\alpha,u-1} + P_{\alpha,u} + \mu_1 P_{\alpha,u}}{\mu_1}. \tag{3.19}$$

Note that any value of μ_1 will give piecewise curve that tangentially smooth at joining point but the behaviour of the curve after passing through joining point will be different with each value of μ. C^1 continuity is obtained whenever $\mu_1 = 1$. Basically in addition of same tangent direction, both curves also share same tangent value at joining point.

Lastly, we will introduce G^2 continuity or curvature vector continuity. The connecting curve will have same curvature vector direction at the joining point. It can be achieved if G^0 and G^1 continuity are satisfied with an addition condition which is

$$B_{\alpha}^{(2)}(1) = \mu_1^2 B_{\beta}^{(2)}(0) + \mu_2 B_{\beta}^{(1)}(0); \mu_1 > 0, \mu_2 \in \mathbb{R} \tag{3.20}$$

Different value for μ_1 will affect the curve the same way as in G^1 case while maintaining the curvature value at joining point. However, μ_2 is an arbitrary value and for simplicity, we can let $\mu_2 = 0$. As a result, Equation (3.20) will become

$$B_{\alpha}^{(2)}(1) = \mu_1^2 B_{\beta}^{(2)}(0); \mu_1 > 0. \tag{3.21}$$

which gives

$$P_{\beta,2} = \frac{P_{\alpha,u-2} - 2P_{\alpha,u-1} + P_{\alpha,u} - \mu_1^2 P_{\alpha,u} + 2\mu_1^2 P_{\beta,1}}{\mu_1^2}. \tag{3.22}$$

C^2 continuity is obtained whenever $\mu_1 = 1$. Basically in addition of same curvature direction, both curves also share same curvature value at joining point

Continuity condition of higher degree is possible but in practicality, second order continuity (G^2 or C^2) can produce visually pleasing curve and this thesis concern is only up to G^2 continuity in producing desirable piecewise curve.

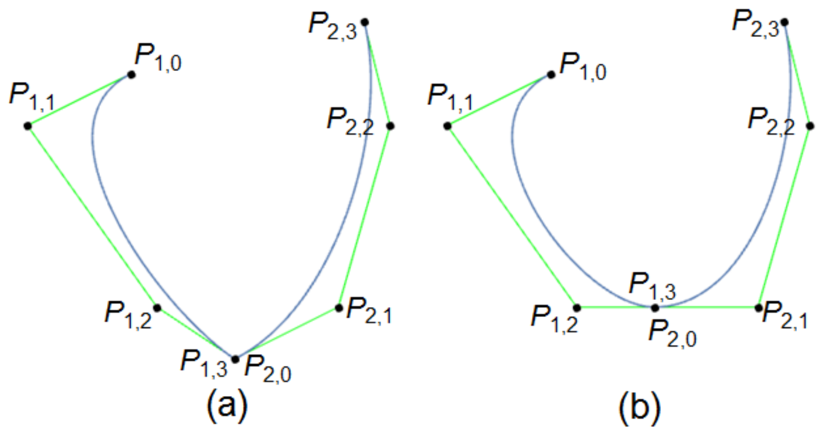

Figure 3.7: Piecewise Bezier consists of 2 curve segments; first curve (made up from $P_{1,0}$, $P_{1,1}$, $P_{1,2}$, and $P_{1,3}$) and second curve (made up from $P_{2,0}$, $P_{2,1}$, $P_{2,2}$, and $P_{2,3}$).