Fast computation of half-integral weight modular forms

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Abstract

To study statistical properties of modular forms, including for instance Sato-Tate like problems, it is essential to have a large number of Fourier coefficients. In this article, we exhibit three bases for the space of modular forms of any half-integral weight and level 4, which have the property that many coefficients can be computed (relatively) quickly on a computer.

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Coefficients of modular forms, especially of Hecke eigenforms, carry a lot of arithmetic information. This is true for both integral weight and half-integral weight. Many aspects that could be completely understood in the case of integral weight, such as the resolution of the Sato-Tate Conjecture [BLGHT11], are open in the half-integral weight case. Even the equidistribution of signs is still open despite many recent results (for example, [IW13], [AdRIW15], [IW16], [KKT18], [Kum13]). In the absence of many tools that work in integral weight, such as the description of modular forms as differential forms on modular curves that admit models over the rational field, a natural approach in view of clarifying the expectations in half-integral weight is to perform numerical studies of modular forms. Consequently, it is very important to know modular forms to a very high precision. For most cases, however, it is very time (and memory) consuming to compute q-expansions up to a high power of q. This statement already applies to integral weight modular forms. It is even more true in the half-integral weight case, which in many treatments is reduced to the integral one.

The purpose of this article is to introduce and study bases of spaces of half-integral weight modular forms such that 'many' coefficients of the standard q-expansions for each form in the basis can be calculated relatively quickly. We focus on achieving high weights, but work in the lowest possible level $\Gamma_0(4)$. This is comparable to the classical case of integral weight modular forms of level 1, where a 'fast' basis is given by the standard Eisenstein series (see, for instance, the Miller basis in [Ste07, Lemma 2.20]). These bases do not easily generalise to the case of higher levels, which can be considered an open question.

We build on existing results from the literature, particularly, the papers of Cohen [Coh75] and Kohnen [Koh80], and exploit them in view of our aims. The general approach chosen here, contrary to the way half-integral weight modular forms are implemented in Magma [BCP97] and Pari/GP [The19] (the latter has more features, such as Hecke operators; see [BC18]), is to use only modular forms that are easy and quick to write down, such as Eisenstein series and theta series, and to multiply power series. Note that the theta series used are lacunary and that computing Eisenstein series costs little.

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Hence, this is very simple to implement and does not rely on more complex algorithms such as the modular symbols algorithms (see e.g. [Ste07] or [Wie19])). However, the bottleneck will be the multiplication of power series for which good algorithms (e.g. those depending on the Fast Fourier Transformation) should be used. The algorithms have been implemented in Magma, which provides such fast algorithms. See [Wie20] for the corresponding Magma package.

We chose to present three kinds of bases, which we name the *standard basis* (see Section 2), the *Kohnen basis* (see Section 3) and the *Rankin-Cohen basis* (see Section 4), respectively. The systematic study of the distribution of Fourier coefficients in [IDOTW21] is a concrete example of the use of the Rankin-Cohen basis allowing the relatively quick computation of $2 \cdot 10^8$ Fourier coefficients of some modular form of half-integral weight.

In Section 5, we make a simple theoretical analysis of the three algorithms and also compare their performance experimentally. One can summarise the findings by stating that in small weights $k + \frac{1}{2}$ with even k, the Rankin-Cohen basis performs best for computing the Kohnen plus-space. In all other cases, the plus-space is best computed by the Kohnen basis. For the computation of the full space, the standard basis always behaves very well.

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1 Background on modular forms

Let k be a non-negative integer. Denote by $M_k(N)$ (resp. $S_k(N)$) the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of weight k and level $\Gamma_0(N)$ for a positive integer N. Furthermore, write $M_{k+1/2}(4)$ (resp. $S_{k+1/2}(4)$) for the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of half-integral weight $k + 1/2$ and level $\Gamma_0(4)$. These contain the *Kohnen plus-space* $M^+_{k+1/2}(4)$ (resp. $S^+_{k+1/2}(4)$) consisting of those modular forms (resp. cusp forms) f such that $a_n(f) = 0$ whenever $(-1)^k n \equiv 2, 3 \pmod 4$. Here $a_n(f)$ denotes the n-th Fourier coefficient (for $n \in \mathbb{Z}$) of the Fourier expansion of f at the standard cusp ∞. The Kohnen plus-space can be described as the eigenspace for a certain eigenvalue of an explicit linear operator. See [Koh80, Proposition 2]. We shall, however, not need this operator in our computations.

