Does the coexistence of online and offline firms improve welfare?

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Xiwei Zhu* and Toshitaka Gokan+

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Abstract

This study shows how the coexistence of online and offline firms affects consumer welfare. By introducing two dimensions of heterogeneity in productivity and quality, we find that the consumers' indirect utility under the coexistence of online and offline firms is higher than that of only offline firms. Specifically, we show that: (1) if the initial investment of online firms is small enough or if the initial investment of offline firms is large enough, or (2) if the fixed costs of offline firms are sufficiently large under the general distribution of productivity and quality. Additionally, we find that the cutoff productivity level of domestic online firms increases due to the cost-saving of the fixed costs among online exporting firms, leading to the higher indirect utility compared to the indirect utility without cost-saving.

Keywords: Heterogeneous firms; Online; Offline

JEL classification: D04, R12

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1 Introduction

We observe that the coexistence of online sales and offline sales is normal and common in many countries. The purpose of this study is to contribute to a better understanding of this phenomenon by exploring the main heterogeneities on online and offline markets and the cost-saving stemming from the property of the online market.

Melitz (2003) explained that productivity differences may reflect cost differences as well as differences in consumer valuations of the good. However, the distinction between the two differences is not clear in Melitz (2003) due to the single channel of heterogeneity. Bekkers (2016), Hallak and Sivadasan (2013), and Johnson (2012) explicitly addressed these differences. However, the distinction between online firms and offline firms has not yet been explored. Bekkers (2016) as well as Hallak and Sivadasan (2013) assume that firms' production costs depend on product quality. Following Johnson (2012), we assume that there is no relationship between production cost and product quality. However, we employ a general distribution of two heterogeneities rather than a special distribution. By introducing these two dimensions of heterogeneity explicitly, this study allows us to investigate how the combination of imperfect information and the cost-saving of exporting online firms affects the market equilibrium and its welfare.

Electronic commerce contains many aspects, as explained by Borenstein and Saloner (2001). Online sales are characterized by easy access to numerous types of information, asynchronized communication, and tailored information. As a result, better matching between consumers and sellers can be achieved, and the costs related to product handling, theft, rents and selling costs are saved. Furthermore, geographically dispersed offline stores incur inventory costs, whereas online firms may enjoy the economies of centralized inventories. The uncertainty or imperfect information can be considered the primary shortcoming of online shopping, because some information of an item may not be transmitted smoothly via the Internet. Furthermore, consumers may have no patience to wait for the delivery of an item bought from an online market.

Online sellers outsource many tasks to the selling platform (such as Amazon.com) in order to avoid the activities for which make it difficult for a single seller to achieve economies of scale. Thus, online sellers can enjoy more outputs and higher labor productivity as demonstrated in a study on outsourcing IT (Han, Kauffman, and Nault, 2011). The development and operation costs may also decrease if a large number of sellers gather on the platform, as development costs are shared equally among platform owners, as discussed by Nocke, Peitz, and Sthal (2007). Furthermore, Hounde, Newberry, and Seim

(2017) has clarified that the economies of density works in Amazon's delivery network.

We consider firms with heterogeneous product quality to express imperfect information only in the online market. To understand the impact of these two channels, we avoid the search and matching, and the waiting costs associated with online purchases among the characteristics of online sales, although Goldmanis, Hortaçsu, Syverson, and Emre (2009) and Williams (2018) analyze the former, and Loginova (2009) focuses on the latter. Furthermore, we consider that online exporting firms require the same fixed costs even if the number of regions increases while offline firms need to incur the same fixed costs when entering each region.

The assumption on the imperfect information of the online market in this study is similar to that of Chen, Hu, and Li (2017). Firms of heterogeneous quality choose an online or offline market, and then the quality of the products is disclosed in the offline market while remaining hidden in the online market. Furthermore, the higher fixed costs of the offline market correspond to the cost of disclosing information. The analytical framework of Chen, Hu, and Li (2017) concerns vertical product differentiation under oligopolies following the literature on industrial organization. In contrast, our analytical framework is based on the Dixit-Stiglitz model of monopolistic competition, which is popular in trade, economic growth and economic geography. Furthermore, Chen, Hu, and Li (2017) do not consider two channels.

The importance of sensory examination differs among products. Using the results of a consumer survey on clothes, books and digital cameras in online and offline markets, Gruber (2009) shows that offline (resp. online) channels of clothes (resp. digital cameras) generally reveal more price dispersion, while books take up a moderate position. Higher price dispersion can be regarded as an indicator of differentiation with quality or services. A case that relies heavily on sensory examination, for example, would be an art auction. Kazumori and McMillan (2005) show that higher-value items are more likely to be sold live than an online auction. Furthermore, they show that a lower valuation uncertainty leads sellers to choose online auctions both theoretically and empirically. Low-value uncertainty can be interpreted as low-quality products. Thus, our model illustrates the market for a product in which there is a huge gap in information between online and offline markets, such as for clothes.

Our main findings are as follows. The indirect utility with coexisting online and offline firms is higher than that of only offline firms if the initial investment of online firms is small enough or if the initial investment of offline firms is large enough under the general distribution of productivity and quality. As the consumption share of the varieties produced by offline firms increases, the threshold values of initial investment by online or offline firms become larger or smaller depending on the size of the consumption share, as well as the elasticities of substitution among online firms. Furthermore, the large fixed costs of offline firms provide that the indirect utility with coexisting online and offline firms is larger than the indirect utility in the offline case only. Additionally, we find that the cut-off productivity of firms selling for the home market only becomes higher due to the cost saving of the fixed costs among online firms, which leads to the higher indirect utility compared with the indirect utility without the cost-saving of exporting online firms. Furthermore, we find that the measures to improve the lowest quality of goods in the online market, such as comments by customers, improve indirect utility.

Cost-saving in this study is related to cost-sharing, as in Krautheim (2012) which introduces cost-sharing in the Dixit-Stiglitz monopolistic competition model for heterogeneous firms, accounting for the fixed costs of exporting, which, in turn, decreases with the number of exporters. To determine the number of exporters, the study assumes that the total number of firms in an industry is fixed, and under these conditions, firms' entry and exit in an industry are not affected. On the other hand, we endogenize firms' entry and exit.

The remainder of this paper is organized as follows. Section 2 explains the one-region model. Section 3, 4, and 5 analyze the economy of sole offline firms, sole online firms, and coexistence of offline and online firms. Section 6 compares the welfare under an economy with only offline firms to those with both online and offline firms. Section 7 analyzes the multiple-region model. Section 8 provides concluding remarks.

2 The model

2.1 Basic setup

A country comprises a continuum of firms producing horizontally differentiated products under the Dixit-Stiglitz monopolistic competition. We denote the population in the economy at the aggregate level as L. Each individual inelastically supplies one unit of labor, which is the only production factor. Without loss of generality, we take labor as the numéraire, which implies a unit wage $w = 1$. Thus, the individual income, y, and regional income, Y, are, respectively, given by $y = 1$ and $Y = yL = L$.

2.1.1 Demand

We consider an economy with online (type N) and offline (type F) firms. All consumers share the same homothetic preference and the utility function given by:

$$U \equiv \frac{1}{\mu^{\mu}(1-\mu)^{(1-\mu)}} C_F^{\mu} C_N^{1-\mu}, \tag{1}$$

where C_F is the consumption of the composite manufactured good produced by offline firms, C_N is the consumption of the composite manufactured good produced by online firms, μ is the share of C_F, and $1-\mu$ is the share of C_N. We define C_o as follows:

$$C_o \equiv \left\{ \int_{\omega \in \Omega_o} \left[q_o(\omega) \varphi_o(\omega) \right]^{\frac{\sigma_o - 1}{\sigma_o}} d\omega \right\}^{\frac{\sigma_o}{\sigma_o - 1}}, o \in \{F, N\}$$

where Ω_o is the set of available varieties produced by all o-type firms, $q_o(\omega)$ is the quantity of the consumption of variety ω produced by an o-type firm, $\varphi_o(\omega)$ is the *product quality index* of variety ω produced by an o-type firm, and $\sigma_o > 1$ is the common elasticity of substitution between any two o-type firms.

Each consumer's budget constraint is

$$\sum_{o \in \{F,N\}} \int_{\omega \in \Omega_o} p_o(\omega) q_o(\omega) d\omega = 1$$

where $p_o(\omega)$ is the price of variety ω produced by an o-type firm.

Utility maximization yields the total demand for variety ω given by:

$$q_o(\omega) = \frac{R_o}{\mathcal{P}_o} \varphi_o(\omega)^{\sigma_o - 1} \left[\frac{p_o(\omega)}{\mathcal{P}_o} \right]^{-\sigma_o}, \qquad o \in \{F, N\}, \tag{2}$$

where R_o is the aggregate expenditure of o-type firms[1], and the price index of varieties produced by all o-type firms, $\mathcal{P}_o$, is given by:

$$\mathcal{P}_o \equiv \left[\int_{v \in \Omega_o} \varphi_o(v)^{\sigma_o - 1} p_o(v)^{1-\sigma_o} dv \right]^{\frac{1}{1-\sigma_o}}. \tag{3}$$

Note that (2) implies that the higher the quality, the larger the demand. Thus, each

[1] Labor market clearing condition implies that $R_N + R_F = L$ holds.

consumer's indirect utility, V, is determined by

$$V = \frac{1}{\mathcal{P}_F^{\mu} \mathcal{P}_N^{1-\mu}}.$$

2.1.2 Production

Following Melitz and Ottaviano (2008), we consider a static (one-period) model. Prior to entry, firms are identical. Each firm faces uncertainty about its productivity level ψ and quality level φ. To become an o-type firm, each firm must make an initial investment. In other words, entry as an o-type firm requires a sunk cost of $\mathcal{F}_o$ units of labor. Once this cost is paid, firms observe their productivity $\psi \in (0, +\infty)$ and quality $\varphi \in (0, +\infty)$ from the common joint probability density function $h(\psi, \varphi)$ which has positive supports over $(0, +\infty) \times (0, +\infty)$ and has the joint cumulative distribution $H(\psi, \varphi)$. There are $\mathcal{M}_o$ potential o-type firms, who draw the lottery, and M_o active o-type firms, which produce differentiated products under increasing returns to scale technology with different productivity levels. Prior to selling its product, each o-type firm incurs a fixed labor requirement $f_o > 0$ in production. Specifically, $f_o = f$ holds for online firms and $f_o = F > f$ holds for offline firms, respectively. [2] Furthermore, there is no economies of scope in production. Thus, each firm produces a single variety and each variety is produced by a single firm. For simplicity, we assume that the two variables ψ and φ are independent and drawn from the same density function $g(\cdot)$, which implies that $h(\psi, \varphi) = g(\psi)g(\varphi)$ holds. To produce a variety for an o-type firm (ψ, φ), it needs a marginal requirement c_o/ψ units of labor with $c_o > 0$. Choosing the unit of each variety, we set $c_o = (\sigma_o - 1)/\sigma_o$.

