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FINITE DIFFERENCE METHODS FOR LINEAR FUZZY TIME FRACTIONAL DIFFUSION AND ADVECTION-DIFFUSION EQUATION

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by

HAMZEH HUSNI RASHEED ZUREIGAT

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It is very important to learn to be self-sufficient and independent, but the truth is that we cannot do it alone, and at times, we probably should not. We need supporters in every aspect of our life, personal and professional, to be there for us when we need words of encouragement, a vote of confidence. But when one is walking in the journey of life, you just start to thank those who walked beside you and helped you throughout your way.

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LIST OF ABBREVIATIONS

FPDEs	Fuzzy partial differential equations
FFPDEs	Fuzzy fractional partial differential equations
TFDE	Time fractional diffusion equation
TFADE	Time fractional advection diffusion equation
FTFDE	Fuzzy time fractional diffusion equation
FTFADE	Fuzzy time fractional advection diffusion equation
FTFDEs	Fuzzy time fractional diffusion equations
FTFADEs	Fuzzy time fractional advection diffusion equations
FDM	Finite difference schemes
FTCS	Forward time center space
BTCS	Backward time center space
CFTCS	Compact forward time center space
CBTCS	Compact backward time centre space
C-N	Crank-Nicolson

KAEDAH BEZA TERHINGGA UNTUK MASALAH RESAPAN DAN RESAPAN-OLAKAN PECAHAN MASA KABUR LINEAR

ABSTRAK

Persamaan pembezaan pecahan telah mendapat banyak perhatian dalam dekad yang lalu. Ini terbukti daripada bilangan penerbitan mengenai persamaan ini dalam pelbagai bidang sains dan kejuruteraan. Kuantiti yang rapuh dalam persamaan pembezaan pecahan yang tidak tepat dan tidak tentu boleh digantikan oleh kuantiti kabur untuk mencerminkan ketidaktentuan dan ketidakpastian. Persamaan pembezaan separa pecahan kemudian boleh dinyatakan dengan persamaan pembezaan separa pecahan kabur yang dapat memberikan gambaran yang lebih baik untuk fenomena tertentu yang melibatkan ketidakpastian. Penyelesaian analisis persamaan pembezaan separa pecahan kabus biasanya tidak mungkin diperoleh. Oleh itu, wujudnya minat untuk mendapatkan penyelesaian melalui kaedah berangka. Kaedah beza terhingga adalah salah satu kaedah berangka yang kerap digunakan untuk menyelesaikan persamaan pembezaan separa pecahan kerana mudah dan kebolehgunaan sejagatnya. Tumpuan tesis ini adalah pembangunan, analisis dan aplikasi skim beza terhingga dengan kejituan peringkat kedua dan kaedah beza terhingga padat dengan kejituan perngkat empat menyelesaikan persamaan resapan pecahan masa kabur dan persamaan resapan-olakan pecahan masa kabur. Dua teknik komputasi kabur (iaitu nombor kabur tunggal dan gandaan dua parametrik) disiasat. Rumus Caputo digunakan untuk menghitung terbitan pecahan masa kabur. Konsistensi, kestabilan, dan penumpuan kaedah perbezaan terhingga juga diselidiki. Eksperimen berangka dilakukan dan eksperimen menunjukkan keberkesanan dan kelayaka skema yang telah dikembangkan dalam tesis ini. Pendekatan bentuk

parametrik ganda didapati umum, mudah dan pengkomputeran berkesan kerana pemindahan persamaan pentadbir dari ketidakpastian kepada renyah berbanding dengan bentuk tunggal parametrik. Skim Crank Nicolson memberikan hasil yang sedikit lebih tepat daripada kaedah lain yang dipertimbangkan. Ruang pusat masa hadapan adalah kurang tepat berbanding ruang tengah masa ke belakang dan Saulyev adalah paling tidak tepat. Selain itu, skim perbezaan terhingga padat memberikan hasil yang sedikit lebih tepat daripada skim perbezaan terhingga klasik.

FINITE DIFFERENCE METHODS FOR LINEAR FUZZY TIME FRACTIONAL DIFFUSION AND ADVECTION-DIFFUSION EQUATION

ABSTRACT

Fractional differential equations have attracted considerable attention in the last decade or so. This is evident from the number of publications on such equations in various scientific and engineering fields. Crisp quantities in fractional differential equations which are deemed imprecise and uncertain can be replaced by fuzzy quantities to reflect imprecision and uncertainty. The fractional partial differential equation can then be expressed by fuzzy fractional partial differential equations which can give a better description for certain phenomena involving uncertainties. The analytical solution of fuzzy fractional partial differential equations is often not possible. Therefore, there is great interest in obtaining solutions via numerical methods. The finite difference method is one of the more frequently used numerical methods for solving the fractional partial differential equations due to their simplicity and universal applicability. In this thesis, the focus is the development, analysis and application of finite difference schemes of second order of accuracy and compact finite difference methods of fourth order of accuracy to solve fuzzy time fractional diffusion equation and fuzzy time fractional advection-diffusion equation. Two different fuzzy computational techniques (single and double parametric form of fuzzy number) are investigated. The Caputo formula is used to approximate the fuzzy time fractional derivative. The consistency, stability, and convergence of the finite difference methods are investigated. Numerical experiments are carried out and the results indicate the effectiveness and feasibility of the schemes that have been developed. The double parametric form approach is found to be general, easy and

computationally effective due to the transfer of the governing equation from uncertain to crisp as compared with single parametric form. The Crank Nicolson scheme provided slightly more accurate result than the other methods considered. Forward time centre space is less accurate than backward time centre space and Saulyev is the least accurate. Furthermore, the compact finite difference schemes provided slightly more accurate result than classical finite difference schemes.

