



## CYCLIC REFINEMENTS OF THE DISCRETE HÖLDER'S INEQUALITY WITH APPLICATIONS

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*Abstract.* In this paper we obtain refinements of the discrete Hölder's and Minkowski's inequalities for finite and infinite sequences by using cyclic refinements of the discrete Jensen's inequality.

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### 1. INTRODUCTION

One of the most important inequalities concerning convex functions is the following

**Theorem 1** (Discrete Jensen's inequality, [5]). *Let  $C$  be a convex subset of a real vector space  $V$ , and let  $f : C \rightarrow \mathbb{R}$  be a convex function. If  $p_1, \dots, p_n$  are nonnegative numbers with  $\sum_{i=1}^n p_i = 1$ , and  $v_1, \dots, v_n \in C$ , then*

$$f\left(\sum_{i=1}^n p_i v_i\right) \leq \sum_{i=1}^n p_i f(v_i).$$

A number of attempts have been made to refine this inequality (see the book [5] and the references therein).

The following cyclic refinement of the discrete Jensen's inequality is a special case of Theorem 2.1 in the recent paper [6] (see also [1]). To give the result we need the following hypotheses:

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(H<sub>1</sub>) Let  $2 \leq k \leq n$  be integers, let  $\lambda_1, \dots, \lambda_k$  represent a positive probability distribution, and let  $p_1, \dots, p_n$  be positive numbers with  $P_n := \sum_{i=1}^n p_i$ .

(H<sub>2</sub>) Let  $I \subset \mathbb{R}$  be an interval, and  $f : I \rightarrow \mathbb{R}$  be a convex function.

**Theorem 2** (Theorem 2.1 in [6]). *Assume (H<sub>1</sub>) and (H<sub>2</sub>). If  $v_1, \dots, v_n \in I$ , then*

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i v_i\right) \leq C_{dis} = C_{dis}(f, \mathbf{v}, \mathbf{p}, \lambda)$$

$$:= \frac{1}{P_n} \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}\right) f\left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} v_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(v_i) \quad (1.1)$$

where  $i + j$  means  $i + j - n$  in case of  $i + j > n$ .

In this paper we first obtain cyclic refinements of the discrete Hölder’s inequality by using the previous assertion. Then we give some refinements of the discrete Hölder’s inequality for infinite sequences. There are a lot of papers dealing with similar refinements (see e.g. [2–4, 7] and [8]). Our results fit well into the topic of refinements of inequalities corresponding to convex functions, and they give a new approach to have such refinements. Finally, we demonstrate the applicability of our results by means of some new cyclic refinements of the Minkowski’s inequality.

## 2. MAIN RESULTS

Let  $2 \leq k \leq n$ , and let  $i \in \{1, \dots, n\}$  and  $j \in \{0, \dots, k - 1\}$ . In further parts of the paper  $i + j$  always means  $i + j - n$  in case of  $i + j > n$ .

In the first result we give cyclic refinements of the discrete Hölder’s inequality.

**Theorem 3.** *Let  $2 \leq k \leq n$  be integers, and let  $\lambda_1, \dots, \lambda_k$  represent a positive probability distribution. Let  $(w_i)_{i=1}^n$  be a sequence of positive numbers,  $(x_i)_{i=1}^n$  be a sequence of nonnegative numbers, and let  $p > 0$  and  $q \in \mathbb{R} \cup \{\infty\}$  be conjugate exponents that is  $\frac{1}{p} + \frac{1}{q} = 1$  (if  $p = 1$ , we define its conjugate exponent to be  $q = \infty$ ).*

(a<sub>1</sub>) *If  $(y_i)_{i=1}^n$  is a sequence of nonnegative numbers and  $p > 1$ , then*

$$\sum_{i=1}^n w_i x_i y_i \leq \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q\right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{i=1}^n w_i x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n w_i y_i^q\right)^{\frac{1}{q}}. \quad (2.1)$$

(a<sub>2</sub>) If  $(y_i)_{i=1}^n$  is a sequence of nonnegative numbers, then

$$\begin{aligned} \sum_{i=1}^n w_i x_i y_i &\leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j} \right) \cdot \max_{j \in \{0, \dots, k-1\}} y_{i+j} \\ &\leq \left( \sum_{i=1}^n w_i x_i \right) \cdot \max_{i \in \{1, \dots, n\}} y_i. \end{aligned}$$

(a<sub>3</sub>) If  $(y_i)_{i=1}^n$  is a sequence of positive numbers and  $0 < p < 1$ , then the reverse inequalities hold in (2.1).

