Matrix Algebras, Irreducible Representation Spaces, and Relation to Particle Physics

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Contents

List of Figures

Zusammenfassung

In dieser Dissertation studieren wir die simultanen Realisierungen mehrerer irreduzibler Representationsräume innerhalb von Matrizenalgebren. Dabei zeigen wir, wie Relationen zwischen irreduziblen Representationsräumen als Konsequenz davon entstehen, fundamentale und adjungierte Representationsräume als linear unabhängige Unterräume auszudrücken. Unsere Arbeit gliedert sich in zwei Teile. In beiden Fällen arbeiten wir mit der Algebra M(8,$\mathbb{C}$), welche den Raum der acht-mal-acht-Matrizen aufspannt. Wir wählen diesen Raum aufgrund seiner Eigenschaften: es ist der kleinstmögliche Raum, der die verschiedenen Eichrepräsentationen des Standardmodells beinhaltet, er ist isomorph zur komplexen Cliffordalgebra $\mathbb{C}l(6)$ sowie eine Realisierung linearer Abbildungen auf den komplexifizierten Oktonionen. Im ersten Teil präsentieren wir eine explizite Einbettung der Eichgruppen des Standardmodells. Im zweiten Teil zeigen wir die Induktion einer direkten Summenzerlegung der Matrizenalgebra zu einem Satz irreduzibler Repräsentationsräume. Wir diskutieren die Eigenschaften der irreduziblen Repräsentationsräume innerhalb der Matrizenalgebra sowie deren Beziehungen zueinander. Wir vergleichen unsere Ergebnisse zu den Eigenschaften von Supersymmetrie, Großer Vereinheitlichter Theorien und nichtkommutativer Geometrie. Unsere Arbeit stellt keine Herleitung oder Erklärung der Eichrepräsentationen des Standardmodells dar. Stattdessen schlägt unsere Arbeiten einen neuen Zugang vor, Kombinationen irreduzibler Repräsentationsräume zu studieren. Insofern untersucht unsere Arbeit die Einführung linearer Unabhängigkeit zwischen irreduziblen Repräsentationsräumen, die Implikationen dieser zusätzlichen Struktur, wie sie in endlichdimensionalen Verktorräumen verwirklicht ist, und verbindet unser Ergebnis zu den irreduziblen Repräsentationsräumen des Standardmodells.

Abstract

In this thesis we study simultaneous realizations of multiple irreducible representations spaces within matrix algebras. In so doing we show how relations between irreducible representation spaces arise as a consequence of expressing fundamental and adjoint representation spaces as linearly independent subspaces. Our work proceeds in two parts. In both cases we work with the algebra M(8,$\mathbb{C}$), which spans the space of eight by eight complex matrices. This space is chosen as it is: the smallest possible space to simultaneously incorporate the different gauge representations of the Standard Model, isomorphic to the complex Clifford algebra $\mathbb{C}$l(6), and a realization of the linear maps on the complexified Octonions. In the first part we present an explicit embedding of the Standard Model gauge groups. Second, we show the induction of a direct sum decomposition of the matrix algebra into a set of irreducible representation spaces. We discuss the features of and relationships between the irreducible representation spaces in the matrix algebra, and compare our results to features of Supersymmetry, Grand Unified Theories, and Noncommutative Geometry. Our work is not intended to be a derivation or explanation of Standard Model gauge representations. Instead, our work proposes a novel approach to studying combinations of irreducible representation spaces. As such this work explores the introduction of linear independence between irreducible representation spaces, the implications of this additional structure as realized in finite dimensional vector spaces, and relates our results to the Standard Model's irreducible representation spaces.

Chapter 1

Introduction

Our current understanding of physics has been forged over centuries of hard work by a multitude of people from all around the world. In simplest terms one could say our understanding of modern physics is based on theories that describe the interactions of fields. However, this statement barely begins to scratch the surface of what these theories truly contain. There are many properties which are not immediately apparent, and hidden features which are only fully understood much later. Symmetries, for example, are central to invariants and conserved quantities, as discovered over a century ago through the work of Emmy Noether, Ref. [1]. Often these symmetries are encoded in the building blocks of theories, like how diffeomorphism invariance is encoded in the use of tensors in Einstein's General Relativity, Ref. [2]. In other cases, symmetries are hidden, only becoming apparent after careful investigation. A prime example of this is the hidden SU(3) symmetry behind the appearance and interactions of mesons and baryons, whose identification helped lead to the development of quantum chromodynamics, Ref. [3, 4].

Symmetries, of course, only account for some of the hidden features of field theories. For example, the way in which theories are expressed is not unique, and reformulations can provide new perspectives and insights into existing theories. Additionally, another property encoded in these theories are the constraints which determine the physical degrees of freedom of fields, and whose formulation requires a great deal of mathematical tools to uncover. Dirac and Bergmann independently started the process of studying the constraints of field theories in the 1950's, Ref. [5, 6, 7, 8]. This process of identifying constraints for first order theories was generalized in the late 1970's, Ref. [9], with much of the formalism cemented in the 1990's, Ref. [10]. Even so constraints in field theories continues to be a topic of research to this day. For example, the relationships between constraints and gauge generators is an active area of active research, Ref. [12], and is demonstrated for the case of Electromagnetism by Pitts, Ref. [11].

My doctorate has focused on such hidden structures of field theories. My earlier work, Ref. [13], studied a reformulation of bimetric theory. Bimetric theory is a theory of two (or more) interacting metric fields, i.e. spin-2 fields, with a unique low energy interaction structure, Ref. [14]. For consistency only one metric may interact with matter, referred to as the physical metric. My contribution to Ref. [13] was to integrate out the "nonphysical" metric and derive the resultant ghost free infinite derivative theory of gravity. This reformulation incorporated the effect of the additional, not directly observable, metric as an infinite series of gravitational self-interactions. This new, but classically equivalent, theory allowed for a physically consistent interpretation of the ghost degree of freedom in Weyl squared gravity, as originating from a finite truncation of this infinite series of gravitational self-interactions. This work received interest from String Theory, whose low-energy gravitational sector also describes interactions of spin-2 fields, Ref. [15, 16].

From there, I proceeded to study the construction of field theories from their constraint algebra. This study considered first order field theories, for spin-1 abelian fields, in a Minkowski spacetime. My contribution to the resultant pair of publications, Ref. [17, 18], was a construction mechanism of the most general Lagrangian compatible with the constraint algebra; i.e. the construction of all ghost-free interactions of the abelian spin-1 fields from the constraint enforcing relations. This allowed for model independent definitions of both Maxwell and Proca fields in terms of their constraint structure, and led to the most general first-order theory of interacting Maxwell and Proca fields in a Minkowski spacetime.

My later work takes a different approach to studying underlying structures of field theories. Instead of working with specific theories and constraint structures, my later publications, Ref. [19, 20],[1] focus on studying irreducible representation spaces as realized in matrix algebras. These works extend the idea of studying hidden structures of field theories to studying combinations of irreducible representation spaces of groups, which are the building blocks of modern particle theory. In particular this research concerns the relations and restrictions on a set of irreducible representation spaces when simultaneously realized as subspaces of a larger, but finite, dimensional vector space. The following dissertation is based on this idea.

1.1 Motivation and Research Question

The research presented in this dissertation is based on the idea of having relations between particles at the level of their representation spaces. This idea is not new, and has already been explored in formulations such as Supersymmetry (SUSY), Grand Unified Theories (GUTs), and Non-commutative Geometry (NCG). We will refer to these methods of establishing relationships between irreducible representation spaces as "unification approaches", and provide a brief review in section 1.3. In these unification approaches the different representations of the symmetry groups exist in different spaces. For example, fundamental representations are often described by column vectors while adjoint representations tend to be realized as matrices which act on these column vectors. As these are distinct vector spaces we have linear independence within our adjoint representations and within our fundamental representations, but not between the adjoint and fundamental representation spaces. We wish to extend this property of linear independence of elements within a representation space to linear independence of elements in distinct representation

[1]For Ref. [19] I was responsible for all calculations and most analysis.

spaces. This implies the simultaneous realization of adjoint and fundamental gauge[2] representations within one space; or in other words, as linearly independent subspaces. In this context we will also be considering multiplicities of representations as the appearance of multiple linearly independent subspaces transforming identically. Of course, in an infinite dimensional vector space one can include infinitely many linearly independent irreducible representation spaces. Instead, we will be interested in studying finite dimensional vector spaces.

For this to be possible we must study vector spaces which can realize both fundamental and adjoint representations. As mentioned, fundamental representations are realized as column vectors. However, as adjoint representations are realized by matrices which act on column vectors we cannot use column vectors to describe both the fundamental and adjoint representations. Instead, we require a vector space which also has a well defined operation of composition of elements. This is required not only to describe the action of adjoint representations on fundamental representations, but also to describe the action of adjoint representations on themselves. The simplest choice for such a space is a matrix algebra, with composition of elements described by matrix multiplication. Matrix algebras are a natural choice because any matrix algebra can always be embedded as a subalgebra of a larger dimensional matrix space, making them ideal for containing adjoint representations. We can also represent some N-dimensional fundamental representation by either a $N \times 1$ or $1 \times N$ matrix, i.e. a column or row vector respectively. These simple relations lead us to the research question we will be investigating in this dissertation:

> **What features are associated with simultaneous realizations of multiple representation spaces within a single matrix algebra?**

We stress that the purpose of this work is neither to present a unified theory of particle physics nor to propose any unification scheme. Instead, we will simply study features that arise when trying to simultaneously realize the adjoint and fundamental representations, as the building blocks of modern particle theory, in a single vector space. The immediate aim of this paper is therefore to study features and relationships between the representation spaces that are simultaneous realized within our matrix algebra. To provide a connection to modern particle theory, we will compare the resultant representation spaces to those of the Standard Model. We believe that the results of this work will help yield a deeper understanding of how to incorporate relationships between representation spaces. This supports the long term goal of this research, and its further developments, to identify structures and relationships in sets of representation spaces as to aid the development of unification approaches in particle theory.

[2]Note that we will in general refer to gauge transformations and gauge representations without any reference to a spacetime structure. We abuse the terminology of "gauge groups" in this thesis to instead denote groups not associated to the transformations of spacetime. In practice the groups we will be using are the unitary groups, which have a natural connection to the Standard Model gauge groups.

1.2 Outline

In section 1.3 we introduce the unification approaches of Supersymmetry (SUSY), Grand Unified Theories (GUTs), and Noncommutative Geometry (NCG). Highlighting features inherent to these different approaches provides a solid basis for understanding various results and features of our work.

Next, in section 1.4, we provide further motivation for our choice of matrix algebras as the space in which to study simultaneous realization of gauge representations. We motivate this choice in the context of Clifford algebras and matrix realisations of Standard Model representations. In subsection 1.4.2 we also introduce the space of eight by eight complex matrices, M(8,$\mathbb{C}$), as the smallest matrix algebra which may simultaneously realize all the different Standard Model gauge representations, including multiplicities and conjugate representations. This will be the primary algebra of focus for the entirety of the thesis. Proceeding onto our own work, we show in chapter 2 how one may embed the Standard Model gauge representations, including three generations, within our matrix algebra. In this chapter the selection of groups and irreducible representation spaces is ad-hoc, as we seek only an explicit identification with the Standard Model representations.

We comment on emergent features and relationships of the irreducible representations of this Standard Model embedding in chapter 3, and discuss how these could be implemented as principles in a bottom up construction approach to identifying irreducible representation spaces in a matrix algebra. This provides the basis for the work of chapter 4 where we show how to induce a direct sum decomposition of the matrix algebra M(8,$\mathbb{C}$), as a realization of the space of linear maps on the complex Octonions $\mathbb{C} \otimes \mathbb{O}$, into a set of irreducible representation spaces. We split these subspaces into three distinct classes, which we compare to the Standard Model representations in subsections 4.5.1 - 4.5.3. In subsection 4.5.4 we comment on uniqueness of the induced representation spaces. In section 4.6 we discuss the equivalence, from the perspective of the linear maps M(V) on V, of interpreting the space V as either a vector space or as an algebra. This presents avenues for generalizing our work to view the matrix algebra as representing linear maps on larger classes of spaces.

Having presented two ways in which the simultaneous realization of gauge representations is achieved, in section 4.7 we compare results and implications of these two approaches. Here we highlight advantages of the approach in chapter 4 from the perspective of understanding how certain conditions on our linear maps imply specific relations on the irreducible representation spaces. In particular, we highlight that the Standard Model only contains specific combinations of gauge and Lorentz representations, i.e. no spinors transforming in adjoint representations of gauge groups. This leads us to the following work of section 4.8, where we show how irreducible Lorentz representations be identified in the space of linear maps on M(2,$\mathbb{C}$).

In section 5.1 we return to the unification approaches introduced in section 1.3, discussing how our results compare to SUSY, GUT, and NCG. We discuss generalizations to our construction in section 5.2, and comment on potential implications. We summarize our work and provide concluding remarks in chapter 6.

1.3 Established Unification Approaches

In this section we discuss three approaches to imposing relations between irreducible representation spaces. There is a vast literature of various approaches to unification, and we comment only on a few here in this thesis. We will discuss their motivations and some features inherent to their formulation. In addition we will show how all these approaches, to some degree, incorporate linear independence between different irreducible representation spaces. This further motivates the idea of extending linear independence to be applicable between all irreducible representation spaces. The studies of SUSY, GUT, and NCG are well developed and so we are only able to comment on some of the many features and applications of these approaches. We provide reference to supplementary works that elaborate on points not discussed here.

1.3.1 Supersymmetry (SUSY)

SUSY describes a set of transformation, compatible with Lorentz and gauge symmetries while satisfying the Coleman-Mandula no-go theorem of Ref. [21], which relate the bosonic and fermionic Lorentz representations of the Standard Model. Historically, according to the review by Zumino in Ref. [22], SUSY was discovered three times independently, Ref. [23, 24, 25]. Of these, the first discovery of SUSY transformations was motivated by extending spacetime symmetries to restrict the action beyond one which is simply Lorentz invariant, Ref. [23]. This was made possible because SUSY introduces transformations between bosonic and fermionic degrees of freedom, and the combinations of terms which respect such transformations are a restricted subset of those which are Lorentz invariant.

Of course, in modern physics there are other motivations for studying SUSY. In particular its application to String Theory results in Superstring Theory, Ref. [26], which yields both bosonic and fermionic degrees of freedom. Additionally the Minimally Symmetric Standard Model, the minimal application of SUSY to the Standard Model particle content, both provides a close gauge coupling unification at large energies and reduces the sensitivity of the Higgs mass to radiative corrections, Ref. [27]. A defining property of SUSY is that it is a spacetime symmetry, not a gauge symmetry. As such it only has the potential of relating different Lorentz representations with identical gauge representations. However, Standard Model vector fields transform under different gauge representations to the fermionic fields. This means that the Minimally Symmetric Standard Model does not provide any relations between Standard Model particles, but instead doubles the particle content. To date there are no detections of supersymmetric particles.

A common description for SUSY involves the use of the superspace formulation. This is a vector space that includes both bosonic and Grassmann, or fermionic, dimensions, Ref. [27]. As such the superspace formulation is a method of incorporating different irreducible representation spaces as linearly independent subspaces of a larger space, i.e. the superspace. This is conceptually very similar to the work of this paper.[3] For more

[3]The superspace of SUSY is not a matrix algebra, so while conceptually very similar ideas their imple-

information on SUSY we refer the reader to Ref. [27]

1.3.2 Grand Unified Theories

GUTs have a different approach than SUSY to providing relationships between irreducible representation spaces. Instead of trying to relate the different Lorentz representations, GUTs try to describe the different Standard Model representations as originating from irreducible representations of some larger unified group. This is achieved by selecting a suitable group of which the Standard Model gauge group $G_{SM} = \mathrm{SU}(3) \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is a subgroup. The Standard Model symmetry transformations are then recovered by spontaneously breaking this larger group through the use of additional scalar degrees of freedom. Therefore, GUTs appear as a natural extension of the idea of spontaneous symmetry breaking in the Higgs sector of the Standard Model. Further, by calculating the beta functions of the electroweak and strong forces, in the Standard Model there is an apparent convergence of the gauge coupling values at large energies of $\Lambda_{\mathrm{GUT}} \approx 10^{16}$ GeV, Ref. [32].[4] The quantization of electromagnetic charge can also be explained through the use of GUTs, Ref. [29]. Together these observations motivate the idea that at the scale Λ_{GUT} all the different Standard Model forces should unify into a single force, and thus be described by a single gauge group. While there are several different proposals for the unifying group of GUTs, in this section we will chose to focus only on two prominent GUTs: SU(5) and SO(10). We will discuss the Pati-Salam model, Ref. [30], in the context of SO(10), as it is itself not a true GUT.

The first GUT was proposed by Howard Georgi and Sheldon Glashow in 1974, Ref. [31], and was based on the gauge group SU(5). This proposed theory unified the distinct gauge symmetries of the Standard Model into one group, and provided the different lepton and quark gauge representations as originating from restricted irreducible representations of SU(5). As such the theory incorporates certain combinations of Standard Model representations into a larger space. For example, when considering the restriction of the 10 representation of SU(5) we have that

$$10 \to (3,2) \oplus (\bar{3},1) \oplus (1,1) \tag{1.1}$$

of irreducible SU(3)×SU(2) representation spaces, Ref. [31]. Here (a,b) denotes a vector space transforming in the a representation of SU(3) and the b representation of SU(2). Clearly, the SU(5) GUT does achieve linear independence between some of the irreducible representation spaces contained in the Standard Model, but not between all representation spaces.

The SU(5) GUT is not capable of predicting the multiplicity of the fundamental representations, i.e. generations of leptons and quarks. Further, due to the existence of gauge bosons in SU(5) not belonging to the G_{SM} gauge group, the theory predicts proton decay.

mentations are rather different.

[4]Note that this apparent convergence is not as close as the one predicted from the Minimally Symmetric Standard Model, discussed in subsection 1.3.1.

Proton decay is also predicted by the scalar sector which spontaneously breaks the SU(5) gauge symmetry. Current limits on proton decay have ruled out the minimal SU(5), but not other variants like flipped SU(5) and SUSY SU(5), Ref. [33]. For more information on SUSY applied to SU(5) we refer the reader to Ref. [34, 35]

The other GUT we will mention is SO(10). This name is only by convention as the gauge group under consideration is actually Spin(10), the double cover of SO(10), Ref. [36]. This GUT is particularly interesting, because the irreducible **16** representation exactly encodes one generation of left handed particles and antiparticles. Similarly, the conjugate representation $\overline{\mathbf{16}}$ encodes one generation of right handed particles and antiparticles, including the right handed neutrino. In this context we should also discuss the SU(2)×SU(2)×SU(4) gauge group of the Pati-Salam model, Ref. [30]. This model is based on a gauge group which can be written as a direct product of simply connected Lie groups, as such it does not have one unifying group structure and therefore cannot be called a GUT. However, it still presents a partially unified structure through the SU(4) group. Specifically, the lepton becomes interpreted as the fourth colour of SU(4), and is mixed with the three colours of SU(3)⊂SU(4). Additionally, Pati-Salam introduces a symmetry between left handed and right handed fermions by the inclusion of another copy of SU(2) which only acts on right handed particles. Neither SO(10) nor Pati-Salam theories predict the generations of fermions.[5] Both the SU(5) GUT and the SU(2)×SU(2)×SU(4) Pati-Salam theory can be recovered from the SO(10) GUT.

Clearly GUTs are very attractive unification approaches, in that they attempt to explain the observed particle content of matter simply from restricted representations of a single group. In this way GUTs indeed provide relationships between the irreducible representation spaces of the Standard Model, by having the different particles identified with restricted subspaces of irreducible representation spaces of the grand unified group. This is certainly a nice feature of a theory, but struggles to incorporate the different generations of the Standard Model. This is because GUTs only consider the different representations of groups, but offer no guidelines on their multiplicities. One could of course gauge the flavour symmetry of the generations by having the three generations lie in a three dimensional representation of some group. In the context of GUTs this would require larger unification groups and thus many more bosonic degrees of freedom, Ref. [38]. For more information on GUTs we refer the interested reader to the above cited works.

1.3.3 Noncommutative Geometry

The main idea behind NCG is to geometrize the origin of the Standard Model gauge group, as to have one theory capable of reproducing both the Standard Model particle content and gravitational effects. This requires going from a picture of geometry using real variables

[5] It is worth noting that in the original proposal of Pati-Salam only two of the three generations were known. Their original proposal, built on SU(4)×SU(4)×SU(4) actually included all the two generations known at the time. Here the two generations of SU(2) representation spaces formed the $\mathbb{C}^4$ representation of SU(4), Ref. [30, 37]. However such a set-up fails due to the existence of a third generation, and thus “modern” Pati-Salam does not predict the three generations.

to a picture involving self adjoint functions on a Hilbert space, referred to as a spectral geometry, Ref. [39]. This requires an algebra of coordinates, described by the self-adjoint functions, which map to a noncommutative space, as detailed in the above reference. Then by studying the automorphisms of this algebra one recovers the diffeomorphism group of the manifold as the outer automorphisms. The inner automorphisms are instead identified with the "internal" symmetries, i.e. gauge symmetries. The action for the theory is constructed from the spectral decomposition of the algebra. For a thorough review we refer the reader to Ref. [39, 40, 41].

A crucial feature of NCG is that the different fermionic representations span a finite-dimensional Hilbert space, and there is an operator algebra acting on this Hilbert space as matrices acting on a vector space. In this way NCG obtains fermionic particles in fundamental representations. The bosons, as internal fluctuations of the operator algebra, also receive the correct transformation rules. By bosons we mean all integer spin particles, i.e. the graviton, gauge fields, and Higgs field. Indeed the Higgs field arises in the same way as gauge fields, but differs in its representation due to left and right handed fermions lying in distinct elements of a finite dimensional space. The Higgs boson is then the "gauge field" that corresponds to a finite separation of points, as opposed to the vector gauge fields that correspond to infinitesimal separations of points.

NCG can be viewed as a unification approach where the different bosons are unified and the different fermions are unified, but bosons are not unified with fermions as they lie in different spaces. Therefore linear independence is achieved between certain irreducible representation spaces, but not all. As a side note, SUSY has been employed in this context to see whether one could also formulate a NCG where fermions and bosons are on equal footing, Ref. [42, 43, 44]. This collection of three papers conclude that it is possible to obtain a theory with particle content matching the Minimally Symmetric Standard Model. However the coefficients of interaction terms are such that the standard action functional, the method used in NCG to generate an action, is in fact not supersymmetric.

There are many attractive features of NCG. For example the relation of diffeomorphisms and gauge transformations as outer and inner automorphisms results in the graviton as the outer fluctuations and gauge bosons plus Higgs as the inner fluctuations of the operator algebra, Ref. [41]. This makes NCG particularly interesting in that it addresses both the gauge and spatial nature of fields. Additionally, NCG formulates the entire action from the action functional, which depends only on the spectrum of the operator algebra. Thus NCG not only presents a particle content, but also a system for describing the interactions of the constituent particles. However, similar to GUTs, NCG offers no explanation for the origin of precisely three generations of fermionic particles.

n	$\mathbb{C}$l(n)
2m	M(2^m, $\mathbb{C}$)
2m+1	M(2^m, $\mathbb{C}$)$\oplus$M(2^m, $\mathbb{C}$)

Figure 1.1: Relationship between Clifford and matrix algebras over $\mathbb{C}$.

1.4 Particle Representations in Matrix Algebras

1.4.1 Clifford Algebras as Matrix Algebras

We start our discussion of particle representations in matrix algebras by first discussing Clifford algebras. Clifford algebras are also often referred to as Geometric algebras, because they allow for geometric interpretations of algebraic operations. For an introduction to Clifford algebras we refer the reader to Ref. [45]. In simple terms, the structure of a Clifford algebra can be understood from some generating vector space V, over the field $\mathbb{F}$, and a quadratic form $Q : V \times V \to \mathbb{F}$. This quadratic form can then be extended to act on all of $\bigwedge V$, the exterior space of V. Combining this with the exterior product yields the Clifford product as a bilinear map $\cdot$ such that, for $p \geq q$,

$$\cdot : \bigwedge^{p} V \times \bigwedge^{q} V \to \bigoplus_{i=-q}^{q} \bigwedge^{p+i} V. \tag{1.2}$$

For detailed description of Clifford algebras we refer the reader again to Ref. [45].

Clifford algebras appear in Standard Model physics and in unification approaches such as GUTs, to be elaborated on later in this section. As these Clifford algebras are finite-dimensional associative algebras over fields, they are all representable by a product of finitely many matrix algebras over division algebras, Ref. [46]. For Clifford algebras over the complex numbers the identification with matrix algebras is simply given by figure 1.1, Ref. [47].

As the algebras are isomorphic, we can extend ideas and motivations appearing from Clifford algebras directly to matrix algebras. It is worth noting that there is additional structure in Clifford algebras, such as the natural grading inherited from the exterior algebra, which is not apparent in matrix algebras.[6] Therefore one may ask: why work with matrices and not directly with the Clifford algebras themselves? The answer is two-fold. First matrices are more well known to the general physics community, and makes the following work more easily accessible. Second, the work of chapter 4 centres on the study of linear maps, which are representable as matrices but do not in general contain a natural grading. In this paper the matrix algebra of focus, M(8,$\mathbb{C}$), has a Clifford algebra equivalent. This is elaborated on in chapter 2.

[6] Of course one can always identify these structures in matrix algebras with a suitable isomorphism.

Clifford Algebras in the Standard Model

Different Clifford algebraic structures appear in the Standard Model. Most notably, the fermionic sector generates the structure of a Clifford algebra through use of the gamma matrices $\{\gamma_\mu\}_{\mu=0}^3$. These have a quadratic form given by

$$Q(\gamma_\mu, \gamma_\nu) := \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \tag{1.3}$$

where $\{\cdot, \cdot\}$ denotes the anticommutator. From this quadratic form we can identify the relevant algebra as Cl(1,3), the real Clifford algebra generated by the vector space $\mathbb{R}^{1,3}$. This algebra is commonly referred to as the Clifford algebra of spacetime or, within Clifford algebras, simply as the Spacetime algebra. For Dirac spinors the generators for Lorentz transformations are represented by the antisymmetric product of gamma matrices

$$S_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]. \tag{1.4}$$

When viewing the γ_μ as generating elements of the Clifford algebra of spacetime, the Lorentz generators (1.4) correspond to $\gamma_\mu \wedge \gamma_\nu \in \bigwedge^2 \mathbb{R}^{1,3}$. That is, they are bi-vectors in the Clifford algebra and can be viewed as oriented planes in $\mathbb{R}^{1,3}$, i.e. as spacetime rotations.

This makes it clear that Clifford algebraic relations are implicit in the formulation of Standard Model physics. This appearance can also be made explicit, as it is possible to write the Dirac Lagrangian purely in terms of Clifford algebra elements. This is the case even if the Dirac spinor is coupled to an electromagnetic field, Ref. [52]. In this setup the vector space representation of the electromagnetic field and the spinor representation of the Dirac field appear simultaneously, i.e. as linearly independent subspaces. This is a further motivation for our exploration of simultaneous realisations of distinct gauge representations in a matrix algebra.

For completeness we mention that the Clifford algebra Cl(1,3) has also been used to simply formulate a gauge theory of gravitational interactions, where displacements and Lorentz rotations are gauged separately, Ref. [48, 49].

Clifford Algebras for Unification

Besides the implicit use of Clifford algebras within the Standard Model, Clifford algebras also receive attention from unification approaches due to the idea of a "binary code" for the fermionic particles. The idea is discussed in detail in Ref. [50] with connection to GUTs. Therefore we will only provide a brief sketch of the arguments here, and refer the interested reader to the above cited work.

The idea of a binary code appears naturally from studying the restricted representations of SU(5) on $\bigwedge \mathbb{C}^5$ as describing one generation of Standard Model particles. To see this lets denote a basis for $\mathbb{C}^5$ by u, d, r, g, b where u, d denotes up-type, down-type and r, g, b denotes red, green, blue. Then the "colourless" basis elements of $\bigwedge \mathbb{C}^5$ are

$$1,\ u,\ d,\ u \wedge d,\ r \wedge g \wedge b,\ u \wedge r \wedge g \wedge b,\ d \wedge r \wedge g \wedge b,\ u \wedge d \wedge r \wedge g \wedge b. \tag{1.5}$$

These eight colourless basis elements are identified with the left and right handed electron, positron, neutrino, and antineutrino. Similarly the "coloured" basis elements, which span the rest of the space, are identified with the left handed, right handed, and respective antiparticles of the up and down quarks.