CHAPTER 1

INTRODUCTION

1.1 General Background

Many phenomena in science and engineering can be expressed as mathematical models. Some of these problems can be modelled by classical ordinary differential equation and some by classical partial differential equations. However, there are certain phenomena that can be more fully and comprehensively described by fractional differential equations. Fractional calculus essentially is differentiation and integration to an arbitrary order and the foundation of fractional calculus was established by well-known mathematicians such as G. W. Leibniz, P. S. Laplace, L. Euler, M. Caputo, J. Louiville, B. Riemann, N. H. Abel, H. Hardy and J. Fourier. In recent years, interest in the fractional calculus has been revived and it has been used to develop models of real-life problems which cannot be adequately modelled using classical differential equations. Fractional partial differential equations can be used for modeling scientific problems in several fields including biology, physics, chemistry and engineering. Fractional partial differential equations, for example, can be utilized for modeling anomalous diffusion in which a cloud of particles spreads at a rate incompatible with the classical Brownian motion pattern (Meershchaert and Tadjeran, 2006). Diffusing species are moving in fractal media and diffusing particle are jostled by collisions with other particles and molecules which prevent the particles from following a straight line and may cause species to diffuse at a slower rate. Existence of attractors in the media, on the other hand, may cause diffusion at a faster rate. For modelling problems such as these, scientists have used derivatives with fractional order instead of integer order in the governing differential equations (Das, 2011). In diffusion equations when the first-order time derivative is replaced by a fractional derivative we obtain the time fractional diffusion equation which describes some problems in physical environments: bacteria cells moving in biofilms, charge carrier motion in amorphous semiconductors and diffusion in critical percolation networks (Klemm, 2002; Kordt et al., 2015; D'Souza and Nagler, 2015).

The anomalous or fractional diffusion equation is given by $\frac{\partial^{\alpha}}{\partial t^{\alpha}} = D \frac{\partial^{2}}{\partial x^{2}}$ (Podlubny, 1998). When $\alpha = 1$, we recover the classical diffusion in which the motion of the diffusion particles is linearly proportional to the time t. Anomalous sub-diffusion ($0 < \alpha < 1$) occurs when the diffusing particles is stuck and hindered for a long time. When the diffusion process is faster than the Brownian motion, this leads to anomalous super-diffusion ($\alpha > 1$). The time fractional diffusion equation with drift component is called time fractional advection-diffusion equation. The fractional advection-diffusion equation can be utilized to describe both speed and movement of particles which are incompatible with the classical Brownian motion pattern. Many natural problems can be modelled by fractional advection-diffusion equations which can give a better description for the problems as compared with classical advection-diffusion equations. These include groundwater hydrology; the applications arise in aerodynamics, groundwater fluid flows in a porous medium and physiology (Meerschaert and Sikorskii, 2012; Horvat and Horvat, 2016; Johnsen et al., 2017).

In reality, real-life problems are often vague and contain uncertainties. This vagueness is known as "stochastic uncertainty" and can be found in certain fields of engineering, manufacturing, medicine, meteorology, and others (Nemati and Matinfar, 2008; Muhamediyeva, 2014; Faran et al., 2011; Hung and Elbert, 2006). The fuzziness can appear in the measurement process, the data collection, experimental part and also it can arise when calculating the initial conditions. These fuzzy aspects can occur when collecting data about materials such as water, microbial populations, soil, etc as described in (Mondal and Roy, 2015; Behzadi et al., 2016; Zhou et al., 2017). Fuzzy sets can be an important tool for handling such problems and to achieve a better understanding of main phenomena. The intention of early research in fuzzy set theory that was carried out by Zadeh (1975) was to generalize the classical concept of a set and provide a proposal to explain the fuzziness. Fuzzy set theory is considered as a tool for modelling vague systems and processing uncertain information in mathematical models. These include using the fuzzy differential equations instead of deterministic differential equations. Studies of the theory of fuzzy partial differential equations (FPDEs) have increased manifold in recent times; the FPDEs are used in modelling robotics, quantum optics, engineering, medicine, gravity and intelligence tests (Omer and Omer, 2013; Angela and Nieto, 2006; Faran et al., 2011; Long et al., 2017). Therefore, both fractional partial differential equations and uncertainty play an essential role in solving mathematical problems. These lead to the fuzzy fractional partial differential equations (FFPDEs).

The analytical solution of FFPDEs is often impractical due to the complexity of the model. Therefore, increasing the interest in obtaining approximate solutions via numerical methods. The understanding and analysis of problems described can be enhanced by the use of numerical methods of solutions. One of the important types of numerical methods is the finite difference methods (FDM) which involves replacing the derivatives with finite difference approximation. As a result, the differential equations will be converted to algebraic equations involving dependent variables at discrete points.

The aim of this thesis is to develop, analyse and apply finite difference methods to solve FFPDEs. In particular, fuzzy time fractional diffusion equations (FTFDEs) and fuzzy time fractional advection diffusion equations (FTFADEs).

1.2 Motivation and significance

Fractional partial differential equations are a generalization of classical partial differential equation which can give a better description of certain complex phenomena (Li and Ding, 2014). Crisp quantities in the fractional partial differential equations which are deemed imprecise and uncertain can be replaced by fuzzy quantities to reflect imprecision and uncertainty. This can be expressed by FFPDEs which have been discussed in numerous scientific articles (Khan and Razzaq, 2015; Long et al., 2017; Senol et al., 2019). Some researchers have resorted to fuzzy fractional diffusion and fuzzy fractional advection-diffusion equation to model their complex physical problems in an efficient way. The FTFDEs differs from the fuzzy conventional diffusion equation in that the first-order time derivative is substituted by a fractional derivative so as to make the fuzzy phenomena global in time. FTFDEs with drift component are called FTFADEs. Recently some authors have studied FTFDE and FTFADE and the details of the techniques used can be found in these papers (Salah et al.,2013; Chakraverty and Tapaswini,2014; Ghazanfari and Ebrahimi, 2015; Abu-Saman and El-Zeri, 2016; Huang et al., 2018). FDM is one of the more frequently used methods for solving the FFPDEs due to their simplicity and universal applicability. From our review of the literature, most of the papers on the solution of fuzzy fractional diffusion equations involve approximate analytical methods as opposed to numerical methods. The primary aim of this thesis is to develop, analyse and apply the finite difference schemes of second order of accuracy and compact finite difference methods of fourth order of accuracy to solve fuzzy fractional partial differential equations particularly, FTFDE and FTFADE. The availability of a wide array of the numerical methods to solve FTFDE and FTFADE will assist researchers in the study of phenomena which can be modelled using FTFDE and FTFADE. Therein lies the main motivation. Another motivation is the increasing use of double parametric form of fuzzy number. In the single parametric form of fuzzy number, a fuzzy equation converts to two crisp equations for the solution. Here, we have to solve the two crisp equations separately to obtain the lower and upper bounds of the solution which increase the computational cost. Hence to reduce the computational cost and increase the accuracy of the solution, the use of the double parametric form of fuzzy number will be investigated. This is also another motivation in this work. The significance of this research lies in the fact that new solution tools are developed. There tools add to the list of technique available to solve fuzzy fractional differential equations.