Let $q := \exp(2\pi i z)$ for the complex variable $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, the upper half-plane. For an even integer $k \geq 4$, we let

$$E_k(q) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

denote the (q-expansion of the) standard normalised Eisenstein series of weight k and level 1, where $\sigma_{k-1}(n) = \sum_{0<d|n} d^{k-1}$ and B_k denotes the standard k-th Bernoulli number. More generally, for

any positive integer k and any pair of primitive Dirichlet characters χ_1, χ_2, let

$$E_k^{\chi_1,\chi_2}(q) = \frac{-B_k^{\chi_1}}{2k} + \sum_{n=1}^{\infty} \big(\sum_{0<d|n} \chi_1(d)\chi_2(n/d)d^{k-1} \big) q^n$$

denote the (q-expansion of the) normalised Eisenstein series of weight k attached to the characters χ_1, χ_2, where $B_k^{\chi_1}$ denotes the k-th generalised Bernoulli number for the character χ_1. The level of $E_k^{\chi_1,\chi_2}$ is the product of the conductors of χ_1 and χ_2. We also let

$$F_2 := \sum_{n \geq 1 \text{ odd}} \sigma_1(n) q^n \in M_2(4)$$

be the standard Eisenstein series of weight 2 and level 4.

We shall also need the result that the algebra of all integral weight modular forms of level 1 is generated by E_4 and E_6 and that a $\mathbb{C}$-basis of the vector space of modular forms of level 1 and weight k is given by $E_4^a E_6^b$ where $a, b \in \mathbb{Z}_{\geq 0}$ run through all possibilities for $k = 4a + 6b$. For more details, see Chapter 8 and Chapter 10.6 of [CS17].

The for our purposes most important modular form of half-integral weight is the *standard ϑ-series* defined as

$$\vartheta := \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}.$$

It is a modular form of weight $1/2$ for the group $\Gamma_0(4)$. Finally, for every integer $k \geq 2$, let $H_{k+1/2}$ be the modular form which is explicitly described in the proof of [Coh75, Theorem 3.1] as a linear combination of two linearly independent Eisenstein series in $M_{k+1/2}(4)$.

2 The standard basis

We recall [Coh75, Proposition 1.1].

Proposition 2.1. *The natural embedding*

$$\mathbb{C}[\vartheta, F_2] \to \bigoplus_{\ell \in \frac{1}{2}\mathbb{Z}} M_\ell(4)$$

is an isomorphism of graded algebras, where ϑ and F_2 are the modular forms of weight $1/2$ and 2, respectively, that are described above.

Corollary 2.2. *Let $\ell \in \frac{1}{2}\mathbb{Z}$. Then the modular forms*

$$\vartheta^a F_2^b \text{ for all } a, b \in \mathbb{Z}_{\geq 0} \text{ such that } \ell = \frac{a}{2} + 2b$$

form a basis of $M_\ell(4)$, which we call the standard basis.

The standard basis is computed by writing down ϑ and F_2 explicitly as power series (using their definition) and then multiplying power series.

The standard basis is a basis for the full space of modular forms of weight $\ell \in \frac{1}{2}\mathbb{Z}$ and level 4. We now describe how to compute the Kohnen plus-space as a subspace for $\ell = k + \frac{1}{2}$ with $k \in \mathbb{Z}$. Instead of using the operator introduced in [Koh80], we solve this as a linear algebra problem. Take a basis $f_1, \ldots, f_m$ of the full space (with precision D), e.g. the standard basis, and, for each $1 \leq i \leq m$, write the coefficients $a_n(f_i)$ for all $0 \leq n < D$ such that $(-1)^k n \equiv 2, 3 \pmod 4$ into a vector v_i. Then take the matrix M of these vectors and compute a basis $b_1, \ldots, b_r$ of its kernel. Then a basis of the Kohnen plus-space is given by $g_i = \sum_{j=1}^m b_{i,j} f_j$ for $1 \leq i \leq r$ where $b_{i,j}$ is the j-th entry of the vector b_i.