With this identification, the restricted representations of SU(5) on the exterior algebra $\bigwedge \mathbb{C}^5$ yield the gauge transformations of 16 fermions and 16 anti-fermions of one generation of the Standard Model, with the inclusion of right handed neutrinos. This is referred to as a "binary code" because of the simple identification with a binary string. Specifically, let a 1 denote the appearance of a basis element and a zero denote its absence in the exterior product, then we have the identification

$$n_1 n_2 n_3 n_4 n_5 \sim u^{n_1} \wedge d^{n_2} \wedge r^{n_3} \wedge g^{n_4} \wedge b^{n_5}, \tag{1.6}$$

where $n_1 n_2 n_3 n_4 n_5$ is a five bit string and the power on the right hand side is defined in terms of the exterior product. The range of each n_i is $\{0, 1\}$, and so we can describe the different basis elements, which are identified with particles, by their 5 bit string, i.e. their binary code.

Now clearly $\bigwedge \mathbb{C}^5$ is not a space of matrices, but by inclusion of a Clifford product we can map $\bigwedge \mathbb{C}^5$ to $\mathbb{C}l(5)$, the complex Clifford algebra of dimension 32, see table 1.1. This idea of a binary code is not restricted to $\mathbb{C}l(5)$, but can be generalized to any complex Clifford algebra. Indeed, the u, d, r, g, b basis elements of the generating space $\mathbb{C}^5$ are identified as representing charges of SU(2) and SU(3). Thus for a gauge theory with charges $\{c_i\}_{i=1}^n$, of unitary groups, one may construct the algebra $\mathbb{C}l(n)$ and identify particle representations with irreducible representation spaces appearing in the algebra.

In this text we neither aim to construct a GUT from a Clifford algebra nor to present an approach to unification. Further, in this idea of a "binary code" one implements only fermions, and keeps the gauge representations as separate. This differs from our goal of studying these representations as linearly independent subspaces. Nevertheless, this binary code exemplifies the potential of considering different irreducible representation spaces as subspaces of a matrix algebra. We have shown how Clifford algebras, and therefore matrix algebras, relate to Standard Model representations. Next, we proceed with the explicit realization of Standard Model representations as matrices.

1.4.2 Matrices for Standard Model Representations

Matrix Representation of Individual Particles

Having discussed matrices through the use of Clifford algebras, we turn our attention to employing matrices explicitly in the Standard Model. Matrices are relevant to Standard Model representations, as they can describe both the adjoint and fundamental representations. As mentioned, the fundamental representation, seen as a column or row vector, can be incorporated as the column or row of a matrix, i.e. as a left or right ideal of a matrix algebra. Of course, in the Standard Model, particles transform under more than one gauge group. There are three different symmetry groups in the gauge sector, namely

SU(3), $SU(2)_L$, and $U(1)_Y$. The subscript L on SU(2) denotes that this group only acts on left handed fermions, and the Y on the U(1) group denotes that this is the group generated by the weak hypercharge. A crucial property is that any particle will be at most in the fundamental representation of two non-abelian gauge groups. This is significant as we can then encode the fundamental representation of SU(3) as acting from the left on a matrix and of SU(2) as acting from the right. That is SU(3) and SU(2) would preserve left and right ideals respectively; we refer to this matrix representation, whose transformation is the simultaneous left and right action of two groups, as a bi-representation.[7]

Let us make these statements explicit. The adjoint representations are: the traceless Hermitian 2×2 matrices, i.e. the Pauli matrices, spanning the adjoint representation of $SU(2)_L$; and the traceless Hermitian 3×3 matrices, i.e. the Gell-Mann matrices, spanning the adjoint representation of SU(3). For the fundamental representations we have left handed fermions, right handed fermions, and the Higgs doublet. Left handed quarks exist as a doublet of $SU(2)_L$ and a triplet of SU(3); they can be represented by 3×2 matrices with SU(3) acting from the left and SU(2) acting from the right. Left handed leptons also exist as a doublet of $SU(2)_L$, but a singlet of SU(3). These can be expressed as 1×2 matrices. In contrast, right handed fermions are singlets of $SU(2)_L$. Therefore to simultaneously describe a left handed up and a down type quark one requires two sets of 3×1 matrices, i.e. a 3×1 matrix for each left handed quark. We also have the right handed electron, as a 1×1 matrix. Adding the right handed neutrino would require another 1×1 matrix. Additionally, there are three generations of fermions. Thus to simultaneously describe the transformation of all Standard Model particles would necessitate three copies of the above-mentioned matrices describing fundamental representations. Finally, the Higgs doublet as a singlet of SU(3) and doublet of $SU(2)_L$ requires a 1×2 matrix to describe its gauge representation. In addition we also have the $U(1)_Y$ field, which does not transform under $SU(2)_L$ and SU(3). The $U(1)_Y$ field is represented by a scalar, i.e. a 1×1 matrix, when acting on any of the above-mentioned irreducible representation spaces.

The Smallest Matrix Space for Particle Representations

Clearly all the different gauge transformations of the Standard Model can independently be represented by matrices. However, just being able to express the Standard Model gauge structures in terms of matrices is not enough to study the simultaneous realization of distinct gauge representations. In order to study these representations simultaneously we will consider them as distinct sub-matrices of a larger matrix algebra. To do so we must first select a relevant matrix algebra to work with. It is clear that with a large enough algebra one can embed practically any collection of irreducible representation spaces. We wish to proceed by selecting a minimal algebra, i.e. the algebra of smallest dimension while still incorporating all Standard Model gauge group representation spaces including multiplicities.

[7] This terminology is similar to that used when discussing "bi-unitary" transformations which diagonalize the Yukawa couplings.

The simplest requirement we can impose on the algebra is its dimension. As we wish to study simultaneous realization of gauge representations it is clear that the different representations must be linearly independent, just as was the case for fundamental SU(5) representations in $\bigwedge \mathbb{C}^5$, discussed in subsection 1.4.1. Thus we need a space which is large enough such that it can fit all the necessary particle representations as linearly independent subspaces. For the Standard Model this implies considering all the gauge representations of three generations of fermions and anti-fermions, a Higgs doublet, and our gauge groups. Lets preform this counting. Each generation of fermions appears in 1 (quark) triplet and 1 (lepton) singlet of SU(3). All fermions come in pairs where for left-handed fermions and the SU(3) singlet Higgs these pairs are in a doublet of SU(2). Additionally, fermions possess independent antiparticles transforming in conjugate representations. The adjoint representations of the Standard Model are further spanned by 12 gauge generators. In total, this adds up to $12+2\cdot 2\cdot(3\cdot(3+1))+2 = 62$ dimensions required to describe all the different representation spaces, including multiplicities, of the Standard Model gauge group. Note that actually, the above counting includes right handed neutrinos, which are not part of the Standard Model particle content. Excluding the three right handed neutrinos and their anti-particles from the count would leave us with 56 dimensions. However, as

$$\mathrm{Dim}\{\mathrm{M}(7,\mathbb{C})\} < 56 < 62 < \mathrm{Dim}\{\mathrm{M}(8,\mathbb{C})\}, \tag{1.7}$$

the inclusion of right handed neutrinos does not affect our choice of M(8,$\mathbb{C}$) as the smallest possible complex matrix algebra for incorporating all Standard Model representations.

In this paper we choose to use complex matrix algebras due to the appearance of complex numbers both in the adjoint and fundamental representations. In principle one could employ real matrix algebras by inclusion of a linear complex structure to take the place of the unit complex imaginary. However, if this operator cannot be expressed as an element of the matrix algebra itself, we cannot formulate our representation spaces as subspaces of the matrix algebra; preventing the goal of this research. Additionally, there would be uncertainty in how to define this complex structure on our space. So, for simplicity and definiteness we choose to work with complex matrix algebras.

In chapter 2 we show explicitly how all the different gauge representations of the Standard Model can be realized simultaneously in M(8,$\mathbb{C}$) subalgebras of $\mathrm{M}(2,\mathbb{C}) \otimes \mathrm{M}(8,\mathbb{C}) \cong \mathrm{M}(16,\mathbb{C})$, where $\mathrm{M}(n,\mathbb{C})$ is the space of $n \times n$ complex matrices. The tensor product with M(2,$\mathbb{C}$) is to incorporate Lorentz representations, which are needed in order to produce the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ transformations that distinguish based on chirality. The Lorentz representations in M(2,$\mathbb{C}$) are elaborated on in section 2.2.

Chapter 2

Simultaneous Realization of Standard Model Representations

2.1 Preliminaries: Matrix-Clifford Equivalence

In subsection 1.4.1 we commented on the Clifford algebra equivalence of our matrix algebra of choice, which for describing the gauge representations will be M(8,$\mathbb{C}$). This algebra is isomorphic to the complex 64 dimensional Clifford algebra $\mathbb{C}l(6)$, introduced in Appendix B.1, which we will use as a starting point for our work. Specifically, using the isomorphism defined in Appendix B.2, there exists a $\mathbb{C} \oplus \mathbb{C}^3 \oplus (\mathbb{C} \oplus \mathbb{C}^3)^*$ vector space structure of left ideals of this algebra, where complex conjugation changes between the triplet and conjugate triplet representations of SU(3). The structure of this decomposition is also identical to the $\mathbb{C}^4 \oplus \mathbb{C}^{4*}$ SU(4)-structure of the Pati-Salam model, shown in Ref. [50], after the breaking of SU(4) into U(1)×SU(3). For an analysis of the SU(3) representations that appears in $\mathbb{C}l(6)$ we refer the reader to Ref. [51].

This presents a convenient way of describing SU(3) transformations of leptons and quarks along with their antiparticles, which has the SU(3) structure $\mathbb{C} \oplus \mathbb{C}^3$ for one generation of leptons and quarks and $\mathbb{C}^* \oplus \mathbb{C}^{3*}$ for the respective antiparticles. Complex conjugation in the Clifford algebra $\mathbb{C}l(6)$ does not correspond to complex conjugation in the matrix algebra M(8,$\mathbb{C}$). Instead we obtain a new operator $\bar{*}$ in M(8,$\mathbb{C}$) which is identified with complex conjugation * in the Clifford algebra, via the isomorphism described in Appendix B.2. We will refer to the $\bar{*}$ operation as complex conjugation for the remainder of chapter 2.

We note that while we indeed draw these properties from the Clifford algebra $\mathbb{C}l(6)$, the entirety of this section is written from the standpoint of M(8,$\mathbb{C}$) and does not require an understanding of the Clifford algebra structure. Thus the Clifford algebra isomorphism simply motivates our starting point for how to describe the SU(3) transformations of our irreducible representation spaces, and introduces an operator $\bar{*}$ to describe the transition between complex conjugate representation spaces.

2.2 Preliminaries: Lorentz Representations

In this chapter we investigate the embedding of the Standard Model gauge representations within the matrix algebra M(16,$\mathbb{C}$). Specifically, we examine what features arise as a consequence of a simultaneous realization of all Standard Model representations within the same matrix algebra. In particular, because we wish to have the different gauge structures of left and right handed fields, this requires the inclusion of Lorentz representations. The smallest space which can incorporate simultaneous left and right handed chiral fields is of complex dimension 4. Since we are interested in matrix algebras we will therefore consider M(2,$\mathbb{C}$) as describing the Lorentz representations of SL(2,$\mathbb{C}$). We note that M(2,$\mathbb{C}$) is isomorphic to the complexified Pauli-algebra, along with an identity element, and as such is also isomorphic to the real "Algebra of Physical Space", Cl(3), which describes spinors, scalars, and vectors; Ref. [52]. However, for convenience we will use the isomorphism to $\mathbb{C} \otimes \mathbb{H}$, the algebra of complexified Quaternions. This is because the complex Quaternions yield simple forms for describing Lorentz representations and their respective conjugate representations. For example, within the complex Quaternions linearly independent left and right handed spinor representation spaces are simply related by complex conjugation. This draws a simple connection to how complex conjugation relates conjugate representations in $\mathbb{C}$l(6).

The specific details of the Quaternions are not needed to understand the following appearance of representations, and so we present only the most basic ingredients here. For more details on the Quaternions and their complexification, as well as the isomorphism to M(2,$\mathbb{C}$), we refer the reader to Appendix A. The Quaternions are spanned by four basis elements. This basis can be expressed as the identity element and three $\{\varepsilon_i\}_{i=1}^{3}$, which are imaginary roots of -1 and have commutation relations isomorphic to $\mathfrak{so}(3)$. Under complexification we can separate the algebra into two $\mathbb{R}^{1,3}$ vector spaces

$$\mathrm{M}(2,\mathbb{C}) \cong \mathbb{C} \otimes \mathbb{H} = \mathrm{Span}_{\mathbb{R}}\{1, i\varepsilon_1, i\varepsilon_2, i\varepsilon_3\} \oplus \mathrm{Span}_{\mathbb{R}}\{i, \varepsilon_1, \varepsilon_2, \varepsilon_3\}. \tag{2.1}$$

where for any a in either of the $\mathbb{R}^{1,3}$ vector subspaces of (2.1) and $\Lambda \in \mathrm{SL}(2,\mathbb{C}) \subset \mathbb{C} \otimes \mathbb{H}$, a transformation of the type

$$a \to \Lambda a \Lambda^{\dagger} \tag{2.2}$$

describes the transformations of a spacetime vector. Here the † operation in $\mathbb{C} \otimes \mathbb{H}$ is such that it reproduces Hermitian conjugation in M(2,$\mathbb{C}$), see Appendix A. For the left and right handed spinors we note that the projector onto left handed spinors P and the projector onto right handed spinors $\bar{P}$ are related by complex conjugation. These project onto distinct left ideals of the algebra $\mathbb{C} \otimes \mathbb{H}$. Thus by describing one ideal as contining all left handed spinors we naturally obtain right handed spinors in the second ideal, with the correct transformation properties under complex conjugation in $\mathbb{C} \otimes \mathbb{H}$.[1] Finally, scalar particles appear as the centre of $\mathbb{C} \otimes \mathbb{H}$ which is just $\mathbb{C}$ itself, as expected.

[1]Note that these different left ideals correspond to distinct columns in M(2,$\mathbb{C}$).

This demonstrates the ability to encode all the necessary Lorentz representations for describing Standard Model representation spaces within only M(2,$\mathbb{C}$). So while for gauge transformations we are interested in the matrix algebra M(8,$\mathbb{C}$), we are viewing it as a subalgebra of M(16,$\mathbb{C}$) = $\mathrm{M}(2,\mathbb{C})\otimes_{\mathbb{C}}\mathrm{M}(8,\mathbb{C})$; where $\otimes_{\mathbb{F}}$ denotes the tensor product over the field $\mathbb{F}$.[2] However, because of the simple tensor structure of the product between the space responsible for gauge representations and the one responsible for Lorentz representations, we will be able to restrict our attention predominantly to M(8,$\mathbb{C}$). Indeed linear independence of our irreducible representation spaces in M(16,$\mathbb{C}$) reduces to linear independence of the irreducible representation spaces in M(8,$\mathbb{C}$). This is not a general feature of the algebra M(16,$\mathbb{C}$), but rather a consequence of how we chose to express the different irreducible subspaces in M(8,$\mathbb{C}$). Specifically, this behaviour is a consequence of letting the gauge representation spaces associated with left handed spinor transformations and those associated with right handed spinor transformations occupy the same subspace in M(8,$\mathbb{C}$). This is possible as they occupy different ideals in M(2,$\mathbb{C}$) and are therefore linearly independent in M(16,$\mathbb{C}$). Thus in the following work the only reference to M(2,$\mathbb{C}$) will be in ensuring distinct transformations between left and right handed spinors. This implies that we will be considering right handed neutrinos in the construction. However, as these are singlet representations, one can always project out their components without affecting the structure of the rest of the representation spaces. This demonstrates that the inclusion of right handed neutrinos will not affect the validity of our claims about this Standard Model embedding.

Additionally, in our embedding of Standard Model irreducible representation spaces we will not be associating a field to the generators of Lorentz representations, i.e. we will not be including the spin connection field. This is equivalent to how the Lorentz group only acts globally in the Standard Model.

2.3 Summary of Results

The result of our embedding of Standard Model representations is the direct sum decomposition of the matrix space M(8,$\mathbb{C}$) as

$$\mathrm{M}(8,\mathbb{C}) = \mathbb{C} \otimes \Big[\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)\Big] \oplus 3 \cdot \Big[F_3 \oplus F_{\bar{3}} \oplus F_1 \oplus F_{\bar{1}}\Big] \oplus F_\phi \oplus P_{\mathrm{Add}} \,. \tag{2.3}$$

The first three subspaces correspond to the adjoint representations which satisfy the commutation relations dictated by their respective Lie algebras. Specifically, as we are considering complex matrix algebras, we find the complexification of the Lie algebras of the Standard Model. Together with the two sets of Lorentz vector representations described in section 2.2, this implies that considering the complexification of the Lie algebra in M(8,$\mathbb{C}$) is equivalent to considering both the hermitian and anti-hermitian vector spaces in M(16,$\mathbb{C}$).

[2]Note that for brevity, $\otimes$ without a subscript denotes the tensor product over the real numbers.

It is currently not clear whether there is any significance in this appearance of double vector representations. Indeed, while we are here only concerned with representation spaces, in a proper theory construction it is not possible to say whether all representations will even be present in the action. For example, we refer the reader to subsection 5.1.3 for comments on how only one real adjoint representation is recovered in NCG.

The F_i subspaces denote the SU(3) transformations of our fundamental representations. For left handed fermions these are shown to all transform as doublets under SU(2) for right handed fermions they are singlets of SU(2). F_ϕ is instead the Higgs SU(2)-doublet representation. Under SU(3) F_3 transforms as a triplet, $F_{\bar{3}}$ as an anti-triplet, and F_1, $F_{\bar{1}}$, F_ϕ as singlets. Correct hypercharge assignments stem from application of the $\mathfrak{u}(1)$ generator of U(1).

In the following we discuss in detail how the space $\mathrm{M}(8,\mathbb{C})$ decomposes, as per (2.3), into the direct sum of the irreducible representation spaces of the gauge group of the Standard Model. As mentioned before, and part of the counting in subsection 1.4.2, for the transformations associated to fermionic particles we will be simultaneously realizing both the fundamental representations and their respective conjugates. As expected from our counting of subsection 1.4.2, we find a two dimensional subspace, denoted P_{Add}, which is not identified with any of the Standard Model representation spaces or right handed neutrinos.

2.4 Technical Summary

2.4.1 Choosing a basis of $\mathrm{M}(8,\mathbb{C})$

Let R_I with $I = 1, \dots, 8$ be a complete set of basis vectors of $\mathbb{C}^8$, chosen such that their inner product is[3]

$$R_I{}^\dagger R_J = \delta_{IJ}\,. \tag{2.4}$$

We furthermore define 8 vectors $\{V_a^+, V_a^-\}$ with $a = 0, \dots, 3$ and express them as linear combinations of the above basis vectors,

$$V_a^\pm = \sum_{I=1}^{8} a_{aI}^\pm R_I\,. \tag{2.5}$$

The complex coefficients $a_{aI}^\pm$ are chosen such that the $V_a^\pm$ are linearly independent and their inner products satisfy[4]

$$(V_a^\pm)^\dagger V_b^\pm = \delta_{ab}\,, \qquad (V_a^\pm)^\dagger V_b^\mp = 0\,. \tag{2.6}$$

[3]These basis vectors also satisfy $R_I{}^{\bar{*}} := \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} R_I{}^* = R_{I+4}$, where $\eta = \mathrm{diag}(1,-1,-1,-1)$. The $\bar{*}$ is an operation in $\mathrm{M}(8,\mathbb{C})$ that corresponds to complex conjugation in $\mathbb{C}\mathrm{l}(6)$, as described in appendix B.2.

[4]Additionally, we will require that $(V_a^\pm)^{\bar{*}} = V_a^\mp$ which implies $\left(a_{aI}^+\right)^* = a_{a(I+4)}^-$.

Note that the vectors $\{V_a^+, V_a^-\}$ form another orthonormal basis of $\mathbb{C}^8$.

Out of the basis vectors R_I one can construct a basis M_{IJ} of $\mathrm{M}(8,\mathbb{C})$ using the outer product,

$$M_{IJ} = R_I R_J^\dagger \,, \qquad I,J = 1,\ldots,8 \,. \tag{2.7}$$

The irreducible representation spaces of the Standard Model gauge group will be identified with 62 linearly independent combinations of these basis elements.

2.4.2 Identification with particles

Using the basis vectors defined in the previous subsection, we identify the Standard Model representation spaces within the algebra $\mathrm{M}(8,\mathbb{C})$ as follows.

The SU(3) generators. The $\mathfrak{su}(3)$ Lie algebra is spanned by the generators,

$$\begin{aligned}
\lambda_1 &= R_2\,(R_1)^\dagger + R_1\,(R_2)^\dagger - R_6\,(R_5)^\dagger - R_5\,(R_6)^\dagger \ , \\
\lambda_2 &= iR_2\,(R_1)^\dagger - iR_1\,(R_2)^\dagger + iR_6\,(R_5)^\dagger - iR_5\,(R_6)^\dagger \ , \\
\lambda_3 &= R_1\,(R_1)^\dagger - R_2\,(R_2)^\dagger - R_5\,(R_5)^\dagger + R_6\,(R_6)^\dagger \ , \\
\lambda_4 &= R_1\,(R_3)^\dagger + R_3\,(R_1)^\dagger - R_5\,(R_7)^\dagger - R_7\,(R_5)^\dagger \ , \\
\lambda_5 &= iR_3\,(R_1)^\dagger - iR_1\,(R_3)^\dagger + iR_7\,(R_5)^\dagger - iR_5\,(R_7)^\dagger \ , \\
\lambda_6 &= R_3\,(R_2)^\dagger + R_2\,(R_3)^\dagger - R_7\,(R_6)^\dagger - R_6\,(R_7)^\dagger \ , \\
\lambda_7 &= iR_3\,(R_2)^\dagger - iR_2\,(R_3)^\dagger + iR_7\,(R_6)^\dagger - iR_6\,(R_7)^\dagger \ , \\
\lambda_8 &= \frac{1}{\sqrt{3}}\left[R_1\,(R_1)^\dagger + R_2\,(R_2)^\dagger - 2R_3\,(R_3)^\dagger - R_5\,(R_5)^\dagger - R_6\,(R_6)^\dagger + 2R_7\,(R_7)^\dagger\right] \ .
\end{aligned} \tag{2.8}$$

Using the orthonormality relations (2.4), it is easy to verify that these elements indeed satisfy the $\mathfrak{su}(3)$ commutation relations.

The U(1) generator. The element of M(8,$\mathbb{C}$) responsible for $\mathrm{U}(1)_Y$ transformations is,

$$Q = -R_4(R_4)^\dagger - \frac{1}{3}\sum_{I=1}^{3} R_I(R_I)^\dagger - \frac{2}{3}\sum_{I=5}^{7} R_I(R_I)^\dagger. \tag{2.9}$$

Note that this basis element commutes with $\mathfrak{su}(3)$, which as we will show is crucial for a consistent formulation of charge distributions.

The SU(2) generators. The $\mathfrak{su}(2)$ Lie algebra is spanned by the generators,

$$T_1 = \sum_{a=0}^{3} V_a^- \left(V_a^+\right)^\dagger + V_a^+ \left(V_a^-\right)^\dagger \,,$$
$$T_2 = \sum_{a=0}^{3} iV_a^+ \left(V_a^-\right)^\dagger - iV_a^- \left(V_a^+\right)^\dagger \,,$$
$$T_3 = \sum_{a=0}^{3} V_a^+ \left(V_a^+\right)^\dagger - V_a^- \left(V_a^-\right)^\dagger \,. \tag{2.10}$$

As for $\mathfrak{su}(3)$, the orthonormality relations (2.6) imply the desired $\mathfrak{su}(2)$ commutation relations.

The fermions. The 16 elements of $\mathrm{M}(8,\mathbb{C})$ which transform as one generation of particles and antiparticles under the gauge groups correspond to

$$\left\{ R_I \left(V_a^\pm\right)^\dagger \,\middle|\, I = 1, ..., 8 \right\}, \tag{2.11}$$

where the index $a \in \{1,2,3\}$ labels the generation. The elements with $I \in \{1,2,3,4\}$ correspond to particle representations, i.e. fundamental representations, while those with $I \in \{5,6,7,8\}$ correspond to antiparticle representations, i.e. anti-fundamental representations.

The Higgs. The 2 elements describing the Higgs doublet are

$$\left\{ R_4 \left(V_\phi^\pm\right)^\dagger \right\}, \tag{2.12}$$

for a linear combination,

$$V_\phi^\pm = \sum_{a=0}^{3} h_a^\pm V_a^\pm \,, \tag{2.13}$$

where $h_a^\pm \in \mathbb{C}$, and $\left(h_a^+\right)^* = h_a^-$ with $h_0^\pm \neq 0$.

The transformation laws. The transformation laws for the different subspaces of M(8,$\mathbb{C}$) follow directly from identifying the correct transformations of the corresponding Standard Model irreducible representation spaces. As these transformations are all well known, we will not be presenting their explicit forms here. For every symmetry transformation there are corresponding elements in M(8,$\mathbb{C}$) whose action on subspaces of M(8,$\mathbb{C}$) reproduces the transformations of Standard Model particles. Additionally, for fundamental representation spaces related by complex conjugation, their transformation properties are also complex conjugate of each other. This implies anti-particle transformations are given by the complex conjugate representation rather than the hermitian conjugate representation. As the complex conjugate and the dual representations coincide for the groups under consideration, Ref. [53], this is consistent with the Standard Model representations. The symmetry transformations of the irreducible representation spaces are described in subsection 2.5.1 for the groups SU(3), $U(1)_Y$, and $SU(2)_L$.

Linear independence. The necessary and sufficient conditions for linear independence of (2.8)-(2.12) are

$$\begin{aligned}
&\left(a^+_{01} \quad a^+_{02} \quad a^+_{03}\right)^{\mathrm{T}} \not\propto \left((a^+_{05})^* \quad (a^+_{06})^* \quad (a^+_{07})^*\right)^{\mathrm{T}} , \\
&\left(a^+_{01} \quad a^+_{02} \quad a^+_{03}\right)^{\mathrm{T}} \not\perp \left((a^+_{05})^* \quad (a^+_{06})^* \quad (a^+_{07})^*\right)^{\mathrm{T}} , \qquad h^\pm_0 \neq 0 ,
\end{aligned} \tag{2.14}$$

where neither of the two $\mathbb{C}^3$ vectors above may be the zero vector, along with the condition

$$|a^+_{8,4}|^2 \neq S(|a^+_{4,8}|^2) \quad \text{for} \quad S(x) := \frac{1}{6}\Big(-1 - 10x + \sqrt{25 + 8x + 64x^2}\Big). \tag{2.15}$$

Separately, care must be taken to chose elements a^+_{aI}, for $a \in \{0, 1, 2, 3\}$, such that the orthonormality relations in (2.6) are satisfied. The conditions (2.14) and (2.15), which are derived in section 2.6, are very weak and still allow for a lot of freedom in choosing our basis elements. In fact, the set $\{a^\pm_{aI}\}$ still contains 28 real parameters after all conditions for orthogonality, conjugation and linear independence have been imposed. The linear independence of (2.8)–(2.12) ensures the direct sum decomposition of M(8,$\mathbb{C}$) in (2.3).

2.5 Identifying Representation Spaces

We now show how to arrive at the set of linearly independent representation spaces associated to the different Standard Model particles. As we are embedding these representation spaces within the algebra M(8,$\mathbb{C}$), we will start simply by assigning a set of 24 linearly independent elements of M(8,$\mathbb{C}$) to all weak isospin doublets of the Standard Model associated to fermionic particles. Demanding these representation spaces be linearly independent from their complex conjugates yields another 24 weak isospin doublets associated to fermionic antiparticles. After including the Higgs doublet in a similar fashion, we identify the relevant gauge generators by demanding correct transformation laws for our irreducible representations as given by the matrix algebra acting on itself.

Fermions and Higgs

Starting from the basis vectors introduced in subsection 2.4.1, consider the following elements of M(8,$\mathbb{C}$),

$$\Big\{ R_I \left(V^+_a\right)^\dagger \Big| I = 1, ..., 4 \Big\} . \tag{2.16}$$

For $a = 1, 2, 3$, these will be identified with the fermions of the Standard Model. As will become clear later, the "+" index denotes that the particles corresponding to these elements have weak isospin value "up". Their companion particles of weak isospin "down" are denoted by

$$\left\{R_I\left(V_a^-\right)^\dagger \middle| I = 1, ..., 4\right\}. \tag{2.17}$$

Hence the set of elements

$$\left\{R_I\left(V_a^\pm\right)^\dagger \middle| I = 1, ..., 4\right\} \tag{2.18}$$

describes one generation of weak isospin doublets, where the generation is labelled by $a \in \{1, 2, 3\}$.