1.3 Objective

The general objective of this thesis is to develop finite difference method for FFDEs.

The specific objectives of this study are:

1. To formulate and apply finite difference methods for solving the one-dimensional fuzzy time fractional diffusion equation by using two different fuzzy computational techniques based on single parametric form and double parametric form of fuzzy number.

2. To extend the formulation of these finite difference methods to solve the one-dimensional fuzzy time fractional advection-diffusion equations by using a fuzzy computational technique based on the double parametric form of fuzzy numbers.

3. To formulate, analyze, and apply the compact finite difference methods for solving fuzzy time fractional diffusion and advection-diffusion equations in the double parametric form of fuzzy numbers.

4. To establish the consistency, stability convergence and accuracy for each one of these developed methods (the classical finite difference methods and the compact finite difference methods).

1.4 Methodology

The methodology of this study is as follows:

1) The finite difference methods will be studied for crisp time fractional diffusion equation. Then, the FTFDE will be converted to crisp form using two different fuzzy computational techniques (single parametric and double parametric forms of fuzzy number). Next, four finite difference methods which are, forward time centre space (FTCS), Saulyev's, backward time centre space (BTCS) and Crank Nicolson methods are formulated and applied to obtain the uncertain bound of the solution of FTFDE in single and double parametric form. The fractional derivative in the considered equation is estimated using the Caputo formula. The obtained results by the proposed methods are compared with the exact solution. Numerical experiments will be conducted for these methods using wolfram Mathematica10.

2) The FDM will be studied for crisp time fractional advection-diffusion

equation. Then, the FTFADE will be converted to crisp form using double parametric forms of fuzzy number. Next, four finite difference methods which are. FTCS, Saulyev's, BTCS and Crank Nicolson methods are formulated and applied to obtain the uncertain bound of the solution of FTFADE in double parametric form. The fractional derivative in the considered equation is estimated using the Caputo formula. The obtained results by the proposed methods are compared with the exact solution. Numerical experiments will be conducted for these methods using wolfram Mathematica10.

3) The compact finite difference methods will be studied for crisp time fractional diffusion and advection-diffusion equations. Then, the FTFDE and FTFADE will be converted to crisp form using double parametric forms of fuzzy number. Next, four compact finite difference methods which are, compact forward time centre space (CFTCS), compact Saulyev's, compact backward time centre space (CBTCS) and compact Crank Nicolson methods are formulated and applied to obtain the uncertain bound of the solution of FTFDE and FTFADE in double parametric form. After that, the obtained results will be compared with exact solution. Numerical experiments will be conducted for these methods using wolfram Mathematica10.

4) Consistency, stability, and convergence analysis will be implemented by using established techniques to check the feasibility and reliability of the methods.

1.5 Thesis Outline

A description of the chapters contained in this thesis is as follows

Chapter 2: The basic concepts which are required in this study are described.

Chapter 3: We review the literature on numerical method for solving fractional diffusion equations in both crisp and fuzzy form.

Chapter 4: the finite difference schemes used for the solution of one dimensional FTFDE for two different fuzzy computational techniques are presented and discussed in this chapter. A comparative study between the proposed schemes is carried out.

Chapter 5: the finite difference schemes used for the solution of one dimensional FTFADE for double parametric form of fuzzy number are presented and discussed in this chapter. A comparative study between the proposed schemes is carried out.

Chapter 6: the compact finite difference schemes used for the solution of one dimensional FTFDE for double parametric form of fuzzy number are presented and discussed in this chapter. A comparative study between the proposed schemes is carried out.

Chapter 7: the compact finite difference schemes used for the solution of one dimensional FTFADE for double parametric form of fuzzy number are presented and discussed in this chapter. A comparative study between the proposed schemes is carried out.

Chapter 8: The consistency, stability, and convergence analysis of the numerical methods that was developed in this study are investigated.

Chapter 9: The conclusion of our study is discussed in chapter 9

CHAPTER 2

BASIC CONCEPTS AND BACKGROUND

2.1 Introduction

The idea of the fuzzy fractional differential equations (FFDEs) is to form a suitable setting for mathematical modeling of certain real life phenomena. FFDEs take into consideration that the information about the behavior of a dynamical system may be uncertain or involve vague parameters so as to obtain a more practical and flexible model (Buckley and Feuring, 2001). In this chapter, we introduce the basic concepts of fuzzy sets and fractional calculus so as to provide the necessary backdrop for this thesis. We also give a brief overview of some theoretical aspects of the finite difference method.

2.2 Fuzzy set

Fuzzy set theory was introduced by Zadeh (1965) and is considered as a generalization of crisp (classical) set theory (Zadeh, 1975). In crisp sets theory the membership of elements in relation to a set is assessed in binary terms - an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in relation to a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets are an extension of classical set theory since, for a certain universe, a membership function may act as an indicator function, mapping all elements to either 1 or 0, as in the classical notion (Salazar et al., 2012) . A crisp set is normally defined as a collection of elements or objects $x \in X$ which can be, countable, or not countable. Each single element can be either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belongs to A" is true, whereas in the latter case the statement is false (Zadeh, 1965).

Definition 2.1 (Salazar et al., 2012)**:** If x is a collection of objects denoted generally by X, then a $\tilde{A}$ fuzzy set in X is expressed as a set of order pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\},$$

where $\mu_{\tilde{A}}(x) : x \rightarrow [0,1]$ is a membership function of the fuzzy set $\tilde{A}$.