(b<sub>1</sub>) If  $(y_i)_{i=1}^n$  is a sequence of nonnegative numbers and If  $p > 1$ , then

$$\begin{aligned} \sum_{i=1}^n w_i x_i y_i &\leq \left( \sum_{i=1}^n w_i y_i^q \right)^{\frac{1}{q}} \left( \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{1-p} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j} \right)^p \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{i=1}^n w_i x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n w_i y_i^q \right)^{\frac{1}{q}}. \end{aligned} \quad (2.2)$$

If  $y_i = y_{i+1} = \dots = y_{i+k-1} = 0$  for some  $i \in \{1, \dots, n\}$ , then

$$\left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{1-p} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j} \right)^p$$

means

$$\left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} \right)^{1-p} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j} \right)^p.$$

(b<sub>2</sub>) If  $(y_i)_{i=1}^n$  is a sequence of nonnegative numbers, then

$$\begin{aligned} \sum_{i=1}^n w_i x_i y_i &\leq \max_{i \in \{1, \dots, n\}} y_i \cdot \left( \sum_{i=1}^n \frac{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j}}{\max_{j \in \{0, \dots, k-1\}} y_{i+j}} \right) \\ &\leq \left( \sum_{i=1}^n w_i x_i \right) \cdot \max_{i \in \{1, \dots, n\}} y_i. \end{aligned}$$

If  $y_i = y_{i+1} = \dots = y_{i+k-1} = 0$  for some  $i \in \{1, \dots, n\}$ , then

$$\frac{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j}}{\max_{j \in \{0, \dots, k-1\}} y_{i+j}}$$

means

$$\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}.$$

(b<sub>3</sub>) If  $(y_i)_{i=1}^n$  is a sequence of positive numbers and  $0 < p < 1$ , then the reverse inequalities hold in (2.2).

*Proof.* Consider the power functions

$$f_s : ]0, \infty[ \rightarrow \mathbb{R}, \quad f_s(x) = x^s, \quad s \neq 0, 1. \quad (2.3)$$

It is well known that  $f_s$  is strictly convex for each  $s \in ]-\infty, 0[ \cup ]1, \infty[$ , and strictly concave for each  $s \in ]0, 1[$ . By applying Theorem 2 to the function  $f_s$  and to the positive numbers  $v_1, \dots, v_n$ , we obtain that

$$\left( \frac{1}{P_n} \sum_{i=1}^n p_i v_i \right)^s \leq \frac{1}{P_n} \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} \right)^{1-s} \left( \sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} v_{i+j} \right)^s \leq \frac{1}{P_n} \sum_{i=1}^n p_i v_i^s, \quad (2.4)$$

if  $s \in ]-\infty, 0[ \cup ]1, \infty[$ , and the reverse inequality holds if  $s \in ]0, 1[$ .

(a<sub>1</sub>) Assume first that  $x_i, y_i$  ( $i = 1, \dots, n$ ) are positive numbers. By using the substitutions

$$s = \frac{1}{p}, \quad p_i = w_i y_i^q, \quad v_i = x_i^p y_i^{-q}$$

in (2.4), and by taking into account that  $f_{1/p}$  is concave, we obtain

$$\begin{aligned} \left( \frac{1}{\sum_{i=1}^n w_i y_i^q} \right)^{\frac{1}{p}} \left( \sum_{i=1}^n w_i x_i^p \right)^{\frac{1}{p}} &\geq \frac{1}{\sum_{i=1}^n w_i y_i^q} \cdot \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right) \cdot \left( \frac{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p}{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q} \right)^{\frac{1}{p}} \\ &\geq \frac{1}{\sum_{i=1}^n w_i y_i^q} \sum_{i=1}^n w_i x_i y_i, \end{aligned}$$

and this is equivalent to (2.1).