Consumers have perfect information on firms' productivity ψ and quality φ in the offline market, while they have imperfect information on the firms' quality φ and perfect information on productivity ψ in the online market. The reason behind this is because consumers can identify firms' productivity by observing firms' prices. In other words, observing online firm's price $p(\psi)$, the consumer can induce its productivity ψ under the markup pricing strategy. On the other hand, since the quality φ and productivity ψ are independent, consumers have only a common expected value of online firms' quality $\mathbb{E}\varphi$, which is defined as follows:

$$\mathbb{E}\varphi \equiv \int \int_{ON} \varphi \mu_N(\psi, \varphi) d\psi d\varphi, \tag{4}$$

[2] The fixed production cost f_o can also be explained as the entry cost of online market and offline market, respectively.

where ON is the set of firms who choose online sales, and $\mu_N(\psi, \varphi)$ is the conditional probability of online firm (ψ, φ). That is, we assume that all consumers have rational expectations on ON, which is also common knowledge for all firms. Thus, both consumers and firms make their optimal decisions based on the same $\mathbb{E}\varphi$. Note that each firm's behavior does not impact on the other firms under monopolistic competition. Similarly, we can assume that each firm's behavior has no impact on consumers' choices, but the aggregate behavior of firms affects each consumer's expectations and choices.

The demand of an online firm (ψ, φ) is given by:

$$q_N(\psi, \varphi) = \frac{R_N}{\mathcal{P}_N} \mathbb{E}\varphi^{\sigma_N - 1} \left[\frac{p_N(\psi, \varphi)}{\mathcal{P}_N} \right]^{-\sigma_N}. \tag{5}$$

Accordingly, the profit of an online firm (ψ, φ) is given by:

$$\pi_N(\psi, \varphi) = \left[p_N(\psi, \varphi) - \frac{c_N}{\psi} \right] q_N(\psi, \varphi) - f, \tag{6}$$

where $q_N(\psi, \varphi)$ is given by (5). Profit maximization yields an online firm (ψ, φ)'s optimal price:

$$p_N(\psi, \varphi) = \frac{\sigma_N c_N}{(\sigma_N - 1)\psi} = \frac{1}{\psi}. \tag{7}$$

Substituting (7) into (5) yields the demand and revenue of online firm (ψ, φ) as follows:

$$\begin{aligned} q_N(\psi, \varphi) &= \frac{R_N}{\sigma_N} \frac{(\mathbb{E}\varphi)^{\sigma_N - 1}}{\mathcal{P}_N^{1-\sigma_N}} \psi^{\sigma_N}, \text{ and} \\ r_N(\psi, \varphi) &= \frac{R_N}{\sigma_N} \frac{(\mathbb{E}\varphi)^{\sigma_N - 1}}{\mathcal{P}_N^{1-\sigma_N}} \psi^{\sigma_N - 1}. \end{aligned}$$

Substituting (5) and (7) into (6) yields the online firm (ψ, φ)'s profit given by:

$$\pi_N(\psi, \varphi) = \frac{R_N}{\sigma_N} \frac{(\mathbb{E}\varphi)^{\sigma_N - 1}}{\mathcal{P}_N^{1-\sigma_N}} \psi^{\sigma_N - 1} - f, \tag{8}$$

which implies that the profit of online firm (ψ, φ) increases in its productivity ψ and the quality of online firms expected by consumers $\mathbb{E}\varphi$. However, it is independent of its quality φ.

Correspondingly, the profit of the offline firm (ψ, φ) is given by:

$$\pi_F(\psi, \varphi) = \left[p_F(\psi, \varphi) - \frac{c_F}{\psi} \right] q_F(\psi, \varphi) - F, \tag{9}$$

where $q_F(\psi, \varphi)$ is given by

$$q_F(\psi, \varphi) \equiv \frac{R_F}{\mathcal{P}_F} \varphi^{\sigma_F - 1} \left[\frac{p_F(\psi, \varphi)}{\mathcal{P}_F} \right]^{-\sigma_F}. \tag{10}$$

The profit maximization yields the optimal price of the offline firm (ψ, φ):

$$p_F(\psi, \varphi) = \frac{\sigma_F}{\sigma_F - 1} \frac{c_F}{\psi} = \frac{1}{\psi}. \tag{11}$$

Thus, we have $p_F(\psi, \varphi') = p_N(\psi, \varphi''), \forall \varphi', \varphi''$. In other words, the difference in productivity changes the price, but the difference in quality does not affect the price. Substituting (10) and (11) into (9) yields the offline firm (ψ, φ)'s demand, revenue, and profit, given by:

$$\begin{aligned} q_F(\psi, \varphi) &= R_F \frac{\varphi^{\sigma_F - 1}}{\mathcal{P}_F^{1-\sigma_F}} \psi^{\sigma_F}, & (12) \\ r_F(\psi, \varphi) &= R_F \frac{\varphi^{\sigma_F - 1}}{\mathcal{P}_F^{1-\sigma_F}} \psi^{\sigma_F - 1}, & (13) \\ \pi_F(\psi, \varphi) &= \frac{R_F}{\sigma_F} \frac{\varphi^{\sigma_F - 1}}{\mathcal{P}_F^{1-\sigma_F}} \psi^{\sigma_F - 1} - F. & (14) \end{aligned}$$

Thus, the offline firm (ψ, φ)'s profit increases in both its productivity ψ and quality φ.

The ratios of any two online firms' outputs and revenues depend on the ratio of their productivity levels:

$$\frac{q_N(\psi_1, \mathbb{E}\varphi)}{q_N(\psi_2, \mathbb{E}\varphi)} = \left(\frac{\psi_1}{\psi_2} \right)^{\sigma_N}, \qquad \frac{r_N(\psi_1, \mathbb{E}\varphi)}{r_N(\psi_2, \mathbb{E}\varphi)} = \left(\frac{\psi_1}{\psi_2} \right)^{\sigma_N - 1}.$$

The ratios of any two offline firms' outputs and revenues depends on the ratio of the combination of their productivity and quality levels:

$$\frac{q_F(\psi_1, \varphi_1)}{q_F(\psi_2, \varphi_2)} = \left(\frac{\varphi_1}{\varphi_2} \right)^{\sigma_F - 1} \left(\frac{\psi_1}{\psi_2} \right)^{\sigma_F}, \qquad \frac{r_F(\psi_1, \varphi_1)}{r_F(\psi_2, \varphi_2)} = \left(\frac{\varphi_1}{\varphi_2} \right)^{\sigma_F - 1} \left(\frac{\psi_1}{\psi_2} \right)^{\sigma_F - 1}.$$

That is, a more productive online firm and a more productive and qualified offline firm will be bigger and earn a higher profit than a less productive online firm and a less productive and qualified offline firm.

3 Only offline firms

Let us first consider the case in which there is only an offline sector in the economy, setting $\mu = 1$. Thus, this case is a variation of Melitz (2003) with two dimensions of heterogeneity. We now define the iso-profit condition of offline firms as

$$\mathrm{I}(\Phi) \equiv \left\{(\psi, \varphi) \in \mathbb{R}_+^2 : \varphi^{\sigma_F - 1}\psi^{\sigma_F - 1} = \Phi > 0\right\}.$$

We further define *the zero cutoff profit condition of offline firms* as $\pi_F(\psi, \varphi) = 0$, which is equivalent to:

$$\text{ZCP-F} \equiv \left\{(\psi, \varphi) \in \mathbb{R}_+^2 : \varphi^{\sigma_F - 1}\psi^{\sigma_F - 1} = \underline{\Phi} \equiv \left(\frac{\sigma_F F}{R_F}\right)\left(\frac{1}{\mathcal{P}_F}\right)^{\sigma_F - 1}\right\} = \mathrm{I}(\underline{\Phi}). \tag{15}$$

Accordingly, the offline firm $(\psi, \varphi) \in \mathrm{I}\left(\underline{\Phi}^+\right) \equiv \left\{(\psi, \varphi) \in \mathbb{R}_+^2 : \varphi^{\sigma_F - 1}\psi^{\sigma_F - 1} > \underline{\Phi}\right\}$ has a positive profit, which will be an active offline firm. The offline firm $(\psi, \varphi) \in \mathrm{I}\left(\underline{\Phi}^-\right) \equiv \left\{(\psi, \varphi) \in \mathbb{R}_+^2 : \varphi^{\sigma_F - 1}\psi^{\sigma_F - 1} < \underline{\Phi}\right\}$ does not produce and quits the market immediately after observing its own productivity and quality.

The ex-ante probability of successful entry for an offline firm, pe_F, is determined by

$$pe_F \equiv \int\int_{\mathrm{I}\left(\underline{\Phi}^+\right)} g(\psi)g(\varphi)d\psi d\varphi.$$

Accordingly, the conditional distribution of the productivity and quality of the offline firm (ψ, φ) operating in the market is given by

$$\mu_F(\psi, \varphi) = \begin{cases} \dfrac{g(\psi)g(\varphi)}{pe_F} & \text{if } (\psi, \varphi) \in \mathrm{I}(\underline{\Phi}^+), \\ 0 & \text{otherwise.} \end{cases}$$

We also have $M_F = pe_F \mathcal{M}_F$.