Also membership function can be called a degree of compatibility or degree of truth such that fuzzy $\tilde{A}$ set is totally characterized by this membership function, and the range of membership function is a subset of the non-negative real numbers whose supremum is finite. As we can see in Fig.2.1, The membership function $\mu_{\tilde{A}}(x)$ quantifies the grade of membership of the elements x to the fundamental set X. An element mapping to the value 0 means that the member is not included in the given set, 1 describes a fully included member. Values strictly between 0 and 1 characterize the fuzzy members (Salazar et al., 2012).

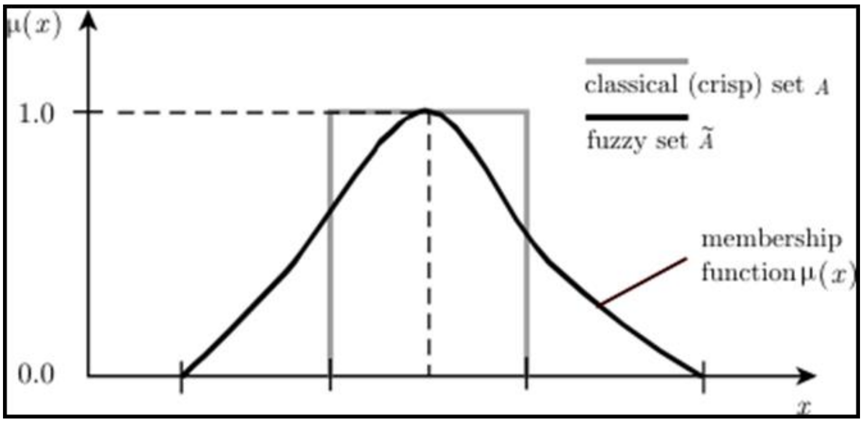

Figure 2.1 Fuzzy set A˜ with classical crisp set

Definition 2.2 (Support of a fuzzy set) (George and Bo, 1995)**:** The support of a fuzzy set $\tilde{A}$ within the universal set X is the set

$$Supp(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\}.$$

The support of a fuzzy set $\tilde{A}$ is the set $Supp(\tilde{A})$ that contains all the elements in X that have nonzero membership grades in $\tilde{A}$.

Definition 2.3 (Convex fuzzy set) (Dong et al., 2010)**:** Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $E(\mathbb{R}^n) = \tilde{E}$ denote the set of all nonempty fuzzy sets in $\mathbb{R}^n$.

A fuzzy set with membership function $\mu_{\tilde{E}}: \mathbb{R}^n \to [0,1]$ is called convex if

$$\mu(\theta x_1 + (1-\theta)x_2) \geq min\{\mu(x_1), \mu(x_2)\},$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $\theta \in (0,1)$. According to (Mahdaoui et al., 2011) a fuzzy set with membership function $\mu: \mathbb{R}^n \to [0,1]$, is called a cone if $\mu(\theta x) = \mu(x)$, for all $x \in \mathbb{R}^n$ and$\theta > 0$. A convex fuzzy cone is a fuzzy cone, which is also a convex fuzzy set.

2.3 The Extension Principle

One of the most fundamental concepts of fuzzy set theory, which can be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle which it used for the fuzzification process (convert the governing equation from the crisp case to fuzzy case) in chapter 4 and chapter 5.

Definition 2.4 (Gerla and Scarpati, 1998)**:** Let X be the Cartesian product of universes $X_1, X_2, \ldots, X_n$ which is denoted by X and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ be n-fuzzy subsets in $X_1, X_2, \ldots, X_n$, respectively, with Cartesian product $\tilde{A} = \tilde{A}_1 \times \tilde{A}_2 \times \ldots \times \tilde{A}_n$ and f is a mapping from X to a universe $Y, \left(y = f(x_1, x_2, \dots, x_n)\right)$. Then, the extension principle allows defining a fuzzy subset $\tilde{B} = f\left(\tilde{A}\right)$ in Y by:

$$\tilde{B} = \left\{\left(y, \mu_{\tilde{B}}(y)\right): y = f(x_1, x_2, \dots, x_n), x_1, x_2, \dots, x_n \in X\right\},$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x_1, x_2, \dots, x_{n \in f^{-1}(y)}} min\left\{\mu_{\tilde{A}_1}(x_1), \dots, \mu_{\tilde{A}_n}(x_n)\right\}, if\ f^{-1}(y) \neq \emptyset \\ 0 \qquad otherwise \end{cases}$$

f^{-1} is the inverse image of f.

Remark (2.1):

For n = 1, the extension principle will be:

$$\tilde{B} = \left\{\left(y, \mu_{\tilde{B}}(y)\right): y = f(x), x \in X\right\},$$

where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_{\tilde{A}}(x)\}, \qquad if\ f^{-1}(y) \neq \emptyset \\ 0 \qquad otherwise \end{cases}$$

which is one of the definitions of a fuzzy function (Ahmad et al., 2013).

2.4 The *r*-Level Sets

The *r*-level sets can be used to prove some results that are satisfied in ordinary sets are also satisfied in fuzzy sets.

Definition 2.5 (Bodjanova, 2006)**:** The *r*-level (or *r*-cut) set of a fuzzy set $\tilde{A}$, labeled as $\tilde{A}_r$, is the crisp set of all x in X such that $\mu_{\tilde{A}} \geq r$ i.e.,

$$\tilde{A}_r = \{x \in X | \mu_{\tilde{A}} > r, r \in [0,1]\},$$

Remark (2.2) (Bodjanova, 2006)**:**

One can also define the strong *r*-level sets as:

$$A_r^+ = \{x \in X | \mu_{\tilde{A}} > r, r \in [0,1]\}$$

The importance of strong r-level set is can cover and satisfied more properties of fuzzy set theory as compared to the classical r-level set. In the strong r-level set it is easily checked that the following properties are satisfied for all r, $s \in [0, 1]$:

1. $\left(\tilde{A} \cup \tilde{B}\right)_r = \tilde{A}_r \cup \tilde{B}_r$.
2. $\left(\tilde{A} \cap \tilde{B}\right)_r = \tilde{A}_r \cap \tilde{B}_r$
3. $\tilde{A} \subseteq \tilde{B}$ gives $\tilde{A}_r \subseteq \tilde{B}_r$
4. If $r \leq s$, then $\tilde{A}_r \supseteq \tilde{A}_s$.
5. $\tilde{A} = \tilde{B}$ if and only if $\tilde{A}_r = \tilde{B}_r$, $\forall\ r \in [0, 1]$.
6. $\tilde{A}_r \cap \tilde{A}_s = \tilde{A}_s$ and $\tilde{A}_r \cup \tilde{A}_s = \tilde{A}_r$, if $r \leq s$.