In the general case let  $\varepsilon > 0$ , apply the proved inequality to the positive sequences  $(x_i + \varepsilon)_{i=1}^n$  and  $(y_i + \varepsilon)_{i=1}^n$ , and then take the limit as  $\varepsilon \rightarrow 0+$ .

(a<sub>2</sub>) The inequality can be obtained by taking the limit as  $q \rightarrow \infty$  in (2.1).

(a<sub>3</sub>) The reverse inequalities in (2.1) can be proved exactly as in (a<sub>1</sub>) by considering that  $f_{1/p}$  is convex.

(b<sub>1</sub>) and (b<sub>2</sub>) can be proved similarly to (a<sub>1</sub>) and (a<sub>2</sub>) by using the substitutions

$$s = p, \quad p_i = w_i y_i^q, \quad v_i = x_i y_i^{1-q}$$

in (2.4).

(b<sub>3</sub>) The reverse inequalities in (2.2) can be proved exactly as in (b<sub>1</sub>) by considering that the function  $f_p$  is concave.

The proof is complete.  $\square$

The following new inequalities follow from the previous result.

**Corollary 1.** *Let  $2 \leq k \leq n$  be integers, and let  $\lambda_1, \dots, \lambda_k$  represent a positive probability distribution. Let  $(w_i)_{i=1}^n$  and  $(y_i)_{i=1}^n$  be sequences of positive numbers,  $(x_i)_{i=1}^n$  be a sequence of nonnegative numbers, and let  $p > 0$  and  $q \in \mathbb{R} \cup \{\infty\}$  be conjugate exponents.*

(a) *If  $p > 1$ , then*

$$\sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{1-p} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j} \right)^p \leq \sum_{i=1}^n w_i x_i^p,$$

while if  $0 < p < 1$ , then the reverse inequality holds.

(b) *If  $p = 1$ , then*

$$\sum_{i=1}^n \frac{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j} x_{i+j}}{\max_{j \in \{0, \dots, k-1\}} y_{i+j}} \leq \sum_{i=1}^n w_i x_i.$$

*Proof.* Theorem 3 (b<sub>1</sub>), (b<sub>2</sub>) and (b<sub>3</sub>) imply these inequalities.  $\square$

As an immediate consequence of Theorem 3 we obtain cyclic refinements of the Cauchy-Schwarz-Bunyakovskij inequality.

**Corollary 2.** *Let  $2 \leq k \leq n$  be integers, let  $\lambda_1, \dots, \lambda_k$  represent a positive probability distribution, and let  $(x_i)_{i=1}^n$  and  $(y_i)_{i=1}^n$  be sequences of real numbers. Then*

(a)

$$\begin{aligned} \left| \sum_{i=1}^n w_i x_i y_i \right| &\leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{i=1}^n w_i x_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n w_i y_i^2 \right)^{\frac{1}{2}}. \end{aligned}$$

(b)

$$\begin{aligned} \left| \sum_{i=1}^n w_i x_i y_i \right| &\leq \left( \sum_{i=1}^n w_i y_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n \frac{\left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} |y_{i+j}| |x_{i+j}| \right)^2}{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^2} \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{i=1}^n w_i x_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n w_i y_i^2 \right)^{\frac{1}{2}}. \end{aligned}$$

If  $y_i = y_{i+1} = \dots = y_{i+k-1} = 0$  for some  $i \in \{1, \dots, n\}$ , then

$$\frac{\left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} |y_{i+j}| |x_{i+j}|\right)^2}{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^2}$$

means

$$\frac{\left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} |x_{i+j}|\right)^2}{\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j}}.$$

*Proof.* Choose  $p = 2$  in Theorem 3 (a<sub>1</sub>) and (b<sub>1</sub>). □

By Theorem 3, some refinements of the Hölder’s inequality for infinite sequences can be obtained. We have to be very careful, because a change in  $n$  causes a corresponding change in  $i + j$ , that is  $i + j$  depends on  $n$ . We illustrate this by the generalization of inequality (2.1).