We define the aggregate combination of productivity and quality levels for active offline firms as follows:

$$\widetilde{\Phi}(\underline{\Phi}^+) \equiv \int_0^\infty \int_0^\infty \varphi^{\sigma_F - 1}\psi^{\sigma_F - 1}\mu_F(\psi, \varphi)d\psi d\varphi = \frac{1}{pe_F}\int\int_{\mathrm{I}\left(\underline{\Phi}^+\right)} \varphi^{\sigma_F - 1}\psi^{\sigma_F - 1}g(\psi)g(\varphi)d\psi d\varphi. \tag{16}$$

Using (3) and (7), we obtain the following price index:

$$\begin{aligned}
\mathcal{P}_F^{1-\sigma_F} &\equiv \int\int_{\mathrm{I}(\underline{\Phi}^+)} M_F \varphi^{\sigma_F-1} p_F(\psi,\varphi)^{1-\sigma_F}\mu_F(\psi,\varphi)d\psi d\varphi \\
&= \int\int_{\mathrm{I}(\underline{\Phi}^+)} M_F \varphi^{\sigma_F-1}\psi^{\sigma_F-1}\mu_F(\psi,\varphi)d\psi d\varphi \\
&= M_F\widetilde{\Phi}. \qquad (17)
\end{aligned}$$

The average revenue and profit of active offline firms have the following relationship with the revenue and profit of firm $(\psi,\varphi)\in \mathrm{I}(\widetilde{\Phi})$:

$$\bar{r}_F = \int\int_{\mathrm{I}(\underline{\Phi}^+)} R_F \frac{\varphi^{\sigma_F-1}}{\mathcal{P}_F^{1-\sigma_F}}\psi^{\sigma_F-1}\mu_F(\psi,\varphi)d\psi d\varphi = \frac{R_F}{\mathcal{P}_F^{1-\sigma_F}}\widetilde{\Phi} = r_F(\psi,\varphi)\big|_{(\psi,\varphi)\in \mathrm{I}(\widetilde{\Phi})} \equiv r_F(\widetilde{\Phi}),$$

$$\bar{\pi}_F = \int\int_{\mathrm{I}(\underline{\Phi}^+)} \left(\frac{R_F}{\sigma_F}\frac{\varphi^{\sigma_F-1}}{\mathcal{P}_F^{1-\sigma_F}}\psi^{\sigma_F-1} - F\right)\mu_F(\psi,\varphi)d\psi d\varphi = \frac{\bar{r}_F}{\sigma_F} - F = \frac{r_F(\widetilde{\Phi})}{\sigma_F} - F \equiv \pi_F(\widetilde{\Phi}).$$

The free entry condition is $\mathcal{F}_F = pe_F\bar{\pi}_F$. Meanwhile, the market-clearing condition for labor is given by

$$L = \mathcal{M}_F\mathcal{F}_F + M_F\frac{\sigma_F-1}{\sigma_F}\bar{r}_F + M_F F.$$

Thus, the mass of active offline firms can be obtained as follows:

$$M_F = \frac{R_F}{\bar{r}_F} = \frac{L}{\sigma_F\left(\bar{\pi}_F + F\right)}.$$

Combining (15) and (17), the welfare measured by the indirect utility of the representative consumer is determined by

$$V_F = \frac{1}{\mathcal{P}_F} = \left(M_F\widetilde{\Phi}\right)^{\frac{1}{\sigma_F-1}} = \left(\frac{R_F\underline{\Phi}}{\sigma_F F}\right)^{1/(\sigma_F-1)}. \qquad (18)$$

Equilibrium is obtained by using the zero-cutoff condition and the free entry condition. We need to derive the equilibrium value of $\underline{\Phi}$. Using the zero-cutoff condition for offline firms, $\pi_F(\underline{\Phi}) = 0$, we obtain

$$\bar{\pi}_F = \left(\frac{\widetilde{\Phi}}{\underline{\Phi}} - 1\right)F. \qquad (19)$$

Substituting (19) into the free entry condition yields

$$\frac{\mathcal{F}_F}{F} = pe_F\left(\frac{\widetilde{\Phi}}{\underline{\Phi}} - 1\right). \qquad (20)$$

We define

$$\begin{aligned} k_F(\underline{\Phi}) &\equiv \frac{\widetilde{\Phi}}{\underline{\Phi}} - 1 \text{ and} \\ j_F(\underline{\Phi}) &\equiv pe_F k_F(\underline{\Phi}). \end{aligned}$$

We have

$$\begin{aligned} j_F(\underline{\Phi}) &= \frac{\widetilde{\Phi}}{\underline{\Phi}} pe_F - pe_F \\ &= \frac{1}{\underline{\Phi}} \int\int_{\mathrm{I}(\underline{\Phi}^+)} \varphi^{\sigma_F-1}\psi^{\sigma_F-1} g(\psi)g(\varphi)d\psi d\varphi - \int\int_{\mathrm{I}(\underline{\Phi}^+)} g(\psi)g(\varphi)d\psi d\varphi \\ &= \frac{1}{\underline{\Phi}} \int_0^\infty \int_{\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}}^\infty \varphi^{\sigma_F-1}\psi^{\sigma_F-1} g(\psi)g(\varphi)d\psi d\varphi \\ &\quad - \int_0^\infty \int_{\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}}^\infty g(\psi)g(\varphi)d\psi d\varphi \end{aligned}$$

Its derivative is as follows:

$$\begin{aligned} j_F'(\underline{\Phi}) &= \frac{1}{\underline{\Phi}} \frac{\partial \left[\int_0^\infty \int_{\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}}^\infty \varphi^{\sigma_F-1}\psi^{\sigma_F-1} g(\psi)g(\varphi)d\psi d\varphi\right]}{\partial \underline{\Phi}} \\ &\quad - \frac{\widetilde{\Phi}}{\underline{\Phi}} pe_F \frac{1}{\underline{\Phi}} - \frac{\partial \left[\int_0^\infty \int_{\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}}^\infty g(\psi)g(\varphi)d\psi d\varphi\right]}{\partial \underline{\Phi}}. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} &\frac{\partial \left[\int_0^\infty \int_{\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}}^\infty \varphi^{\sigma_F-1}\psi^{\sigma_F-1} g(\psi)g(\varphi)d\psi d\varphi\right]}{\partial \underline{\Phi}} \\ = \ &-\frac{1}{\sigma_F-1} \underline{\Phi}^{-\sigma_F/(\sigma_F-1)} \underline{\Phi} \int_0^\infty g(\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}) g(\varphi)\varphi^{-(\sigma_F-1)/(\sigma_F-1)} d\varphi \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial \left[\int_0^\infty \int_{\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}}^\infty g(\psi)g(\varphi)d\psi d\varphi\right]}{\partial \underline{\Phi}} \\ = \ &-\frac{1}{\sigma_F-1} \underline{\Phi}^{-\sigma_F/(\sigma_F-1)} \int_0^\infty g(\underline{\Phi}^{1/(\sigma_F-1)}\varphi^{-(\sigma_F-1)/(\sigma_F-1)}) g(\varphi)\varphi^{-(\sigma_F-1)/(\sigma_F-1)} d\varphi. \end{aligned}$$

Thus, we have

$$j_F'(\underline{\Phi}) = -\frac{\widetilde{\Phi}}{\underline{\Phi}^2} pe_F < 0.$$

Furthermore, we have $\lim_{\underline{\Phi}\to 0} j_F(\underline{\Phi}) = \infty$ and $\lim_{\underline{\Phi}\to\infty} j_F(\underline{\Phi}) = 0$. Therefore, equation $j_F(\underline{\Phi}) = \mathcal{F}_F/F$ has a unique solution $\underline{\Phi}^* \in (0,\infty)$. As a result, the zero-cutoff condition of offline firms provides the equilibrium value of $\mathcal{P}_F^*$, which leads to the equilibrium value of V_F. Furthermore, we obtain the equilibrium value of pe_F using this definition. Then, the free entry condition provides the equilibrium value of $\bar{\pi}_F$. Thus, we obtain the equilibrium value of M_F and then the equilibrium value of $\bar{r}_F$ and $\mathcal{M}_F$.

Note that $k_F(\underline{\Phi}) \equiv \frac{\widetilde{\Phi}}{\underline{\Phi}} - 1$ holds; thus, we obtain the following:

$$\frac{j_F'(\underline{\Phi})\underline{\Phi}}{j_F(\underline{\Phi})} = -\frac{\widetilde{\Phi}}{\underline{\Phi}}\frac{pe_F}{j_F(\underline{\Phi})} = -\frac{\widetilde{\Phi}}{\underline{\Phi}}\frac{1}{k_F(\underline{\Phi})} = -\left(1 + \frac{1}{k_F(\underline{\Phi})}\right) < -1.$$

Investigating (20), we find that an increase in $\mathcal{F}_F$ results in a decrease in $\underline{\Phi}$, which means that the average profit level of offline firms increases to hold the free entry condition of offline firms, which ultimately leads to a decrease in $\underline{\Phi}$ holding the zero-cutoff profit condition of offline firms. Since a decrease in $\underline{\Phi}$ leads to an increase in pe_F, the net value of entry on offline firms is maintained. Then, (15) shows that a decrease in $\underline{\Phi}$ is fulfilled with milder competition for the offline firms under the cut-off profit, which implies an increase in $\mathcal{P}_F^*$ and then a decrease in V_F from the definition of the indirect utility.

Investigating (19), we find that an increase in F results in a larger value of $\underline{\Phi}$, which means that an increased average profit level of offline firms (to hold the zero-cutoff condition of offline firms) leads to an increase in $\underline{\Phi}$ to hold the free entry condition of offline firms. The same reasoning for an increase in $\mathcal{F}_F$ indicates that an increase in F results in a decrease in $\mathcal{P}_F^*$, an increase in V_F, and a decrease in pe_F.

4 Only online firms

We now consider the case in which there is only an online market in the economy, setting $\mu = 0$. Thus, the iso-profit condition of online firms can be defined as follows:

$$\mathrm{I}(\Psi) \equiv \left\{(\psi,\varphi) \in \mathbb{R}_+^2 : \psi = \Psi > 0\right\}.$$

We further define *the zero cutoff profit condition of online firms* as $\pi^N(\psi,\varphi) = 0$, which is equivalent to:

$$\text{ZCP-N} \equiv \left\{(\psi,\varphi) \in \mathbb{R}_+^2 : \psi = \left[\frac{\sigma_N f}{R_N(\mathbb{E}\varphi)^{\sigma_N - 1}}\right]^{\frac{1}{\sigma_N - 1}} \mathcal{P}_N^{-1} \equiv \underline{\Psi}\right\} = \mathrm{I}(\underline{\Psi}), \qquad (21)$$

where $\mathcal{P}_N$ is the price index of the available varieties produced by only online firms, and R_N is the aggregate expenditure for online firms only. Thus, the offline firm $(\psi,\varphi) \in \mathrm{I}\left(\underline{\Psi}^{+}\right) \equiv \left\{(\psi,\varphi) \in \mathbb{R}_{+}^{2} : \psi > \underline{\Psi}\right\}$ has a positive profit and become an active online firm. The offline firm $(\psi,\varphi) \in \mathrm{I}\left(\underline{\Psi}^{-}\right) \equiv \left\{(\psi,\varphi) \in \mathbb{R}_{+}^{2} : \psi < \underline{\Psi}\right\}$ does not produce and exits the market immediately after observing its own productivity and quality.

The ex-ante probability of successful entry for online firms, pe_N, is

$$pe_N \equiv \int\int_{\mathrm{I}(\underline{\Psi}^{+})} g(\psi)g(\varphi)d\psi d\varphi.$$

Accordingly, the conditional distribution of productivity and quality of the online firm $(\psi,\varphi) \in \mathrm{I}\left(\underline{\Psi}^{+}\right)$ operating in the market is given by

$$\mu_N(\psi,\varphi) = \begin{cases} \dfrac{g(\psi)g(\varphi)}{pe_N} & \text{if } (\psi,\varphi) \in \mathrm{I}(\underline{\Psi}^{+}) \ , \\ 0 & \text{otherwise.} \end{cases}$$

We also have $M_N = pe_N \mathcal{M}_N$.