If $\tilde{A}$ is a fuzzy set, $\{\tilde{A}_r\}$, $\forall\ r \in [0, 1]$ is a family of subsets of the universal set X, then:

$$\tilde{A} = \bigcup_{r \in [0,1]} r\,\tilde{A}_r,$$

This means that all *r*-levels corresponding to any fuzzy set form a family of nested crisp sets, as visually depicted in Figure 2.2.

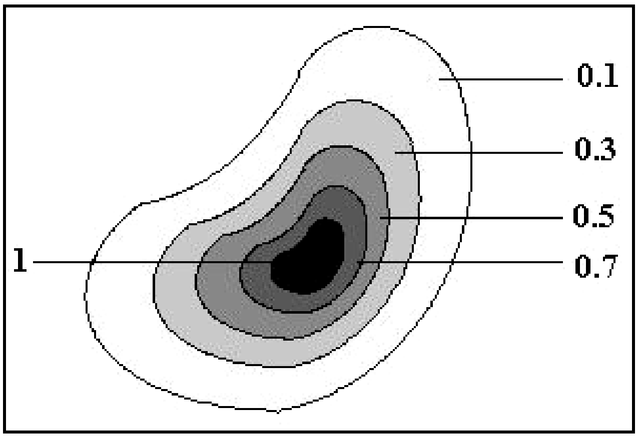

Figure 2.2 Nested r-level sets (Bodjanova, 2006)

2.5 Fuzzy Numbers

Fuzzy numbers are subsets of the real numbers set and represent vagueness values (Dong et al., 2010). Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. Also a fuzzy number can be expressed as a fuzzy set defining a fuzzy interval in the real number $\mathbb{R}$. Since the boundary of this interval is ambiguous, the interval is also a fuzzy set. Generally a fuzzy interval is represented by two endpoints a_1 and a_3 peak point a_2 as $[a_1, a_2, a_3]$ (Figure 2.3). The r-cut operation can be also applied to the fuzzy number. If we denote r-cut interval for fuzzy number $\tilde{A}$ as $[\tilde{A}]_r$, the obtained interval $[\tilde{A}]_r$, is defined as $[\tilde{A}]_r = [a_1{}^{(r)}, a_3{}^{(r)}]$, we can also know that it is an ordinary crisp interval (Figure 2.3).

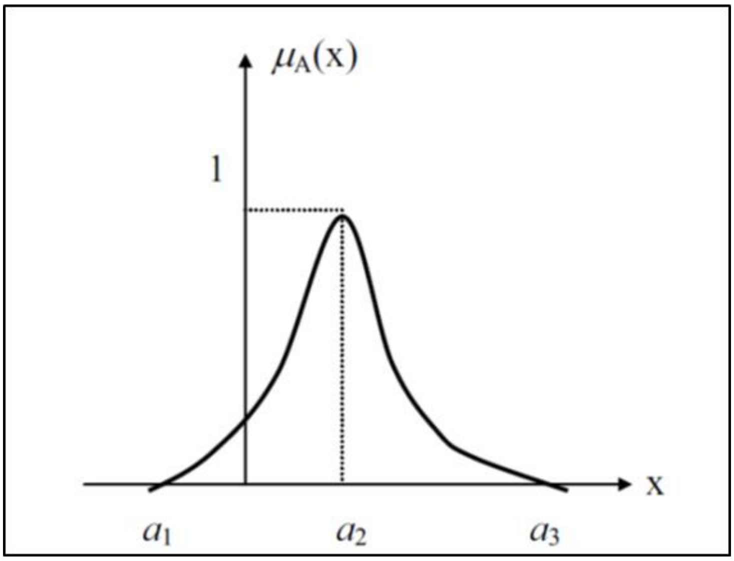

Figure 2.3 Fuzzy Number $\tilde{A} = [a_1, a_2, a_3]$

Definition 2.6 (Kanagarajan and Sambath, 2010)**:** Let $\tilde{E}$ be the set of all upper semi-continuous normal convex fuzzy numbers with r-level bounded intervals such that:

$$[\mu]_r = \{t \in \mathbb{R}: \mu \geq r\} .$$

An arbitrary fuzzy number is represented by an ordered pair of membership functions $[\tilde{\mu}(t)]_r = \left[\underline{\mu}(t), \overline{\mu}(t)\right]_r$ for all $r \in [0,1]$ which is satisfying

1. $\mu(t)$ is normal, i.e $\exists t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$.
2. $\mu(t)$ is convex fuzzy set ,i.e.$\mu(\lambda t + (1-\lambda)s) \geq \min\{\mu(t) , \mu(s)\} \quad \forall t, s \in \mathbb{R}, \lambda \in [0,1]$.
3. $\forall \mu \in \tilde{E}, \mu$ is upper semi continuous on R; $\{x \in R; \mu(t) > 0 \}$ is compact.
4. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over [0,1].
5. $\overline{\mu}(t)$ is a bounded left continuous non-increasing function over [0,1].
6. $\underline{\mu}(t) \leq \overline{\mu}(t)$, for all $r \in [0, 1]$.

The r-level sets of any fuzzy number are much more effective as representation forms of fuzzy set than the above properties. Also, according to

(Zadeh, 1978), fuzzy sets can be defined by the families of their r-level sets based on the resolution identity theorem.