**Theorem 4.** Let  $k \geq 2$  be an integer, and let  $\lambda_1, \dots, \lambda_k$  represent a positive probability distribution. Let  $(w_i)_{i=1}^\infty$  be a sequence of positive numbers, and let  $(x_i)_{i=1}^\infty$  and  $(y_i)_{i=1}^\infty$  be sequences of nonnegative numbers such that

$$\sum_{i=1}^\infty w_i x_i^p < \infty, \quad \sum_{i=1}^\infty w_i y_i^q < \infty,$$

where  $p > 0$  and  $q \in \mathbb{R} \cup \{\infty\}$  are conjugate exponents.

(a) If  $p > 1$ , then

$$\begin{aligned} \sum_{i=1}^\infty w_i x_i y_i &\leq \sum_{i=1}^\infty \left(\sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l x_l^p\right)^{\frac{1}{p}} \left(\sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l y_l^q\right)^{\frac{1}{q}} \\ &\quad + \sum_{m=1}^{k-1} \left(\sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} x_{m-k+1+j}^p\right)^{\frac{1}{p}} \left(\sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} y_{m-k+1+j}^q\right)^{\frac{1}{q}} \\ &\leq \left(\sum_{i=1}^\infty w_i x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^\infty w_i y_i^q\right)^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

(b) If  $y_i > 0$  ( $i = 1, 2, \dots$ ) and  $0 < p < 1$ , then the reverse inequalities hold in (2.5).

*Proof.* (a) It follows from Theorem 3 (a<sub>1</sub>) that for each fixed  $n \geq k$

$$\sum_{i=1}^n w_i x_i y_i \leq \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q\right)^{\frac{1}{q}}$$

$$\begin{aligned}
 &= \sum_{i=1}^{n-k+1} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{\frac{1}{q}} \\
 &+ \sum_{i=n-k+2}^n \left( \sum_{j=0}^{n-i} \lambda_{j+1} w_{i+j} x_{i+j}^p + \sum_{j=n-i+1}^{k-1} \lambda_{j+1} w_{i+j} x_{i+j}^p \right)^{\frac{1}{p}} \\
 &\cdot \left( \sum_{j=0}^{n-i} \lambda_{j+1} w_{i+j} y_{i+j}^q + \sum_{j=n-i+1}^{k-1} \lambda_{j+1} w_{i+j} y_{i+j}^q \right)^{\frac{1}{q}} \\
 &= \sum_{i=1}^{n-k+1} \left( \sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l x_l^p \right)^{\frac{1}{p}} \left( \sum_{l=i}^{i+k-1} \lambda_{l-i+1} w_l y_l^q \right)^{\frac{1}{q}} \\
 &+ \sum_{m=1}^{k-1} \left( \sum_{j=0}^{k-m-1} \lambda_{j+1} w_{m+n-k+1+j} x_{m+n-k+1+j}^p + \sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} x_{m-k+1+j}^p \right)^{\frac{1}{p}} \\
 &\cdot \left( \sum_{j=0}^{k-m-1} \lambda_{j+1} w_{m+n-k+1+j} y_{m+n-k+1+j}^q + \sum_{j=k-m}^{k-1} \lambda_{j+1} w_{m-k+1+j} y_{m-k+1+j}^q \right)^{\frac{1}{q}}.
 \end{aligned}$$

Now, as  $n$  tends to infinity, we obtain the result.

(b) We can prove as in (a) by using Theorem 3 (a<sub>3</sub>).

The proof is complete. □

### 3. APPLICATIONS

In the first result a cyclic refinement of the Minkowski's inequality is given.

**Theorem 5.** *Let  $2 \leq k \leq n$  be integers, let  $\lambda_1, \dots, \lambda_k$  represent a positive probability distribution, and let  $(x_i)_{i=1}^n$  and  $(y_i)_{i=1}^n$  be sequences of real numbers. If  $p > 1$  and  $q > 1$  are conjugate exponents, and there exists  $i \in \{1, \dots, n\}$  such that  $x_i + y_i \neq 0$ , then*

$$\begin{aligned}
 \left( \sum_{i=1}^n w_i |x_i + y_i|^p \right)^{\frac{1}{p}} &\leq \frac{1}{\left( \sum_{i=1}^n w_i |x_i + y_i|^p \right)^{\frac{1}{q}}} \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} |x_{i+j} + y_{i+j}|^p \right)^{\frac{1}{q}} \\
 &\cdot \left( \left( \sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} |x_{i+j}|^p \right)^{\frac{1}{p}} + \left( \sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} |y_{i+j}|^p \right)^{\frac{1}{p}} \right) \\
 &\leq \left( \sum_{i=1}^n w_i |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n w_i |y_i|^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

*Proof.* A similar argument as in the proof of the classical Minkowski's inequality can be applied.