Next we will assign SU(3) charges to these basis elements. We choose, arbitrarily, to assign the SU(3) charges (red, green, blue) to $I = (1, 2, 3)$ respectively, and to make $I = 4$ a singlet of SU(3). For hypercharges consistent with the Standard Model charge allocations we must include considerations of their Lorentz representation, so we postpone this until we have described all the relevant SU(2) and SU(3) charge assignments.[5]

Having described three generations of particles, we now identify their respective antiparticles. Since we want particles and antiparticles to have opposite electroweak charge and be related via the complex conjugation operation $\bar{*}$, we choose the vectors $V_a^\pm$ such that,

$$\left(V_a^\pm\right)^{\bar{*}} = V_a^\mp . \tag{2.19}$$

Note that this relationship implicitly uses that the 2 and 2^* representations of SU(2) are related by a similarity transformation. We then have that,

$$\left(R_I\left(V_a^\pm\right)^\dagger\right)^{\bar{*}} = \left(R_I\right)^{\bar{*}}\left(V_a^\mp\right)^\dagger . \tag{2.20}$$

Since in our construction the representation spaces associated to particle and antiparticle transformations are described by linearly independent subspaces, this implies (2.20) must be linearly independent from (2.18). We thus demand our basis vectors to satisfy,

$$\left(R_I\right)^{\bar{*}} = R_{I+4}. \tag{2.21}$$

Then the antiparticle states corresponding to (2.18) are

$$\left\{R_I\left(V_a^\mp\right)^\dagger \middle| I = 5, ..., 8\right\}. \tag{2.22}$$

[5] Note that for Hypercharge transformations we have non-trivial actions on left and right handed spinors. For $SU(2)_L$ transformations case is simpler to implement, as since it acts trivially on right handed spinors incorporating the effect of chiral discrimination only involves the inclusion of the projector P onto left handed spinors.

To summarize, for each $a \in \{1,2,3\}$ we have one full generation of fundamental and anti-fundamental representation spaces of the form

$$\left\{ R_I \left(V_a^{\pm}\right)^{\dagger} \middle| I = 1, ..., 8 \right\}. \tag{2.23}$$

Here, $I = 8$ denotes a singlet of SU(3) and $I = (5,6,7)$ are assigned the SU(3) charges (anti-red, anti-green, anti-blue) respectively.[6]

Next we turn to the Higgs doublet representation which must have the same charge assignment as the SU(3) singlet of (2.18). This leads us to elements of the type

$$\left\{ R_4 \left(V_\phi^{\pm}\right)^{\dagger} \right\}. \tag{2.24}$$

Here $V_\phi^{\pm}$, as defined in (2.13), are linear combinations of the $V_a^{\pm}$ which must include $V_0^{\pm}$ for (2.24) to be linearly independent from (2.23). The complex conjugate of (2.24) yields another pair of elements,

$$\left\{ R_8 \left(V_\phi^{\mp}\right)^{\dagger} \right\}. \tag{2.25}$$

While these basis elements have charge assignments opposite to those of the Higgs doublet representation, they cannot be made linearly independent from the rest of the irreducible representation spaces identified with the Standard Model particle content. This means that for the Higgs doublet representation we cannot not have an independent conjugate representation when embedded in M(8,$\mathbb{C}$).

This condition agrees with the implementation of the Higgs doublet in the Standard Model. While the Yukawa interactions require both the Higgs doublet and its conjugate, the same two complex parameters which describe the Higgs doublet appear in the conjugate doublet (see, for instance, Ref. [54]). Therefore, in the Standard Model the conjugate doublet is not associated to an independent particle, by construction. Conversely, in this embedding of Standard Model representations within the algebra M(8,$\mathbb{C}$) the lack of an independent conjugate Higgs doublet is unavoidable, following directly from the identification of irreducible representation spaces as linearly independent basis elements.

2.5.1 The gauge generators

Here we identify the gauge generators by imposing the desired gauge transformation properties of the particle states (2.18) and (2.24). Before we begin, let us denote an arbitrary gauge transformation, given by some operator $\mathcal{O}$. In the Standard Model particles and antiparticles are acted upon by the same gauge generators, albeit in different representations. Specifically for a fundamental transformation

$$K \to \mathcal{O}K \tag{2.26}$$

[6] We note of course that it is arbitrary to which elements we associate the different colour charges, and we assign an allocation only for definiteness.

and the hermitian conjugate transformation

$$K^\dagger \to (\mathcal{O}K)^\dagger = K^\dagger \mathcal{O}^\dagger, \tag{2.27}$$

$\mathcal{O}$ and $\mathcal{O}^\dagger$ can be expressed via the same basis of generators. However, here the conjugate fundamental representations here span complex conjugate basis elements, which do not necessarily lie in the same space. Thus, we seek transformation operators $\mathcal{O}$ which, for any K in (2.18) or (2.24), satisfy

$$(\mathcal{O}K)^{\bar{*}} \stackrel{!}{=} \mathcal{O}K^{\bar{*}}. \tag{2.28}$$

In the following this will be used to identify the form of our gauge generators.[7] We note that the Higgs doublet space transforms in the same way as a left handed lepton doublet of SU(2).

SU(3) transformations To identify the $\mathfrak{su}(3)$ generators, let us consider again the fermionic particles in (2.18), which span an SU(3) invariant subspace. Hence, an SU(3) transformation acting on these states should be of the form,

$$R_I \left(V_a^\pm\right)^\dagger \longmapsto \sum_J c_{IJ} R_J \left(V_a^\pm\right)^\dagger, \tag{2.29}$$

with $c_{IJ} \in \mathbb{C}$. The transformation matrix itself will be a linear combination of the full set of basis elements (2.7). These basis elements act from left on the fundamental representations as,

$$R_K {R_J}^\dagger R_I \left(V_a^\pm\right)^\dagger = \delta_{IJ} R_K \left(V_a^\pm\right)^\dagger. \tag{2.30}$$

We can then deduce the form of SU(3) generators by demanding that their action leaves the elements of (2.18) with $I = 4$ invariant. The most general expression satisfying this is,

$$\bar{\lambda}_I = \sum_{K,L=1}^{3} \Omega_{IKL} R_K (R_L)^\dagger, \qquad \Omega_{IKL} \in \mathbb{C}, \quad I = 1, \ldots, 8. \tag{2.31}$$

The coefficients Ω_{IJK} are fixed by assigning the SU(3) colour charges to the index $I = 1, 2, 3$ of our fermions. Then the $\bar{\lambda}_I$ become maps between the different colour charges, as desired.

The generators acting on the anti-particle states (2.22) can then be identified by using (2.28). Clearly by complex conjugation the SU(3) generators that transform the conjugate fundamental representations must be $\bar{\lambda}^{\bar{*}}$. We need a transformation which reduces to λ_I when acting on the fundamental representation spaces and $\bar{\lambda}^{\bar{*}}$ when acting on the anti-fundamental representation spaces.

[7] Note while the *operations* of gauge transformations should satisfy this criteria, it is not required of the gauge generators themselves.

Now, due to (2.21), $\bar{\lambda}_I$ and $\bar{\lambda}_I^{\bar{*}}$ are linearly independent. Further still, $\bar{\lambda}_I$ and $\bar{\lambda}_I^{\bar{*}}$ commute and $\bar{\lambda}_I$ annihilates the anti-fundamental representation spaces while $\bar{\lambda}_I^{\bar{*}}$ annihilates fundamental representation spaces under left multiplication. Thus we may instead identify one transformation given by the set of generators,

$$\lambda_I = \bar{\lambda}_I - \bar{\lambda}_I^{\bar{*}}, \qquad I = 1, \ldots, 8, \tag{2.32}$$

which satisfies (2.28). This gives precisely the expressions in (2.8). Now, $e^{i\lambda_I}$ correctly transform both particle and anti-particle states, and the generators λ_I will be the ones associated to the gauge field of $\mathfrak{su}(3)$. This simple identification between transformation operator and generator of the symmetry transformation is possible simply because left and right handed fermions in the Standard Model have the same SU(3) representations. This is not the case for $\mathrm{U}(1)_Y$ and $\mathrm{SU}(2)_L$ representations, as we will show in the remainder of this subsection.

Note that the Higgs elements (2.24) and the corresponding complex conjugate elements are automatically invariant under SU(3) transformations, just like the singlet states of (2.23).

$\mathbf{U}_Y(1)$ transformations Similarly to the SU(3) charges, hypercharges must also be assigned to the index I and so the hypercharge transformation must also act from the left. Since the $\mathrm{U}_Y(1)$ transformation does not transition between colours, it must be constructed only out of elements of the form $R_I(R_I)^\dagger$, which takes the index I to the index I. However, here we must take a bit more care than with the implementation of SU(3) transformations, namely because the $\mathrm{U}(1)_Y$ transformations discriminate between left and right handed chiral fermions. We also have discrimination between up-type and down-type particles for right handed particles, i.e. particles of different weak charge, which exist in distinct left ideals. It is clear that for U(1) transformations we cannot write down any one generator which acts by the same action on all the elements in (2.18), unlike the case for SU(3) transformations. However, we still need one element in M(8,$\mathbb{C}$) which can describe U(1) transformations in order to be able to make claims about linear independence of our irreducible representation spaces.

We will be able to achieve the formulation of such a U(1) generator by incorporating four different projectors in M(8,$\mathbb{C}$). We have a pair of projects which specify particle vs. anti-particle gauge representations, these are:

$$\mathcal{R} := \sum_{I=1}^{4} R_I(R_I)^\dagger \tag{2.33}$$

$$\bar{\mathcal{R}} := \mathcal{R}^{\bar{*}} = \sum_{I=5}^{8} R_I(R_I)^\dagger \tag{2.34}$$

where via left action $\mathcal{R}$ singles out particle transformations and $\bar{\mathcal{R}}$ singles out anti-particle transformations. Similarly, we require the projectors

$$\mathcal{V} := \sum_a V_a^-(V_a^-)^\dagger \tag{2.35}$$

$$\bar{\mathcal{V}} := \mathcal{V}^{\bar{*}} = \sum_a V_a^+(V_a^+)^\dagger, \tag{2.36}$$

where via right action $\mathcal{V}$ will single out "up-type" particles and $\bar{\mathcal{V}}$ will single out "down-type" particles. Since we will require these projectors to act either from the left or the right we will define simultaneous multiplication on M(8,$\mathbb{C}$) from the left by X and from the right by Y, for some $X, Y \in$ M(8,$\mathbb{C}$), as the operation $X|Y$. Explicitly, for any $K \in$ M(8,$\mathbb{C}$), $(X|Y)K := XKY$. Then, defining the element

$$Q := -R_4(R_4)^\dagger - \frac{1}{3}\sum_{I=1}^{3} R_I(R_I)^\dagger - \frac{2}{3}\sum_{I=5}^{7} R_I(R_I)^\dagger, \tag{2.37}$$

we may write a general U$(1)_Y$ transformation operator, of the (anti-)fundamental representations associated to fermionic sector, as the action via the operator

$$\hat{Y} := \left[Q\mathcal{R}|1 + Q\bar{\mathcal{R}}|\left(2\mathcal{V} - \bar{\mathcal{V}}\right)\right] P \tag{2.38}$$

$$+ \left[Q^{\bar{*}}\bar{\mathcal{R}}|1 + Q^{\bar{*}}\mathcal{R}|\left(\mathcal{V} - 2\bar{\mathcal{V}}\right)\right] \bar{P}. \tag{2.39}$$

Here we see for the first time the appearance of the chirality projectors P and $\bar{P}$ as introduced in section 2.2. Thus we have the basis element Q in M(8,$\mathbb{C}$) acting on the different subspaces of (2.18) generate the correct transformations, and that the different actions on these subspaces are described by the operator $\hat{Y}$. That is, to accurately describe U$(1)_Y$ transformations we must include these projects ad-hoc. Note that although we have introduced an operator here, this is just for compactness, indeed all the U$(1)_Y$ transformations of M(8,$\mathbb{C}$) are described in terms of the element Q or its conjugate. We emphasize that this relationship between operator for transformations and element of M(8,$\mathbb{C}$) is also present for SU(3) transformations, albeit much more trivial.

Since SU(3) transformations act from the left and commute with the projectors $\mathcal{R}$ and $\bar{\mathcal{R}}$, and do not distinguish based on chirality, this would imply that the relevant operator would be

$$\hat{\lambda}_I := \mathcal{R}\bar{\lambda}_I|1 - \bar{\mathcal{R}}\bar{\lambda}_I^{\bar{*}}|1 = \left(\bar{\lambda}_I - \bar{\lambda}_I^{\bar{*}}\right)|1 = \lambda_I|1, \tag{2.40}$$

where we have used $\bar{\lambda}_I$ as defined in (2.31). At this stage one may wonder why we chose to consider λ_I as the SU(3) generators, instead of choosing $\bar{\lambda}_I$ and employing similar transformation rules via projectors as in (2.40). The necessity of considering λ_I instead of $\bar{\lambda}_I$ arises from the need for linear independence of our irreducible representation spaces, section 2.6.

Both the basis element Q, and the corresponding operator $\hat{Y}$, defined in this way automatically commutes with the $SU(3)$ transformations. This is crucial since action of both the groups SU(3) and U$_Y(1)$ transformations are described by matrix multiplication from the left and we need to be able to treat them as independent transformations.

SU(2)$_L$ transformations Finally, we turn our attention to the SU(2)$_L$ transformations. These transformations act on any irreducible representation space in (2.18) and (2.24) as,

$$R_I \left(V_a^{\pm}\right)^{\dagger} \longmapsto \quad c_+ R_I \left(V_a^{+}\right)^{\dagger} + c_- R_I \left(V_a^{-}\right)^{\dagger}, \tag{2.41}$$

with $c_{\pm} \in \mathbb{C}$. Again we would like to express this operation in terms of matrix multiplication of the elements. However, in this case we are forced to use right multiplication, as the elements in (2.18) and (2.24) with V_a^+ and V_a^- are in different left-invariant subspaces, i.e. different left ideals. Therefore, there is no matrix multiplication on the left which could transition between weak-isospin $\pm$ states.

When acting from the right with $V_a^{\pm}\left(V_a^l\right)^{\dagger}$ on these matrix elements, we have that,

$$R_I \left(V_a^{\pm}\right)^{\dagger} V_b^{\pm} \left(V_c^l\right)^{\dagger} = \delta_{ab} R_I \left(V_c^l\right)^{\dagger}. \tag{2.42}$$

where we have used (2.6). These transformations only affect the $\pm$ index, and not the generation index a on $V_a^{\pm}$ as required.

The SU(2)$_L$ transformations must act differently on matrix elements $R_I(V_a^{\pm})^{\dagger}$ depending on whether $I \in \{1,2,3,4\}$ or $I \in \{5,6,7,8\}$ respectively, as fundamental SU(2) and SU(3) representations are correlated in the Standard Model. Further, they must also discriminate based on the chiral representation of particles. Thus we know we will need to employ projectors in a similar fashion to what was done for the operator $\hat{Y}$ describing U(1)$_Y$ transformations. Indeed, we have that SU(2)$_L$ transformations are described via the operators

$$\hat{T}_j := \left(\mathcal{R}|T_j\right) P - \left(\bar{\mathcal{R}}|T_j^{\bar{*}}\right) \bar{P} \tag{2.43}$$

such that SU(2)$_L$ transformations of (2.18) can collectively be written as

$$K \to e^{i\hat{T}_j} K, \tag{2.44}$$

with T_i as given by (2.10). This ensures that only left handed particles, and the corresponding right handed antiparticles, transform under SU(2)$_L$. Similar to the U(1)$_Y$, case the SU(2)$_L$ transformations reduce to actions of T_j on the various subspaces of M(8,$\mathbb{C}$). Having identified the different irreducible representation subspaces of M(8,$\mathbb{C}$) we then proceed to find the conditions for their linear independence.

2.6 Conditions for Linear independence

We now verify that all elements assigned to the 62 different particle types of the Standard Model can be made linearly independent. We emphasise in our embedding of the Standard Model gauge representation spaces in M(8,$\mathbb{C}$), we consider only complex subspaces and their linear independence. This means we will be using the complexification of the Lie algebras for linear independence.

The relevant subspaces are spanned by elements of the form:

$$\text{Generations \& anti-generations}: \quad \left\{R_I\left(V_a^\pm\right)^\dagger\right\}_{I=1}^{8} \quad \text{with } a=1,2,3 \tag{2.45a}$$

$$\text{Higgs doublet}: \quad \left\{R_4\left(V_\phi^\pm\right)^\dagger\right\} \tag{2.45b}$$

$$\text{SU(3) generators}: \quad \left\{\lambda_I\right\}_{I=1}^{8} \tag{2.45c}$$

$$\text{SU(2) generators}: \quad \left\{T_j\right\}_{j=1}^{3} \tag{2.45d}$$

$$\text{Hypercharge generator}: \quad Q \tag{2.45e}$$

By construction all generations and anti-generations are linearly independent from each other. We use that

$$V_0^+ = \sum_{I=1}^{8} a_I R_I\,, \qquad V_0^- = \sum_{I=1}^{4}(a_{I+4})^* R_I + \sum_{I=5}^{8}(a_{I-4})^* R_I\,, \tag{2.46}$$

where, for notational simplicity, we have renamed $a_{0I}^+ \to a_I$. Our requirement (2.19) fixes the coefficients in V_0^-. Due to the orthogonality of the $V_a^\pm$, any element K in (2.45a) satisfies

$$KV_0^\pm = 0\,. \tag{2.47}$$

This observation provides us with a necessary and sufficient condition for any linear combination S of elements in (2.45b)-(2.45e) to be linearly independent both from each other and from (2.45a). Namely we must have that at least one of SV_0^+ and SV_0^- does not vanish. In order to achieve this, we simply need to exclude those sets of $\{a_I\}$ for which there exists at least one linear combination S such that $SV_0^\pm = 0$.

To this end, note that in the $R_I(R_J)^\dagger$ basis we may write any complex linear combination of $\mathfrak{su}(3)$ generators in block-diagonal matrix form as

$$\lambda = \text{Diagonal}\left\{M, 0, -M^{\mathrm{T}}, 0\right\} \tag{2.48}$$

where the zeros are just scalars, or 1×1 matrices, and M is a general 3×3 complex traceless matrix. We then write an arbitrary linear combination of (2.45b)-(2.45e) as

$$S = \lambda + c_j T_j + d^+ R_4\left(V_\phi^+\right)^\dagger + d^- R_4\left(V_\phi^-\right)^\dagger + gQ, \tag{2.49}$$

with $c_i, d^\pm, g \in \mathbb{C}$. We now need to find those a_I for which $SV_0^\pm = 0$ if and only if $\lambda = 0$ and $c_j = d^\pm = g = 0$.

Conditions for $c_j = d^\pm = g = 0$

Let us define the two following vectors in $\mathbb{C}^3$,

$$a := \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}^{\mathrm{T}}, \qquad \bar{a} := \begin{pmatrix} a_5 & a_6 & a_7 \end{pmatrix}^{\mathrm{T}}. \tag{2.50}$$

The equation $SV_0^+ = 0$ is equivalent to two vector and two scalar equations,

$$Ma + (c_1 + ic_2)\,\bar{a}^* + \left(c_3 + \frac{2}{3}g\right) a = 0\,, \tag{2.51a}$$

$$-M^{\mathrm{T}}\bar{a} + (c_1 + ic_2)\,a^* + \left(c_3 + \frac{1}{3}g\right) \bar{a} = 0\,, \tag{2.51b}$$

$$(c_1 + ic_2)\,a_8^* + c_3 a_4 + d^+ \left(h_0^+\right)^* = 0\,, \tag{2.51c}$$

$$(c_1 + ic_2)\,a_4^* + (c_3 + g)\,a_8 = 0\,. \tag{2.51d}$$

Similarly, $SV_0^- = 0$ gives,

$$M\bar{a}^* + (c_1 - ic_2)\,a + \left(-c_3 + \frac{2}{3}g\right) \bar{a}^* = 0\,, \tag{2.52a}$$

$$-M^{\mathrm{T}}a^* + (c_1 - ic_2)\,\bar{a} + \left(-c_3 + \frac{1}{3}g\right) a^* = 0\,, \tag{2.52b}$$

$$(c_1 - ic_2)\,a_4 - c_3 a_8^* + d^- h_0^+ = 0\,, \tag{2.52c}$$

$$(c_1 - ic_2)\,a_8 + (-c_3 + g)\,a_4^* = 0\,. \tag{2.52d}$$

From equations (2.51a) and (2.51b) we find that,

$$(c_1 + ic_2)\left(|a|^2 + |\bar{a}|^2\right) = -\,(2c_3 + g)\,(a^{\mathrm{T}}\bar{a})\,, \tag{2.53}$$

while equations (2.52a) and (2.52b) yield,

$$(c_1 - ic_2)\left(|a|^2 + |\bar{a}|^2\right) = -\,(-2c_3 + g)\,(a^{\dagger}\bar{a}^*)\,. \tag{2.54}$$

Together we then have that,

$$\left(|a|^2 + |\bar{a}|^2\right)\Big(-ic_1\,\mathrm{Im}(a^{\mathrm{T}}\bar{a}) + ic_2\,\mathrm{Re}(a^{\mathrm{T}}\bar{a})\Big) = -2c_3|\bar{a}^{\mathrm{T}}a|^2\,, \tag{2.55}$$

and

$$\left(|a|^2 + |\bar{a}|^2\right)\Big(c_1\,\mathrm{Re}(a^{\mathrm{T}}\bar{a}) + c_2\,\mathrm{Im}(a^{\mathrm{T}}\bar{a})\Big) = -g|\bar{a}^{\mathrm{T}}a|^2\,. \tag{2.56}$$

We have four equations (2.51d), (2.52d), (2.55) and (2.56) which are only dependent on the four parameters $\{c_i\}$ and g. We combine a_4^*(2.51d) and a_8(2.52d) such that, with a bit of manipulation, we get the pair of equations

$$c_1\Big(|a_4|^2a_4^*a_8^* + |a_8|^2a_4a_8\Big) + ic_2\Big(|a_4|^2a_4^*a_8^* - |a_8|^2a_4a_8\Big) = -2g|a_4a_8|^2 \tag{2.57}$$

$$c_1\Big(|a_4|^2a_4^*a_8^* - |a_8|^2a_4a_8\Big) + ic_2\Big(|a_4|^2a_4^*a_8^* + |a_8|^2a_4a_8\Big) = -2c_3|a_4a_8|^2\,. \tag{2.58}$$

Since V_0^+ and V_0^- are orthonormal vectors, this implies that,

$$a^{\mathrm{T}}\bar{a} = -a_4 a_8 \quad \text{and} \quad |a|^2 + |\bar{a}|^2 = 1 - |a_4|^2 - |a_8|^2. \tag{2.59}$$

From (2.55-2.58) we obtain two distinct equations relating the parameters c_1 and c_2. While the actual forms of these equations are rather messy, we can schematically write them as

$$f_1 c_1 = f_2 c_2 \quad \text{and} \quad f_3 c_1 = f_4 c_2\,, \tag{2.60}$$

where $f_i = f_i(a_4, a_8)$ are functions of a_4 and a_8. The pair of equations (2.60) then have two possible solutions. Either

$$f_1 f_4 = f_2 f_3\,, \tag{2.61}$$

and the two equations relating c_1 and c_2 are equivalent; or

$$f_1 f_4 \neq f_2 f_3\,, \tag{2.62}$$

in which case the only valid solution for the parameters c_1 and c_2 is $c_1 = c_2 = 0$. The case (2.62) is the one of interest, as if c_1 and c_2 both vanish it is evident that by equations (2.51c), (2.51d), (2.52c), and (2.52d) that $c_3 = g = d^\pm = 0$. Condition (2.62) simplifies to

$$|a_4|^2|a_8|^2\Bigg(2 - |a_4|^2 - |a_8|^2 - 10|a_4|^2|a_8|^2 - 3|a_4|^4 - 3|a_8|^2\Bigg) \neq 0 \tag{2.63}$$

which implies that

$$|a_4|, |a_8| \neq 0\,, \tag{2.64}$$

and for $|a_{4,8}|^2 \in (0, \frac{2}{3})$

$$|a_{8,4}|^2 \neq S(|a_{4,8}|^2)\,, \tag{2.65}$$

where

$$S(x) := \frac{1}{6}\Big(-1 - 10x + \sqrt{25 + 8x + 64x^2}\Big)\,. \tag{2.66}$$

Thus we have found the necessary and sufficient conditions for $c_j = d^\pm = g = 0$, without considering the final parameter λ encoded by the 3×3 matrix M. We now turn our attention to ensuring that this parameter must also vanish for equations (2.51a)-(2.52d) to hold.

Conditions for $\lambda = 0$

With $c_j = d^\pm = g = 0$, the equations (2.51a)-(2.52d) reduce to $\lambda V_0^\pm = 0$. Writing any linear combination λ in terms of its generators as

$$\lambda = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_5\lambda_5 + b_6\lambda_6 + b_7\lambda_7 + b_8\lambda_8\,, \qquad b_i \in \mathbb{C}\,, \tag{2.67}$$

we use the explicit form for λ_I in (2.8) and define the two matrices,

$$m_{\mathrm{S}} \equiv \begin{pmatrix} b_3+b_8 & b_1 & b_4 \\ b_1 & b_8-b_3 & b_6 \\ b_4 & b_6 & -2b_8 \end{pmatrix}, \qquad m_{\mathrm{A}} \equiv \begin{pmatrix} 0 & -b_2 & -b_5 \\ b_2 & 0 & -b_7 \\ b_5 & b_7 & 0 \end{pmatrix}. \tag{2.68}$$

It is straightforward to verify that the matrix equations $\lambda V_0^{\pm} = 0$ are equivalent to,

$$a\,,\,\bar{a}^* \quad \in \quad \mathrm{Kern}\big(m_{\mathrm{S}} + i m_{\mathrm{A}}\big) \cap \mathrm{Kern}\big(m_{\mathrm{S}} - i m_{\mathrm{A}}\big)\,. \tag{2.69}$$

Here, Kern(S) denotes the kernel of S, and $\cap$ denotes the intersection of the two kernels. This in turn implies,

$$a\,,\,\bar{a}^* \quad \in \quad \mathrm{Kern}\big(m_{\mathrm{S}}\big) \cap \mathrm{Kern}\big(m_{\mathrm{A}}\big)\,. \tag{2.70}$$

It is easy to convince oneself that, since m_S is a traceless symmetric matrix and m_A is antisymmetric, neither of them can have rank 1.[8] Hence, they must have rank 3, 2 or 0. If the matrices in (2.68) are both trivial this implies $\lambda = 0$, and thus we have achieved linear independence. If either of the matrices has rank 3 then $\lambda V_0^{\pm} \neq 0$ for any $V_0^{\pm} \neq 0$, and again we have achieved linear independence. This leaves us with the case where at least one of the matrices has rank 2 and a kernel of dimension 1. Then for both vectors in (2.70) to lie in the same one dimensional space we would require $a \propto \bar{a}^*$, as neither vector may be the zero vector for the solutions $c_j = d^{\pm} = g = 0$ to hold. Ergo, if $a \not\propto \bar{a}^{\bar{*}}$ then linear independence of (2.45a)-(2.45e) ensured when also taking into account conditions (2.64) and (2.65).

All together the sufficient and necessary conditions for (2.45a)-(2.45e) to be linearly independent are:

$$\begin{aligned} \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}^{\mathrm{T}} \quad &\not\propto \quad \begin{pmatrix} a_5^* & a_6^* & a_7^* \end{pmatrix}^{\mathrm{T}}\,, \\ \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}^{\mathrm{T}} \quad &\not\perp \quad \begin{pmatrix} a_5^* & a_6^* & a_7^* \end{pmatrix}^{\mathrm{T}}\,, \end{aligned} \tag{2.71}$$

and

$$|a_{8,4}|^2 \neq S(|a_{4,8}|^2)\,, \qquad h_0^{\pm} \neq 0\,, \tag{2.72}$$

with $S(x)$ as in (2.66).

Note that neither vector a nor $\bar{a}^*$ may be the zero vector, as this would imply orthogonality of the two vectors. Further, the orthogonality condition (2.71) of a and $\bar{a}^*$ is equivalent to condition (2.64) by equation (2.59).

[8]If the symmetric matrix has rank 1, it only has 1 non-vanishing eigenvalue and can therefore not be traceless. As the antisymmetric matrix m_A must have zeros on its diagonal component, it can trivially not have rank 1.

2.7 Main Features

Having provided the explicit realization of all Standard Model gauge representations within M(8,ℂ), as a subalgebra of M(16,ℂ), we comment on some of the main features which are present in our direct sum decomposition.

Starting with the adjoint representation spaces in M(8,ℂ), we have that gauge transformations are described by matrix multiplication. This is also the case in the Standard Model. However, instead of the matrices acting on column vectors they here act on other matrices in the same space. Thus symmetry transformations are described by the matrix algebra acting on itself in such a way that it preserves the irreducible representation spaces and their invariant products. Therefore the inclusion of fundamental and adjoint representations in the same space excludes the need to define operator algebras acting on the spaces of our fermions.[9] THe symmetry transformations can also be seen as transformations on the basis elements of the matrix algebra which leaves the irreducible representation spaces invariant. In this way the symmetry transformations become redundancies in our direct sum decomposition.