Definition 2.7 (Triangular fuzzy Number): A fuzzy number μ, is called a triangular fuzzy number if defined by three numbers $\alpha < \beta < \gamma$ where the graph of μ (x) is a triangle with the base on the interval $[\alpha, \gamma]$ and vertex at $x = \beta$, as can see in Fig 2.4 and its membership function has the following form (Kanagarajan and Sambath, 2010):

$$\mu\ (x;\alpha,\beta,\gamma) = \begin{cases} 0, & if\ \ x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & if\ \ \alpha \le x \le \beta \\ \dfrac{\gamma-x}{\gamma-\beta}, & if\ \ \beta \le x \le \gamma \\ 0, & if\ \ x > \gamma \end{cases}$$

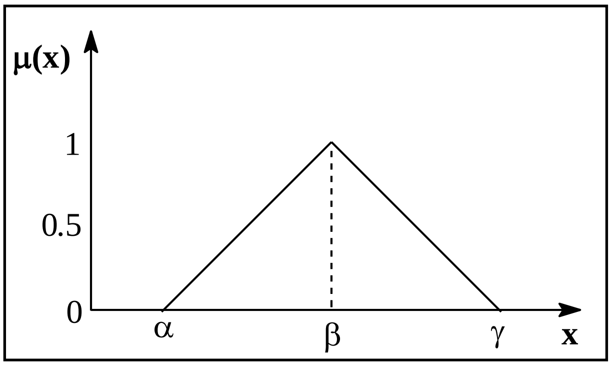

Figure 2.4 Triangular Fuzzy Number

and its r-level is

$$[\tilde{\mu}]_r = [\alpha + r\ (\beta - \alpha),\ \gamma - r\ (\gamma - \beta)],\ r \in [0, 1].$$

Definition 2.8 (Gaussian fuzzy Number):

The membership function $\mu_U(x)$ of an arbitrary asymmetrical Gaussian fuzzy number, $U = \{\delta, \sigma_1, \sigma_r\}$ is defined as followes **(**Chakraverty and Tapaswini, 2014):

$$\mu_U(x) = \begin{cases} e^{-\frac{(x-\delta)^2}{2\sigma_1{}^2}} , & for\ x \leq \delta \\ e^{-\frac{(x-\delta)^2}{2\sigma_r{}^2}} , & for\ x \leq \delta \end{cases} \quad \forall x \epsilon R$$

where $\delta, \sigma_1, \sigma_r$ are the modal value, denote the left-hand and right-hand spreads (fuzziness) corresponding to the Gaussian distribution.

Definition 2.9 (Trapezoidal fuzzy Number):

A fuzzy number μ is called a Trapezoidal fuzzy number if defined by four numbers $\alpha < \beta < \gamma < \delta$ where the graph of $\mu\ (x)$ is a Trapezoid with the base on the interval $[\alpha, \delta]$ and vertex $x = \beta$, $x = \gamma$ as can be seen in Fig. 2.5 and its membership function has the following form (Dubois and Prade, 1980):

$$\mu\ (x;\alpha,\beta,\gamma,\delta) = \begin{cases} 0, & \text{if } x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq x \leq \beta \\ 1, & \text{if } \beta \leq x \leq \gamma \\ \dfrac{\delta-x}{\delta-\gamma}, & \text{if } \gamma \leq x \leq \delta \\ 0, & \text{if } x > \delta \end{cases}$$

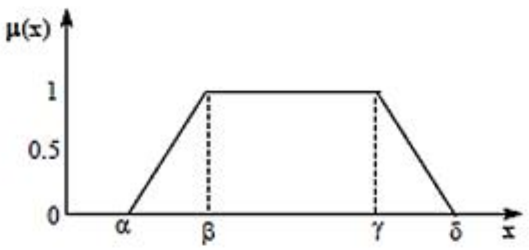

Figure 2.5: Trapezoidal Fuzzy Number

The r-level set of trapezoidal fuzzy number can be defined as follows:

$$\forall\ r \in [0,1], [\tilde{\mu}]_r = [(\beta - \alpha)r + \alpha, \delta - (\delta - \gamma)r]$$

Definition 2.10: double parametric form of fuzzy numbers (Chakraverty and Tapaswini, 2014**)**

Using the single parametric form, we have $\widetilde{U} = [\underline{u}(r), \overline{u}(r)]$. Now this may can be written as crisp number using double parametric form

$$\widetilde{U}(r, \beta) = \beta\left[\overline{u}(r) - \underline{u}(r)\right] + \underline{u}(r) \text{ where } r \text{ and } \beta \in [0,1].$$

2.6 Fuzzy Function

Fuzzifying a crisp function of crisp variable is a function which produces images of crisp domain in a fuzzy set.

Definition 2.8 (Fard, 2009)**:** A mapping $\tilde{f}: T \to \tilde{E}$ (or $\tilde{P}(E)$) for some interval $T \subseteq \tilde{E}$ Is called a fuzzy function or fuzzy process with non-fuzzy variable (crisp), and we denote *r*-level set by:

$$[\tilde{f}(t)]_r = \left[\underline{f}(t;r), \overline{f}(t;r)\right], t \in T, r \in [0,1],$$

where $\tilde{E}$ defined in (Fard, 2009). That is to say, the fuzzifying function is a mapping from a domain to a fuzzy set of range. Fuzzifying function and the fuzzy relation coincide with each other in the mathematical manner.