We define aggregate productivity and quality level as

$$\widetilde{\Psi}(\underline{\Psi}^{+}) = \int_0^\infty \int_0^\infty (\mathbb{E}\varphi)^{\sigma_N-1} \psi^{\sigma_N-1} \mu_N(\psi,\varphi)d\psi d\varphi = \frac{1}{pe_N}\int\int_{\mathrm{I}(\underline{\Psi}^{+})} (\mathbb{E}\varphi)^{\sigma_N-1} \psi^{\sigma_N-1} g(\psi)g(\varphi)d\psi d\varphi. \tag{22}$$

Using (3) and (11), the price index of the sole online market case can be written as:

$$\begin{aligned} \mathcal{P}_N^{1-\sigma_N} &= \int\int_{\mathrm{I}(\underline{\Psi}^{+})} M_N (\mathbb{E}\varphi)^{\sigma_N-1} p_N(\psi,\varphi)^{1-\sigma_N} \mu_N(\psi,\varphi)d\psi d\varphi \\ &= \int\int_{\mathrm{I}(\underline{\Psi}^{+})} M_N (\mathbb{E}\varphi)^{\sigma_N-1} \psi^{\sigma_N-1} \mu_N(\psi,\varphi)d\psi d\varphi \\ &= M_N \widetilde{\Psi}. \end{aligned}$$

The expected revenue and profit of online firms are, respectively, determined by the following:

$$\begin{aligned} \bar{r}_N &= \int\int_{\mathrm{I}(\underline{\Psi}^{+})} R_N \frac{(\mathbb{E}\varphi)^{\sigma_N-1}}{\mathcal{P}_N^{1-\sigma_N}} \psi^{\sigma_N-1} \mu_N(\psi,\varphi)d\psi d\varphi = \frac{R_N}{\mathcal{P}_N^{1-\sigma_N}} \widetilde{\Psi}, \\ \bar{\pi}_N &= \int\int_{\mathrm{I}(\underline{\Psi}^{+})} \left(\frac{R_N}{\sigma_N} \frac{(\mathbb{E}\varphi)^{\sigma_N-1}}{\mathcal{P}_N^{1-\sigma_N}} \psi^{\sigma_N-1} - f \right) \mu_N(\psi,\varphi)d\psi d\varphi = \frac{r_N(\widetilde{\Psi})}{\sigma_N} - f. \end{aligned}$$

The free entry condition is expressed as $\mathcal{F}_N = pe_N \bar{\pi}_N$. Meanwhile, the market-clearing condition for labor is given by

$$L = \mathcal{M}_N \mathcal{F}_N + M_N \frac{\sigma_N - 1}{\sigma_N} \bar{r}_N + M_N f.$$

Thus, the mass of producing online firms can be obtained as follows:

$$M_N = \frac{R_N}{\bar{r}_N} = \frac{L}{\sigma_N \left(\bar{\pi}_N + f\right)}.$$

Using the zero-cutoff condition of online firms, the welfare measured by the representative consumer is given as follows:

$$V_N = \frac{1}{\mathcal{P}_N} = \left(M_N \widetilde{\Psi}\right)^{1/(\sigma_N - 1)} = \left[\frac{R_N \left(\mathbb{E}\varphi\right)^{\sigma_N - 1} \underline{\Psi}^{\sigma_N - 1}}{\sigma_N f}\right]^{1/(\sigma_N - 1)}, \tag{23}$$

where $\mathbb{E}\varphi$ is determined by:

$$\mathbb{E}\varphi = \int \int_{ON} \varphi \mu_N(\psi, \varphi) d\psi d\varphi = \frac{\int \int_{ON} \varphi g(\psi) g(\varphi) d\psi d\varphi}{pe_N} = \frac{\int_0^\infty \varphi g(\varphi) d\varphi}{\int_0^\infty g(\varphi) d\varphi}.$$

Using the zero cutoff condition and the free entry condition of online firms, we derive the equilibrium value of $\underline{\Psi}$. Combining the zero-cutoff condition of online firms, $\pi_N(\underline{\Psi}) = 0$, and $\bar{\pi}_N$, we obtain

$$\bar{\pi}_N = \left[\frac{\widetilde{\Psi}}{\left(\mathbb{E}\varphi\right)^{\sigma_N - 1} \underline{\Psi}^{\sigma_N - 1}} - 1\right] f. \tag{24}$$

Substituting this expression into the free entry condition yields:

$$\frac{\mathcal{F}_N}{f} = pe_N \left(\frac{\widetilde{\Psi}}{\left(\mathbb{E}\varphi\right)^{\sigma_N - 1} \underline{\Psi}^{\sigma_N - 1}} - 1\right).$$

We define:

$$\begin{aligned} k_N(\underline{\Psi}) &\equiv \frac{\widetilde{\Psi}(\underline{\Psi})}{\left(\mathbb{E}\varphi\right)^{\sigma_N - 1} \underline{\Psi}^{\sigma_N - 1}} - 1 \text{ and} \\ j_N(\underline{\Psi}) &\equiv k(\underline{\Psi}) pe_N. \end{aligned}$$

Thus, we have:

$$\frac{\mathcal{F}_N}{f} = j_N(\underline{\Psi}). \tag{25}$$

Therefore, $j_N(\underline{\Psi})$ can be rewritten as:

$$j_N(\underline{\Psi}) = \left[\frac{\widetilde{\Psi}(\underline{\Psi})}{(\mathbb{E}\varphi)^{\sigma_N-1}\,\underline{\Psi}^{\sigma_N-1}} - 1\right] pe_N = \frac{1}{\underline{\Psi}^{\sigma_N-1}} \int_0^\infty \int_{\underline{\Psi}}^\infty \psi^{\sigma_N-1} g(\psi)g(\varphi)d\psi d\varphi - \int_0^\infty \int_{\underline{\Psi}}^\infty g(\psi)g(\varphi)d\psi d\varphi.$$

Taking the derivative of $j_N(\underline{\Psi})$ with respect to $\underline{\Psi}$ yields:

$$\begin{aligned} j_N'(\underline{\Psi}) =& \frac{\partial \int_0^\infty \int_{\underline{\Psi}}^\infty \psi^{\sigma_N-1} g(\psi)g(\varphi)d\psi d\varphi}{\partial \underline{\Psi}} \frac{1}{\underline{\Psi}^{\sigma_N-1}} \\ & - (\sigma_N - 1)\frac{\widetilde{\Psi}(\underline{\Psi})}{(\mathbb{E}\varphi)^{\sigma_N-1}\,\underline{\Psi}^{\sigma_N-1}} \frac{1}{\underline{\Psi}} pe_N - \frac{\partial \int_0^\infty \int_{\underline{\Psi}}^\infty g(\psi)g(\varphi)d\psi d\varphi}{\partial \underline{\Psi}} \\ =& -(\sigma_N - 1)\frac{\widetilde{\Psi}(\underline{\Psi})}{(\mathbb{E}\varphi)^{\sigma_N-1}\,\underline{\Psi}^{\sigma_N}} pe_N < 0. \end{aligned}$$

Furthermore, we have $\lim_{\underline{\Phi}\to 0} j_N(\underline{\Phi}) = \infty$ and $\lim_{\underline{\Phi}\to\infty} j_N(\underline{\Phi}) = 0$. Therefore, $j_N(\underline{\Psi}) = \mathcal{F}_N/f$ has a unique root of $\underline{\Psi}^* \in (0, \infty)$.

Thus, the zero-cutoff condition of online firms provides the equilibrium value of $\mathcal{P}_N^*$, which leads to the equilibrium value of V_N. Using the equilibrium value of $\underline{\Psi}^*$, we can obtain the equilibrium value of pe_N using the definition of pe_N. Then, the free entry condition provides the equilibrium value of $\bar{\pi}_F$. Accordingly, we obtain the equilibrium value of M_N and then the equilibrium value of $\bar{r}_N$ and $\mathcal{M}_N$.

Note that

$$\begin{aligned} [k_N(\underline{\Psi}) + 1]\frac{(\mathbb{E}\varphi)^{\sigma_N-1}}{\underline{\Psi}} &= \frac{\widetilde{\Psi}(\underline{\Psi})}{\underline{\Psi}^{\sigma_N}} \text{ and} \\ j_N(\underline{\Psi}) &\equiv k(\underline{\Psi})pe_N. \end{aligned}$$

and thus we obtain

$$\frac{j_N'(\underline{\Psi})\underline{\Psi}}{j_N(\underline{\Psi})} = -(\sigma_N - 1)\left(1 + \frac{1}{k(\underline{\Psi})}\right) < -(\sigma_N - 1).$$

Similar to the impact of $\mathcal{F}_F$, investigating (25), we find that an increase in $\mathcal{F}_N$ results in a smaller $\underline{\Psi}^*$, which means that the increased average profit level of online firms due to an increase in $\mathcal{F}_N$ leads to a decrease in $\underline{\Psi}^*$ to hold the zero-cutoff profit condition for online firms. Because a decrease in $\underline{\Psi}^*$ leads to an increase in pe_N, the net value of entry on online firms is maintained. Then, (21) shows that a decrease in $\underline{\Psi}^*$ is realized with milder competition for the online firm with the cut-off profit, which implies an increase in $\mathcal{P}_N^*$ and then a decrease in V_N from the definition of the indirect utility.

Investigating (24), we find that an increase in f results in an increase in $\underline{\Psi}^*$,[3] which means that the increased average profit level of online firms (to hold the zero-cutoff condition of online firms) leads to an increase in $\underline{\Psi}^*$ to hold the free entry condition. Furthermore, the reasoning on an increase in $\mathcal{F}_N$ provides that an increase in f results in a decrease in $\mathcal{P}_N^*$, an increase in V_N, and a decrease in pe_N.