Of course, in our derivation we found the complexified Lie algebras, while Standard Model physics only considers the real $\mathfrak{su}(N)$ Lie algebras for the gauge groups. While we commented on this earlier, it is prudent to emphasise that at the level of simply embedding representation spaces into M(16,ℂ) there is no fundamental reason for why only the real part of these complex Lie algebras should be physical. This complexification may be understood as the simultaneous appearance of two vector representations in M(2,ℂ), which are the space of hermitian vectors V_H and anti-hermitian vectors iV_H. Regardless, this explicit realization of Standard Model representations offers no mechanism for which to select only the real Lie algebras.

Another relevant point for the discussion of gauge transformations is the appearance of two sets of basis vectors $\{R_I\}$ and $\{V_a^{\pm}\}$ for $\mathbb{C}^8$. While there is a lot of freedom in how to define these two sets of basis elements, indeed only certain combinations of values are not allowed, the two sets may not be identical while still achieving linear independence of our irreducible representation spaces. This is evident from (2.71) alone, which implies each of the basis elements $V_0^{\pm}$ must be the linear combination of at least two elements of $\{R_I\}$. As a result all our fundamental representations appear as outer products of two different sets of basis elements of $\mathbb{C}^8$, while our adjoint representations are all expressed as the outer product of only one set of basis elements. The requirement that our two sets of basis elements be non-identical is interesting from the perspective of inducing representations. It suggests that the matrices describing fundamental representations map between different copies of $\mathbb{C}^8$, while the matrices describing adjoint representations preserve map within a copy of $\mathbb{C}^8$. This idea will be employed as a basic ingredient for the work in chapter 4. This line of reasoning will then offer a natural explanation for why we do not consider SU(2) transformations of SU(3) or visa versa; indeed, while these generators span linearly

[9] Note that the use of operators in defining our symmetry transformations does not imply our adjoint representations belong to an operator algebra. Rather this simply describes how our symmetry generators act on the different representation spaces.

independent subspaces of the same matrix algebra, they do not commute. This construction differs from the formulation of Lie algebras in the Standard Model, which are elements of different spaces and act on different objects and trivially commute.

Apart from the form of the Lie algebras, the subspace describing the Higgs doublet representation is interesting from the perspective of understanding coupling constants in theories. It is clear that, for most choices of parameters, the contraction of $V_\phi^\pm$ with any of the vectors $V_a^\pm$ is non-zero. This provides a setup in which the different coupling strength between generations of fermions and the Higgs can be encoded into the form of the Higgs doublet field itself.

Chapter 3

Octonions and the Algebra of Linear Maps

In the previous chapter we showed an embedding of the Standard Model gauge representations within M(16,$\mathbb{C}$), where the gauge transformations were associated to the algebra M(8,$\mathbb{C}$). The starting point for this embedding was the identification of the matrix algebra with the Clifford algebra $mathdsC\text{l}(6)$, from which we identified SU(3) transformations resulting in a $\mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^* \oplus \mathbb{C}^{3*}$ decomposition of column vectors within the matrix algebra. Alternatively we could have viewed the space M(8,$\mathbb{C}$) as representing linear maps acting on some vector space with a $\mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^* \oplus \mathbb{C}^{3*}$ decomposition under SU(3).

There is in fact an eight dimensional complex vector space which naturally contains such an SU(3) decomposition, namely the algebra $\mathbb{C} \otimes \mathbb{O}$ of complexified Octonions. Furthermore this algebra also has natural decompositions under SU(2), albeit of a different form than presented in chapter 2. Therefore, while the choice of representations in the previous section was ad-hoc to embed Standard Model representations, considering M(8,$\mathbb{C}$) as representing endofunctions of the complexified Octonions will allow us to induce irreducible representation spaces in M(8,$\mathbb{C}$). However, before proceeding with such an induction we first provide a basic introduction to the Octonions and their complexification.

3.1 Overview and Matrix Algebra

The Octonions form an eight dimensional non-associative and non-commutative algebra. In simplest terms: it is a unital algebra of seven square roots of -1, denoted $\{e_i\}_{i=1}^{7}$, with unity often denoted 1 or e_0. In this thesis we will always denote the unity element by e_0, and reserve 1 for the SU(3) singlet representation. The algebra's first appearance in a publication was by A. Cayley in 1845, as a result the Octonions are also referred to as Cayley numbers or the Cayley algebra, Ref. [55]. There are several different and equivalent ways to define the multiplication rules for the Octonions, which is to say several equivalent ways to define a basis. Here we chose a particular basis which makes the multiplicative structure simpler:

$$\begin{aligned} e_ie_j = -e_je_i \text{ for } i \neq j; \qquad & e_i^2 = -1, \quad e_1e_2 = e_4; \\ e_ie_j = e_k \implies \qquad & e_{i+1}e_{j+1} = e_{k+1} \\ e_ie_j = e_k \implies \qquad & e_{2i}e_{2j} = e_{2k}, \text{ mod } 7. \end{aligned} \tag{3.1}$$

where $i, j, k \in \{1, ..., 7\}$. Note that by mod 7 we mean $e_{i+7n} := e_i$, for all $i \in \{1, ..., 7\}$ and $n \in \mathbb{N}$, such that $e_7 = e_7$, $e_8 = e_1$, etc. This definition of the Octonions is the same as the one used in Ref. [56], and can be visualised through the Fano plane, see figure 3.1. The Fano plane should be understood such that all lines wrap around on themselves and contain three elements, e.g. the line from e_4 to e_6 continues to connect e_6 to e_3. Multiplication is understood from this diagram pairwise, by defining e_ie_j as going from e_i through e_j to the next element on the line. For example, e_1e_2 takes e_1 through e_2 to yield e_4. Going against the direction of the arrows yields the same rule but with an overall minus sign, i.e. $e_2e_1 = -e_1e_2 = -e_4$.

With these multiplication rules one can see that all the Quaternion subalgebras of $\mathbb{O}$ are described by any three elements lying on the same line in the diagram. Another core feature of the Octonions, which is prominent in this diagram, is their non-associativity. This can be seen by looking at the multiplication of any three elements not on the same line in the Fano plane. For example, the two expressions

$$e_1(e_3e_6) = e_1e_4 = e_2 \quad \text{and} \quad (e_1e_3)e_6 = e_7e_6 = -e_2 \tag{3.2}$$

differ by an overall sign. Associativity is preserved only if multiplying elements within Quaternionic subalgebras of $\mathbb{O}$, i.e. multiplication of any elements lying on a single line in figure 3.1.

The last line of (3.1) describes a discrete automorphism of the Octonions we will refer to as index doubling, a term only valid under definition (3.1) of the multiplicative structure. This automorphism forms the cyclic group C_3 as performing the index doubling thrice returns the same element. This transformation can be visualized in the Fano plane by rotating the triangle by $2\pi/3$ counter clockwise. The continuous automorphism group of the Octonions is G_2, the smallest of the exceptional Lie groups. There is a lot of literature about this group and its properties, and we will only mention a few of relevance to this paper. For a more comprehensive overview of the Octonions, as well as their relation to all the exceptional Lie groups, we refer the reader to Ref. [57].

One feature which is of interest to us is, as mentioned, the SU(3) subgroup of G_2. There are several distinct, but overlapping, SU(3)$\subset G_2$ and each can be identified with the subgroup of G_2 which leaves invariant some unit imaginary element e_i. As the automorphisms must trivially leave the identity e_0 invariant, this implies that selecting a SU(3) subalgebra of G_2 is equivalent to restricting to only the transformations which preserve a complex subalgebra of $\mathbb{O}$, as $\text{Span}_{\mathbb{R}}\{e_0, e_i\} \cong \mathbb{C}$. The remaining space, $\text{Span}_{\mathbb{R}}\{e_j | j \neq 0, i\}$ is then a six dimensional real subspace which transforms irreducibly under SU(3). This space must naturally transform in some triplet representation, as the only representations of real

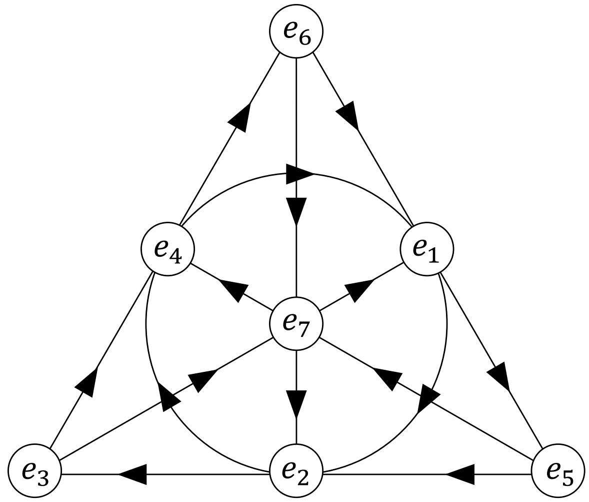

Figure 3.1: The Fano Plane for Octonionic Multiplication

dimension 6 which are irreducible representation of SU(3) are the triplet representations 3 and 3^*. Under complexification this six dimensional real space is separated into a pair of three dimensional complex spaces. Here SU(3) acts on $\text{Span}_{\mathbb{C}}\{e_j : j \neq 0, i\}$ as the direct sum $3 \oplus 3^*$ of irreducible representation spaces of SU(3). As the complex subspace of $\mathbb{O}$ is invariant under SU(3), it is clear that SU(3) acts on $\mathbb{C} \otimes \mathbb{O}$ as $1 \oplus 3 \oplus 1^* \oplus 3^*$, where the symbol * both denotes that the spaces transform in the complex conjugate representation of SU(3) and that these subspaces are themselves related by complex conjugation. Note that the complex conjugation operation describing both relations between vector subspaces and representation spaces in $\mathbb{C} \otimes \mathbb{O}$ is possible since the SU(3) group action on $\mathbb{C} \otimes \mathbb{O}$ commutes with complex conjugation. This SU(3) decomposition of the complexified Octonions is of interest since it exactly mirrors the SU(3) representation of one generation of leptons and quarks along with their antiparticles, as mentioned in section 2.1.

There are also several SU(2) subgroups of G_2, some of these appear as SU(2)$\subset$SU(3). In chapter 4 we consider SU(2) subgroups of G_2 which are such that they preserve element wise a Quaternionic subalgebra of $\mathbb{O}$. These SU(2)$\subset G_2$ fall into three sets when under index doubling. There are three SU(2) subgroups which are related by index doubling and preserve e_7, three SU(2) subgroups which are related by index doubling but do not preserve e_7, and one SU(2) subgroup which is invariant under index doubling and preserves the elements e_1, e_2, e_4. This follows immediately from the 7 different lines of Fano plane, figure 3.1.

Thus, by using the SU(3) and SU(2) subgroups of G_2, we show that it is possible to induce a direct sum decomposition of $\text{M}(\mathbb{C} \otimes \mathbb{O})$ into irreducible representations of SU(2)$\times$SU(3). This allows for a more natural way to study simultaneous realizations of gauge representations in the space of matrices M(8,$\mathbb{C}$), while still considering groups which are of interest to particle theory.

3.2 Past Appearances in Physics

The use of Octonions in physics is not new. For example, the Octonions were seriously studied in the 1970's, Ref. [58, 59], as an alternative approach to the then newly proposed QCD scheme of quarks. The idea behind using the Octonions in this setting was to explain the non-observability of quarks. Having quarks describe states which lie outside the space of observables was achieved though the use of a "fictitious Hilbert space" based on the Octonions. The non-observability of the quark states then follows from the fact that an algebra of observable states requires a Hilbert space over an associative algebra, which the Octonions are not.[1] We refer the reader to the above references for simple examples of why non-associativity fails to yield a Hilbert space of observables. The Hilbert space of observables was instead the Hilbert space of SU(3) singlets. This lead to Hadrons being the only observables of the theory, as single quark states are charged under SU(3).

The Octonions also appear as ingredients of the unique exceptional Euclidean Jordan algebra, J_3. This algebra has itself been the central object of study in several papers trying to identify the Standard Model symmetries as related to the exceptional Lie groups, Ref. [60, 61, 62, 63].

There are several relations between the Octonions and String Theory. In particular we comment on the work of John C. Baez and John Huerta, Ref. [64, 65], which relate the dimensions of String Theory to those of the normed division algebras. The Octonions as an eight dimensional division algebra are then related to either 10 dimensional minimal super Yang-Mills theories or 11 dimensional classical super-2-brane theories. These relationships are established well beyond the level of dimensional arguments and we refer the interested reader to the above citations. See also Ref. [66] for a formulation of M-theory algebra in terms of Octonions and Refs. [67, 68] for additional relations between G_2, the Octonions and M-Theory. Further, the Octonion algebra itself can arise in String Theory from string R-flux algebras, Ref. [69]. While we will not be using these relationships to String Theory in our present work, it highlights the range and depth of applications of the Octonions as the largest of the normed division algebras.

For the purpose of this dissertation, we have chosen to work with the algebra of the complexified Octonions simply because: its restricted symmetry groups are those of the Standard Model; its decomposition under SU(3) contains the same SU(3) representation spaces as one generation of leptons and quarks; and the resultant space of maps from the algebra to itself is that of eight by eight complex matrices, which as we showed in chapter 2 is the minimal matrix algebra for encoding the set of Standard Model representations. As a side benefit we note that the Octonions have been seen to arise in many different

[1] Note that as we will be focused on the maps M($\mathbb{C}\otimes\mathbb{O}$) and not on the complexified octonions themselves this would not be an issue of our formulation.

fields. As such it is our belief that working with such a proliferous algebra will help make our results and ideas more applicable to other fields of physics.

Chapter 4

Inducing Gauge Transformations in Matrix Algebras

Here we go beyond a simple embedding of Standard Model gauge representations, and discuss how irreducible representation spaces can be induced in a matrix algebra. The purpose of this line of reasoning is not to reproduce Standard Model physics, but instead to induce a direct sum decomposition of a matrix algebra into irreducible representation spaces. The subspaces will then be analysed and compared with those of the Standard Model.

To this end we will be using the complexified Octonions, as introduced in chapter 3. Consequently we will not be needing to take the route through $\mathbb{C}l(6)$ to obtain the matrix algebra $\mathrm{M}(8,\mathbb{C})$. Therefore we will be defining the operation of complex conjugation directly in $\mathbb{C}\otimes\mathbb{O}$, and by extension also in $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$, and have no need for two different operations $\bar{*}$ and $*$; unlike the case for chapter 2. Thus, in the following section there is only one operation of complex conjugation which will be denoted by the standard operation $*$. We will describe how this operation acts on the vector space $\mathbb{C}\otimes\mathbb{O}$, and the resultant effect in the matrix algebra.

4.1 Method of induction

Before we begin, we must explain what we mean by inducing representations, what this implies, and how we will achieve such an induction. First, by inducing representations we mean using the features of some underlying space V to generate irreducible representation spaces in the space of maps $\mathrm{M}(V)$ on V. Here we will be using the complexified Octonions as our underlying space, i.e. $V = \mathbb{C}\otimes\mathbb{O}$. The features we will be using for our direct sum decomposition are the restricted representations of $\mathbb{C}\otimes\mathbb{O}$ under action of G_2, the continuous automorphism group of the Octonions.

Of course, we also need a mechanism for which we will use the restricted representations of $\mathbb{C}\otimes\mathbb{O}$ to generate irreducible representations in $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$. This first requires considering what types of representation we wish to generate through our process of induction. This is important, as the space $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ is realized simply by eight by eight dimensional matrices, which we could interpret as a set of 64 singlet states unless we have a construction which specifies how irreducible representations should be identified. In other words, the algebra $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ is just that, an algebra without any prior relation to particle theory. So it is clear that one would necessarily have to impose additional conditions, or structure, on the algebra in order to allow for an identification of irreducible representation spaces which can be compared to Standard Model particles. In our work we will refer to this imposition of conditions on the algebra $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ as "principles" of our construction, i.e. the principles which allow for the induction of a direct sum decomposition into irreducible representation spaces.

As mentioned, the purpose of this paper is to investigate how the different types of irreducible representation spaces of the Standard Model can simultaneously be realized in the same space. This implies that we wish for a construction which will yield at least adjoint representations of non-abelian Lie algebras and fundamental representations of these same algebras.[1] To ensure the appearance of adjoint and fundamental representation spaces of non-abelian Lie groups, we draw inspiration from some of the main features of our explicit embedding highlighted in section 2.7. Specifically, we found that the fundamental representations were most easily expressed in terms of matrices composed of two distinct set of basis vectors. Additionally, our $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ Lie algebras appeared as endofunctions on these basis elements that preserved certain decomposition of some eight dimensional row or column vector. I.e. preserved irreducible representation spaces of some $\mathbb{C}^8$. In order to recover adjoint representations we will consider subspaces of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ which act as endofunctions on $\mathbb{C}\otimes\mathbb{O}$ while preserving restricted representations. To obtain fundamental representations we will use maps between restricted representations on $\mathbb{C}\otimes\mathbb{O}$. This is detailed as principle 2 in section 4.2. For notational simplicity and ease of reading we will refer to the space V, here $\mathbb{C}\otimes\mathbb{O}$, as the "base space" upon which the maps $\mathrm{M}(V)$ act. Additionally we will also demand that invariants can be formed via the contraction of elements in the resultant matrix algebra, as the formation of invariants is central to the construction of any gauge theory.

Finally, we also demand that we provide a complete decomposition of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ into irreducible representation spaces of the relevant restricted group actions on $\mathbb{C}\otimes\mathbb{O}$. One of the less attractive features of the explicit embedding of chapter 2 is the appearance of the subspace P_{Add}. This space describes the residual components left over once the Standard Model gauge representations, as well as those of right handed neutrinos, have been embedded in $\mathrm{M}(8,\mathbb{C})$. These components were not problematic for our explicit embedding, by dimensional counting their existence was inevitable. However, when inducing representations we do not have any reason why there would be additional components. Indeed, why should only a certain part of the algebra be understood in terms of irreducible representation spaces? Nor do we have any limit on the dimensionality of the space spanned by these additional components. Thus allowing for an arbitrary P_{Add} in our induced di-

[1]Note that this only demands which types of transformations must be present, and does not exclude the appearance of other representations.

rect sum decomposition prevents making any definite claims about the set of irreducible representation spaces. So, for definiteness and consistency of interpretation, we will only permit sets of irreducible representation spaces whose direct sum is precisely the entirety of the space $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$.

In the following section we formulate the above points into four construction principles, which we will use to construct a direct sum decomposition of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ into irreducible representation spaces of SU(2)×SU(3).

4.2 The Setup

In the previous section we introduced and motivated the notion of principles of construction that we will use to induce a direct sum decomposition of M(8,$\mathbb{C}$) into irreducible representation spaces. Here we present these principles in a concise form.

1. We consider only spaces of maps transforming in irreducible representations of the subgroups of G_2 that element-wise preserves some definite complex or Quaternionic subalgebra of $\mathbb{O}$.

2. We consider two distinct sets of maps:

   · Those which map from $\mathbb{C}\otimes\mathbb{O}|_{\mathbb{C}}$ to $\mathbb{C}\otimes\mathbb{O}|_{\mathbb{H}}$.

   · Those corresponding to a Lie algebra of maps describing a redundancy in the decomposition of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$.

3. The Lie algebras corresponding to automorphisms of $\mathbb{O}|_{\mathbb{C}}$ and $\mathbb{O}|_{\mathbb{H}}$ must be present in the decomposition.

4. Invariants may be formed using map composition and trace operations of elements and their Hermitian conjugates.[2]

For principle 2, we have defined $\mathbb{O}|_{\mathbb{X}}$ be the algebra $\mathbb{O}$ with symmetry group $G'\subset G_2$ such that $G'(a)=a,\ \forall a\in\mathbb{X}\subset\mathbb{O}$, i.e. G' is a restricted representation of G_2 on $\mathbb{O}$.

Let us elaborate on these principles. For principle 1, automorphisms of $\mathbb{O}$ which preserve some complex subalgebra, $\mathbb{O}|_{\mathbb{C}}$, and those which preserve some Quaternionic subalgebra, $\mathbb{O}|_{\mathbb{H}}$, form the groups SU(3) and SU(2) respectively, as subgroups of G_2. So we are not employing ad-hoc defined groups, as was the case in chapter 2, but those corresponding to transformations which are inherent to the multiplicative structure of the algebra. In simple terms, the first principle provides a process by which definite subgroups of G_2 are selected as transformation groups of our direct sum decomposition. Principle 2 then provides a

[2]Note that we could not chose to form invariants with inverse matrix elements, as not all irreducible representation spaces have matrix inverses. Second we cannot form invariants using transposition of matrix elements, as this does not form SU(N) invariants and we already have two special unitary transformations present. Thus Hermitian conjugation is the only matrix operation applicable for forming invariants between our irreducible representation spaces.

process by which $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ is broken down into a direct sum decomposition of irreducible representation spaces of these same subgroups. The choice to incorporate maps from $\mathbb{O}|_{\mathbb{C}}$ to $\mathbb{O}|_{\mathbb{H}}$, and not visa-versa, both restricts the freedom in identifying irreducible representation spaces and simplifies the calculations of linear independence.

From just these two principles one may not ensure uniqueness of the direct sum decomposition. This is evident from the fact that the third principle implies a starting point for the decomposition. Otherwise we could simply span the space $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ by maps from $\mathbb{C}\otimes\mathbb{O}|_{\mathbb{C}}$ to $\mathbb{C}\otimes\mathbb{O}|_{\mathbb{H}}$. Then we would have a set of only fundamental representation spaces, and no adjoint representations. To ensure we find a unique set of SU(2)×SU(3) irreducible representation spaces in our direct sum decomposition we must minimally have a unique starting point. Principles 3 and 4 are together simply the statement that we wish to study representations relevant for gauge theories. This implies minimally that we need the appearance of adjoint representations in the direct sum decomposition, which we will take as our starting point, and the ability to form invariants.

On another note, the results of chapter 2 relied on defining fundamental representations, and then identifying the Lie Algebras elements which described the transformation of these subspaces. Here the approach is different not only in that we derive our group structures from an underlying algebra, but also in the starting point of imposing the group structure, here SU(2)×SU(3), and then identifying the various irreducible representation spaces of these transformations.

4.3 Results

Before proceeding with the process of identifying the irreducible representation spaces of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$, we first present the complete set of these subspaces for conciseness and ease to the reader.

The adjoint representations which appear are those which correspond to the complexified Lie algebras:

- $\mathbb{C}\otimes\mathfrak{u}(1)$
- $\mathbb{C}\otimes\mathfrak{su}(2)$
- $\mathbb{C}\otimes\mathfrak{su}(3)$

The complexified SU(2) and SU(3) Lie algebras naturally correspond to the appearance of the SU(2) and SU(3) subgroups of G_2, as discussed in previous sections. However, the complexified U(1) Lie algebra[3] corresponds not to any automorphism group of G_2, but

[3] Normally in particle theory we do not discuss generators of a U(1) group or a $\mathfrak{u}(1)$ Lie algebra. This is because when restricted to irreducible representation spaces the generator for a U(1) group becomes just a scalar and the Lie algebra is trivial. While the Lie algebra is indeed trivial in our construction, the generator takes on a specific form in $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ which describes the charge distribution of the irreducible representation spaces. This makes it constructive to refer to a U(1) generator, as well as grouping its representation space into the set of complexified Lie algebras.

instead to a "residual redundancy" in how to define the different irreducible representation spaces of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$. Here the term "residual redundancy" denotes a redundancy in the decomposition which does not arise from the algebra $\mathbb{C}\otimes\mathbb{O}$ itself, but rather as a further redundancy of the SU(2)×SU(3) invariants that may be formed from elements in $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ as per principle 4. As a result the relative charges of the corresponding U(1) generator will not be fixed.

Similar to the work of chapter 2, there is no apparent process by which to select only the real Lie algebras. Thus we find the complexified Lie algebras as the linearly independent irreducible adjoint representation spaces. Even so, it is the Lie groups generated by the real Lie algebras $\mathfrak{u}(1)$, $\mathfrak{su}(2)$, and $\mathfrak{su}(3)$ that describe redundancies of the direct sum decomposition of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$.

In the maps from $\mathbb{C}\otimes\mathbb{O}|_\mathbb{C}$ to $\mathbb{C}\otimes\mathbb{O}|_\mathbb{H}$ we find:

- Eight singlets of SU(2)×SU(3)
- Two singlets under SU(3) which transform as 2 under SU(2)
- Two singlets under SU(3) which transform as 2^* under SU(2)
- Four singlets under SU(2) which transform as 3 under SU(3)
- Four singlets under SU(2) which transform as 3^* under SU(3)
- Two sets of 12-dimensional real vector spaces which transform in the doublet representation of SU(2) and the triplet representation of SU(3).

While fundamental representations are common in the Standard Model, the 12-dimensional real representations seem quite out of place. We will discuss these representation spaces in subsection 4.5.3. As the relative charges of our U(1) transformation are not fixed we have not included charge assignments for the above representation spaces.

All the fundamental representations are contained in a set of 4 reducible representation spaces $\mathcal{F}_1$, $\mathcal{F}_{1^*}$, $\mathcal{F}_3$, and $\mathcal{F}_{3^*}$. Note the $\mathcal{F}$-spaces here are distinct from the ones introduced in chapter 2, i.e. the F_i spaces. As we are talking about two different constructions there should be no confusion about to which direct sum decomposition these subspaces belong. A similar notation has been kept simply as they describe the SU(3) transformations of the representation spaces. Here the spaces $\mathcal{F}_1$ and $\mathcal{F}_{1^*}$ contain all singlets of SU(3), with $\mathcal{F}_3$ and $\mathcal{F}_{3^*}$ describing only fundamental representations of SU(3) and their conjugates. The 12-dimensional real representation spaces are denoted H_1^α and H_2^β, where α and β are complex parameters describing the exact structure of our spaces. The condition $\alpha,\beta\neq\pm1$ is required for linear independence. All together this yields the direct sum decomposition

$$\mathrm{M}(\mathbb{C}\otimes\mathbb{O})=\mathbb{C}\otimes\Big(\mathfrak{u}(1)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(3)\Big)\oplus\mathcal{F}_3\oplus\mathcal{F}_{3^*}\oplus\mathcal{F}_1\oplus\mathcal{F}_{1^*}\oplus H_1^\alpha\oplus H_2^\beta. \qquad (4.1)$$

In the following subsection we will show how we stepwise construct the above direct sum decomposition of our matrix algebra. A detailed analysis of properties of these different irreducible representation spaces is given in section 4.5. Note, that by properties of the irreducible representation spaces we naturally mean features of their realization within $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$, i.e. those which are besides the properties associated purely to their representation.

4.4 Construction

Here we show how to induce a direct sum decomposition of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ into a unique set of irreducible representation spaces of SU(2)×SU(3). We start by defining the adjoint representations associated with our SU(2) and SU(3) symmetries, as per principle 3.

4.4.1 The Non-Abelian Adjoint Representations

The SU(3) subgroup of G_2 splits our 8-dimensional complex vector space into 4 irreducible representation spaces:

$$V = \big(1 \oplus 3\big) \oplus \big(1^* \oplus 3^*\big), \tag{4.2}$$

where again the complex conjugate denotes both the subspaces' transformations under SU(3), and that the different subspaces are themselves related by complex conjugation. Here 1 is a singlet and 3 is a triplet of SU(3), as discussed in section 3.1. Similarly we have the irreducible representation spaces of SU(2)

$$V = \big(\underline{1} \oplus \underline{1}' \oplus 2\big) \oplus \big(\underline{1}^* \oplus \underline{1}'^* \oplus 2^*\big). \tag{4.3}$$

where we have introduced both an underline and a prime notation on our SU(2) singlet states to emphasize that, in general, they do not correspond to the singlet states of SU(3), see Appendix C. The only difference between the singlet states is that they constitute different subspaces of V, this will be important to ensure linear independence of our maps. For brevity we will group the singlet states of SU(2) into a four dimensional complex vector space 1_2. The subspace 2 transforms as a doublet of SU(2).

Considering maps from the vector space V to itself it is clear that the Lie algebras corresponding to the SU(3) and SU(2) Lie groups are themselves realized as endofunction on V that preserve the irreducible representations. I.e.

$$\mathfrak{su}(3) : \big(1 \oplus 3\big) \oplus \big(1^* \oplus 3^*\big) \to \big(1 \oplus 3\big) \oplus \big(1^* \oplus 3^*\big), \tag{4.4}$$

and similarly

$$\mathfrak{su}(2) : 1_2 \oplus 2 \oplus 2^* \to 1_2 \oplus 2 \oplus 2^* \tag{4.5}$$

where the Lie algebras satisfy their respective commutation relations, and transform in the adjoint representation of their associated Lie groups.

Further, $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ are here linearly independent subalgebras of $\mathfrak{g}_2$ and consequently linearly independent subspaces of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$. This is because the representation of $\mathfrak{g}_2$ acting on some vector space V is naturally a subset of $\mathrm{GL}(V) \subset \mathrm{M}(V)$. We then find the adjoint representations $\mathbb{C} \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(3))$ as the starting point for our direct sum decomposition of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$.