2.7 Fuzzy Differentiation

Definition 2.11 (Salahshour, 2011)**:** Let $D_H\left(\left[\tilde{a}, \tilde{b}\right]_r\right)$ be the Hausdorff distance between two fuzzy set (or fuzzy numbers) $\tilde{a}, \tilde{b} \in \tilde{E}$ such that

$$D\left(\left[\tilde{a}, \tilde{b}\right]\right) = sup\left\{D_H\left(\left[\tilde{a}, \tilde{b}\right]_r\right) \middle| r \in [0,1]\right\},$$

and $(\tilde{E}, D)$ is a complete metric space. $\tilde{E}$ is the set of all upper semi-continuous normal convex fuzzy numbers with bounded *r*-level set. Since the *r*-cuts of fuzzy numbers are always closed and bounded, such that the intervals are $[\tilde{\mu}(t)]_r = \left[\underline{\mu}(t), \overline{\mu}(t)\right]_r$, $t \in \mathbb{R}$, $\forall\, r \in [0,1]$ .Let $[\tilde{a}(r)] = [\underline{a}(r), \overline{a}(r)], [\tilde{b}(r)] = [\underline{b}(r), \overline{b}(r)]$ be two fuzzy numbers in definitions (2.7-2.8), for $s \geq 0$. According to (Lowen, 1980), we can define the addition and multiplication between to fuzzy numbers by s as follows

1- $(\underline{a+b})(r) = \left(\underline{a}(r) + \underline{b}(r)\right)$

2- $(\overline{a+b})(r) = \left(\overline{a}(r) + \overline{b}(r)\right)$

3- $(\underline{sa})(r) = s.\underline{a}(r), (\overline{sa})(r) = s.\overline{a}(r)$.

Now let: $\tilde{E} \times \tilde{E} \to \mathbb{R} \cup \{0\}$,

$D\left(\left[\tilde{a}, \tilde{b}\right]_r\right) = sup_{\gamma \in [0,1]} Max\{|\underline{a}(r) - \underline{b}(r)|, |\overline{a}(r) - \overline{b}(r)|\}$ be the Hausdorff distance between fuzzy numbers where the following properties are well-known (Rebecca, 2009):

i. $D\left(\left[\tilde{a} + \tilde{c}, \tilde{b} + \tilde{c}\right]_r\right) = D\left(\left[\tilde{a}, \tilde{b}\right]_r\right), \ \forall \tilde{a}, \tilde{b}, \tilde{c} \in \tilde{E}$.

ii. $D\left(\left[s.\tilde{a}, s.\tilde{b}\right]_r\right) = |s| D\left(\left[\tilde{a}, \tilde{b}\right]_r\right), \ \forall s \in \mathbb{R}, \forall \tilde{a}, \tilde{b} \in \tilde{E}$.

iii. $D\left(\left[\tilde{a} + \tilde{b}, \tilde{c} + \tilde{d}\right]_r\right) \leq D\left(\left[\tilde{a}, \tilde{b}\right]_r\right) + D\left(\left[\tilde{c}, \tilde{d}\right]_r\right), \ \forall \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \tilde{E}$.

Definition 2.12 (Seikkala, 1987)**:** Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exist $\tilde{z} \in \tilde{E}$ such that $\tilde{x} = \tilde{y} + \tilde{z}$, then z is called the Hukuhara difference of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition 2.13 (Mansouri and Ahmady, 2012)**:** If $\tilde{f}: I \to \tilde{E}$ and $y_0 \in$ I, where $I \in [t_0, T]$. $\tilde{f}$ is said to be Hukuhara differentiable at y_0, if there exists an element $\left[\widetilde{f'}\right]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), $\tilde{f}(y_0 + h; r) \ominus$

$\tilde{f}(y_0;r)$and $\tilde{f}(y_0;r) \ominus \tilde{f}(y_0-h;r)$exists with the limits are taken in the metric space$(\tilde{E}\ ,D)$

$$\lim_{h\to 0+} \frac{\tilde{f}(y_0+h;r) \ominus \tilde{f}(y_0;r)}{h} = \lim_{h\to 0+} \frac{\tilde{f}(y_0;r) \ominus \tilde{f}(y_0-h;r)}{h}$$

The fuzzy set $\left[\widetilde{f'}(y_0)\right]_r$ is called the Hukuhara derivative of $\left[\widetilde{f'}\right]_r$ at y_0.

These limits are taken in the space $(\tilde{E}\ ,D)$ if t_0 or T, then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r-level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r$, $\forall\, r \in [0,1]$. By consideration of the definition of the metric D all the r-level set $\left[\tilde{f}(0)\right]_r$ are Hukuhara differentiable at y_0, with Hukuhara derivatives $\left[\widetilde{f'}(y_0)\right]_r$,when $\tilde{f}: I \to \tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $\left[\widetilde{f'}(y_0)\right]_r$ lead to that $\tilde{f}$ is Hukuhara differentiable for all $r \in [0,1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0+\alpha, T]$ then all its r-levels $\left[\widetilde{f'}(t)\right]_r$ are Hukuhara differentiable at t_0.

Theorem 2.1 (Stefaninia et al., 2006)**:** Let $\tilde{f}: [t_0+\alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote

$$\left[\widetilde{f'}(t)\right]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t;r), \overline{f}'(t;r)\right].$$

Then the boundary functions $\underline{f}'(t;r), \overline{f}'(t;r)$ are both differentiable

$$\left[\widetilde{f'}(t)\right]_r = \left[\left(\underline{f}(t;r)\right)', \left(\overline{f}(t;r)\right)'\right], \forall\, r \in [0,1]$$

Theorem 2.2 (Mansouri and Ahmady, 2012)**:** Let $\tilde{f}: [t_0+\alpha, T] \to \tilde{E}$ be Hukuhara differentiable and denote

$\left[\widetilde{f'}(t)\right]_r = \left[\underline{f}'(t), \overline{f}'(t)\right]_r = \left[\underline{f}'(t;r), \overline{f}'(t;r)\right]$. Then both of the boundary functions $\underline{f}'(t;r), \overline{f}'(t;r)$ are differentiable, we can write for nth order fuzzy derivative

$$[\tilde{f}^{(n)}(t)]_r = \left[\left(\underline{f}^{(n)}(t;r)\right)', \left(\overline{f}^{(n)}(t;r)\right)'\right], \forall\, r \in [0,1] .$$

From the proof that mentioned in (Mansouri and Ahmady, 2012) of theorem (2.1) we can define the fuzzy Hukuhara differentiability of n times as following:

Definition 2.14: Define the mapping $\tilde{f}': I \rightarrow \tilde{E}$ and $y_0 \in$ I, where $I \in [t_0, T]$. We say that $\tilde{f}'$ is Hukuhara differentiable for $t \in \tilde{E}$, if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r), \tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}\,, D)$

$$\lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h\to 0+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$.