5 Online and offline firms

We now turn to the economy with both online and offline firms, setting $\mu \in (0,1)$. In this case, we denote the expenditure for the varieties produced by offline firms, R_F, and that for the varieties produced by online firms, R_N, as follows:

$$R_F = \mu R \text{ and } R_N = (1-\mu)R.$$

Using the zero cutoff conditions, we obtain the profits of an online firm (ψ, φ) and an offline firm (ψ, φ), respectively, given by:

$$\begin{aligned} \pi_F(\psi, \varphi) &= \left(\frac{\varphi^{\sigma_F - 1} \psi^{\sigma_F - 1}}{\underline{\Phi}} - 1 \right) F, \\ \pi_N(\psi, \varphi) &= \left(\frac{\psi^{\sigma_N - 1}}{\underline{\Psi}^{\sigma_N - 1}} - 1 \right) f. \end{aligned}$$

We further have $M_F = pe_F \mathcal{M}_F$ and $M_N = pe_N \mathcal{M}_N$. Meanwhile, the market-clearing condition for labor is given by

$$\begin{aligned} L &= \mathcal{M}_F \mathcal{F}_F + M_F \frac{\sigma_F - 1}{\sigma_F} \bar{r}_F + M_F F \\ &\quad + \mathcal{M}_N \mathcal{F}_N + M_N \frac{\sigma_N - 1}{\sigma_N} \bar{r}_N + M_N f. \end{aligned}$$

Since $\mathcal{F}_F = pe_F \bar{\pi}_F$, $\mathcal{F}_N = pe_N \bar{\pi}_N$, $\bar{r}_F/\sigma = \bar{\pi}_F + F$, and $\bar{r}_N/\sigma = \bar{\pi}_N + f$, the market-clearing condition for labor can be rewritten as the following:

$$L = \sigma_F pe_F \mathcal{M}_F \left(\bar{\pi}_F + F \right) + \sigma_N pe_N \mathcal{M}_N \left(\bar{\pi}_N + f \right). \tag{26}$$

[3] Differentiating $\mathcal{F}_N = j_N(\underline{\Psi}) f$, we obtain

$$0 \;=\; j_N(\underline{\Psi}) \partial f + f j_N'(\underline{\Psi}) \partial \underline{\Psi} \Leftrightarrow \frac{\partial \underline{\Psi}}{\partial f} = \frac{F_{eN}}{\sigma_N - 1} \frac{\underline{\Psi}^{\sigma_N}}{\tilde{\Psi}(\underline{\Psi})} \frac{1}{f^2} > 0.$$

Furthermore, the equilibrium values of $\underline{\Psi}$ and $\underline{\Phi}$ are, respectively, determined by:

$$\frac{\mathcal{F}_N}{f} = j_N(\underline{\Psi})$$

and

$$\frac{\mathcal{F}_F}{F} = j_F(\underline{\Phi}).$$

Thus, we obtain the equilibrium values of $\underline{\Psi}^*$ and $\underline{\Phi}^*$. Accordingly, we obtain the equilibrium values of the price indices,

$$\mathcal{P}_F = \left(\frac{\sigma_F F}{\mu R \underline{\Phi}^*}\right)^{\frac{1}{\sigma_F - 1}}$$

and

$$\mathcal{P}_N = \left[\frac{\sigma_N f}{(1-\mu)R(\mathbb{E}\varphi)^{\sigma_N - 1}}\right]^{\frac{1}{\sigma_N - 1}} \frac{1}{\underline{\Psi}^*}$$

where $\mathbb{E}\varphi = \int_0^\infty \varphi g(\varphi) d\varphi / \int_0^\infty g(\varphi) d\varphi$. Using the definitions of pe_F and pe_N, we can obtain $pe_F^*(\underline{\Phi}^*)$ and $pe_N^*(\underline{\Psi}^*)$, respectively. Accordingly, we obtain $\widetilde{\Phi}(\underline{\Phi}^*)$ and $\widetilde{\Psi}(\underline{\Psi}^*)$, respectively. Therefore, we obtain the following:

$$M_F^* = \mathcal{P}_F^{1-\sigma_F} / \widetilde{\Phi}(\underline{\Phi}^*) \qquad \text{and} \qquad M_N^* = \mathcal{P}_N^{1-\sigma_N} / \widetilde{\Psi}(\underline{\Psi}^*).$$

Combining zero cut-off conditions for online firms and for offline firms with $M_F^* = pe_F^*(\underline{\Phi}^*)\mathcal{M}_F$ and $M_N^* = pe_N^*(\underline{\Psi}^*)\mathcal{M}_N$, we obtain

$$\mathcal{M}_F \frac{(\underline{\Psi}^*)^{\sigma_N - 1}(\mathbb{E}\varphi)^{\sigma_N - 1}}{(1-\mu)\sigma_N f \widetilde{\Psi}(\underline{\Psi}^*) pe_N^*(\underline{\Psi}^*)} = \mathcal{M}_N \frac{\underline{\Phi}^*}{\mu \sigma_F F \widetilde{\Phi}(\underline{\Phi}^*) pe_F^*(\underline{\Phi}^*)}. \tag{27}$$

Combining (26) and (27) yields $\mathcal{M}_F^*$ and $\mathcal{M}_N^*$ in equilibrium as follows:

$$\mathcal{M}_N^* = \frac{L}{\sigma_N pe_N^*(\underline{\Psi}^*) \left[\frac{(1-\mu)f}{\mu F} \frac{\underline{\Phi}^* \widetilde{\Psi}(\underline{\Psi}^*)}{\widetilde{\Phi}(\underline{\Phi}^*)(\underline{\Psi}^*)^{\sigma_N - 1}(\mathbb{E}\varphi)^{\sigma_N - 1}} (\bar{\pi}_F + F) + \bar{\pi}_N + f\right]},$$

and

$$\mathcal{M}_F^* = \frac{L}{\sigma_F pe_F^*(\underline{\Phi}^*) \left[\frac{\mu F}{(1-\mu)f} \frac{\widetilde{\Phi}(\underline{\Phi}^*)(\underline{\Psi}^*)^{\sigma_N - 1}(\mathbb{E}\varphi)^{\sigma_N - 1}}{\underline{\Phi}^* \widetilde{\Psi}(\underline{\Psi}^*)} (\bar{\pi}_N + f) + \bar{\pi}_F + F\right]}.$$

Finally, the indirect utility of a representative consumer is given as:

$$V = \left(\frac{\mu R\underline{\Phi}^*}{\sigma_F F}\right)^{\frac{\mu}{\sigma_F - 1}} \left(\left[\frac{(1-\mu)R(\mathbb{E}\varphi)^{\sigma_N - 1}}{\sigma_N f}\right]^{\frac{1}{\sigma_N - 1}} \underline{\Psi}^*\right)^{1-\mu}. \tag{28}$$

We examine the impact of F or f on V as the following. Since

$$\frac{\partial \frac{\underline{\Phi}}{F}}{\partial F}\frac{F}{\frac{\underline{\Phi}}{F}} = -\frac{\frac{1}{k_F(\underline{\Phi})}}{1+\frac{1}{k_F(\underline{\Phi})}} \qquad \text{and} \qquad \frac{\partial\left(\left(\frac{1}{f}\right)^{\frac{1}{\sigma_N - 1}}\underline{\Psi}^*\right)}{\partial f}\frac{f}{\left(\frac{1}{f}\right)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*} = -\frac{\frac{1}{k_N(\underline{\Psi})}}{1+\frac{1}{k_N(\underline{\Psi})}}\frac{1}{\sigma_N - 1},$$

we obtain:

$$\frac{\partial V}{\partial f} = (1-\mu)\frac{V}{f}\left[-\frac{\frac{1}{k_N(\underline{\Psi})}}{1+\frac{1}{k_N(\underline{\Psi})}}\right]\frac{1}{\sigma_N - 1} < 0$$

and

$$\frac{\partial V}{\partial F} = \frac{\mu}{\sigma_F - 1}\frac{V}{F}\left[-\frac{\frac{1}{k_F(\underline{\Phi})}}{1+\frac{1}{k_F(\underline{\Phi})}}\right] < 0,$$

which yields:

$$\frac{\partial V}{\partial f} + \frac{\partial V}{\partial F} = 0$$

$\Leftrightarrow$

$$\left.\frac{\partial F}{\partial f}\right|_{\overline{V}} = -\frac{1-\mu}{\mu}\frac{\sigma_F - 1}{\sigma_N - 1}\frac{F}{f}\frac{\frac{\frac{1}{k_N(\underline{\Psi})}}{1+\frac{1}{k_N(\underline{\Psi})}}}{\frac{\frac{1}{k_F(\underline{\Phi})}}{1+\frac{1}{k_F(\underline{\Phi})}}} < 0$$

where $\overline{V}$ is the given value of V. In other words, an increase in F or f leads to a lower indirect utility.

We now turn to examine the impact of $\mathcal{F}_N$ or $\mathcal{F}_F$ on V. Using $\mathcal{F}_N/f = j_N(\underline{\Psi})$ and $\mathcal{F}_F/F = j_F(\underline{\Phi})$, we obtain:

$$\frac{\partial \underline{\Psi}}{\partial \mathcal{F}_N} = \frac{1}{j_N'(\underline{\Psi})f} \qquad \frac{\partial \underline{\Phi}}{\partial \mathcal{F}_F} = \frac{1}{j_F'(\underline{\Phi})F},$$

which leads to:

$$\frac{\partial V}{\partial \mathcal{F}_N} = (1-\mu)\frac{V}{\underline{\Psi}}\frac{1}{j_N'(\underline{\Psi})f} > 0$$

and

$$\frac{\partial V}{\partial \mathcal{F}_F} = \frac{\mu}{\sigma_F - 1}\frac{V}{\underline{\Phi}^*}\frac{1}{j_F'(\underline{\Phi})F} > 0.$$

Thus, we obtain:

$$\frac{\partial V}{\partial \mathcal{F}_N}+\frac{\partial V}{\partial \mathcal{F}_F}=0$$

$\Leftrightarrow$

$$\left.\frac{\partial \mathcal{F}_N}{\partial \mathcal{F}_F}\right|_{\overline{V}}=-\frac{1-\mu}{\frac{\mu}{\sigma_F-1}}\frac{\Phi^*}{\underline{\Psi}}\frac{j_F'(\underline{\Phi})F}{j_N'(\underline{\Psi})f}<0.$$

In other words, an increase in $\mathcal{F}_N$ or $\mathcal{F}_F$ leads to a higher indirect utility.