We note that $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ are real subalgebras of $\mathfrak{g}_2$, and so a generator $\tau \in \mathfrak{su}(2)$ or $\mathfrak{su}(3)$ satisfies $\tau^* = -\tau$.[4] Additionally, we have only considered the groups SU(N), even though the decompositions (4.2) and (4.3) are left invariant under the respective GL(N,$\mathbb{C}$) groups, i.e. the groups of general linear transformations on $\mathbb{C}^N$. This relates to principle 4, and we will comment on the appearance of additional U(1) transformations in subsection 4.4.4.

4.4.2 Fundamental and Singlet Representations

Consider then the maps which are trivially linearly independent from $\mathfrak{su}(3)$. These are the maps $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} : 1 \oplus 1^* \to \mathbb{C} \otimes \mathbb{O}$. But are these maps also linearly independent from $\mathfrak{C} \otimes (\mathfrak{su}(3) \oplus \mathfrak{su}(2))$? To investigate this we note that the overlap of $1 \oplus 1^*$ and $2 \oplus 2^*$ is a one dimensional space containing the unit imaginary $d \in \mathbb{O}$ used in defining the complex subalgebra kept invariant by SU(3), see Appendix C. The space of maps $\{d\} \to \mathbb{C} \otimes \mathbb{O}$ appears as the subset of $\mathcal{F}_1 \oplus \mathcal{F}_{1^*}$ which share a non-trivial intersection with the domain of the maps in $\mathbb{C} \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(3))$. Thus we need only ensure that there is no map in these complexified Lie algebras which is such that it acts only on $\{d\}$. In other words we need only ensure that no map in $\mathbb{C} \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(3))$ is rank 1 for linear independence with $\mathcal{F}_1 \oplus \mathcal{F}_{1^*}$.

Now, $\mathfrak{su}(2)$ is defined as the Lie algebra of the Lie subgroup of G_2 that element-wise preserves some Quaternionic subspace $\mathrm{Span}_{\mathbb{R}}\{e_0, a, b, c\} \subset \mathbb{O}$. Then it is evident that $\mathfrak{su}(2)$ may not act on d without also acting on some $dx \in \mathrm{Span}_{\mathbb{C}}\{da, db, dc\} \notin (1 \oplus 1^*)$. This follows immediately from the observation that any element in $\mathbb{C} \otimes \mathfrak{su}(2)$ is an even-rank matrix. To see this note that for some element $\tau \in \mathbb{C} \otimes \mathfrak{su}(2)$, we have

$$\sigma = \tau|_2 \quad \Longleftrightarrow \quad -\sigma^T = \tau|_{2^*} \tag{4.6}$$

where $X|_y$ is the $\dim(y) \times \dim(y)$ matrix representation of element $X \in \mathrm{M}(\mathbb{C} \otimes \mathbb{O})$ restricted to act on a subspace $y \subset \mathbb{C} \otimes \mathbb{O}$, and σ is some complex linear combination of the Pauli matrices. This then implies that any element in $\mathbb{C} \otimes \mathfrak{su}(2)$ must act non-trivially on both 2 and 2^*, implying it must be an even rank matrix as matrix transposition does not affect rank.

Now a similar relationship as (4.6) holds for $\tau \in \mathbb{C} \otimes \mathfrak{su}(3)$,

$$\sigma = \tau|_3 \quad \Longleftrightarrow \quad -\sigma^T = \tau|_{3^*} \tag{4.7}$$

where now the σ matrices describe complex linear combinations of the Gell-Mann matrices. These matrices are also complex linearly independent, and clearly any element of $\mathbb{C} \otimes \mathfrak{su}(3)$

[4] This uses the physics convention of the fundamental representation of SU(N) generated by hermitian elements. In the mathematics convention where SU(N) is generated by skew-hermitian elements we have instead $\tau^* = \tau$.

must also be an even rank matrix. The sum of two even rank matrices is another even rank matrix. Consequently, $\mathcal{F}_1 \oplus \mathcal{F}_{1^*}$ is linearly independent from $\mathbb{C} \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(3))$.[5]

Now lets repeat this line of thought, but starting with the maps which are trivially linearly independent from $\mathfrak{su}(2)$. Excluding any overlap with the spaces $\mathcal{F}_1$ and $\mathcal{F}_{1^*}$, we have the spaces of maps $\mathcal{F}_3 : 3 \to 1_2$ and $\mathcal{F}_{3^*} : 3^* \to 1_2$. The domain and range of $\mathcal{F}_3 \oplus \mathcal{F}_{3^*}$ which intersects with the domain and range of $\Xi := \mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathbb{C} \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(3))$ is then the space of maps $3 \oplus 3 \to \text{Span}_{\mathbb{C}}\{a, b, c\}$.

As $\mathbb{C} \otimes \mathfrak{su}(3)$ is the only element of Ξ which acts on the same domain as $\mathcal{F}_3 \oplus \mathcal{F}_{3^*}$, we first establish that these two subspaces are linearly independent. These spaces are linearly independent if there is no map in $\mathbb{C} \otimes \mathfrak{su}(3)$ which is such that its range is entirely contained in 1_2. Consider then some elements $x \in 3$ and $y \in 3^*$ such that there exists a map $M \in \mathbb{C} \otimes \mathfrak{su}(3)$ for which $M(x + y) = M(x) + M(y) \in \text{Span}_{\mathbb{C}}\{a, b, c\}$. Let, without loss of generality, 3 be spanned by the basis elements $\{\alpha_i\}$ and 3^* spanned by $\{\alpha_i^*\}$ such that $\alpha_i + \alpha_i^* \in \text{Span}_{\mathbb{C}}\{a, b, c\}$ and $\alpha_i - \alpha_i^* \in \text{Span}_{\mathbb{C}}\{da, db, dc\}$, see Appendix C. Then our map M must be such that

$$M(x) = \sum_i \gamma_i \alpha_i \quad \text{and} \quad M(y) = \sum_i \bar{\gamma}_i \alpha_i^* \tag{4.8}$$

where $\gamma_i \in \mathbb{C}$, and $\gamma_i - \bar{\gamma}_i = 0$. Then the same map M will be such that $M(x - y) = M(x) - M(y) \in \text{Span}_{\mathbb{C}}\{da, db, dc\} \notin \text{Range}\{\mathcal{F}_3 \oplus \mathcal{F}_{3^*}\}$. Therefore, we cannot construct any map in $\mathbb{C} \otimes \mathfrak{su}(3)$ which is such that it corresponds to a map in $\mathcal{F}_3 \oplus \mathcal{F}_{3^*}$, and thus these two spaces are linearly independent.

There is no map in $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathbb{C} \otimes \mathfrak{su}(2)$ with domain $3 \oplus 3^*$ and range $\text{Span}_{\mathbb{C}}\{1, a, b, c\}$. To ensure linear independence of the two subspaces $\mathcal{F}_3 \oplus \mathcal{F}_{3^*} \oplus \mathbb{C} \otimes \mathfrak{su}(3)$ and $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathbb{C} \otimes \mathfrak{su}(2)$, we need only ensure there exists no map in $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathbb{C} \otimes \mathfrak{su}(2)$ which has domain and range in $\text{Span}_{\mathbb{C}}\{da, db, dc\}$.

The space $\mathbb{C} \otimes \mathfrak{su}(2)$ describes the only set of maps in $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathbb{C} \otimes \mathfrak{su}(2)$ acting on $\text{Span}_{\mathbb{C}}\{da, db, dc\}$, and this space acts on 2 and 2^* in complex conjugate representations. As 2 and 2^* can be defined, without loss of generality, to be eigenstates of a, and generators of $\mathfrak{su}(2)$ satisfy (4.6), there is no map in $\mathbb{C} \otimes \mathfrak{su}(2)$ which maps some dx to dy without also mapping some dz to d, for $dx, dy, dz \in \text{Span}_{\mathbb{C}}\{da, db, dc\}$. Since the maps in $\mathcal{F}_1 \oplus \mathcal{F}_{1^*}$ do not act on $\text{Span}_{\mathbb{C}}\{da, db, dc\}$, this implies that there is no map in $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathbb{C} \otimes \mathfrak{su}(2)$ with domain and range in $\text{Span}_{\mathbb{C}}\{da, db, dc\}$. As a result, we have linear independence of $\mathcal{F}_3 \oplus \mathcal{F}_{3^*} \oplus \mathbb{C} \otimes \mathfrak{su}(3)$ from $F_1 \oplus F_{1^*} \oplus \mathbb{C} \otimes \mathfrak{su}(2)$, which is to say that all irreducible representation spaces identified so far are linearly independent.

4.4.3 The 12-Dimensional Real Vector Subspaces

Together the set of maps $\mathbb{C} \otimes \mathfrak{su}(2)$, $\mathbb{C} \otimes \mathfrak{su}(3)$, $\mathcal{F}_3$, $\mathcal{F}_{3^*}$, $\mathcal{F}_1$, and $\mathcal{F}_{1^*}$ span a 51 $\mathbb{C}$-dimensional subspace of the 64 $\mathbb{C}$-dimensional space $\text{M}(\mathbb{C} \otimes \mathbb{O})$. Therefore there exists a 13 dimensional

[5] Note that this argument for linear independence relies on the full-rank nature of any $M \in \mathbb{C} \otimes \mathfrak{su}(2)$ when restricted to act on 2 or 2^*. As we will see in subsection 4.4.4, this is precisely what demands our additional U(1) transformations must act on the same decomposition as SU(3).

space linearly independent maps not encapsulated by the irreducible representation spaces thus far described. The only subspace of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ which has not been fully spanned is the space of maps $3\oplus 3^*\to 2\oplus 2^*$. The rest of the space being fully spanned by the maps $\mathcal{F}_3$, $\mathcal{F}_{3^*}$, $\mathcal{F}_1$, and $\mathcal{F}_{1^*}$.

The maps in the subspace of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$ which map $3\oplus 3^*$ to $2\oplus 2^*$ break down into four distinct ideals corresponding to the spaces of maps:

$$\underline{\phi}: 3\to 2,\quad \underline{\phi}^*: 3^*\to 2^*,\quad \underline{\psi}: 3^*\to 2,\quad \underline{\psi}^*: 3\to 2^*. \tag{4.9}$$

As these maps belong to different ideals than $\mathcal{F}_1$, $\mathcal{F}_{1^*}$, $\mathcal{F}_3$, and $\mathcal{F}_{3^*}$, linear independence between these subspaces is trivial. Further, any map in $\underline{\phi}\oplus\underline{\phi}^*\oplus\underline{\psi}\oplus\underline{\psi}^*\oplus\mathbb{C}\otimes(\mathfrak{su}(2)\oplus\mathfrak{su}(3))$ is linearly independent from $\mathcal{F}_1\oplus\mathcal{F}_{1^*}\oplus\mathcal{F}_3\oplus\mathcal{F}_{3^*}$ if and only if the projection of this map on the space $3\oplus 3^*\to 2\oplus 2^*$ is non-zero.

Any generator, i.e. basis element, γ of our Lie algebras satisfies $\gamma^*=-\gamma$. So clearly our complexified Lie group elements can be written as a combination of basis elements $\{\kappa_i\}$ which satisfy $\kappa_i^*=\pm\kappa_i$. Since the operation of projecting on the space of maps $3\oplus 3^*\to 2\oplus 2^*$ is invariant under complex conjugation, this implies that after projecting the generators κ_i on this space we still have basis elements which are invariant, up to an overall sign, under complex conjugation. Thus to ensure that any map in $\underline{\phi}\oplus\underline{\phi}^*\oplus\underline{\psi}\oplus\underline{\psi}^*$ is linearly independent from our complexified Lie algebras, we need only ensure our linear combination is not expressible in terms of basis elements which are invariant, up to an overall sign, under complex conjugation.

Let the space $\underline{\phi}$ be spanned by basis elements ϕ_i as a 12-dimensional real vector space. Then we can span $\underline{\phi}^*$ by the basis elements ϕ_i^*. Similarly we span $\underline{\psi}$ and $\underline{\psi}^*$ by the 12 basis elements ψ_i and ψ_i^* respectively. That is, we treat our spaces of complex dimension 6 as spaces of real dimension 12. In addition to requiring linear independence with the projection of our complexified Lie algebras, we also require that our spaces are irreducible representations of the Lie groups. Correspondingly, we must find which sets of maps, out of all the maps in $3\oplus 3^*\to 2\oplus 2^*$, span irreducible representation spaces, to be denoted by the letter H and referred to as H-spaces, of SU(2)×SU(3) while still being linearly independent. This greatly restricts our choice of relevant spaces, H, as all the subspaces in (4.9) are all irreducible representation spaces of SU(2)×SU(3). Thus for any relevant subspace H, we must have that the projection of H on any of the subspaces in (4.9) is either the zero element or the entire space.

The argument for linear independence of our H-spaces and the complexified Lie algebra centres on these H-spaces not containing any elements A that satisfy $A^*=\pm A$. As such we can analyse subspaces of $\underline{\phi}\oplus\underline{\phi}^*$ and $\underline{\psi}\oplus\underline{\psi}^*$ independently. Thus we will only present the following analysis for the subspace $\underline{\phi}\oplus\underline{\phi}^*$, with the arguments for $\underline{\psi}\oplus\underline{\psi}^*$ following analogously. Clearly, we cannot use the entirety of $\underline{\phi}\oplus\underline{\phi}^*$, as it is no difficult task to find basis elements in this space which are their own complex conjugate. Indeed, we can at most span a 12-dimensional real subspace of the 24-dimensional real subspace $\underline{\phi}\oplus\underline{\phi}^*$. This follows immediately from the observation that choosing 13 or more basis elements in the 24-dimensional real space requires choosing some $\phi\in\underline{\phi}$ and its corresponding $\phi^*\in\underline{\phi}^*$, such that $\phi+\phi^*$ is a basis element which is its own complex conjugate.

Additionally, since the projection of H on $\underline{\phi}$ is either $\{0\}$ or $\underline{\phi}$, and similarly for $\underline{\phi}^*$, H must be spanned by basis elements

$$\{\phi_i + f(\phi_i)\}_{i=1}^{12}, \tag{4.10}$$

where $f : \underline{\phi} \to \underline{\phi}^*$ is either the zero map (i.e. maps all elements to the zero element) or some bijection. This bijection must be such that for $\phi \to U\phi V$, with $U \in \mathrm{SU}(3)$ and $V \in \mathrm{SU}(2)$, we have $f(\phi) \to f(U\phi V) \equiv U^* f(\phi) V^*$. As the 3 and 3^* representations of SU(3) are distinct representations they cannot be transformed into each other via a linear map, i.e. matrix multiplication. From this we know that f must contain the operation of complex conjugation. Then any other operation in f must commute with the group elements U and V, which implies multiplication by an overall scalar.

Thus we can write our relevant subspaces as

$$H^\alpha := \mathrm{Span}_{\mathbb{R}}\{\phi_i + \alpha\phi_i^*\} \tag{4.11}$$

for some $\alpha \in \mathbb{C}$. Note that we are considering this space as a real vector space, as opposed to our previous analysis which has always focused on complex vector spaces. This is because for some complex coefficient $\gamma \in \mathbb{C}$ in front of one of our basis elements,

$$\gamma(\phi_i + \alpha\phi_i) = (\gamma\phi_i) + \alpha(\gamma^*\phi_i)^* \tag{4.12}$$

does not yield a vector of the form $\phi + \alpha\phi^*$ unless $\gamma \in \mathbb{R}$, and so is an element of $H^\alpha \iff \gamma \in \mathbb{R}$. This feature arises because f is an anti-linear map, and so commutes only with multiplication by real numbers. Ergo, the space H^α is only a vector space under real scalar multiplication.

Next we must identify the range of values we may chose for $\alpha \in \mathbb{C}$ while still retaining linear independence from the projection of the complexified Lie algebras. To do this we need only exclude those values for α for which there exists some $A \in H^\alpha$ for which we also have $A^* \in H^\alpha$. Without loss of generality, we write $A = \phi_1 + \alpha\phi_1^*$ for some $\phi_1 \in \underline{\phi}$. We may then span H^α by the basis elements $\{A, \phi_i + \alpha\phi_i^*\}_{i=2}^{12}$. H^α has a non-trivial intersection with the projection of the complexified Lie groups if and only if there exists some set of real parameters $\{d_i\}$ such that

$$A^* = \phi_1^* + \alpha^*\phi_1 = \sum_i d_i\,(\phi_i + \alpha\phi_i^*) \in H_1^\alpha. \tag{4.13}$$

Now since all the basis elements ϕ_i are linearly independent this implies a non-trivial intersection with the projection of the complexified Lie algebras if and only if

$$1 = d_1\alpha \quad \text{and} \quad \alpha^* = d_1. \tag{4.14}$$

This implies clearly that we may only have a non-trivial intersection if $\alpha = \pm 1$, as $d_1 \in \mathbb{R}$ demands $\alpha \in \mathbb{R}$. Employing the same arguments straightforwardly to the space

$\underline{\psi} \oplus \underline{\psi}^*$, we find the pair of 12-dimensional real vector spaces

$$H_1^\alpha := \mathrm{Span}_{\mathbb{R}}\Big\{h \in \mathrm{M}(V);\ h = \phi_1 + \phi_2 \| \phi_1 : 3 \to 2,\ \phi_2 : 3^* \to 2^*;\ \text{s.t. } \phi_2 = \alpha\phi_1^*\Big\}, \quad (4.15)$$

$$H_2^\beta := \mathrm{Span}_{\mathbb{R}}\Big\{h \in \mathrm{M}(V);\ h = \psi_1 + \psi_2 \| \psi_1 : 3^* \to 2,\ \psi_2 : 3 \to 2^*;\ \text{s.t. } \psi_2 = \beta\psi_1^*\Big\}, \quad (4.16)$$

for some $\alpha, \beta \in \mathbb{C}$, where $\alpha, \beta \neq \pm 1$ is required for linear independence. Note that while the spaces H_1^α and H_2^β describe spaces of complex matrices, they each have the structure of a 12-dimensional real vector space. Thus together, $H := H_1^\alpha \oplus H_2^\beta$ span a 24 dimensional real vector subspace of M(8,$\mathbb{C}$).

Let us also take a brief moment to comment on the limits of $\alpha, \beta \in \mathbb{C}$. Clearly as $|\alpha|, |\beta| \to 0$ we have that H_1^α and H_2^β go to $\underline{\phi}$ and $\underline{\psi}$ respectively. In these limits we recover a fundamental 6-dimensional complex vector space representations of SU(2)×SU(3), i.e. the tensor product of a doublet of SU(2) and triplet of SU(3). On the other hand, in (4.15) and (4.16) we could, for any $|\alpha|, |\beta| > 0$, have used the relations $\phi_1 = \alpha^{-1}\phi_2^*$ and $\psi_1 = \alpha^{-1}\psi_2^*$. So in this case we see that the limits $|\alpha|, |\beta| \to \infty$ implies H_1^α and H_2^β go to $\underline{\phi}^*$ and $\underline{\psi}^*$ respectively. In this limit we recover the complex conjugate representation spaces to the limit $|\alpha|, |\beta| \to 0$. This relationship of our limits is a general feature of the interesting behaviour of these H-spaces, namely under complex conjugation $H_1^\alpha \leftrightarrow H_1^{(\alpha^{-1})}$ and $H_2^\beta \leftrightarrow H_2^{(\beta^{-1})}$. This differs from our complexified Lie algebras, which are invariant under complex conjugation, and from our fundamental representations $\mathcal{F}_n$, which all have corresponding complex conjugate spaces $\mathcal{F}_n^* \equiv \mathcal{F}_{n^*}$. In subsection 4.5.3 we further discuss this behaviour under complex conjugation as well as other interesting features about these representation spaces.

We comment here on the identification of these H-spaces as the spaces which are not expressible in terms of complex conjugate invariant basis elements. Naturally this may seem like a sufficient, but not necessary, condition for linear independence with our complexified Lie algebras. In reality this condition is both sufficient and necessary. When projected onto the space $3 \oplus 3^* \to 2 \oplus 2^*$, our complexified Lie algebras $\mathbb{C} \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(3))$ span an 11-dimensional complex subspace. On the other hand, the set of all elements $\gamma^* = -\gamma$ together span a 12-dimensional complex subspace. Thus at first glance the condition that no elements h in our H-spaces are of the form $h^* = \pm h$ may seem too restrictive. However, as we are only interested in irreducible representations it is clear that our H-spaces must be of real dimension divisible by four.[6] As such we may minimally have irreducible representation spaces of complex dimension 6 or real dimension 12. This implies that having any one element $h^* = \pm h$ in our H-spaces demands having some 12-dimensional real subspace of elements transforming as $h^* = \pm h$. Clearly this is not possible without overlapping with elements in $\mathbb{C} \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(3))$, and thus we must only consider maps which are not expressible as $h^* = \pm h$.

[6]The space of maps $3 \to 2$ is a six dimensional complex map but a 12 dimensional real map. The even nature of complex spaces then manifests itself as the divisibility of four when considered in terms of real dimensions.

Additionally, we comment on the vector space structure of our H-spaces. While these $H-$spaces are real vector spaces under scalar multiplication, they do transform as a triplet of SU(3) and a doublet of SU(2), as is evident purely from dimensional counting. However due to the different transformations of ϕ_i and ϕ_i^* the complex structure cannot simply be represented by multiplication via i, the complex unit imaginary. Instead, the complex linear structure describing the complex nature of the fundamental transformations of the space H_1^α becomes multiplication by i on $\underline{\phi}$ and by $-i$ on $\underline{\phi}^*$. This is evident from the appearance of the anti-linear bijection f.

In total we now have only a one dimensional complex subspace left to span the entirety of M(8,$\mathbb{C}$). This remaining subspace is found in the following subsection.

4.4.4 An Additional Symmetry Arises

The maps defined in subsections 4.4.1, 4.4.2, and 4.4.3 together span a 126 $\mathbb{R}$-dimensional subspace of M(8,$\mathbb{C}$).[7] We only lack two real vectors in M($\mathbb{C} \otimes \mathbb{O}$) to fully span the space of maps. These maps must be invariant under transformations of SU(2)×SU(3). Further, since we have exhausted the set of possible maps from the decomposition (4.2) to (4.3), it is clear that this remaining map must be an endofunction on one of the decompositions. The only type of map which satisfies all the criteria is a map which acts as a U(1) charge generator on *either* of the decompositions (4.2) or (4.3).

However, unlike the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ Lie algebras, which correspond to SU(2) and SU(3) subgroups of G_2 respectively, there is no U(1) subgroup of G_2 which preserves either decomposition of $\mathbb{C}\otimes\mathbb{O}$. Thus this U(1) group does not describe any redundancy in defining subalgebras $\mathbb{C} \otimes \mathbb{O}$. Instead it describes a redundancy in the direct sum decomposition of M($\mathbb{C} \otimes \mathbb{O}$). Indeed, looking at the gauge-invariant terms we could construct with the so far defined irreducible representation spaces in

$$\mathbb{C} \otimes \Big(\mathfrak{su}(2) \oplus \mathfrak{su}(3)\Big) \oplus \mathcal{F}_3 \oplus \mathcal{F}_{3^*} \oplus \mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus H_1^\alpha \oplus H_2^\beta \subset \mathrm{M}(8, \mathbb{C}), \tag{4.17}$$

it is clear that any SU(2) invariant subspace could be extended to a U(2) invariant subspace, and similarly for SU(3) invariant subspaces. This is because the maps of our irreducible representation spaces are not invertible. That is, these maps are not bijections on $\mathbb{C} \otimes \mathbb{O}$. Thus the only operation which yields SU(N) invariant terms is matrix multiplication of a map in (4.17) with the hermitian conjugate of another map in (4.17).[8] Ergo, our invariants can at most have a redundancy under unitary transformations. This relates directly to principle 4 and the absence of GL(N,$\mathbb{C}$) transformations in subsection 4.4.1. So we must then find what type of U(1) transformations leave the so far defined irreducible representation spaces invariant. This is most easily done by focusing on the generator Y of our U(1) transformations. Note that this Y will be an element of M($\mathbb{C} \otimes \mathbb{O}$) and is not associated with the operator $\hat{Y}$ introduced in chapter 2.

[7] We here are referring to real, as opposed to complex, dimensions due to the appearance of the real vector spaces H_1^α and H_2^β.

[8] Note that we require hermitian conjugation due to the appearance of our SU(N) transformations.

Consider the condition for the spaces H_1^α and H_2^α to be invariant under U(1) transformations. Both of these spaces involve the use of an anti-linear function in relating the complex conjugate subspaces of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$. It is clear that for $\widetilde{Y}$ describing the projection of Y on the space of maps $3\oplus 3^* \to 2\oplus 2^*$, we must have $\widetilde{Y}^* = -\widetilde{Y}$ for our U(1) transformations to leave $H_1^\alpha \oplus H_2^\alpha$ invariant. This condition also yields linear independence of the spaces $\mathbb{C}\otimes(\mathfrak{u}(1)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(3))$ and $H_1^\alpha\oplus H_2^\alpha$, as discussed in subsection 4.4.3. While it may be appealing to have Y act on all of $\mathbb{C}\otimes\mathbb{O}$ in such a way that $Y^* = -Y$, as this implies complex conjugate basis elements have opposite U(1) charges, within our setup there is no reason to impose this restriction. Thus we do not know the relative charge assignment of Y, only that the generator acts as multiplication by a scalar on any of the irreducible representation spaces of (4.2) or (4.3), dependent on which decomposition this charge generator acts.

We first discuss the case in which the U(1) transformations act on the same decomposition as SU(2), i.e. on (4.3). Let us then investigate the linear independence of $\mathbb{C}\otimes\big(\mathfrak{u}(1)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(3)\big)$ with $\mathcal{F}_1\oplus\mathcal{F}_{1^*}\oplus\mathcal{F}_3\oplus\mathcal{F}_{3^*}$. This is equivalent to investigating the linear independence of all vectors in $\mathbb{C}\otimes\big(\mathfrak{u}(1)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(3)\big)$ when projected on the space of maps $3\oplus 3^*\to 2\oplus 2^*$. In this case the spaces 2 and 2^* are eigenspaces of Y, with opposite eigenvalues. Also, 2 and 2^* can each be spanned by a pair of eigenvectors, also with opposite eigenvalues, for some generator of $\mathfrak{su}(2)$. With a linear combination of Y and the diagonal generator of $\mathfrak{su}(2)$, we may then define a map on $2\oplus 2^*$ which becomes a rank one diagonal real matrix when acting only on 2 or 2^*. This is important, because it allows us to create maps $M\in\mathfrak{u}(1)\oplus\mathfrak{su}(2)$ which have both domain and range in $\mathrm{Span}_{\mathbb{C}}\{da, db, dc\}$, and is rank 1 when restricted to act on 2 or 2^*. Then as $M|_2 = -M|_{2^*}$, for basis elements $dx - idy \in 2$ this implies $M(dx) = -idy$ and $M(dy) = idx$, with $dx, dy \in \{da, db, dc\}$.[9] Now in $\mathfrak{su}(3)$ we have the basis of 3 in terms of elements $\alpha_i = \epsilon_i - id\epsilon_i$ for any $\epsilon_i\in\{a,b,c\}$ as these are eigenvectors under action of $d\in\mathbb{O}$, Appendix C. We introduce the pair of basis elements $\{x - idx, y - idy\}$ in 3 and $\{x + idx, y + idy\}$ in 3^*. On these pairs of basis elements there exists a map $\widetilde{M}\in\mathfrak{su}(3)$ which acts as

$$\begin{aligned}\widetilde{M}(x\pm idx) = -i(y\pm idy) \quad &\text{and} \quad \widetilde{M}(y\pm idy) = i(x\pm idx)\\ \implies \widetilde{M}(dx) = -idy \quad &\text{and} \quad \widetilde{M}(dy) = idx\end{aligned} \tag{4.18}$$

That is, we have the relationship $M = \widetilde{M}$. Therefore $\widetilde{M} - M = 0$, and so $\widetilde{M} - M$ is a linear combination of elements in $\mathbb{C}\otimes(\mathfrak{u}(1)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(3))$ which is vanishing on $3\oplus 3^*\to 2\oplus 2^*$. In other words M and $\widetilde{M}$ are not linearly independent on $3\oplus 3^*\to 2\oplus 2^*$. Ergo, we cannot have a linearly independent generator Y of a U(1) transformation which acts on the same decomposition, (4.3), as SU(2).