2.8 Fractional Calculus

The Fractional calculus involves the differentiation and integration to an arbitrary order. This area of study emerged in 1695 after the Leibniz created the notation $\frac{\partial^n y}{\partial x^n}$ when he was asked by L'Hopital (what if $n = \frac{1}{2}$?) at which he replied: "It will lead to a paradox". Later, Leibniz stated that differential calculus might be used for achieving this result. Leibniz was referring to Wallis's infinite product for $\pi/2$ which utilized the notation $\partial^{\frac{1}{2}} y$ (Dold and Eckmann, 1975). Numerous mathematicians have studied this area further, which is now known as the fractional calculus.

2.8.1 Fractional integrals

The fractional integrals indicate to the integrals of arbitrary order (Podlubny, 1999). For any dependent function $f(t)$, the fractional integral operator of order, $\alpha > 0$, can be represented as

$$ {}_cD_t^{-\alpha} f(t) \ or \ {}_cI_t^{\alpha} f(t), $$

wherein, c and x represent the two limits of a fractional integral operator and these are generally called as the terminals of the fractional integral (Podlubny, 1999).

In 1874, the mathematician Riemann obtained a formula for the fractional integration by applying a generalization of Taylor series in the following manner:

$$ {}_cI_x^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^x \frac{f(t)}{(x-1)^{1-\alpha}} \partial t + \Psi(x), \ Re(\alpha) > 0, \quad (2.5) $$

where $\Psi(x)$ is a complementary function was introduced by Riemann since he did not fix the lower integration limit c (Miller and Rose, 1993).

Sonin and Barus (1968) presented the Riemann-Liouville definition in his research article. Furthermore, he used the Cauchy integral scheme for the integral order derivatives of the complex domain which is given by (Weilbeer, 2005).

$$ D^n f(z) = \frac{n!}{2\pi i} \int_c \frac{f(t)}{(t-z)^{n+1}} \ dt. \quad (2.6) $$

The Riemann-Liouville integral is define as

$$ {}_cI_x^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^x \frac{f(t)}{(x-1)^{1-\alpha}} \partial t \ , \ Re(\alpha) > 0. \quad (2.7) $$

Also, the Riemann-Liouville integral can be derived in another way by considering the n-fold for any function $f(x)$ as follows (Dold and Eckmann, 1975):

$$ {}_cI_x^{\alpha} f(x) = \int_c^x dx_1 \int_c^{x_1} dx_2 \ldots \int_c^{x_{n-1}} f(x_n) dx_n $$

From Dirlichlet's approach, the n-fold integral can be considered as a single integral

$$ {}_cI_x^{\alpha} f(x) = \frac{1}{(n-1)!} \int_c^x \frac{f(x_n))}{(x-x_n)^{1-n}} dx_n. \quad (2.8) $$

Eq.(2.8) can be thought as the general formula of equation Eq. (2.7) by replacing $n \; with \; \alpha$ and assuming $x_n = t$.

2.8.2 Fractional derivatives

The short name for derivatives of arbitrary order is fractional derivatives. Usually the notation $D_x^{\alpha} f(t)$ used to express the derivative of order α of function $f(t)$. Here α is an arbitrary positive real number and α and x denote the two limits related to the operation of fractional differentiation.

Fractional derivatives can be defined in different ways. The most common definition is Riemann-Liouville definition as follows (Klages et al., 2008):

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{f(\xi)}{(t-\xi)^{\alpha}} \partial\xi, \qquad 0 < \alpha < 1 \qquad (2.9) $$

The general form of Eq. (2.9) is written as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial t}\right)^n \int_a^t \frac{f(\xi)}{(t-\xi)^{\alpha-n+1}} \partial\xi, \qquad n-1 < \alpha < n \qquad (2.10) $$

The fractional derivative was also defined by Caputo (1967) as follows:

$$ {}_0^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial t} f(\xi)}{(t-\xi)^{\alpha}} \partial\xi, \qquad 0 < \alpha < 1 \qquad (2.11) $$

The Eq. (2.9) and (2.11) are linked to the Riemann-Liouville integral as follows

$$ {}_0D_t^{\alpha} f(t) = {}_0D_t^{1} \, {}_0I_t^{1-\alpha}, \qquad 0 < \alpha < 1 $$

$$ {}_0^cD_t^{\alpha} f(t) = {}_0I_t^{1-\alpha} \, {}_0D_t^{1}, \qquad 0 < \alpha < 1 $$

Based on Eq. (2.11) the time fractional derivative term can be approximated as: (Zhuang and Liu, 2006)

$$ \frac{\partial^{\alpha} u(x,t)}{\partial^{\alpha} t} = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{n} b_j \left(u_i^{n+1-j} - u_i^{n-j}\right), \qquad (2.12) $$

where $b_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}$, $j = 0,1,2, \ldots$

Also, time fractional derivative term can be written in the following form (Karatay et al., 2011):

$$\frac{\partial^\alpha u(x,t)}{\partial^\alpha t} = \Delta t^{-\alpha} \sum_{j=0}^{n} v_j \left(u_i^{n-j} - u_i^0\right), \tag{2.13}$$

where $v_0 = 1,\ v_j = \left(1 - \frac{\alpha+1}{j}\right) v_{j-1}$, $j = 1,2, \ldots.$

The Caputo fractional derivative is a good candidate to model the processes of real-life problems governed by fractional differential equation. In the formulation of the problem, the Caputo fractional derivative allows initial and boundary conditions to be included. Furthermore, its derivative at a constant value is zero (Karatay et al., 2011).

2.9 Fractional Diffusion equation

The basic process in the diffusion phenomena is the flow of fluid from a region of higher density to the one of lower density. The anomalous or fractional diffusion equation differs from the classical diffusion equation when the integer order derivatives are replaced by the fractional order derivatives. The anomalous diffusion happens when the cloud of particles spreads at a rate incompatible with the classical Brownian motion pattern. There are three types of fractional diffusion equation, time fractional diffusion equation, space fractional diffusion equation and time-space fractional diffusion equation. In diffusion equations when the first-order time derivative is replaced by a fractional derivative we obtain the time fractional diffusion equation.