6 Comparison: Offline case and online-offline case

To compare the offline case with the online-offline case, we assume that σ_F, F, and $\mathcal{F}_F$ are the same in both cases. Thus, we obtain the same value of $\underline{\Phi}$ in equilibrium for both cases. Furthermore, we set $R_F=\mu R$ in the online-offline case and $R_F=R$ in the offline case. We have the following:

$$V>V_F \Leftrightarrow \left\{\left[\frac{(1-\mu)R(\mathbb{E}\varphi)^{\sigma_N-1}}{\sigma_N f}\right]^{\frac{1}{\sigma_N-1}}\underline{\Psi}\right\}^{1-\mu}\left(\frac{R\underline{\Phi}}{\sigma_F F}\right)^{-\frac{1-\mu}{\sigma_F-1}}>(1-\mu)^{-\frac{1-\mu}{\sigma_N-1}}\mu^{\frac{-\mu}{\sigma_F-1}}.$$

First, we examine the impact of $\mathcal{F}_N$ and $\mathcal{F}_N$ on $V>V_F$ as the following. Using $\mathcal{F}_N/f=j_N(\underline{\Psi})$ and $\mathcal{F}_F/F=j_F(\underline{\Phi})$, we obtain

$$\frac{\partial\underline{\Psi}}{\partial\mathcal{F}_N}<0,\qquad \frac{\partial\underline{\Phi}}{\partial\mathcal{F}_N}=0.$$

Because of $\mathbb{E}\varphi=\int_0^\infty \varphi g(\varphi)d\varphi/\int_0^\infty g(\varphi)d\varphi$, we obtain $\partial\mathbb{E}\varphi/\partial\underline{\Psi}=0$. Therefore, if $\mathcal{F}_N$ approaches to zero, $\underline{\Psi}$ goes to infinity, while $\mathbb{E}\varphi$ and $\underline{\Phi}$ remain unchanged. Accordingly, there exists a threshold value of $\widehat{\mathcal{F}}_N$ to hold $V=V_F$. That is, $V>V_F$ holds if $\mathcal{F}_N<\widehat{\mathcal{F}}_N$.

Similarly, $\mathcal{F}_N/f=j_N(\underline{\Psi})$ and $\mathcal{F}_F/F=j_F(\underline{\Phi})$ provide

$$\frac{\partial\underline{\Psi}}{\partial\mathcal{F}_F}=0,\qquad \frac{\partial\underline{\Phi}}{\partial\mathcal{F}_F}<0.$$

Therefore, if $\mathcal{F}_F$ approaches to zero, $\underline{\Phi}$ goes to infinity, while $\mathbb{E}\varphi$ and $\underline{\Psi}$ remain unchanged. Thus, a threshold value $\widehat{\mathcal{F}}_F$ to hold $V=V_F$ exists. That is, $V>V_F$ holds if $\mathcal{F}_F>\widehat{\mathcal{F}}_F$. We now obtain the following Proposition.

Proposition 1 *Assuming σ_F, F, and $\mathcal{F}_F$ are the same as in the offline case, as well as that in the online-offline case, the indirect utility in the online-offline case is higher than that of the offline case if $\mathcal{F}_N<\widehat{\mathcal{F}}_N$ or if $\mathcal{F}_F>\widehat{\mathcal{F}}_F$.*

As we obtained in the case of only online firms, a decrease in $\mathcal{F}_N$ results in smaller $\mathcal{P}_N^*$ because of the tougher competition among online firms. On the other hand, an increase in $\mathcal{F}_F$ results in an increase in $\mathcal{P}_F^*$ because of the milder competition among offline firms. Thus, if $\mathcal{F}_N$ decreases, V increases, but V_F remains unchanged because $\mathcal{F}_F$ remains unchanged. Meanwhile, if $\mathcal{F}_F$ increases, V decreases more than V_F.

We now further examine the impact of the consumption share on welfare. Since

$$\begin{aligned} V &> V_F \Leftrightarrow \\ R^{\frac{1}{\sigma_N-1}-\frac{1}{\sigma_F-1}}\left\{\left[\frac{(\mathbb{E}\varphi)^{\sigma_N-1}}{\sigma_N f}\right]^{\frac{1}{\sigma_N-1}}\underline{\Psi}\right\}\left(\frac{\Phi}{\sigma_F F}\right)^{-\frac{1}{\sigma_F-1}} &> (1-\mu)^{-\frac{2}{\sigma_N-1}}\mu^{\frac{-\mu}{(\sigma_F-1)(1-\mu)}}, \end{aligned} \tag{29}$$

we obtain

$$\frac{\partial(1-\mu)^{-\frac{2}{\sigma_N-1}}\mu^{\frac{-\mu}{(\sigma_F-1)(1-\mu)}}}{\partial\mu} = \frac{(1-\mu)^{-\frac{2}{\sigma_N-1}}\mu^{\frac{-\mu}{(\sigma_F-1)(1-\mu)}}}{(\sigma_N-1)(\sigma_F-1)}\left[(1-\mu)(2\sigma_F-1-\sigma_N)+(\sigma_N-1)\ln\mu\right] \gtreqless 0. \tag{30}$$

Therefore, we obtain the following proposition.

Proposition 2 *As the consumption share of varieties produced by offline firms, μ, increases, a smaller $\widehat{\mathcal{F}}_N$ and/or larger $\widehat{\mathcal{F}}_F$ are needed to hold $V > V_F$ if $(1-\mu)(2\sigma_F-1-\sigma_N)+(\sigma_N-1)\ln\mu > 0$, whereas larger $\widehat{\mathcal{F}}_N$ and/or smaller $\widehat{\mathcal{F}}_F$ are needed to hold $V > V_F$ if $(1-\mu)(2\sigma_F-1-\sigma_N)+(\sigma_N-1)\ln\mu < 0$.*

If $\sigma_F = \sigma_N$, we have $(1-\mu)(2\sigma_F-1-\sigma_N)+(\sigma_N-1)\ln\mu = (\sigma_N-1)[1-\mu+\ln\mu] < 0$, which is equivalent to $\partial(1-\mu)^{-\frac{2}{\sigma_N-1}}\mu^{\frac{-\mu}{(\sigma_F-1)(1-\mu)}}/\partial\mu < 0$. That is, if $\sigma_F = \sigma_N$, as the consumption share of varieties produced by offline firms increases, obtaining $V > V_F$ becomes easier. If σ_F is sufficiently large compared with σ_N, and if μ is large enough, we obtain $\partial(1-\mu)^{-\frac{2}{\sigma_N-1}}\mu^{\frac{-\mu}{(\sigma_F-1)(1-\mu)}}/\partial\mu > 0$. That is, as the consumption share of varieties produced by offline firms increases, obtaining $V > V_F$ becomes more difficult.

In other words, if the product differentiation is the same between online firms and offline firms, as the online market becomes more dominant than the offline market, it becomes harder to have $V > V_F$. This is simply because the difference between V and V_F decreases. However, if the products of offline firms are less differentiated than those of online firms, and if the consumption share of offline products is large, it becomes easier to hold $V > V_F$ as the consumption share of varieties produced by offline firms increases. This is because the exponential of μ in the LHS of (29) becomes smaller.

Since $\sigma_N > 1$ and $\sigma_F > 1$, we rewrite the bracket of (30) as follows:

$$\begin{aligned} 2(1-\mu)(\sigma_F - 1) + (\sigma_N - 1)\left[\ln \mu - (1-\mu)\right] &\gtreqless 0 \Leftrightarrow \\ \frac{\sigma_F - 1}{\sigma_N - 1} &\gtreqless \frac{(1-\mu) - \ln \mu}{2(1-\mu)} = \frac{1}{2} - \frac{\ln \mu}{2(1-\mu)} \equiv z(\mu). \end{aligned}$$

We find that $z'(\mu) < 0$, $\lim_{\mu \to 0} z(\mu) = \infty$, and $\lim_{\mu \to 1} z(\mu) = 1$. Since it is easy to compare prices in the online market with those in the offline market, a consumer's relative choices over varieties in online market may change more than that in offline market as the relative prices of varieties change, which implies $\sigma_F < \sigma_N$. Thus, if we assume $\sigma_F < \sigma_N$, we have $\frac{\sigma_F - 1}{\sigma_N - 1} < 1$. That is, as the consumption share of varieties produced by online firms, $1-\mu$, increases, a smaller $\widehat{\mathcal{F}}_N$ or larger $\widehat{\mathcal{F}}_F$ is required to hold $V > V_F$.

Next, we examine the impact of fixed costs on $V > V_F$. Because f changes f itself and $\underline{\Psi}^*$ in the LHS of (29), we obtain:

$$\frac{\partial \left(\left(\frac{1}{f} \right)^{\frac{1}{\sigma_N - 1}} \underline{\Psi}^* \right)}{\partial f} \gtreqless 0 \Leftrightarrow \frac{\partial \underline{\Psi}^*}{\partial f} \frac{f}{\underline{\Psi}^*} \gtreqless \frac{1}{\sigma_N - 1}.$$

Using $\mathcal{F}_N = j_N(\underline{\Psi}) f$, we have:

$$\frac{\partial \underline{\Psi}}{\partial f} = -\frac{j_N(\underline{\Psi})}{j_N'(\underline{\Psi}) f} = -\frac{j_N(\underline{\Psi})}{j_N'(\underline{\Psi}) f}.$$

Since

$$\frac{j_N'(\underline{\Psi})}{j_N(\underline{\Psi})} = -(\sigma_N - 1)\left(1 + \frac{1}{k_N(\underline{\Psi})}\right)\frac{1}{\underline{\Psi}},$$

we obtain:

$$\frac{\partial \underline{\Psi}}{\partial f} = \frac{\underline{\Psi}}{(\sigma_N - 1)\left(1 + \frac{1}{k_N(\underline{\Psi})}\right) f}.$$

Thus, we obtain:

$$\frac{\partial \underline{\Psi}^*}{\partial f} \frac{f}{\underline{\Psi}^*} \gtreqless \frac{1}{\sigma_N - 1} \Leftrightarrow 1 \gtreqless 1 + \frac{1}{k_N(\underline{\Psi})}.$$

That is, we have:

$$\frac{\partial \left(\left(\frac{1}{f} \right)^{\frac{1}{\sigma_N - 1}} \underline{\Psi}^* \right)}{\partial f} < 0.$$

We also obtain:

$$\frac{\partial\left(\left(\frac{1}{f}\right)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\right)}{\partial f}\frac{f}{\left(\frac{1}{f}\right)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*}=-\frac{\frac{1}{k_N(\underline{\Psi})}}{1+\frac{1}{k_N(\underline{\Psi})}}\frac{1}{\sigma_N-1},$$

which is smaller than -1 if σ_N is larger than 2. Thus, a lower f provides a larger V, which leads to a wider gap between V and V_F. Using l'Hopital's rule, we obtain:

$$\lim_{f\to 0}\left(\frac{1}{f}\right)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*=\lim_{f\to 0}\left(\frac{1}{f}\right)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\times\lim_{f\to 0}\frac{1}{1+\frac{1}{k_N(\underline{\Psi})}}.$$

There are two possibilities: $\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*=0$ or $\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\neq 0$. If $\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*=0$, there is a contradiction with $\partial\left((1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\right)/\partial f<0$. Thus, we have $\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\neq 0$. We find that $\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\neq 0$ and $\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*=\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\times\lim_{f\to 0}\frac{1}{1+\frac{1}{k_N(\underline{\Psi})}}$ implies $\lim_{f\to 0}\frac{1}{1+\frac{1}{k_N(\underline{\Psi})}}=1\Leftrightarrow\lim_{f\to 0}1+\frac{1}{k_N(\underline{\Psi})}=1$. Thus, $\frac{\partial\underline{\Psi}^*}{\partial f}\frac{f}{\underline{\Psi}^*}=\frac{1}{\sigma_N-1}$ holds when $f\to 0$, which implies that $\partial\left((1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*\right)/\partial f=0$ holds when $f\to 0$. Thus, $\lim_{f\to 0}(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*$ becomes the positive maximum value of $(1/f)^{\frac{1}{\sigma_N-1}}\underline{\Psi}^*$ when $f\to 0$.