We then turn our attention to investigating a U(1) generator which acts on the same decomposition as our SU(3) transformations, i.e. on (4.2). Again, we need only ensure

[9] Note that we are not writing the span here, as we are interested just vectors dx, dy in the set of three elements $\{da, db, dc\}$. This is fully general as we have defined our 2 and 2^* states to be eigenstates of one of $\{a, b, c\}$, which implies the existence of basis elements $d\epsilon_j \pm id\epsilon_{j_1}$ where $[\epsilon_j, \epsilon_k] = i\varepsilon_{jkl}\epsilon_k$.

that any non-zero linear combination of elements of our complexified Lie algebras has a non-zero projection on the space of maps $3 \oplus 3^* \to 2 \oplus 2^*$. In this case we have the same situation as in subsection 4.4.2, but with SU(3)→U(3), since we are only interested in the action of the maps on $3 \oplus 3^*$. Note that this greatly simplifies our analysis. Indeed, when analysing linear independence of our irreducible representation spaces in subsection 4.4.2 we used the rank-2 nature of $\mathfrak{su}(2)$ when restricted to act on 2 or 2^*. However, we only used the even-rank nature of $\mathfrak{su}(3)$. Then as none of the arguments in subsection 4.4.2 relied on the special nature of the special unitary group SU(3), it is clear that the arguments for linear independence of $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathcal{F}_3 \oplus \mathcal{F}_{3^*}$ with $\mathbb{C} \otimes \big(\mathfrak{su}(2) \oplus \mathfrak{su}(3)\big)$ extends to linear independence with $\mathbb{C} \otimes \big(\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)\big)$ for U(1) transformations acting on the same decomposition as SU(3).

In other words, the U(1) transformation matrix must act from the left on our states $\mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus \mathcal{F}_3 \oplus \mathcal{F}_{3^*} \oplus H_1^\alpha \oplus H_2^\beta$, and Y may take different values when acting on 1, 1^*, and 3. The relative charges of Y are only fixed between acting on 3 and 3^* as $\widetilde{Y}^* = -\widetilde{Y}$. For the purpose of this dissertation, i.e. studying the simultaneous appearance of irreducible representation spaces, there is no mechanism for which to fix the relative charge assignment of Y. In a full gauge theory, such as the Standard Model, consistent U(1) charge distributions are restricted by gauge anomaly cancellation, Refs. [70, 71, 72].

Having defined our U(1) transformations, up to relative charges, we then have a full direct sum decomposition of our space $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$ of maps as in (4.1), which we rewrite here for convenience:

$$\mathrm{M}(\mathbb{C} \otimes \mathbb{O}) = \mathbb{C} \otimes \Big(\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)\Big) \oplus \mathcal{F}_3 \oplus \mathcal{F}_{3^*} \oplus \mathcal{F}_1 \oplus \mathcal{F}_{1^*} \oplus H_1^\alpha \oplus H_2^\beta \quad (4.19)$$

Clearly this decomposition is valid for any relative charge assignments in Y. Having spent this section predominantly focused on identifying the set of linearly independent irreducible representation spaces, we next turn our attention to analysing some features of these representations.

4.5 Comments on Decomposition

There are three distinct types of spaces in the decomposition (4.1). We will refer to these different types of subspaces, suggestively, as: gauge-like spaces, F-spaces, and H-spaces. Naturally, the gauge-like spaces will be the complexified Lie algebras themselves as, like gauge fields, they transform in the adjoint representation. The F-spaces transform either as a singlet or in the (anti-)fundamental representation of our unitary groups, hence the suggestive "F" to denote either the "fermion-like" gauge transformations. Finally the H-spaces are quite different from the rest of the spaces so far considered. They are real vector spaces in terms of scalar multiplication, yet still transform in the complex fundamental representation of SU(2)×SU(3). We show in subsection 4.5.3 that invariants formed with these irreducible representation spaces can be understood in terms of six dimensional complex vector spaces transforming as triplets of SU(3) and doublets of SU(2).

We will show that due to this feature, the $H-$spaces mirror certain properties of the Standard Model Higgs, but under different gauge groups. Consequently, we have used the letter “H” to denote that these representation spaces, to the extent that we can make comparisons between objects transforming under different groups, are “Higgs-like”.

Decomposition (4.1) only has four parameters in its definition. These are the parameters $\alpha, \beta \neq \pm 1$, and the two parameter describing the relative charge assignment of our U(1) field. The irreducible representation spaces contained in the decomposition remain the same under any choice of these parameters, but this is not the case for invariants formed of different irreducible representation spaces. Indeed, changes to these parameters will be equivalent to scalar multiplication of these invariants. Thus the parameters α and β suggestively encode interaction strengths between irreducible representation spaces. Without a full theory it is of course impossible to make precise statements about interactions; however, we will show in subsection 4.5.3 that the specific realization of the representations does indicate distinct coupling values.

4.5.1 Gauge-like Subspaces

For adjoint representations we have the complexification of the Lie algebras $\mathfrak{u}(1)$, $\mathfrak{su}(2)$, and $\mathfrak{su}(3)$. The $\mathfrak{su}$(N) Lie algebras arise from restricted representations of the automorphism group G_2, and their complexification is required to fully span the algebra $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$. The U(1) transformations do not originate from the automorphism group of the Octonions. Instead, this redundancy corresponds to the fact that invariants constructed out of the irreducible representation spaces of SU(2)×SU(3) in $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$ are invariant under unitary transformation, not just special unitary transformations. This, combined with the requirements of linear independence and a full direct sum decomposition of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$, demands the existence of a U(1) transformation acting on the same decomposition as SU(3) and a corresponding $\mathbb{C} \otimes \mathfrak{u}(1)$ subspace of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$. Resultantly, it is valid to say that, within the setup shown here, the appearance of the SU(2)×SU(3) symmetry structure demands the existence of an additional U(1) symmetry.

One important point of discussion is that gauge fields in the Standard Model take values in the real Lie algebras, and not their complexification. However, as stated before, the corresponding representations are still those of the (special)unitary groups. This implies that, for example, the linearly independent subspaces $\mathfrak{su}(3)$ and $i\mathfrak{su}(3)$ are both irreducible representation spaces of the group SU(3). It is clear that for agreement with Standard Model representations there must then be some selection mechanism by which one of these irreducible representation spaces does not appear as associated to a physical field. However finding such a mechanism would require both the merger of our gauge representations with Lorentz representations, as gauge fields are described by objects which are both spacetime and Lie algebra vectors, and the formulation of a theory in which such a selection mechanism can be defined. As both the formulation of a full theory and simultaneous incorporation of gauge and Lorentz representations are beyond the scope of this paper, so is the discussion of the complexification of our Lie algebras. We do note that the appearance of the complexified Lie algebras is similar to the appearance of gauge fields in

NCG, and we comment on this in subsection 5.1.3.

4.5.2 The F - Spaces

To offer a concise presentation of the irreducible representation spaces contained in the decomposition (4.1), we write out the set of fundamental representation spaces contained in the F - subspaces of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$ as

$$\mathcal{F}_3 \quad \text{contains} \quad 4 \times (3,1) \tag{4.20}$$
$$\mathcal{F}_{3^*} \quad \text{contains} \quad 4 \times (3^*,1) \tag{4.21}$$
$$\mathcal{F}_1 \quad \text{contains} \quad (1,2) + (1,2^*) + 4 \times (1,1) \tag{4.22}$$
$$\mathcal{F}_{1^*} \quad \text{contains} \quad (1,2^*) + (1,2) + 4 \times (1,1) \tag{4.23}$$

with the notation (a, b) describing a space transforming in the a representation of SU(3) and b representation of SU(2). As before we have omitted any U(1) charge assignment, as the relative charges are not fixed. Note that in (4.20)-(4.23) we have no particles simultaneously transforming under SU(3) and SU(2). Thus, it is clear that in this construction not all representation spaces are realized in the direct sum decomposition of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$. Further still the fundamental representations of (4.20)-(4.23) appear in distinct multiplicities.

These distinct multiplicities of the irreducible representation spaces in the $F-$spaces arise as a direct consequence of simultaneously incorporating both adjoint and fundamental representations within the same algebra of linear maps. This can be seen from the origin of our $F-$spaces as the irreducible representation spaces which were trivially linearly independent from either $\mathfrak{su}(2)$ or $\mathfrak{su}(3)$ by virtue of belonging to different ideals. These two Lie algebras themselves belong to ideals of distinct dimensions, i.e. are represented by matrices of different size. Explicitly, in the SU(2) decomposition we had 4 singlets, while in the SU(3) decomposition we only had two singlets. Therefore it is no surprise that we obtain different multiplicities of the different fundamental representation spaces.

In the Standard Model not all fundamental representations of our gauge groups are present. There are no right handed doublets of SU(2). This implies that while we observe the representation $(3, 2)$, and the corresponding conjugate representation, we do not observe any particles transforming in the $(3^*, 2)$, or the conjugate, representation. For comparison with Standard Model representations, it is an attractive feature that not all fundamental representations appear in the direct sum decomposition. Of course, in our set up we do not have either of the representations $(3, 2)$, $(3, 2^*)$, or their conjugates. This, in addition to the lack of Lorentz representations, means we cannot draw a direct comparison with our representations and those of the Standard Model. However, we still find the general feature of different multiplicities of representation spaces. We highlight this natural appearance of distinct multiplicities of gauge groups as a feature of interest in comparing sets of representation spaces with those of the Standard Model.[10]

[10]Note here that having distinct multiplicities also considers representation spaces which do not appear at all in the direct sum decomposition, as these representation spaces have multiplicity zero.

4.5.3 H- Spaces

The remaining spaces to analyse in our decomposition (4.1) are H_1^α and H_2^β. Due to the similarity in the structure of the spaces H_1^α and H_2^β, we will only analyse the space H_1^α, with similar arguments applicable for H_2^α. While the complex structure in H_1^α is not simply multiplication by i, we can understand the space in terms of maps $\phi : 3 \to 2$, where indeed the complex structure is multiplication by i, see subsection 4.4.3. According to the definition of H_1^α in (4.15), elements of this space are maps h on $\mathbb{C} \otimes \mathbb{O}$ which map 3 to 2 and 3^* to 2^* such that

$$h = \phi + \alpha\phi^*, \tag{4.24}$$

where $\phi^* : 3 \to 2$. As ϕ and ϕ^* belong to different left and right ideals of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$, any invariants formed with h and irreducible representation spaces of $\mathcal{F}_i$ reduce to invariants formed with these irreducible representations in $\mathcal{F}_i$ and either ϕ *or* ϕ^*. Then as all the 12 real parameters of H_1^α are encoded in the 6 complex parameters of ϕ, one can fully describe the different possible map compositions with H_1^α by the set of complex parameters in ϕ.

Now as $\alpha \neq \pm 1$, there are no maps in H_1^α whose conjugate map is also H_1^α. Further there is no independent conjugate space to H_1^α in the direct sum decomposition (4.1).[11] However, we still have that for the map ϕ, which transforms in the fundamental representation, there is always a conjugate map ϕ^* transforming in the complex conjugate representation.

In the case one were to write down a theory of interacting fields transforming in irreducible representations, one could then parametrize all the interactions and dynamics of H_1^α in terms of the maps ϕ transforming in the fundamental representation. This would of course require also including ϕ^*, but without independent dynamics and not as a separate field. In other words, ϕ^* would only be required to accurately describe interactions between fields. In the Standard Model, Yukawa interactions require the explicit appearance of both the Higgs doublet Φ and the conjugate Φ^*, which has opposite charge assignments to Φ, Ref. [54]. Thus by viewing our space H_1^α as described by maps ϕ transforming in the fundamental representations of our gauge groups, we recover a picture which, to the extent that we can draw comparisons to field theory, is quite similar to the Standard Model Higgs doublet. Of course in the Standard Model the Higgs doublet and its conjugate are needed to give masses to the different components of the left-handed SU(2) doublets, for example to up- and down-type quarks. This is not possible in our construction as ϕ and ϕ^* not only live in different ideals but also act on the non-similar 3 and 3^* representations of SU(3).[12] Instead the respective maps ϕ and ϕ^* must form contractions between respective conjugate representations.

[11] Note that even for $\alpha = i$, under complex conjugation $H_1^i \to H_1^{-i} = iH_1^i$. The space H_1^α is a real vector space, and it is simple to check that it is $\mathbb{R}$-linearly independent from the space iH_1^α for any $\alpha \in \mathbb{C}$.

[12] The Higgs doublet transforms under SU(2) as its only non-abelian gauge group. Since the conjugate representation 2^* is related by a similarity transformation to the doublet representation 2, this means that the Higgs doublet and its conjugate may still form invariants with fields transforming in the $\bar{2}$ of SU(2). Both of these invariants are required, as after spontaneous symmetry breaking the Higgs doublet gives mass to one component of the fermionic SU(2) doublet while the conjugate Higgs doublet gives mass to the other component of the fermionic SU(2) doublet.

When considering such contractions the factor α may then be interpreted as a relative coupling parameter between ϕ and ϕ^*. That is, as a coupling that incorporates a discrimination between the fundamental and anti-fundamental representations. We highlight this discrimination between conjugate representations as a feature of interest, as nature certainly seems to discriminates between these representations. Indeed, our universe is composed primarily of matter and not anti-matter. In the Standard Model, the Higgs doublet also couples to up- and down-type quarks with different coupling strengths. Therefore it would be interesting to see whether, in another set-up, one may recover similar spaces to H_1^α which only transform under the non-abelian group SU(2). In such a case the parameter α could potentially become interpreted as a relative coupling strength between elements with opposite weak charge. This would present a natural mechanism by which to generate different masses of particles.

In the above analysis we discussed features of the H-spaces in comparison to the Standard Model Higgs doublet. Naturally there are limitations to how close a connection we can make. One feature which is essential to the Higgs doublet is its scalar nature under Lorentz transformations. This is not a feature on which we can provide comparison as our direct sum decomposition contains no Lorentz representations. Further, any discussion about dynamics is also not possible, since we have no theory for our sets of representations. This prevents any discussion regarding spontaneous symmetry breaking. As such, the above arguments are focused only on the statements we can make about representations. These are: for the map ϕ complex conjugation results in a representation space with opposite charges, just as is the case for the Higgs doublet; the maps ϕ and ϕ^* are described by the same real components, just like the Higgs doublet and its complex conjugate; and finally ϕ and ϕ^* form invariants with different representations, this is also the case for the Higgs doublet.[13] Thus for the available points of comparison there is a strong similarity between the H-spaces and Standard Model Higgs doublet. To make or more definitive statements requires inclusion of spacetime symmetries and the formulation of an action.

We conclude our analysis of these H-spaces by noting that their appearance is uniquely required by the appearance of our adjoint representations. This arises as a consequence of the H-spaces belonging to the same left ideals as $\mathbb{C} \otimes \mathfrak{su}(3)$ and the same right ideals as $\mathbb{C} \otimes \mathfrak{su}(2)$. This differs from our F-spaces which share an ideal with at most one of the non-abelian adjoint representations. This suggests the appearance of such H-spaces is not unique to the setup shown here, and could correspond to a more general feature of incorporating irreducible representation spaces as linearly independent maps between restricted representations.

4.5.4 Uniqueness of Decomposition

In this subsection we discuss uniqueness up to linear combination of maps. That is to say, instead of considering $\mathcal{F}_3$ and $\mathcal{F}_{3^*}$ we could consider linear combinations of these

[13] As right handed up- and down-type quarks are singlets of SU(2) with different U(1) charges they are distinct irreducible representation spaces of the Standard Model gauge group.

two spaces. For example, we could have one space spanned by elements A satisfying $A^* = -A$, and span the other space by elements with similar properties to those of our H-spaces. We do not view this as modifying the uniqueness of our decomposition, as it is just forming linear combinations of the established irreducible representations. Therefore in the following we will only discuss distinct decompositions whose irreducible representation spaces, with correct transformations, cannot be recovered as linear combinations of the irreducible representation space in (4.1).

As we have restricted our attention to irreducible representation spaces we know we may only have maps between the irreducible restricted representations of $\mathbb{O}|_{\mathbb{C},\mathbb{H}}$. We demanded by principle 3 the existence of the complexified Lie algebras $\mathbb{C}\otimes(\mathfrak{su}(2) \oplus \mathfrak{su}(3))$. Further, we calculated H_1^α and H_2^β as spanning the two most general irreducible representation spaces which map $3 \oplus 3^*$ to $2 \oplus 2^*$ while maintaining linear independence with the complexified Lie algebras. This in turn demands the existence of our $\mathbb{C} \otimes \mathfrak{u}(1)$ subspace. Thus the question of uniqueness regards the representations contained in our F-spaces, as the projection on $3 \oplus 3 \to 2 \oplus 2$ of the maps $H_1^\alpha \oplus H_2^\beta \oplus \mathbb{C} \otimes (\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3))$ fully spans the space. The sets of irreducible representation spaces in the F-spaces are exactly those which span the remaining ideals of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$, and of course they too are all needed to span the space of maps. However, they may not be needed in their exact form. For example, the map $\{e_0\} \to \{e_0\}$ is contained in $\mathcal{F}_1 \oplus \mathcal{F}_{1^*}$ but is also indistinguishable from maps preserving either the SU(3) decomposition (4.2) or the SU(2) decomposition (4.3). This is because this map is a singlet under SU(2)×SU(3) regardless of whether this map is interpreted as a singlet map between our two $\mathbb{C} \otimes \mathbb{O}|_{\mathbb{C}}$ and $\mathbb{C} \otimes \mathbb{O}|_{\mathbb{H}}$, or as an endofunction on one of these structures. It is clear that the SU(2)×SU(3) representations in the direct sum decomposition (4.1) are unique. However, this does not mean that the singlet states of SU(2)×SU(3) have a unique structure, and could in principle transform under other group actions if such representations could be made linearly independent.

This ambiguity preventing the uniqueness of the direct sum decomposition (4.1), beyond uniqueness SU(2)×SU(3) representations, originates from the existence of multiple singlet states of our non-abelian gauge groups. However, in the Standard Model we do not have any particles which are singlets under all the non-abelian groups if we include Lorentz representations; that is, there are no singlets of both SO(1,3) and SU(2)×SU(3). This highlights a relevance to focus on constructions in which no singlet states exist, for comparison with Standard Model representations. Based on our results here, we expect uniqueness of the direct sum decomposition to be achievable in such a construction.

4.6 Vector Spaces or Algebras?

In the construction presented here we showed how features of the complexified Octonions could be employed to induce a direct sum decomposition on the space of maps $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$. As noted earlier, this space is realized as the space of eight by eight complex matrices $\mathrm{M}(8,\mathbb{C})$ with matrix multiplication corresponding to map composition. Thus from the perspective of the matrix algebra, we cannot distinguish whether the maps act on an algebra or simply a vector space, as the space of linear maps does not capture information about the multiplicative structure of the space on which they act. Therefore one could equivalently work simply with maps on vector spaces instead of maps on an algebra. In this subsection we comment on some advantages and disadvantages of each approach.

The primary advantage in working with vector spaces, and not algebras, is more freedom in how to chose symmetry groups. Indeed we could simply have used some vector space V with transformations given by G instead of focusing on some algebra $\mathbb{A}$ with transformations given by $\mathrm{Aut}\{\mathbb{A}\}$, the automorphism group of $\mathbb{A}$. In this case we would have the ability to incorporate, within the same matrix algebra, more variations of simultaneous realizations of distinct representations. Alternatively one can also let the same group G act on different spaces. For example SL(2,$\mathbb{C}$) can act on $\mathbb{C}^2$, a space of complex dimension 2, but also on M(2,$\mathbb{C}$), a space of complex dimension 4. This freedom in studying maps on vector spaces may ultimately yield direct sum decompositions with properties not achievable when considering only linear maps on algebras.

Of course, the setup of section 4.2, by which we identified the irreducible representation spaces in the space of maps, could not be the same if studying the linear maps on a vector space. This follows immediately from vector spaces not having any defined multiplicative structure, and thus there is no notions of subalgebras by which to define symmetry groups or decompositions as we did for $\mathbb{C} \otimes \mathbb{O}$. Instead, we would require some other process by which to select irreducible representation spaces of V, and maps between them. This is not necessarily an advantage or disadvantage, but does signify a difference in approaching the construction of a direct sum decomposition of the space. For example, one could have multiple copies of V, each transforming under different groups G, and maps on and between these spaces. This is very similar to what was done in chapter 2.

4.7 Inducing Vs. Embedding Representations

In both the approach of chapters 2 and 4 a central property was the use of linear independence between irreducible representation spaces. Naturally this is the core investigation of our research, and as such the appearance of similar features in the two constructions should not come as a complete surprise. Additionally, we identified interesting features in chapter 2, from the perspective of understanding how representation spaces arise and how they compare with the Standard Model, and used ideas gained from these features to identify mechanisms by which we may obtain a set of irreducible representation spaces that are interesting in comparison with modern particle theory. This necessarily implies the existence of many points of similarity between the two approaches. In this section we will discuss merits and advantages of the different approaches in understanding the simultaneous realisation of multiple representation spaces.

The requirement of linear independence in chapter 2 was the origin behind the subspace identified with Higgs doublet transformations not having an independent complex conjugate subspace. Similarly, in the induction process of chapter 4 linear independence implied existence of the $H-$spaces, which we argued showed similar features to the Standard Model implementation of the Higgs doublet representation in interactions. We found two distinct properties under map composition between these "Higgs-like" representations of chapters 2 and 4. In chapter 2 the subspace identified with the Higgs doublet representation had the potential to include a non-zero projection on subspaces describing fermion transformations. As we commented before, this non-zero intersection implies the interaction strengths between the Higgs doublet and different generations of fermions could be incorporated into how these representations appear as subspaces within a matrix algebra. Clearly, the explicit embedding of the Standard Model yielded insights into how to form invariants between representation spaces within M(8,$\mathbb{C}$) and, as all interactions of fields are described in terms of invariants, therefore gave insight into how interactions would appear in such a construction. This feature arose precisely because we demanded the existence of three generations of fermionic representation spaces in chapter 2.

On the other hand, in chapter 4 we found representation spaces, i.e. the $H-$spaces, which necessitated the inclusion of parameter α and β to distinguish map compositions with respective conjugate representations. As discussed, this could potentially describe interactions with matter-antimatter asymmetry, or an asymmetry between interactions of up- and down-type particles,[14] a feature we did not see in the explicit embedding of chapter 2. Both of these features are required to yield the mass spectrum seen in the Standard Model Yukawa interactions. This makes it clear that in the study of simultaneous realizations of representation spaces, both the embedding and induction of these "Higgs-like" representation spaces yielded interesting insight into the encoding of interaction strengths.

This presents the idea that the explicit embedding of Standard Model representation spaces is useful for understanding what relationships between representation spaces can be imposed by the inclusion of linear independence. Inducing representations instead indicates what sets of linearly independent representation spaces are consistent with certain underlying structures, i.e. like the SU(2) and SU(3) decompositions of $\mathbb{C} \otimes \mathbb{O}$ in chapter 4. To exemplify this, note that in chapter 2 there was no fundamental reason, apart from matching with Standard Model representations, to not include higher representations of the gauge groups, i.e. like the 6-dimensional representation of SU(3). If in chapter 4 we had included maps M from 3^* to 3, these maps would have transformed as $M \to UMU^T$ when $3^{(*)} \to U^{(*)}3^{(*)}$, and thus contained the 6 representation of SU(3). However, such maps do not form a Lie algebra describing the redundancy of the direct sum decomposition of M($\mathbb{C} \otimes \mathbb{O}$), i.e. principle 2.[15] So for the particular setup we presented in section 4.2, the appearance of the 6 representation of SU(3) is not possible. This shows that the set of irreducible representation spaces that may arise when inducing these spaces in the matrix algebra is heavily dependent on the principles by which the induction occurs. This is of course to be expected. Indeed we wanted certain features out of our representation spaces, as discussed in chapter 3. Even so, it demonstrates certain relationships which are

[14] We reiterate our disclaimer here that in principle we cannot make definite statements about interaction structures without an action. Our statements originate from an interpretation of features of map composition as a strong indication of interaction structures.

[15] Indeed because maps $M : 3^{(*)} \to 3$ have different right and left ideals the set of such maps do not even form a matrix subalgebra, much less a Lie algebra.

uncovered when inducing representation spaces. We believe more work in this direction could lead to a deeper understanding of how one may simultaneously incorporate different representation spaces within an algebra of linear maps.

Apart from linear independence between irreducible representation spaces, it is interesting to consider how the different groups acted on the irreducible representation spaces in these approaches to studying the simultaneous realization of gauge representations. In both chapters we achieved the fundamental transformations of SU(2)×SU(3) through employing bi-representations, with SU(3) acting from the left and SU(2) acting from the right. However, as chapter 2 was just an embedding of Standard Model representations, it offered no explanation for the origin of adjoint and fundamental representations. This resulted in the use of ad-hoc projectors which ruined the simplicity of the construction. In contrast, when inducing the direct sum decomposition in chapter 4, adjoint representations were seen as describing endofunctions preserving $\mathbb{C}\otimes\mathbb{O}|_{\mathbb{C},\mathbb{H}}$, while fundamental representations were endofunctions of $\mathbb{C}\otimes\mathbb{O}$ which mapped $\mathbb{C}\otimes\mathbb{O}|_{\mathbb{C}}$ to $\mathbb{C}\otimes\mathbb{O}|_{\mathbb{H}}$. Here the idea of projectors can have the natural origin as describing subspaces of different decompositions transforming under different gauge groups. As such the induction of irreducible representation spaces provided a mechanism by which the different adjoint and fundamental representations could both be seen as endofunctions obeying different properties. Further, it also provided a mechanism through which it was evident we could not have SU(3) transformations of SU(2) or visa-versa, something which needed to be imposed additionally in chapter 2. Such a natural inclusion of different transformations could help understand how to construct theories which have only the types of irreducible representations seen in the Standard Model. The ambition of such work is not only to have vector-adjoint representation and spinor-fundamental representations, i.e. the matching of gauge and Lorentz representations seen in the standard model. For example, understanding how representations of these groups combine for different constructions could help uncover structures that result in chiral discrimination of gauge groups, deepening our understanding of this phenomenon.

While the approaches of embedding and inducing representation spaces are quite different, it is clear that they are both advantageous, and have each helped uncover different structures associated to representations in M(8,$\mathbb{C}$). All of this is based on the idea of incorporating linear independence of finite-dimensional spaces as a useful tool for the study of irreducible representation spaces in the context of particle theory. A tool that provides relationships between and restrictions on sets of irreducible representation spaces. Formulating a deeper understanding of how these relationships originate from the choice of base space and setup could offer insights into what types of structures should be present in approaches to unifying particle theory. We conclude this comparison with the emphasis that we are not implying our work, or generalizations, could directly yield the Standard Model's particle content. Rather we suggest that understanding how to incorporate relationships between irreducible representation spaces may guide future approaches to unification in particle theory.

4.8 Towards the Induction of Space-Time Representations

In this paper we have so far focused only on inducing representations associated with gauge groups of the Standard Model. The other symmetry of relevance is the Lie group SO(1,3), or its double cover SL(2,$\mathbb{C}$), generated by the Lie algebra $\mathfrak{so}(1,3)$, which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. In this section we provide some insight into how Lorentz representation spaces can also be incorporated in a space of linear maps.

To proceed in a similar fashion to the induction of adjoint and fundamental representations presented earlier in this section, we will consider an algebra whose automorphism group is SL(2,$\mathbb{C}$). For this choice we will select the algebra we will use $\mathbb{C}\otimes\mathbb{H}\cong\mathbb{C}l(2)\cong$ M$(2,\mathbb{C})$, presented earlier in section 2.2. We will focus on the double cover SL(2,$\mathbb{C}$) of the Lorentz group, as simultaneous left and right action of this group on $\mathbb{C}\otimes\mathbb{H}\cong$M(2,$\mathbb{C}$) produces the vector space representation of SO(1,3) transformations, as presented in chapter 2. However, we will not induce a set of representation spaces in this section. The absence of an induction mechanism is two fold. First, the primary focus of this thesis was to recover the representation spaces concerned with gauge transformations, not Lorentz transformations. Second, it is important to recall that the Lorentz representations identified in $\mathbb{C}\otimes\mathbb{H}$ were not all linearly independent, and linear independence between all the representation spaces was only achieved by the inclusion of M(8,$\mathbb{C}$) describing transformations under the action of the gauge groups. Thus it is not even clear whether we can consistently formulate an induction method based only on the Lorentz transformations themselves.

In section 2.2, we described how vector, spinor, and scalar representations of the Lorentz group can be embedded in the complexified quaternions and equivalently M(2,$\mathbb{C}$). However this leaves out the adjoint, or bi-vector, representation of the group. We note that the scalar transformations are described by

$$a\to\Lambda a\widetilde{\Lambda} \tag{4.25}$$

for some scalar $a\in\text{Center}\{\mathbb{C}\otimes\mathbb{H}\}=\mathbb{C}$, where $\widetilde{\Lambda}\equiv\Lambda^{-1}$ describes Quaternionic conjugation. These automorphisms trivially preserve the centre of the algebra, and so they must also preserve the centre's compliment, i.e. $\mathbb{C}\otimes\text{Im}\{\mathbb{H}\}$.[16] As $\mathbb{C}\otimes\text{Im}\{\mathbb{H}\}\cong\mathfrak{sl}(2,\mathbb{C})$, this subspace of the complexified quaternions transforms in the adjoint representation of SL(2,$\mathbb{C}$) under (4.25).