It is quite fast to compute the standard basis, and the linear algebra step required for calculating a basis for the Kohnen plus-space is also very fast; note that for computing the b_i, one can usually work with a smaller precision than the one that one might like to obtain in the end. However, we will see in the next two sections that there are direct ways to compute the Kohnen plus-space, which do not require the computation of the full space. They are, of course, still faster.

3 The Kohnen basis for the plus-space

The first 'fast' basis for the Kohnen plus-space we present has been studied by Kohnen in the fundamental paper [Koh80], in which he defines the plus-space.

Proposition 3.1 (Kohnen basis – even case)**.** *Let $k \in \mathbb{Z}_{\geq 2}$ be even. Let $a_0 \in \{0, 1, 2\}$ satisfy $k \equiv a_0 \pmod 3$ and put $m = \frac{k-4a_0}{6} - 1$. Then the set consisting of the modular forms*

$$E_4^{a_0+3a+1}(4z) \cdot E_6^{m-2a}(4z) \cdot H_{5/2}(z), \quad E_4^{a_0+3a}(4z) \cdot E_6^{m-2a+1}(4z) \cdot \vartheta(z) \text{ for } 0 \leq a \leq \lfloor \frac{m}{2} \rfloor,$$
$$E_4^{\frac{k}{4}}(4z) \cdot \vartheta(z) \text{ if } 4 \mid k,$$
$$E_6^{\frac{k-2}{6}}(4z) \cdot H_{5/2}(z) \text{ if } 6 \mid (k-2)$$

forms a basis of $M^+_{k+1/2}(4)$.

Proof. The modular forms $E_4^{a_0+3a} E_6^{m-2a}$ for $0 \leq a \leq \lfloor \frac{m}{2} \rfloor$ form a basis of $M_{k-6}(1)$. Multiplying by E_4, this space is mapped injectively into $M_{k-2}(1)$ hitting all standard basis elements of the target space except $E_6^{\frac{k-2}{6}}$ if $6 \mid (k-2)$. Similarly, multiplying by E_6, we obtain a subspace of $M_k(1)$ containing all standard basis elements except $E_4^{\frac{k}{4}}$ if $4 \mid k$. Now it suffices to apply [Koh80, Proposition 1]. $\square$

Proposition 3.2 (Kohnen basis – odd case)**.** *Let $k \in \mathbb{Z}_{\geq 2}$ be odd. Let $a_0 \in \{0, 1, 2\}$ satisfy $k \equiv a_0 \pmod 3$ and put $m = \frac{k-4a_0-9}{6}$. Then the set consisting of the modular forms*

$$E_4^{a_0+3a+1}(4z) \cdot E_6^{m-2a}(4z) \cdot H_{11/2}(z), \quad E_4^{a_0+3a}(4z) \cdot E_6^{m-2a+1}(4z) \cdot H_{7/2}(z) \text{ for } 0 \leq a \leq \lfloor \frac{m}{2} \rfloor$$
$$E_4^{\frac{k-3}{4}}(4z) \cdot H_{7/2}(z) \text{ if } 4 \mid (k-3)$$
$$E_6^{\frac{k-5}{6}}(4z) \cdot H_{11/2}(z) \text{ if } 6 \mid (k-5)$$

forms a basis of $M^+_{k+1/2}(4)$.

Proof. The modular forms $E_4^{a_0+3a} E_6^{m-2a}$ for $0 \leq a \leq \lfloor \frac{m}{2} \rfloor$ form a basis of $M_{k-9}(1)$. Multiplying by E_4, this space is mapped injectively into $M_{k-5}(1)$ hitting all standard basis elements of the target space except $E_6^{\frac{k-5}{6}}$ if $6 \mid (k-5)$. Similarly, multiplying by E_6, we obtain a subspace of $M_{k-3}(1)$ containing all standard basis elements except $E_4^{\frac{k-3}{4}}$ if $4 \mid (k-3)$. Now it suffices to apply [Koh80, Proposition 1]. $\square$

In both cases, the Kohnen bases can be obtained by multiplying power series that can be easily computed. In particular, we use that Cohen's modular forms $H_{5/2}$, $H_{7/2}$ and $H_{11/2}$ can be explicitly given in terms of the standard basis (see [Coh75, Corollary 3.2]).