By Theorem 3 (a<sub>1</sub>),

$$\begin{aligned} \sum_{i=1}^n w_i |x_i + y_i|^p &\leq \sum_{i=1}^n w_i |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n w_i |y_i| |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} |x_{i+j}|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} |x_{i+j} + y_{i+j}|^p \right)^{\frac{1}{q}} \\ &\quad + \sum_{i=1}^n \left( \sum_{j=0}^{k-1} \lambda_{i+j} w_{i+j} |y_{i+j}|^p \right)^{\frac{1}{p}} \left( \sum_{j=0}^{k-1} \lambda_{j+1} w_{i+j} |x_{i+j} + y_{i+j}|^p \right)^{\frac{1}{q}} \\ &\leq \left( \sum_{i=1}^n w_i |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n w_i |x_i + y_i|^p \right)^{\frac{1}{q}} + \left( \sum_{i=1}^n w_i |y_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n w_i |x_i + y_i|^p \right)^{\frac{1}{q}}, \end{aligned}$$

and this implies the result.

The proof is complete.  $\square$

*Remark 1.* By using Theorem 3 (b<sub>1</sub>), we can obtain another cyclic refinement of the Minkowski's inequality.

Related to Theorem 4 we have the following cyclic refinement of the Minkowski's inequality for infinite sums. For simplicity and transparency we consider only the case  $k = 2$ .

**Theorem 6.** Let  $\lambda_1$  and  $\lambda_2$  be positive numbers with  $\lambda_1 + \lambda_2 = 1$ . Let  $(w_i)_{i=1}^{\infty}$  be a sequence of positive numbers, and let  $(x_i)_{i=1}^{\infty}$  and  $(y_i)_{i=1}^{\infty}$  be sequences of real numbers such that

$$\sum_{i=1}^{\infty} w_i |x_i|^p < \infty, \quad \sum_{i=1}^{\infty} w_i |y_i|^q < \infty,$$

where  $p > 1$  and  $q > 1$  are conjugate exponents. If there exists  $i \in \{1, 2, \dots\}$  such that  $x_i + y_i \neq 0$ , then

$$\begin{aligned} \left( \sum_{i=1}^{\infty} w_i |x_i + y_i|^p \right)^{\frac{1}{p}} &\leq \frac{1}{\left( \sum_{i=1}^{\infty} w_i |x_i + y_i|^p \right)^{\frac{1}{q}}} \cdot \left( \sum_{i=1}^{\infty} (\lambda_1 w_i |x_i + y_i|^p + \lambda_2 w_{i+1} |x_{i+1} + y_{i+1}|^p) \right)^{\frac{1}{q}} \\ &\quad \left( (\lambda_1 w_i |x_i|^p + \lambda_2 w_{i+1} |x_{i+1}|^p)^{\frac{1}{p}} + (\lambda_1 w_i |y_i|^p + \lambda_2 w_{i+1} |y_{i+1}|^p)^{\frac{1}{p}} \right) \\ &\quad + (\lambda_2 w_1 |x_1 + y_1|^p)^{\frac{1}{q}} \left( (\lambda_2 w_1 |x_1|^p)^{\frac{1}{p}} + (\lambda_2 w_1 |y_1|^p)^{\frac{1}{p}} \right) \\ &\leq \left( \sum_{i=1}^{\infty} w_i |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{\infty} w_i |y_i|^p \right)^{\frac{1}{p}}. \end{aligned}$$



*Proof.* We can follow the proof of Theorem 5, by using Theorem 4 instead of Theorem 3 (a<sub>1</sub>).  $\square$

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