Clearly the different Standard Model Lorentz representations may all be encoded in $\mathbb{C}\otimes\mathbb{H}$. However, for the setup of section 4.2 we were interested not in the algebra itself, but in the endofunctions on the algebra. Instead, we must find a constructive way to view the algebra M$(\mathbb{C}\otimes\mathbb{H})$. For $\mathbb{C}\otimes\mathbb{O}$ this was in many ways simpler, as we could view the entire algebra from the vector space perspective.[17] However with the algebra $\mathbb{C}\otimes\mathbb{H}$ we

[16] Here the "Im" stands for the imaginary part of the quaternions, and denotes the basis elements ε_j of the quaternions as these basis elements square to minus one.

[17] This in fact stems from left and right multiplication within the Octonions being isomorphic, and thus we can view all maps as matrices acting on the left of the algebra.

have to take into account the appearance of left and right handed ideals at the level of the algebra itself.

In previous literature, Ref. [56], the approach to realize Lorentz representations was to consider that as

$$\mathbb{C}\otimes\mathbb{H}\cong\mathbb{C}\mathrm{l}(2) \tag{4.26}$$

and

$$\mathrm{M}(\mathbb{C}\otimes\mathbb{H})\cong(\mathbb{C}\otimes\mathbb{H})\otimes_{\mathbb{C}}(\mathbb{C}\otimes\mathbb{H})\,, \tag{4.27}$$

we have

$$\mathrm{M}(\mathbb{C}\otimes\mathbb{H})\cong\mathbb{C}\mathrm{l}(2)\otimes_{\mathbb{C}}\mathbb{C}\mathrm{l}(2)\cong\mathbb{C}\mathrm{l}(4)\cong\mathbb{C}\otimes C\mathrm{l}(1,3). \tag{4.28}$$

That is, one may recover the Clifford algebra of spacetime, or rather its complexification, as the algebra describing the space of endofunctions on $\mathbb{C}\otimes\mathbb{H}$. The chain of isomorphisms (4.28) is certainly interesting since, as shown in subsection 1.4.1, the Clifford algebra of spacetime $C\mathrm{l}(1,3)$ is implicit in Standard Model physics. However, the identification (4.28) is not ideal for our construction, as it is not clear how one could induce Lorentz representations in $\mathbb{C}\mathrm{l}(4)$ from irreducible representation spaces in $\mathbb{C}\otimes\mathbb{H}$.

Instead, we will present a different approach to working with $\mathrm{M}(\mathbb{C}\otimes\mathbb{H})$ which is more in line with the ideas of section 4.2. In addition to using isomorphism (4.26), we will also employ the identification

$$\mathbb{C}\otimes\mathbb{H}\cong\mathbb{C}\otimes_{\mathbb{C}}\mathbb{R}^{1,3}, \tag{4.29}$$

where the $\mathbb{R}^{1,3}$ is the real vector space spanned by $\{1, i\varepsilon_1, i\varepsilon_2, i\varepsilon_3\}$ as in (2.1).[18] The validity of this identification is apparent from the separation of $\mathbb{C}\otimes\mathbb{H}$ into hermitian and anti-hermitian real vector spaces as shown in chapter 2. Using (4.27), we have

$$\begin{aligned}\mathrm{M}(\mathbb{C}\otimes\mathbb{H}) &\cong(\mathbb{C}\otimes\mathbb{H})\otimes_{\mathbb{C}}(\mathbb{C}\otimes\mathbb{H})\\ &\cong\left(\mathbb{C}\otimes\mathbb{R}^{1,3}\right)\otimes_{\mathbb{C}}(\mathbb{C}\otimes\mathbb{H})\\ &\cong\quad\mathbb{R}^{1,3}\otimes_{\mathbb{C}}(\mathbb{C}\otimes\mathbb{H})\,.\end{aligned} \tag{4.30}$$

Here we have chosen to incorporate the complex parameters entirely within $\mathbb{C}\otimes\mathbb{H}$, leaving only real parameters in $\mathbb{R}^{1,3}$. Note that isomorphism (4.27) can be visualized as describing $\mathrm{M}(\mathbb{C}\otimes\mathbb{H})$ as consisting of pairs of matrices acting on the left and on the right of $\mathbb{C}\otimes\mathbb{H}\cong\mathrm{M}(2,\mathbb{C})$. In this case $\mathbb{R}^{1,3}$ and $\mathbb{C}\otimes\mathbb{H}$ in (4.30) can be interpreted as the left and the right action on respectively. We will use the isomorphism (4.30) as a starting point for considering endofunctions on $\mathbb{C}\otimes\mathbb{H}$.

[18] Note of course that this only describes the vector space structure of the spaces, indeed we will still have well defined Quaternionic multiplication within $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{R}^{1,3}$. Additionally, tensor product over $\mathbb{C}$ is left in to emphasize that the same unit imaginary i appearing in $\mathbb{C}$ also appears in the construction of $\mathbb{R}^{1,3}$. However, the space $\mathbb{R}^{1,3}$ is still described only in terms of real parameters.

We will focus on maps between the different irreducible representation spaces of Lorentz transformations in $\mathbb{C}\otimes\mathbb{H}$ introduced above. Let V denote the vector representations, V^* denote covector representations, Ψ_L and Ψ_R denote left and right handed spinor representations, Φ denote the scalar representation, ω denote the adjoint representation, and ω^* denote the complex conjugate of the adjoint representation. Further denote a map in $\mathrm{M}\left(\mathbb{C}\otimes\mathbb{H}\right)$, acting on $\mathbb{C}\otimes\mathbb{H}$ as left multiplication of A and right multiplication by B, via (A,B). I.e. for any $A,B,C\in\mathbb{C}\otimes\mathbb{H}$, $(A,B)(C):=ACB$. For this map, A and B will be referred to as the first and second factors of the map respectively. With this notation, we can describe Lorentz representation of maps between irreducible representations spaces of $\mathbb{C}\otimes\mathbb{H}$ as:

$$V\rightarrow\begin{cases}\Psi_R & \text{via maps transforming as} \quad (V^*,\Psi_R)\\ \Phi,\omega^* & \text{via maps transforming as} \quad (V^*,\Phi),(V^*,\omega^*)\\ V^* & \text{via maps transforming as} \quad (V^*,V^*)\end{cases} \tag{4.31}$$

$$\Psi_L\rightarrow\begin{cases}\Psi_R & \text{via maps transforming as} \quad (V^*,\Phi)\\ \Phi,\omega^* & \text{via maps transforming as} \quad \left(V^*,\Psi_L^\dagger\right)\\ V^* & \text{via maps transforming as} \quad \left(V^*,\Psi_R^\dagger\right)\end{cases} \tag{4.32}$$

$$\Phi,\omega\rightarrow\begin{cases}\Psi_R & \text{via maps transforming as} \quad (V^*,\Psi_L)\\ \Phi,\omega^* & \text{via maps transforming as} \quad (V^*,V)\\ V^* & \text{via maps transforming as} \quad (V^*,\Phi),(V^*,\omega)\end{cases} \tag{4.33}$$

The maps between the complex conjugate representations follow directly by complex conjugation:

$$V^*\rightarrow\begin{cases}\Psi_L & \text{via maps transforming as} \quad (V,\Psi_L)\\ \Phi,\omega & \text{via maps transforming as} \quad (V,\Phi),(V,\omega)\\ V & \text{via maps transforming as} \quad (V,V)\end{cases} \tag{4.34}$$

$$\Psi_R\rightarrow\begin{cases}\Psi_L & \text{via maps transforming as} \quad (V,\Phi)\\ \Phi,\omega & \text{via maps transforming as} \quad \left(V,\Psi_R^\dagger\right)\\ V & \text{via maps transforming as} \quad \left(V,\Psi_L^\dagger\right)\end{cases} \tag{4.35}$$

$$\Phi,\omega^*\rightarrow\begin{cases}\Psi_L & \text{via maps transforming as} \quad (V,\Psi_R)\\ \Phi,\omega & \text{via maps transforming as} \quad (V,V^*)\\ V & \text{via maps transforming as} \quad (V,\Phi),(V,\omega^*)\end{cases} \tag{4.36}$$

We have only described maps which are of the form $(V, \cdot)$ or $(V^*, \cdot)$, which is a subset of all possible maps between our irreducible Lorentz representations. This implies we considered only the set of maps between irreducible representation spaces of $\mathbb{C} \otimes \mathbb{H}$ which naturally produces the $\mathbb{R}^{1,3}$ factor seen in (4.30). We considered only maps of this type due to the interpretation of the individual factors: transformations of the first factor is that of some four-vector,[19] and the transformation of the second factor is some irreducible Lorentz representation. With a collection of such maps we could, for example, have a set of elements which span the space of Ψ_L representation in the second factor, with each element of Ψ_L having one or more associated four-vectors in the first factor. This description of a collection of maps contains the necessary ingredients to describe a left handed spinor field in spacetime, as spacetime fields assign a Lorentz representation to each point in $\mathbb{R}^{1,3}$, the space of four-vectors. This argument can be extended to any type of spacetime field transforming in either the spinor, scalar, vector, or adjoint representations.

The interpretation of the first factor describing a four-vector is of course only applicable for the above set of maps. Indeed, mapping any of the above representations to themselves the first factor would transform as $A \to \Lambda A \widetilde{\Lambda}$, and therefore not be consistent with the transformation of four-vectors. This discussion is not an induction of irreducible representations as was the case for maps on $\mathbb{C} \otimes \mathbb{O}$, but instead an example of how certain structures appear within $\mathrm{M}(\mathbb{C} \otimes \mathbb{H})$. More work is required to determine whether there exist mechanisms by which only the above maps are selected. To this end we provide some observations unique to the above set of maps.

We observe that any map in $\mathrm{M}(\mathbb{C} \otimes \mathbb{H})$ can be recovered from the above maps via map composition. This can be seen by noting that the Lorentz representations V, Ψ_L, Φ, and ω are all those which under a Lorentz transformation are acted upon from the left via Λ. Similarly V^*, Ψ_R, Φ, and ω^* are all the Lorentz representations which are acted upon from the left by Λ^*. Since $\Phi \equiv \mathbb{C}$ it commutes with the Lorentz group elements Λ, ergo

$$\Lambda \Phi \widetilde{\Lambda} = \Lambda^* \Phi \Lambda^\dagger = \Phi. \tag{4.37}$$

By having all possible maps between these two sets we can then also recreate any map within these sets via map composition. This itself is not a property unique to the set of maps in (4.31)-(4.36). However, this collection of maps forms the smallest set of maps, with the first factor transforming in a four-vector representation, such that mapping between any of the irreducible Lorentz representations $V, V^*, \Phi, \omega, \omega^*, \Psi_L, \Psi_R \in \mathbb{C} \otimes \mathbb{H}$ is achievable by map composition of two or less maps. In particular mapping any irreducible Lorentz representation space to itself requires the composition of an even number of these maps.

Another interesting feature of these maps is the appearance of scalar representations in $\mathrm{M}(\mathbb{C} \otimes \mathbb{H})$. Clearly any map $(a_1, a_2) \in \mathrm{M}(\mathbb{C} \otimes \mathbb{H})$ with the second factor, a_2, transforming as a bi-vector, i.e. as (4.25), also permits having a scalar representation for a_2 via equation (4.37). However, the converse is not true: maps with a scalar representation for the second

[19] Note that since all we can say for definite is that the first factor transforms in the vector representation of the Lorentz group it is possible to interpret this factor as describing a spacetime vector, momentum vector, or otherwise.

factor do not also imply the existence of a bi-vector representation. To see this discrepancy look at the difference between the top line in (4.32) and the bottom line in (4.33). Focus on the maps which only allow for a scalar representation in the second factor. These are uniquely the maps that take left handed spinor representations of $\mathbb{C}\otimes\mathbb{H}$ to the right handed spinor representations, and visa-versa. Thus these scalar representations arise uniquely in the set (4.31)-(4.36) as the maps which connect left and right handed spinor representations. While this is not necessarily implicative of any underlying connection to particle physics, it has an interesting similarity to how the Higgs doublet in the Standard Model forms invariants between the left and right handed Lorentz spinors. To investigate any potential significance of this observation would require a construction of simultaneous realization of multiple representations transforming under both Lorentz and gauge transformations.

Here we have showed that it is possible to extend the idea of describing representation spaces of gauge groups as endofunctions to describing representation spaces of Lorentz transformations. This resulted in all the different irreducible representation spaces seen in the Standard Model, as a consequence of our algebra of choice $\mathbb{C}\otimes\mathbb{H}$. Furthermore, when using $\mathbb{C}\otimes\mathbb{H}$ as our base space we find not only the irreducible Lorentz representations, but the appearance of a spacetime structure, i.e. the factor of $\mathbb{R}^{1,3}$. While the process of identifying irreducible Lorentz representations in the space of maps $\mathrm{M}(\mathbb{C}\otimes\mathbb{H})$ is similar in spirit to the process defined in section 4.2, i.e. mapping between irreducible representation spaces, they are clearly not directly compatible. In section 4.2 we considered restricted representations of G_2 that preserved certain subalgebras of $\mathbb{C}\otimes\mathbb{O}$. However, there is no way of obtaining the set of maps in (4.31)-(4.36) from transformations of the underlying algebra

$$\mathbb{C}\otimes\mathbb{H}\to\Lambda\left(\mathbb{C}\otimes\mathbb{H}\right)\widetilde{\Lambda} \tag{4.38}$$

by restricting these transformations to preserve subalgebras of $\mathbb{C}\otimes\mathbb{H}$. Therefore we cannot use the same setup as in section 4.2 to induce representation spaces whose transformations are simultaneous gauge and Lorentz representations, if such a construction is even possible. Nevertheless, the set of maps (4.31)-(4.36) together with the earlier results of this chapter demonstrate the applicability of using endofunctions on an vector space as a method of generating both irreducible Lorentz and gauge representation spaces.

Chapter 5

Comparison with Unification Approaches and Outlook

5.1 Comparisons with Established Unification Approaches

In chapters 2 and 4 we detailed two separate approaches to studying the simultaneous realization of multiple gauge representations. This provided a basis for studying consequences which arise when trying to impose linear independence between different irreducible representation spaces. As a result, many of the resultant features that arose from our constructions are reminiscent of those associated with the established approaches of section 1.3. We comment on these features here and compare our results to those of the above referenced unification approaches.

5.1.1 SUSY

As commented in subsection 1.3.1, a crucial feature of SUSY is the appearance of additional Lorentz representations for each irreducible gauge representation, with definitive rules for how these Lorentz representations transform into each other. Since the induced representation spaces of chapter 4 did not consider Lorentz representations, we will focus on the explicit embedding of chapter 2 and the Lorentz representations presented in section 4.8.

A main difference between our work and SUSY is that we are not incorporating any symmetry transformations between the different Lorentz representations. The motivation behind SUSY was to restricted the allowed interaction terms in the Lagrangian. This cannot be achieved by our current work, as we are simply producing representations, and not imposing relations on how they transform into each other. Thus the mechanism by which SUSY achieves approximate gauge coupling unification and reduces sensitivity of Higgs mass radiative corrections is not mirrored by our construction. Since the supersymmetry transformations of SUSY are not pertinent to our discussion we focus instead on its superspace formulation, and compare this construction to the embedding of chapter 2.

In our embedding of Standard Model representation spaces, the main feature of the irreducible Lorentz representation spaces is that they all appeared as subspaces of M(2,$\mathbb{C}$). This matrix algebra is not large enough to incorporate vector and spinor representations as linearly independent subspaces. Instead, the linear independence of the Lorentz representations is ensured by the space M(8,$\mathbb{C}$) of gauge representations. We stress that such a relationship does in no way imply a mixing of the Lorentz and gauge generators. Yet, the linear independence of simultaneous Lorentz and gauge representations realized as subspaces of the larger matrix algebra M(16,$\mathbb{C}$) $\cong \mathrm{M}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{M}(8,\mathbb{C})$ restricts the set of allowed SL(2,$\mathbb{C}$)$\otimes$U(1)$\otimes$SU(2)$\otimes$SU(3) representation spaces. This differs from the superspace formulation of SUSY, where the space of Lorentz representations contains both vector and spinor representations as linearly independent subspaces. In our construction, we provide one unified space for the Standard Model Lorentz representations, by unifying our gauge representations in another space, such that in their tensor product linear independence of irreducible representation spaces is ensured.

The embedding of chapter 2 is more compact than the superspace formulation. While the Minimally Symmetric Standard Model requires a doubling of particle content in its formulation, in our embedding we found only a two dimensional subspace $P_{\mathrm{Add}} \subset \mathrm{M}(8,\mathbb{C})$ not in Standard Model representations, when including right handed neutrinos. However, the purpose of SUSY is not to provide a common space for describing particles, but rather to explain structures of interactions and representations. For example, SUSY answers the question of "why do spinors transform in the fundamental representation and vectors in the adjoint representation under gauge transformations?" by imposing the corresponding particles to these representation spaces as only the low energy excitations. In the true supersymmetry, restored at higher energies, we would then also see vector fields in the fundamental representation and spinor fields in the adjoint representation of gauge groups. This is clearly not possible within M(16,$\mathbb{C}$) by purely dimensional counting.

We saw how Lorentz representations can be described via linear maps on M(2,$\mathbb{C}$) in section 4.8. Here we chose to view the linear maps as pairs of elements of $\mathbb{C} \otimes \mathbb{H}$, which yielded a vector representation for the first factor and some additional irreducible Lorentz representation for the second factor. We showed how in this construction one could interpret the vector representation of the first factor as describing some spacetime, and the second factor as describing the Lorentz representation of fields. However, in principle we could find maps with any type of representations in the first factor. Therefore one could also view M($\mathbb{C}\otimes\mathbb{H}$) as pairs of Lorentz representations and introduce some operation which exchanges these pairs. This provides a setup for which to consider SUSY transformations. Alternatively, we could instead have viewed the linear maps on $\mathbb{C} \otimes \mathbb{H}$ as spanning the Clifford algebra $\mathbb{C}$l(4), allowing for a simultaneous realization of vector and Dirac spinor representations as linearly independent subspaces.

While the work presented here focused on describing simultaneous realizations of representation spaces that were comparable with Standard Model representation spaces, it is clearly possible to generalize our construction in such a way as to incorporate additional combinations of gauge and Lorentz representations. This presents possible avenues for studying the realization of a set of linearly independent representation spaces which respect SUSY transformations. It would be interesting to investigate what types of supersymmetric theories could be constructed in this way.

5.1.2 Comparison with GUTs

In GUTs, the Standard Model symmetry groups all arise from restricted representations of the same larger grand unified group G_{GUT}. This was not the case for the work of chapter 2, where group structures were chosen ad-hoc to reproduce Standard Model gauge transformations. However, the SU(2)×SU(3) symmetry transformations of chapter 4 did appear as restricted representations, in this case of the group G_2. The difference with the appearance of the irreducible representation spaces in chapter 4 is that in GUTs the group G_{GUT} is the true symmetry group of the theory, with the Standard Model groups appearing as the residual, i.e. the effective low energy, symmetries after spontaneous symmetry breaking has occurred. In other words, in GUTs the recovery of the Standard Model gauge groups is a dynamical process. In chapter 4 there is no notion of spontaneous symmetry breaking, as we are simply mapping between restricted representations. Further, while the SU(2) and SU(3) transformations appear as subgroups of G_2 the same is not true for our U(1) transformations, which arise as a redundancy in how to define the direct sum decomposition. Thus, in the work presented here, the relevant symmetry groups are directly related to the redundancy in defining the "particle content", i.e. the irreducible representation spaces, themselves. Of course, spontaneous symmetry breaking is not excluded by the construction of chapter 4. Rather, unlike the case for GUTs, spontaneous symmetry breaking is not required to obtain representations of distinct groups.

In GUTs the different irreducible representation spaces are just that, they are different spaces. So while addition is defined within these irreducible representation spaces, and therefore between the restricted irreducible representation spaces, it is not defined between elements of different irreducible representation spaces of G_{GUT}. This implies there is no demand for a finite set of irreducible representation spaces, and one could have arbitrarily many copies of each irreducible representation and irreducible representations of arbitrarily large dimensions. Indeed, to obtain three generations of fermions in SU(5) GUT one requires three copies of the direct sum $1 \oplus \bar{5} \oplus 10$ of irreducible SU(5) representations.[1] The requirement of multiple copies of irreducible representation spaces is consequently not present within the context of GUTs.

This differs fundamentally from the construction presented here, where the 8 and 3 representations of SU(3) both appear as subspaces of $\mathrm{M}(8,\mathbb{C}) \cong \mathrm{M}(\mathbb{C} \otimes \mathbb{O})$. As a result in order to fully span the space requires a minimum number of irreducible representation spaces. Conversely, because the space of maps is a finite dimensional vector space we also have a limit on the number of irreducible representation spaces that may appear. In other words, as the construction presented here works both with linear independence and irreducibility of representation spaces, within a finite dimensional vector space, it

[1] The singlet state is required if one wishes to include right handed neutrinos, which are singlets under all gauge groups of the Standard Model.

presents a more restrictive appearance of irreducible representation spaces than GUTs. For example, the setup of section 4.2 resulted in the unique the set of SU(2)×SU(3) irreducible representation spaces. Through the work of chapters 2 and 4, it is clear that multiplicities of representation spaces may naturally arise from simultaneous realizations of multiple gauge representations.

We comment that the implementation of the different symmetry transformations is very similar to that of SU(5) GUT. Returning to the restriction of the 10 representation of SU(5) in (1.1), it is clear that the (3,2) representation space transforms as a bi-representation of SU(2)×SU(3). This is not in general the case for GUTs. For example in SO(10) GUT the 16 representation, describing one generation of left handed particles and antiparticles, is the fundamental representation of Spin(10) and does not imply a bi-representation after spontaneous symmetry breaking. Thus the way in which transformations are described within our constructions has a close connection with SU(5) GUT.

This connection with SU(5), however, is only at the level of transformations of our representation spaces. Another emergent feature of our constructions was the possibility to encode interaction strengths into the form of the representation spaces themselves. This was possible precisely because linear independence was extended to hold between all representation spaces, not just between restricted representations as in GUTs. Based on the results of chapters 2 and 4, we find it highly suggestive that under a variation of the constructions presented here one could recover irreducible representation spaces yielding interaction hierarchies under map composition. It would be interesting to investigate such a hierarchy in relation to the mass-hierarchy between generations of fermions in the Standard Model, Ref. [73]. Such considerations are not possible in GUTs without gauging the flavour symmetry of fermions and introducing a plethora of representation spaces not seen in the Standard Model.

5.1.3 Non-commutative Geometry

Unlike GUTs and the work of chapter 4, NCG does not base its gauge structure on restricted representations of some larger group. Therefore it does not derive any symmetry structures, but rather imposes them. In essence this results in the noncommutative Standard Model as a phenomenological model found by incorporating the ideas of noncommutative geometry to Standard Model physics, Ref. [74]. This is different from the work of chapter 4, but is similar in the imposing of symmetry groups to the embedding of chapter 2.

NCG realizes bosons and fermions as objects of different spaces, trivially implying the lack of linear independence between their representation spaces. However, the fermionic states still span a Hilbert space, so linear independence exists between these fundamental representation spaces. Furthermore, this Hilbert space is the product of an infinite space with a finite space, where separation of points in this finite space describes the separation between left and right handed fermions. This separation between left and right handed spinors is what results in the Yukawa interaction matrix in the Higgs sector. In other words, the different parameters in the Yukawa interaction matrix arise as a result of how the fundamental representations, corresponding to fermions, are realized as linearly independent subspaces of the same Hilbert space. This is similar in principle to how matrix multiplication, or equivalently map composition, in our construction yields different interaction strengths precisely because of how our representation spaces are realized linearly independently within the same space of linear maps.

In practice there are of course many differences between the constructions, not the least of which is the Higgs boson in NCG as an operator on the space containing fermionic representations, and not an element of the space itself. However we find that the general idea of having interaction strengths encoded at the level of representation spaces is mirrored in both approaches. Additionally, while our work on Lorentz representations in $\mathrm{M}(\mathbb{C}\otimes\mathbb{H})$ is not yet developed to the same extent as our focus on gauge representations, it is interesting that the maps uniquely representing scalar fields in section 4.8 are maps between left and right chiral fermions. This compares to how the Higgs boson in NCG arrises as a finite difference between left and right handed fields. Thus there is a similar view in the two approaches of Higgs representations as describing differences or connections between left and right handed spinor representations.

Another interesting point for comparison is the appearance of particle and antiparticle representations. In the work presented here, both in chapters 2 and 4, conjugate representations were realized as independent subspaces in our space of fundamental representations. Similarly, in NCG particles and antiparticles appear as different basis elements required to span the Hilbert space of one generation of fermionic states, Ref. [41]. There the particle and antiparticle representations are related by an antilinear isomorphism. In chapter 4 we found that the F-spaces came in pairs related by complex conjugation, i.e. an antilinear isomorphism, and that the complexified Lie algebras were invariant under this operation. Resultantly, the adjoint and F-space representations we have the same feature as NCG, with an equivalence between particle and anti-particle representation spaces. The same is not true when we include the H-spaces, whose complex conjugate representation is not present in the direct sum decomposition (4.1) of $\mathrm{M}(\mathbb{C}\otimes\mathbb{O})$. This presents a contrast between NCG and our work as the inclusion of an antilinear isomorphism was not part of the setup of section 4.2.

We note that the use of this antilinear operator is essential to recovering unitary representations in NCG. Indeed, the operators which are identified with the the bosonic sector in NCG are described by complex valued functions "A". However, the inner fluctuations of the metric D is defined as

$$D \to D + A + JAJ^{-1} \tag{5.1}$$

where J is the antilinear isomorphism and defines a reality structure on the Hilbert space of fermions, Ref. [75]. In NCG it is not the complex valued functions A, but the real functions $A+JAJ^{-1}$ which are present. This process yields the real Lie algebras for the adjoint representations, as opposed to the complexified Lie algebras. Within our construction we only recovered the complexified Lie algebras. To develop this work into a theory with representations similar to those of the Standard Model fields, it may therefore be necessary to introduce a reality structure on the space $\mathrm{M}(8,\mathbb{C})$, or $\mathrm{M}(16,\mathbb{C})$. We do note that J

acts as an involution for the Hilbert space of single particle fermion states in NCG. However, we reiterate that the H-spaces in our construction are not invariant under complex conjugation, unlike our complexified Lie algebras. Thus one must take care if intending to implement reality structure this at the level of construction principles as discussed in section 4.2.

While these are some interesting points of comparison, we note that there are many features of NCG which are not mirrored by our construction. This is in part because the theory of NCG in particle physics is well established within the theoretical community, and has been developed in far greater detail than the work presented here. For example, NCG has a process by which to construct a theory by considering the spectrum of the bosonic fluctuations, which includes spacetime effects. Since our construction does not treat the adjoint representations as arising from operators acting on a Hilbert space of fermionic states, this construction is not applicable to our work. As we have not induced simultaneously Lorentz and gauge representations, constructing a theory is well beyond reach.

5.2 Outlook

5.2.1 Alternative Approaches to Inducing Representation Spaces

In principle there are many different ways one could choose to approach the induction of representation spaces in a matrix algebra. In chapter 4 we presented a relatively simple example, motivated by a set of observations described in chapter 3. The construction principles of section 4.2 are a specific choice of conditions which were imposed in order to allow for a connection to be made between subspaces of $\mathrm{M}(\mathbb{C} \otimes \mathbb{O})$ and Standard Model representations. These can be grouped as

- Selection rules for identifying subspaces: I.e. Principles 1 and 2
- Rules ensuring representations for gauge theory construction: I.e. Principles 3 and 4

However, we could have chosen other principles to describe other variants of these same conditions, or changed the conditions themselves. This would presumably change the resultant direct sum decomposition and its interpretation.

To exemplify this, note that in chapter 4 we decomposed the base space into irreducible representation spaces of the non-abelian symmetries, and imposed these symmetries as redundancies via principle 3. However, we also identified these symmetry groups from the redundancies in the direct sum decomposition. This implies we most likely have a redundancy in the set of conditions imposed on the direct sum decomposition. Indeed, it may be that one needs not to pre-define any symmetry groups at all. This would imply decomposing the base space V via some other conditions, and then deriving the symmetry groups as those which preserve certain redundancies in the direct sum decomposition of $\mathrm{M}(V)$. For example, one could induce the U(N) symmetry groups as the requirement that the redundancies of the decomposition preserve some norm, i.e. like the Frobenius norm. This adds additional structure to the algebra $\mathrm{M}(V)$, as it would also be a normed vector space. In this case different norms could lead to different symmetry transformations and thus different direct sum decompositions.

The abundant freedom in choosing how to impose conditions on the decomposition of the space allows for a lot of different approaches. This is beneficial from the perspective of studying all the different implications of imposing linear independence between irreducible representation spaces. However, the side effect is that it becomes harder to be exhaustive in an investigation, or even to compare approaches that use different principles of construction. This presents a challenge towards a deeper understanding of how linear independence relates different representation spaces. We have not yet found a satisfying solution to this shortcoming.[2] However, underlying similarities of constructions and direct sum decompositions may become apparent with more examples, and this presents a promising strategy for finding a general solution. Therefore we leave this task of comparing different constructions to future work.