Furthermore, we examine the impact of F on $V>V_F$. Because F changes F itself and $\underline{\Phi}^*$ in the LHS of (29), we examine $\underline{\Phi}/F$. We have:

$$\frac{\partial\frac{\underline{\Phi}}{F}}{\partial F}=-\frac{\underline{\Phi}}{F}\frac{1}{F}+\frac{1}{F}\frac{\partial\underline{\Phi}}{\partial F}=\frac{\underline{\Phi}}{F^2}\left(-1+\frac{\partial\underline{\Phi}}{\partial F}\frac{F}{\underline{\Phi}}\right)\gtreqless 0\Leftrightarrow\frac{\partial\underline{\Phi}}{\partial F}\frac{F}{\underline{\Phi}}\gtreqless 1.$$

Using $\mathcal{F}_F=Fj_F(\underline{\Phi})$, we obtain:

$$\frac{\partial\underline{\Phi}}{\partial F}=-\frac{j_F}{Fj_F'(\underline{\Phi})}.$$

Thus, we get:

$$\begin{aligned}\frac{\partial\underline{\Phi}}{\partial F}\frac{F}{\underline{\Phi}}&=-\frac{j_F}{Fj_F'(\underline{\Phi})}\frac{F}{\underline{\Phi}}=-\frac{j_F}{j_F'(\underline{\Phi})}\frac{1}{\underline{\Phi}}=\frac{1}{1+\frac{1}{k_F(\underline{\Phi})}}<1\Leftrightarrow\\ 1&<1+\frac{1}{k_F(\underline{\Phi})}.\end{aligned}$$

That is, we obtain:

$$\frac{\partial \frac{\underline{\Phi}}{F}}{\partial F} < 0.$$

Furthermore, we obtain:

$$\frac{\partial \frac{\underline{\Phi}}{F}}{\partial F} \frac{F}{\frac{\underline{\Phi}}{F}} = -1 + \frac{\partial \underline{\Phi}}{\partial F} \frac{F}{\underline{\Phi}} = -1 + \frac{1}{1 + \frac{1}{k_F(\underline{\Phi})}} = -\frac{\frac{1}{k_F(\underline{\Phi})}}{1 + \frac{1}{k_F(\underline{\Phi})}} < 1.$$

We find that, if F approaches to ∞, $\underline{\Phi}/F$ goes to 0. Thus, we obtain the following proposition:

Proposition 3 *Assuming σ_F, F, and $\mathcal{F}_F$ are the same as in the offline case and the online-offline case, a sufficiently large F holds $V > V_F$, and a smaller f affords to lower F to hold $V > V_F$.*

When F increases, F increases more than $\underline{\Phi}^*$; when f increases, f increases more than $\underline{\Psi}^*$. In other words, the direct impacts of F and f on the indirect utility under the online-offline case surpass their indirect impacts via $\underline{\Phi}^*$ and $\underline{\Psi}^*$. That is, the competition among firms becomes milder with an increase in F or f because of increased average profit level.

7 Multiple Regions

7.1 Setup

We now consider that the economy consists of a number of symmetric regions indexed by $s = 1, 2, ..., S+1$ and $S \geq 1$. Therefore, firms trade with other S regions. Consumers in each region share the same homothetic preferences as in (1). We assume that each region is endowed with L population that supplies L units of labor inelastically. Without loss of generality, we take labor in a region as the numéraire. Symmetricity implies that the equilibrium wage rates in any two different regions are equal, that is, $w_r = w_s = 1$, $\forall r$, $\forall s \neq r$ holds.

Firm heterogeneity $H_r(\psi, \varphi)$ takes the same form among all regions such that $H_r(\psi, \varphi) = H(\psi, \varphi)$, $\forall r$. After paying the same sunk entry costs for o-type firms ($\mathcal{F}_{o,r} = \mathcal{F}_o$, $\forall r$, $o \in \{F, N\}$), a firm in region r draws its productivity index ψ and quality index φ from the cumulative distribution $H(\psi, \varphi)$. We assume that the two variables ψ and φ are independent, and firms draw from the same density function $g(\cdot)$, which implies that $h(\psi, \varphi) = g(\psi)g(\varphi)$ holds.

There exists a fixed labor requirement $f > 0$ for online firms and a fixed labor requirement $F > f$ for offline firms, respectively. Additionally, we assume that all firms are required to pay advertisement costs for their products to be visible in the foreign market. This cost is independent of the amount sold. Furthermore, offline firms need advertisement costs for each region and the advertisement costs of online firms for one region cover that for another region. The advertisement costs of an online firm are f_m units of labor, and that of an offline firm is F_m units of labor. We assume $F_m > f_m$ because online firms use online shopping malls to sell goods not only in the home market but also in foreign markets. Each firm can choose only one region to produce and sell its products to local and foreign consumers.

Each firm incurs iceberg transport costs, as in Samuelson (1954), to sell its goods in a foreign region. The costs are the same for any two regions. In specific, $\tau > 1$ units of goods must be shipped from region r to ensure the delivery of one unit to region $s \neq r$. For simplicity, we assume that the transport costs are zero within a region. Each firm's pricing rule in its domestic market is given by $p_{HF}(\psi, \varphi) = p_{HN}(\psi, \varphi) = 1/\psi$. The firms who export set the higher price $p_{ExF}(\psi, \varphi) = p_{ExN}(\psi, \varphi) = \tau/\psi$. The revenue of an o-type firm depends on its status:

$$r_o(\psi, \varphi) = \begin{cases} r_o^H(\psi, \varphi) & \text{if the } o\text{-type firm does not export,} \\ r_o^H(\psi, \varphi) + S r_o^{Ex}(\psi, \varphi) & \text{if the } o\text{-type firm exports to all regions,} \end{cases}$$

where r_o^H is the revenue of o-type firm from the home country, and r_o^{Ex} is the revenue of o-type firm from the foreign country.

Some active online firms serve consumers in all regions, while other active online firms may serve consumers only in the home market. Using (5), the profits of online firm (ψ, φ) selling their goods in the home market and foreign markets are, respectively, given by

$$\begin{aligned} \pi_N^H(\psi, \varphi) &= \frac{(1-\mu)R}{\sigma_N} \frac{\psi^{\sigma_N - 1} (\mathbb{E}\varphi)^{\sigma_N - 1}}{\mathcal{P}_N^{1-\sigma_N}} - f, \\ \pi_N^{Ex}(\psi, \varphi) &= \frac{(1-\mu)R}{\sigma_N} \frac{S \phi_N \psi^{\sigma_N - 1} (\mathbb{E}\varphi)^{\sigma_N - 1}}{\mathcal{P}_N^{1-\sigma_N}} - f_m, \end{aligned}$$

where $\phi_N \equiv \tau^{1-\sigma_N} \in (0,1)$ represents the trade freeness. Thus, the marginal exporting online firm, Ψ_{EX}, who is indifferent between exporting to the S foreign market simultaneously and only serving the local market, is given as:

$$\frac{(1-\mu)R}{\sigma_N}\frac{S\phi_N(\Psi_{EX})^{\sigma_N-1}(\mathbb{E}\varphi)^{\sigma_N-1}}{\mathcal{P}^{1-\sigma_N}}-f_m=0\Leftrightarrow(\Psi_{EX})^{\sigma_N-1}(\mathbb{E}\varphi)^{\sigma_N-1}=\frac{\sigma_N f_m\mathcal{P}_N^{1-\sigma_N}}{\phi_N(1-\mu)RS}. \tag{31}$$

Note $\mathbb{E}\varphi$ is determined by the system and is given in (31). Thus, we have:

$$\Psi_{EX}=\left[\frac{\sigma_N f_m\mathcal{P}_N^{1-\sigma_N}}{\phi_N S(\mathbb{E}\varphi)^{\sigma_N-1}(1-\mu)R}\right]^{\frac{1}{\sigma_N-1}}.$$

Meanwhile, the zero cutoff profit condition of the online firm yields:

$$(\underline{\Psi})^{\sigma_N-1}(\mathbb{E}\varphi)^{\sigma_N-1}=\frac{\sigma_N f\mathcal{P}_N^{1-\sigma_N}}{(1-\mu)R}. \tag{32}$$

Note $\mathbb{E}\varphi$ is determined by the system and given in (32). Thus, we have:

$$\underline{\Psi}=\left[\frac{\sigma f\mathcal{P}_N^{1-\sigma_N}}{(\mathbb{E}\varphi)^{\sigma_N-1}(1-\mu)R}\right]^{\frac{1}{\sigma_N-1}}.$$

It is natural to assume that the marginal exporting online firm has a higher productivity than that of the online firm with the zero cutoff profit, that is, $\Psi_{EX}>\underline{\Psi}$. That is, the online firm $\psi\in(0,\underline{\Psi})$ leaves the market, the online firm $\psi\in(\underline{\Psi},\Psi_{EX})$ produces exclusively for its domestic market, and the online firm $\psi\in(\Psi_{EX},\infty)$ produces for its domestic market and all foreign markets. Combining (31) and (32) yields:

$$\frac{\Psi_{EX}}{\underline{\Psi}}=\left(\frac{f_m}{\phi_N Sf}\right)^{\frac{1}{\sigma_N-1}}>1\Leftrightarrow f_m>\phi_N Sf. \tag{33}$$

In other words, we assume that not only is f_m sufficiently large, but also that S is sufficiently small to ensure that online firms first serve local markets and then export to foreign markets as the online firms' quality increases.