4 The Rankin-Cohen modular forms

It is well known that the dimension of the cusp space of the Kohnen plus-space of weight $k + 1/2$ equals one for $k = 6, 8, 9, 10, 11, 13$. Therefore, any form in that space is a Hecke eigenform of half-integral weight. We got the inspiration to work with Rankin-Cohen brackets from the nice example $\delta(z)$ [KZ81, p. 177]. That example was good enough to obtain large number of Fourier coefficients. The natural idea is to seek for such nice examples in higher weights.

We recall the definition of the Rankin-Cohen bracket [CS17, Def. 5.3.23], [Coh75] and [Zag94]: Let f, g be two modular forms of weights k and ℓ, respectively, and let $n \in \mathbb{Z}_{\geq 1}$. Put

$$[f, g]_n := \sum_{j=0}^{n} (-1)^j \binom{n+k-1}{j} \binom{n+\ell-1}{n-j} f^{(n-j)} g^{(j)},$$

where $f^{(i)}$ denotes the i-th derivative of f.

For a non-negative integer k, we now define *Rankin-Cohen modular forms* of weight $k + \frac{1}{2}$ and level 4, as follows.

Case 1: k is even. Let $n \in \mathbb{Z}$ satisfy $0 \leq n \leq \frac{k-4}{2}$. Put

$$\Phi_{k,n} := [E_{k-2n}(4z), \vartheta(z)]_n \in M^+_{k+1/2}(4).$$

Case 2: k is odd. Let $n \in \mathbb{Z}$ satisfy $0 \leq n \leq \frac{k-2}{2}$. Put

$$\Phi_{k,n,1} := [E^{1,\chi}_{k-2n}(z), \vartheta(z)]_n, \Phi_{k,n,2} := [E^{\chi,1}_{k-2n}(z), \vartheta(z)]_n \in M^+_{k+1/2}(4)$$

where χ is the Kronecker character of conductor 4 corresponding to $\mathbb{Q}(\sqrt{-1})$.

Definition 4.1. *Let $k \in \mathbb{N}$ be even. Let $d = \dim M^+_{k+1/2}(4)$. If the modular forms*

$$\Phi_{k,n} \text{ for } 0 \leq n \leq d-1$$

are linearly independent, then we call them the Rankin-Cohen basis *of $M^+_{k+1/2}(4)$.*

Let now $k \in \mathbb{N}$ be odd. Let $d = \dim M_{k+1/2}(4)$. If the first d of the following modular forms

$$\Phi_{k,n,1}, \Phi_{k,n,2} \text{ for } 0 \leq n \leq \lceil \frac{d}{2} \rceil$$

are linearly independent, then we call them the Rankin-Cohen basis *of $M_{k+1/2}(4)$.*

We have been unable to prove that the modular forms in the definition are always linearly independent. However, this was true in all cases we computed. It does not seem entirely evident how to write down a basis via Rankin-Cohen brackets for the plus-space if k is odd which is as simple as the one for even k. The Rankin-Cohen bases are straight forward to compute by multiplying and differentiating power series. The sparseness of ϑ positively effects the speed of the computation.

5 Comparison of the complexity and the running time of the algorithms and final comments

The main cost in all three algorithms is the multiplication of power series. It takes significantly more time than adding power series, differentiating them or creating modular forms such as theta series and the simple Eisenstein series we need as power series. We thus use the number of power series multiplications as our measure for the complexity of the algorithms. We disregard finer effects such as how lacunary are the power series (note that the theta series and its derivatives are very lacunary) and the size of the coefficients in the power series. The latter depend on the weight and so similar effects are noticable in all three algorithms.

Straight forward counting for each of the three algorithms (as currently implemented in the package FastBases[Wie20]) yields the following table, showing the number of multiplications of power series (with fixed precision) as a function of the weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ (we assume $k - \frac{1}{2}$ even for the Rankin-Cohen basis) with only minor approximations:

Standard Basis of $M_k(4)$	Kohnen Basis of $M_k(4)^+$	Rankin-Cohen Basis of $M_k(4)^+$
$\frac{3}{2}k + 4$	$\frac{5}{12}k + 12$	$\frac{1}{72}k^2 + \frac{3}{4}k + 10$

We can conclude that the number of multiplications performed for the standard basis and for the Kohnen basis behaves linearly with respect to the weight, whereas the dependence is quadratic for the RC algorithm. Moreover, the Kohnen basis needs roughly a third of the multiplications of the standard basis.