5.2.2 Multiple Decompositions

Above we talked in general about the potential for many different approaches in the construction of a direct sum decomposition of some space $\mathrm{M}(V)$. Here we wish to highlight a particular limiting condition in section 4.2 that we included only for simplicity of construction. This was namely that we only had maps on $\mathbb{C} \otimes \mathbb{O}_{\mathbb{C},\mathbb{O}}$ and maps from $\mathbb{C} \otimes \mathbb{O}_{\mathbb{C}}$ to $\mathbb{C} \otimes \mathbb{O}_{\mathbb{H}}$. There was no fundamental reason to consider only two decompositions, or to consider only maps from one decomposition to the other and not visa-versa.

Visually, one could represent our work in this section via figure 5.1. This construction was sufficient to demonstrate the appearance of relationships between the different irreducible representation spaces as a result of linear independence, while being as simplistic as possible. However, if trying to use this construction to obtain relationships which mimic complex Standard Model features, such as chiral discrimination of gauge groups, more decompositions are needed. This is clear since if each decomposition is associated with one non-abelian gauge group, we need maps to different decompositions for left and right handed fermions. This requires minimally three different decompositions, as in figure 5.2, where for example we have a third decomposition of the base space with some additional symmetry group G', and corresponding Lie algebra g'. This construction with three decompositions is not in principle different from Pati-Salam type constructions. For example, in the case $G' = SU(2)$ we would have the same type of group structure as Pati-Salam type theories after $SU(4) \rightarrow U(1) \times SU(3)$. It is also interesting that here a "Higgs-like" representations could indeed map between the gauge representations associated to left and right handed fermions. This would draw a closer connection between the realization of scalar representations in section 4.8 and the gauge representations of subsection 4.5.3.

[2] We note the obvious solution, which is to only compare decompositions based on similar construction principles. However, such a discrimination could also inhibit the research objective of studying general features of simultaneous realizations of multiple representation spaces.

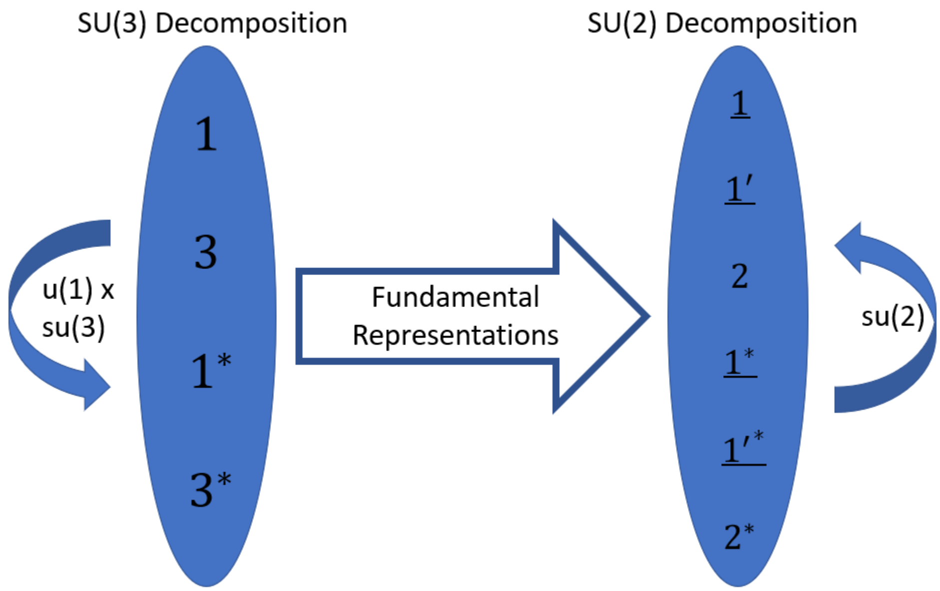

Figure 5.1: Pictographic illustration of maps in chapter 4

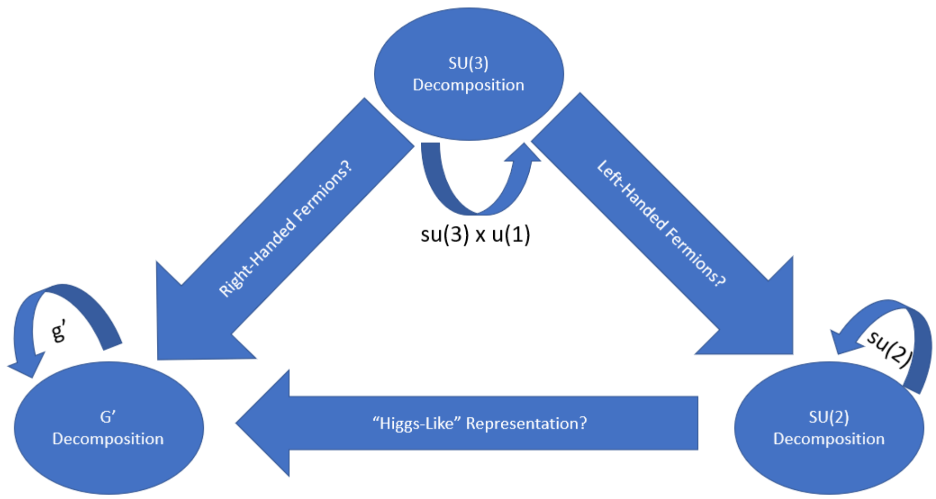

Figure 5.2: Pictographic illustration of maps between three decompositions

Figure 5.2 is of course only suggestive of what may be achievable within the simultaneous realization of multiple representation spaces. Even so, it is clear that there is a large range of applicability of the work presented here. Therefore, it is our belief that generalizations and future developments of this work may yield useful tools for deepening our understanding of how to formulate unification approaches in particle theory.

Chapter 6

Summary and Conclusions

In this thesis we have provided early results in the study of simultaneous realizations of irreducible representation spaces. On the simplest level these results demonstrate explicitly the possibility of spanning a matrix algebra with a set of irreducible representation spaces. This was the results of our complete direct sum decomposition (4.1) of M(8,$\mathbb{C}$) into induced representation spaces. However, there are many additional interesting features on this set of irreducible representation spaces. These originate from the requirement of linear independence. Some of these features were identified in the Standard Model embedding of chapter 2, while others required a less ad-hoc approach and emerged in chapter 4. For brevity and to provide emphasis, we will focus on three main features discussed throughout this text.

The first of feature we will discuss is the compact formulation of Lorentz and gauge representation spaces seen in chapter 2. Specifically, in this section all Lorentz representations were realized in the space, M(2,$\mathbb{C}$), and all gauge representations were realized in the space, M(8,$\mathbb{C}$). A key component to the compact formulation was the lack of linear independence between the different Lorentz representations. Indeed, excluding the adjoint representation associated to gauging the Lorentz group, the only two Lorentz representations which were linearly independent were the left and right handed spinors. As such we found that when realising these Lorentz and gauge representation spaces simultaneously in M(16,$\mathbb{C}$), linear independence of different representation spaces was ensured by the linear independence of their components in M(8,$\mathbb{C}$). This implied that within our construction we could not have both vector and spinor transformations for the same gauge representation space. While this relationship is partly a consequence of our choice of matrix algebras in chapter 2, it is also explicitly linked to the demand of linear independence. This distinct spinor or vector transformation associated with gauge representation spaces is also a central feature of the Standard Model, and we find its emergence within our explicit embedding of representation spaces particularly interesting. As this feature concerned the simultaneous realization of Lorentz and gauge representations we discussed it particularly in the context of SUSY in subsection 5.1.1.

The second feature we will emphasize is the appearance of representation spaces without conjugates. This is the case for both approaches discussed in chapters 2 and 4. In chapter 2 this subspace corresponded to the Higgs doublet representation. The form of the Higgs doublet representation was there included ad-hoc, in order to produce a matching with the representation spaces of the Standard Model. Even so, we found that linear independence demanded the absence of an independent conjugate Higgs doublet representation in the direct sum decomposition (2.3). This feature again appeared when looking at the induced irreducible representation spaces derived in subsection 4.4.3. Here we found two irreducible representation spaces, i.e. our H-spaces, which did not possess linearly independent conjugate representations. The reappearance of this feature in our induced representation spaces is a strong indicator that this could be a general feature of many matrix algebra decompositions. In subsection 4.5.3 we show how these spaces has properties naturally identifiable with the properties of the Standard Model Higgs representation. In particular, we found in both approaches that these "Higgs-like" representations also had the potential of encoding interaction strengths, when viewing interactions from the perspective of map compositions in M(8,$\mathbb{C}$). In chapter 2 this was manifest in the ability to encode different interaction strengths between generations in the choice of parameters in $V_{\phi}^{\pm}$. On the other hand, in chapter 4 we found that the relative interaction strengths between the H-spaces and conjugate irreducible representations in our F-spaces to be described by the parameters α and β. The Standard Model Higgs field has relative interaction strengths both between generations and between conjugate representations, yielding distinct masses for up- and down-type quarks. Therefore the natural appearance of these "Higgs-like" representations in our direct sum decompositions strongly motivates further studies of simultaneous realizations of multiple representation spaces.

The final feature we wish to discuss is the appearance of three distinct types of representation spaces in chapter 4. The three types of representation spaces we found were those which were self-conjugate, i.e. the adjoint representations; those which came in conjugate pairs, i.e. the representations in the F-spaces; and those which neither were self-conjugate nor appeared in conjugate pairs, i.e. our H-spaces discussed above. This feature of three types of representation spaces is reliant on the specific setup of section 4.2. This follows from subspaces of our Matrix algebras only having transformations described by the simultaneous action of at most two other matrices, i.e. matrix multiplication from the left and right. This is the case whether these two matrices correspond to elements of the same group or elements of different groups. So this demand, of Lie algebras as the only endofunctions which return the decomposition upon they act, removes the potential for any other representation described by the simultaneous action of two elements from the same group, like the 6 representation of SU(3). Ergo, we may only transform the rest of our subspaces via matrix multiplication of at most one element of each group, yielding singlet and fundamental representations. We have shown how with only one simple condition, namely endofunctions which preserve decompositions are described by Lie algebras, we automatically recover only the types of representation spaces seen in the Standard Model. This demonstrates that simultaneous realization of representation spaces within a matrix algebra can yield strong restrictions from simple conditions. As the simple condition of Lie algebra endofunctions can result in a parallel with the types of Standard Model transformations, it would be interesting to investigate the other combinations of conditions can result in Standard Model features within these direct sum decompositions of matrix algebras.

While main focus of this paper was on the simultaneous realization of representation spaces of the groups SU(2) and SU(3), in section 4.8 we demonstrated how Lorentz representations could also be encoded as endofunctions between the irreducible representation spaces of the Lorentz group in M(2,$\mathbb{C}$). There we saw how certain sets of maps could yielded the necessary components to represent spacetime fields. There are also many elements of the space M($\mathbb{C} \otimes \mathbb{H}$) which would not describe the structure of spacetime fields. Thus an interesting avenue of research would be to find conditions or constructions which would yield only the maps (4.31)-(4.36). There are several different ways of viewing the endofunctions on M($\mathbb{C} \otimes \mathbb{H}$), from spacetime fields to pairs of representations. For example, as mentioned in subsection 5.1.1, we could consider the $\mathbb{C} \otimes C\text{l}(1,3)$ representation of the endofunctions and look for representations compatible with SUSY transformations. One could also work with other base spaces than $\mathbb{C} \otimes \mathbb{H}$ from which to induce Lorentz representations. Another interesting application of inducing Lorentz representations would be to formulate action construction mechanisms based on these sets of representation spaces. This would open up avenues for further comparisons with NCG and its spectral action principle. Finally, and as emphasized multiple times throughout this text, we comment on inducing combined Lorentz and gauge representation spaces. Such combinations could potentially be used to identify and study constructions that describe chiral discrimination of gauge representations. These structures are clearly of interesting for unification approaches in particle theory, as the Standard Model contains two chirally discriminating gauge groups: $\text{SU}(2)_L$ and $\text{U}(1)_Y$.

It is clear that many interesting results have been found in this exploration of the simultaneous realizations of irreducible representation spaces. Even so, the works presented here encompass only a small part of the available research directions associated with incorporating linear independence between irreducible representation spaces. Indeed, we focused only on maps between vector spaces, in the context of viewing algebras as vector spaces with multiplicative structure. However, one can also study maps on: inner product spaces, Hilbert spaces, metric spaces, etc. Of course, it may turn out that not all of these considerations are relevant to particle theory. Still, based on the findings of this thesis, we speculate that a general understanding of how to incorporate linear independence between irreducible representation spaces may yield insight into structures capable of describing the different representation spaces seen in the Standard Model, and consequently help advance the development of unification approaches in particle theory.

Appendix A

The Complexified Quaternions and M(2,$\mathbb{C}$)

The Quaternions can be defined in many equivalent ways, but here we chose the most direct approach by defining the multiplication rules for elements within the algebra. The Quaternions have three square roots of -1 which satisfy

$$[\varepsilon_j, \varepsilon_k] = \sum_l \epsilon_{jkl}\varepsilon_l \quad \text{and} \quad \varepsilon_j^2 = -1 \tag{A.1}$$

where ε_{jkl} is the completely antisymmetric tensor. The relations (A.1) is identical to the relations for the Pauli matrices $\{\sigma_i\}$ if we identify ε_j with $i\sigma_j$. As the sigma matrices span the hermitian traceless part of M(2,$\mathbb{C}$), under complexification and by including unity it is apparent that $\mathbb{C} \otimes \mathbb{H} \cong \mathrm{M}(2, \mathbb{C})$, with multiplication in $\mathbb{C} \otimes \mathbb{H}$ being identified directly with matrix multiplication in M(2,$\mathbb{C}$). Since the Pauli matrices are hermitian, this means that under hermitian conjugation $(i\sigma_j)^\dagger = -i\sigma_j$ becomes identified with $-\varepsilon_j$. Thus we have that operation of hermitian conjugation in M(2,$\mathbb{C}$) translates simply to $\mathbb{C} \otimes \mathbb{H}$, with $\varepsilon_j^\dagger = -\varepsilon_j$.

Defining complex conjugation in $\mathbb{C} \otimes \mathbb{H}$, we have $\varepsilon_j^* = \varepsilon_j$. Ergo, complex conjugation in $\mathbb{C} \otimes \mathbb{H}$ does not correspond to complex conjugation in M(2,$\mathbb{C}$). Instead we have that complex conjugation in the complexified Quaternions corresponds to some operation $\bar{*}$ in M(2,$\mathbb{C}$) with

$$A^{\bar{*}} := (i\sigma_2)\, A^*\, (i\sigma_2)^\dagger\,, \text{ for } A \in \mathrm{M}(2, \mathbb{C}), \tag{A.2}$$

where * is the standard operation of complex conjugation. Note that this strange form of complex conjugation is rather similar to the form we will be employing in M(8,$\mathbb{C}$), see equation (B.10). This similarity is actually a consequence of the fact that $\mathbb{C} \otimes \mathbb{H}$ is isomorphic to $\mathbb{C}$l(2), the four dimensional complex Clifford algebra generated by $\mathbb{C}^2$. This means we could obtain M(2,$\mathbb{C}$) equivalently from $\mathbb{C}$l(2), just like we obtain M(8,$\mathbb{C}$) from $\mathbb{C}l(6)$ by the use of a nilpotent generating space, see Appendix B.1. We note that the combination of complex and hermitian conjugation results in the anti-automorphism of Quaternionic conjugation $\widetilde{\varepsilon}_j \equiv (\varepsilon_j)^{*\dagger} = -\varepsilon_j$. The Lie algebra for the Lorentz transformations is given by

$$\mathfrak{sl}(2,\mathbb{C}) = \mathrm{Span}_{\mathbb{C}}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \tag{A.3}$$

from which it is evident that $\widetilde{\Lambda} = \Lambda^{-1}$.

For a detailed discussion of the different irreducible Lorentz representation spaces of $\mathbb{C}\otimes\mathbb{H}$ we refer the reader to Ref. [56]. With the above results, the Lorentz representations of $\mathbb{C}\otimes\mathbb{H}$ discussed in chapter 2 and section 4.8 follow directly.

Appendix B

The Matrix Algebra M(8,$\mathbb{C}$)

B.1 The Clifford Algebra $\mathbb{C}l(6)$

We briefly review basic properties of the 6-dimensional Clifford algebra. For the standard basis of the Clifford algebra we could write the generating space as spanned by a set of basis elements $\{\bar{e}_i\}_{i=1}^6$ which satisfy $\{\bar{e}_i, \bar{e}_j\} = -2\delta_{ij}$. While this is the most common way to introduce the generating space for a Clifford Algebra, we will here consider instead two sets of nilpotent basis elements. Then the generating space of $\mathbb{C}$l(6) can be spanned by the two sets of vectors $\{\alpha_i\}_{i=1}^3$ and $\{\alpha_i^\dagger\}_{i=1}^3$, where

$$\alpha_1 := \frac{1}{2}\left(-\bar{e}_5 + i\bar{e}_4\right) \qquad \alpha_2 := \frac{1}{2}\left(-\bar{e}_3 + i\bar{e}_1\right) \qquad \alpha_3 := \frac{1}{2}\left(-\bar{e}_6 + i\bar{e}_2\right), \tag{B.1}$$

and hermitian conjugation † is an anti-automorphism, implying $i^* = -i$ and $\alpha^* = -\alpha^\dagger$. These generating vectors satisfy the anti-commutation relations,

$$\{\alpha_i, \alpha_j^\dagger\} = \delta_{ij}\,, \qquad \{\alpha_i, \alpha_j\} = 0\,, \qquad \{\alpha_i^\dagger, \alpha_j^\dagger\} = 0\,. \tag{B.2}$$

Note that this is identical to the commutation relations of creation and annihilation operators as discussed in Ref. [56].

We will here not work with the elements of the generating space, but rather with the full set of basis elements which span the space $\mathbb{C}l(6)$. To this end we define $\omega := \alpha_1\alpha_2\alpha_3$ and the projectors

$$P_0 := \omega^\dagger\omega\,, \qquad P_i := \alpha_i\omega^\dagger\omega\alpha_i^\dagger\,. \tag{B.3}$$

Let moreover $\bar{P}_a := P_a^*$ for all $a \in \{0,1,2,3\}$. It follows from (B.2) that ω is annihilated by right or left action of any α_i. The projector P_0 was used in [56] to find the Standard Model structure associated to one generation of fermions, but did not employ the use of the projectors P_i introduced here.

The above 8 projectors are linearly independent and split our space into 8 complex linearly independent subspaces. Specifically, each one of these projectors will define a left ideal. The space $\overleftarrow{\mathbb{C}\otimes\mathbb{O}}\,P_b$ can then be spanned by 8 linearly independent basis vectors,

$$B_{ab} := \alpha_a\omega^\dagger\omega\alpha_b^\dagger; \qquad A_{ab} := \alpha_a^\dagger\omega\alpha_b^\dagger \qquad a\in\{0,1,2,3\}\,. \tag{B.4}$$

The basis vectors which span $\overleftarrow{\mathbb{C}\otimes\mathbb{O}}P_b^*$ are found by taking the complex conjugate of (B.4), which we will denote by

$$\bar{B}_{ab} := \alpha_a^\dagger\omega\omega^\dagger\alpha_b \qquad \bar{A}_{ab} := \alpha_a\omega^\dagger\alpha_b \qquad a\in\{0,1,2,3\}\,. \tag{B.5}$$

This provides a compact way of writing all basis elements of $\mathbb{C}l(6)$ in terms of $B_{ab}, \bar{B}_{ab}, A_{ab}, \bar{A}_{ab}$.

B.2 $\mathbb{C}$l(6) ≅ M(8,$\mathbb{C}$)

We will now demonstrate that $\mathbb{C}$l(6) is isomorphic to the algebra of 8×8 complex matrices $\mathrm{M}(8,\mathbb{C})$. Clearly the vector spaces over which the two algebras are defined are isomorphic by virtue of having the same dimension. We thus only need to show that the Clifford product in $\mathbb{C}$l(6) is identified with the matrix product in $\mathrm{M}(8,\mathbb{C})$.

Let $\{M_{IJ}\}$ be a basis of $\mathrm{M}(8,\mathbb{C})$. A basis element M_{IJ} is a matrix with one entry equal to 1 (in the Ith row and Jth column), while all other entries are zero. A general matrix F in $\mathrm{M}(8,\mathbb{C})$ can then be written as $F=\sum_{I,J}F^{IJ}M_{IJ}$. The matrix product of two matrices $F,H\in\mathrm{M}(8,\mathbb{C})$ expressed in this basis reads,

$$FH=\sum_{I,L}\left(\sum_J F^{IJ}H^{JL}\right)M_{IL}\,. \tag{B.6}$$

Next we identify the basis M_{IJ} with the basis elements of $\mathbb{C}$l(6) via

$$M_{IJ}\longleftrightarrow\begin{cases} B_{(I-1)(J-1)} & \text{for } I,J\in\{1,2,3,4\}\\ A_{(I-5)(J-1)} & \text{for } I\in\{5,6,7,8\},\ J\in\{1,2,3,4\}\\ \bar{A}_{(I-1)(J-5)} & \text{for } I\in\{1,2,3,4\},\ J\in\{5,6,7,8\}\\ \bar{B}_{(I-5)(J-5)} & \text{for } I,J\in\{5,6,7,8\}\end{cases} \tag{B.7}$$

Under this identification, we can evaluate the Clifford algebra product of two basis elements and obtain,

$$M_{IJ}M_{KL}=\delta_{JK}M_{IL}\,. \tag{B.8}$$

This reproduces precisely the standard matrix product of M(8,$\mathbb{C}$) in (B.6).

Hermitian conjugation of elements of $\mathbb{C}$l(6) correspond to the usual hermitian conjugation of matrices in $\mathrm{M}(8,\mathbb{C})$, so we will not distinguish between hermitian conjugation in the two algebras and label both the operations by †. Specifically, for any $M\in$ M(8,$\mathbb{C}$) we have that $M^\dagger := (M^*)^{\mathrm{T}}$ where $^{\mathrm{T}}$ is the matrix transpose. The situation is different for the operation of complex conjugation. Due to the Clifford algebra property $\alpha^* = -\alpha^\dagger$, complex conjugation acts on the $\mathbb{C}l(6)$ basis elements (B.4) and (B.5) as,

$$(B_{ab})^* = \sum_{c,d} \eta_{ac} \bar{B}_{cd} \eta_{db} \,, \qquad (A_{ab})^* = \sum_{c,d} \eta_{ac} \bar{A}_{cd} \eta_{db} \,, \tag{B.9}$$

where η is a diagonal matrix with entries $\{1, -1, -1, -1\}$.[1]

From (B.7) and (B.9), it is clear that, in the matrix representation, complex conjugation necessarily affects the index structure. Namely, matrices $M \in \mathrm{M}(8, \mathbb{C})$ satisfy,

$$M^{\bar{*}} := \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} M^* \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} , \tag{B.10}$$

where, in order to distinguish complex conjugation in the two algebras, we have introduced the symbol $\bar{*}$ to denote $\mathbb{C}l(6)$ complex conjugation in the matrix representations. We continue to use * to denote the conjugation of complex numbers.

As our matrix space can be written as the outer product of two vector spaces $\mathbb{C}^8$, this implies that, for any $V \in \mathbb{C}^8$, we have that,

$$V^{\bar{*}} \equiv \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} V^*. \tag{B.11}$$

In assigning basis elements of $\mathbb{C}l(6)$ to Standard Model particle types in chapter 2 we use the matrix representation $\mathrm{M}(8, \mathbb{C})$. This simplifies the analysis of linear independence and makes the paper more accessible to readers less familiar with Clifford algebras. However, we stress that we use properties, like the complex conjugation $\bar{*}$, associated to the complex Clifford algebra $\mathbb{C}l(6)$. We will later see how such a representation of complex conjugation naturally arises when considering maps on $\mathbb{C} \otimes \mathbb{O}$.

[1] Even though η is the same as the Minkowski metric in Cartesian coordinates, this is just an artefact of how we chose to represent our basis elements, and not related to Lorentz transformations.

Appendix C

SU(2) and SU(3) Decompositions of $\mathbb{C} \otimes \mathbb{O}$

We here prove statements given in chapter 4 about the irreducible representation spaces of $\mathbb{C} \otimes \mathbb{O}$ under SU(2) and SU(3). Firstly, we note that any SU(3)$\subset G_2$ appears uniquely as the subgroup of G_2 that preserves some unit imaginary element of $\mathbb{O}$. This is naturally in addition to the identity element which is invariant under G_2. As unit imaginary elements square to -1, SU(3)$\subset G_2$ is the group which preserves some unique complex structure on $\mathbb{O}$. We can extend this idea to preserve two unit imaginary elements of $\mathbb{O}$. However, leaving any two unit imaginary elements $a, b \in \mathbb{O}$ invariant implies that their product $c := ab$ must also be left invariant, where $c \in \mathbb{O}$ must also be an imaginary unit element. Therefore, the next smallest subgroup of G_2 will be such that it element-wise leaves a Quaternionic subalgebra Span$\{1, a, b, c\} \subset \mathbb{O}$ invariant. This group is precisely SU(2)$\subset G_2$.

Note that if one wishes to extend this construction further one immediately encounters that the only subgroup of G_2 which keeps four unit imaginary Octonionic elements invariant is the trivial group $\{\mathrm{I}\} \subset G_2$, where I is the identity element of G_2. This is because given a Quaternionic triplet $\{a, b, c\}$ and some other unit imaginary d, preserving all these elements also implies preserving their products. Or in other words, preserving all elements $\{a, b, c, d, (ad), (bd), (cd)\}$, where ad denotes the Octonionic product of the elements a and d. However, this set spans all unit imaginary elements of $\mathbb{O}$, and thus we are leaving all of $\mathbb{O}$ invariant. This shows the only unique non-trivial subalgebras of G_2 related to element-wise invariant subspaces of $\mathbb{O}$ are SU(3) and SU(2).[1]

Now, the three distinct $\mathbb{C}$-subalgebras of a Quaternionic subalgebra correspond precisely to the fact that each SU(3) contains three distinct SU(2) subgroups. Therefore, to ensure that SU(2) is not a subgroup of SU(3), we must ensure that the Quaternionic subspace preserved by SU(2) does not contain the complex subspace preserved by SU(3). Let the complex subspace preserved by SU(3) be spanned by the basis elements $\{1, d\}$ and the Quaternionic subspace preserved by SU(2) be spanned by basis elements $\{1, a, b, c\}$, such

[1] Note that we could also define U(1) transformations which leave elements of $\mathbb{O}$ invariant, however these U(1) groups would exist as subgroups of either SU(3) or SU(2).

that $d \notin \mathrm{Span}\{1, a, b, c\}$. It becomes clear that the triplet, 3, and conjugate-triplet, 3^*, subspaces of $\mathbb{C} \otimes \mathbb{O}$ are in distinct eigenspaces of left action of d. In particular,

$$3 \quad = \mathrm{Span}\{a - i(da), b - i(db), c - i(dc)\} \tag{C.1}$$

$$3^* \quad = \mathrm{Span}\{a + i(da), b + i(db), c + i(dc)\} \tag{C.2}$$

Similarly the 2 and 2^* subspaces of $\mathbb{C} \otimes \mathbb{O}$ together span

$$2 \oplus 2^* = \mathrm{Span}\{d, (da), (db), (dc)\} \tag{C.3}$$

The subspaces 2 and 2^* are then defined as the conjugate pair of eigen-subspaces of (C.3) under left action of some element in the preserved Quaternionic subalgebra.[2] It is easy to verify that we have $3 \cap 2 = 3 \cap 2^* = \{0\}$. Further, $1_2 \equiv \mathrm{Span}\{1, a, b, c\}$, which is clearly linearly independent from either 3 or 3^*, but not from $3 \oplus 3^*$.

We express the eigenspaces of our SU(2) transformations in a similar way to our SU(3) eigenspaces. Without loss of generality let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \equiv \{a, b, c\}$ be such that $\{\varepsilon_i\}$ satisfies the commutation relation of unit Quaternionic basis elements. Then for some arbitrary ε_j we may define 2 and 2^* as

$$2 \quad := \mathrm{Span}\{d + i(d\varepsilon_j), (d\varepsilon_{j+1}) + i(d\varepsilon_{j+2})\} \tag{C.4}$$

$$2^* \quad := \mathrm{Span}\{d - i(d\varepsilon_j), (d\varepsilon_{j+1}) - i(d\varepsilon_{j+2})\} \tag{C.5}$$

where the indices on ε_j are defined up to mod 3, i.e. $\varepsilon_4 \equiv \varepsilon_1$.

[2] Clearly there is more freedom in how to pick 2 and 2^* than we had freedom in picking our 3 and 3^*. However, picking some arbitrary 2 and 2^* pair only corresponds to a basis choice in the SU(2) decomposition of $\mathbb{C} \otimes \mathbb{O}$.

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