However, the actual running time is not simply proportional to the number of power series multiplications. Other effects play a role. For instance, the size of the coefficients of the power series is important since the computations are exact computations over the rational numbers, and the average size is known to grow with the weight. Moreover, the time needed for a single multiplication of two power series can be significantly lower when at least one of the power series is sparse, which is the case for the powers of θ and their derivatives. In order to see how the algorithms behave in practice, we ran our Magma implementation on a standard laptop computer[1] and obtained the following comparison of the computation times (in seconds) of all coefficients up to the indicated bound for some selected weights.

	Standard Basis			Kohnen Basis			Rankin-Cohen Basis		
weight	10^4	10^5	10^6	10^4	10^5	10^6	10^4	10^5	10^6
25/2	0.16	3.02	55.34	0.12	2.05	34.92	0.13	1.76	31.71
41/2	0.40	6.86	134.19	0.22	3.39	61.96	0.25	4.08	73.91
61/2	0.81	19.60	287.14	0.31	7.26	85.90	0.70	15.11	181.21
81/2	1.39	31.92	563.40	0.54	12.00	200.58	1.19	24.01	381.79
101/2	2.31	50.82	934.79	0.76	17.47	283.91	2.58	51.81	796.42
121/2	3.09	76.02	1226.15	1.04	25.62	380.82	4.15	94.60	1213.17
141/2	4.18	102.95	1917.40	1.37	36.21	565.46	5.20	132.32	2266.65
161/2	4.97	128.07	2339.31	1.75	45.48	796.49	8.62	190.40	3129.62
181/2	7.46	171.61	2882.44	1.85	55.21	869.71	14.44	283.70	4065.21
201/2	8.65	193.83	3486.27	3.58	60.30	1015.79	20.63	358.83	5104.97

[1]Intel Core i5 Dual Core CPU 1.80 GHz, 8 GB 1600 MHz DDR3 RAM

A clear conclusion is that the Kohnen basis is the one to choose for the computation of the Kohnen plus-space unless the weights are small, in which case the Rankin-Cohen basis has a slightly better performance. If the weight is sufficiently high, then even the standard basis with subsequent linear algebra reduction to the plus-space outperforms the Rankin-Cohen basis. The ratio of the number of multiplications between the standard and the Kohnen basis almost becomes visible in the highest weight in the table.

In order to obtain a clearer idea of the complexity of the three algorithms with respect to the weight and also with respect to the number of coefficients to be computed, we used gnuplot [WKm17] for computing functions approximating the computation times. For the behaviour with respect to the weight, the above table was used. In order to understand the behviour with respect to the number of coefficients, also computation times for other numbers of coefficients were measured.

We approximated the running time as a function of the weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ by $f(k) = b \cdot k^a$ with a particular interest in the exponent a. The following table shows the calculated exponents a for the cases of 10^5 and 10^6 coefficients.

nb. coeff.	Standard Basis	Kohnen Basis	Rankin-Cohen Basis
10^5	1.93	1.74	2.77
10^6	1.93	1.81	2.57

These exponents clearly make the advantage of the Kohnen basis over the Rankin-Cohen basis visible. Whereas the exponent a seems to be quite stable with respect to the number of coefficients in the first two cases, one notices a decrease for the Rankin-Cohen basis. For fixed modular forms spaces, we also approximated the running time as a function of the number of coefficients x by the function $g(x) = b \cdot x^a$, again with particular interest in the exponent a. The following table shows the calculated value of a for three different weights.

weight	Standard Basis	Kohnen Basis	Rankin-Cohen Basis
$41/2$	1.16	1.17	1.17
$101/2$	1.31	1.21	1.22
$201/2$	1.22	1.18	1.13

We see that the three algorithms present a similar behaviour of the computation time with respect to the number of coefficients, which is surely only due to the fact that all three rely essentially on multiplications of power series. The data suggests a slight advantage for the Rankin-Cohen basis, which might be caused by the lacunarity of the powers of θ and their derivatives.

We close the paper with some final remarks on higher level cases. The standard and the Kohnen basis rely on an explicit description of a basis in terms of modular forms the q-expansion of which can be computed efficiently to a high precision. We do not know of any such description in any higher level. However, even if only generators (which might have some linear dependence) can be described in such a way, similar algorithms as those presented here will be direct consequences. For the moment, this remains an open problem.

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