The ex-ante probability of successful entry for online firms, pe_N, and the equilibrium distribution of (ψ,φ) for incumbent online firms $\mu_N(\psi,\varphi)$ are the same between the closed and open economies. Ex-ante probability that one of these successful online firms will export is

$$pe_{NX}=\frac{\int\int_{\mathrm{I}(\Psi_{EX}^{+})}g(\psi)g(\varphi)d\psi d\varphi}{pe_N}$$

where $(\psi,\varphi)\in\mathrm{I}\big(\Psi_{EX}^{+}\big)\equiv\big\{(\psi,\varphi)\in\mathbb{R}_{+}^{2}:\psi>\Psi_{EX}\big\}$. Accordingly, the conditional distribution of productivity and quality of online firm $(\psi, \varphi) \in \mathrm{I}\left(\Psi_{EX}^{+}\right)$ exporting is given by

$$\mu_{NX}(\psi, \varphi) = \begin{cases} \dfrac{g(\psi)g(\varphi)}{pe_{NX}} & \text{if } (\psi, \varphi) \in \mathrm{I}(\Psi_{EX}^{+}) \ , \\ 0 & \text{otherwise.} \end{cases}$$

The equilibrium mass of successful online firms is M_N. Then, the equilibrium mass of exporting online firms, M_{NX}, is $M_{NX} = pe_{NX} M_N$. The total mass of varieties by online firms available to consumers in any country is $M_{NT} = M_N + RM_{NX}$.

The aggregate productivity and quality level of successful online firms is the same as in (22). The average productivity and quality level of exporting online firms is

$$\begin{aligned} \widetilde{\Psi}_{EX}(\underline{\Psi}^{+}) &= \int_0^{\infty} \int_0^{\infty} (\mathbb{E}\varphi)^{\sigma_N - 1} \psi^{\sigma_N - 1} \mu_{NX}(\psi, \varphi) d\psi d\varphi \\ &= \frac{1}{pe_{NX}} \int \int_{\mathrm{I}(\underline{\Psi}_{EX}^{+})} (\mathbb{E}\varphi)^{\sigma_N - 1} \psi^{\sigma_N - 1} g(\psi) g(\varphi) d\psi d\varphi. \end{aligned}$$

Then, the average productivity and quality is the following:

$$\widetilde{\Psi}_t = \left\{ \frac{1}{M_{NT}} \left[M_N \widetilde{\Psi} + SM_{NX} \left(\tau^{-1} \widetilde{\Psi}_{EX} \right)^{\sigma_N - 1} \right] \right\}^{\frac{1}{\sigma_N - 1}} .$$

We obtain the price index of varieties produced by online firms:

$$\mathcal{P}_N^{1-\sigma_N} = M_{NT} \widetilde{\Psi}_t.$$

Now we have

$$\begin{aligned} \bar{r}_N &= r_N^H(\widetilde{\Psi}) + pe_{NX} S r_N^{Ex}(\widetilde{\Psi}_{EX}), \\ \overline{\pi}_N &= \pi_N^H(\widetilde{\Psi}) + pe_{NX} S \pi_N^{Ex}(\widetilde{\Psi}_{EX}). \end{aligned}$$

Thus, we obtain $\bar{r}_N/\sigma = \overline{\pi}_N + f + pe_{NX} f_m$.

Using (9), the profits of offline firm (ψ, φ) selling their goods in home market and foreign markets are, respectively, given as:

$$\begin{aligned} \pi_F^H(\psi, \varphi) &= \frac{\mu R}{\sigma_F} \frac{\psi^{\sigma_F - 1} \varphi^{\sigma_F - 1}}{\mathcal{P}_F^{1-\sigma_F}} - F, \\ \pi_F^{Ex}(\psi, \varphi) &= \frac{\mu R}{\sigma_F} \frac{S \phi_F \psi^{\sigma_F - 1} \varphi^{\sigma_F - 1}}{\mathcal{P}_F^{1-\sigma_F}} - SF_m. \end{aligned}$$

where $\phi_F \equiv \tau^{1-\sigma_F} \in (0,1)$ represents the trade freeness.

Thus, the marginal exporting offline firm, $(\psi^{\mathrm{FM}}, \varphi^{\mathrm{FM}})$, who is indifferent between exporting to S foreign markets simultaneously and only serving the local market, is given as:

$$\left(\psi^{\mathrm{FM}}\right)^{\sigma_F-1} \left(\varphi^{\mathrm{FM}}\right)^{\sigma_F-1} \equiv \Phi_{EX} = \frac{\sigma_F F_m \mathcal{P}_F^{1-\sigma_F}}{\mu R \phi_F}. \tag{34}$$

Meanwhile, the zero cutoff profit condition of the offline firm yields the survival offline firm, $(\psi^{\mathrm{FS}}, \varphi^{\mathrm{FS}})$, given as:

$$\left(\psi^{\mathrm{FS}}\right)^{\sigma_F-1} \left(\varphi^{\mathrm{FS}}\right)^{\sigma_F-1} \equiv \underline{\Phi} = \frac{\sigma_F F \mathcal{P}_F^{1-\sigma_F}}{\mu R}. \tag{35}$$

It is also natural to assume that the marginal exporting offline firm has a higher productivity than that of the survival offline firm, that is, $\Phi_{EX} > \underline{\Phi}$. That is, the offline firm $\Phi \in (0, \underline{\Phi})$ exits the market, the offline firm $\Phi \in (\underline{\Phi}, \Psi_{EX})$ produces exclusively for the domestic market, and the offline firm $\Phi \in (\Phi_{EX}, \infty)$ produces for its domestic market and all foreign markets. Combining (34) and (35) yields:

$$\frac{\Phi_{EX}}{\underline{\Phi}} = \frac{F_m}{\phi_F F} > 1. \tag{36}$$

The ex-ante probability of successful entry for offline firms, pe_F, and the equilibrium distribution of (ψ, φ) for incumbent offline firms $\mu_F(\psi, \varphi)$ are the same between the closed and open economies. Ex-ante probability that one of these successful offline firms will export is

$$pe_{FX} = \frac{\int \int_{\mathrm{I}(\Phi_{EX}^{+})} g(\psi) g(\varphi) d\psi d\varphi}{pe_F}$$

where $(\psi, \varphi) \in \mathrm{I}(\Phi_{EX}^{+}) \equiv \left\{(\psi, \varphi) \in \mathbb{R}_{+}^{2} : \varphi^{\sigma_F-1}\psi^{\sigma_F-1} > \Phi_{EX}\right\}$. Accordingly, the conditional distribution of productivity and quality of the offline firm $(\psi, \varphi) \in \mathrm{I}\left(\Phi_{EX}^{+}\right)$ exporting is given by

$$\mu_{FX}(\psi, \varphi) = \begin{cases} \dfrac{g(\psi)g(\varphi)}{pe_{FX}} & \text{if } (\psi, \varphi) \in \mathrm{I}(\Phi_{EX}^{+}) , \\ 0 & \text{otherwise.} \end{cases}$$

The equilibrium mass of successful offline firms is M_F. Then, the equilibrium mass of exporting offline firms, M_{FX}, is $M_{FX} = pe_{FX} M_F$. The total mass of varieties by offline firms available to consumers in any country is $M_{FT} = M_F + S M_{FX}$.

The aggregate productivity and quality level of successful offline firms is the same as in (16). The average productivity and quality level of exporting offline firms is

$$\widetilde{\Phi}_{EX}\left(\underline{\Phi}_{EX}^{+}\right) \equiv \int_0^\infty \int_0^\infty \varphi^{\sigma_F-1}\psi^{\sigma_F-1}\mu_{FX}(\psi,\varphi)d\psi d\varphi = \frac{1}{pe_{FX}} \int\int_{\mathrm{I}\left(\underline{\Phi}_{EX}^{+}\right)} \varphi^{\sigma_F-1}\psi^{\sigma_F-1}g(\psi)g(\varphi)d\psi d\varphi.$$

Then, the average productivity and quality is

$$\widetilde{\Phi}_t = \left\{\frac{1}{M_{FT}}\left[M_F\widetilde{\Phi} + SM_{FX}\left(\tau^{-1}\widetilde{\Phi}_X\right)^{\sigma_F-1}\right]\right\}^{\frac{1}{\sigma_F-1}}.$$

We obtain the price index of varieties produced by offline firms:

$$\mathcal{P}_F^{1-\sigma_F} = M_{FT}\widetilde{\Phi}_t.$$

Now we have

$$\begin{aligned}
\bar{r}_F &= r_F^H(\widetilde{\Phi}) + pe_{FX}Sr_F^{Ex}(\widetilde{\Phi}_{EX}), \\
\overline{\pi}_F &= \pi_F^H(\widetilde{\Phi}) + pe_{FX}S\pi_F^{Ex}(\widetilde{\Phi}_{EX}).
\end{aligned}$$

Thus, we obtain

$$\bar{r}_F/\sigma = \bar{\pi}_F + F + pe_{FX}SF_m.$$

7.2 Equilibrium

Using the zero cutoff conditions, we obtain the profits of an online firm (ψ,φ) and an offline firm (ψ,φ), respectively, given as:

$$\begin{aligned}
\pi_F^H(\psi,\varphi) &= \left(\frac{\varphi^{\sigma_F-1}\psi^{\sigma_F-1}}{\underline{\Phi}} - 1\right)F, \qquad \pi_N^H(\psi,\varphi) = \left(\frac{\psi^{\sigma_N-1}}{\underline{\Psi}^{\sigma_N-1}} - 1\right)f, \\
\pi_F^{Ex}(\psi,\varphi) &= \left(\frac{\varphi^{\sigma_F-1}\psi^{\sigma_F-1}}{\Phi_{EX}} - 1\right)F_m, \qquad \pi_N^{Ex}(\psi,\varphi) = \left(\frac{\psi^{\sigma_N-1}}{\Psi_{EX}^{\sigma_N-1}} - 1\right)f_m.
\end{aligned}$$

The zero cutoff profit conditions $\pi_N^H(\underline{\Psi}^*) = 0$, $\pi_F^H(\underline{\Phi}^*) = 0$, $\pi_N^{Ex}(\Psi_{EX}^*) = 0$ and $\pi_F^{Ex}(\Phi_{EX}^*) = 0$ are, respectively, equivalent to $\pi_N^H(\widetilde{\Psi}) = fk(\underline{\Psi}^*)$, $\pi_F^H(\widetilde{\Phi}) = Fk(\underline{\Phi}^*)$, $\pi_N^{Ex}(\widetilde{\Psi}_{EX}) = f_mk(\Psi_{EX}^*)$ and $\pi_F^{Ex}(\widetilde{\Phi}_{EX}) = F_mk(\Phi_{EX}^*)$ where k is defined as before. Thus, we have

$$\begin{aligned}
\overline{\pi}_N &= fk(\underline{\Psi}^*) + pe_{NX}Sf_mk(\Psi_{EX}^*), \\
\overline{\pi}_F &= Fk(\underline{\Phi}^*) + pe_{FX}SF_mk(\Phi_{EX}^*).
\end{aligned}$$

The free entry conditions are given by $\mathcal{F}_N = pe_N \overline{\pi}_N$ and $\mathcal{F}_F = pe_F \overline{\pi}_F$. Thus, we obtain

$$\begin{aligned} fj(\underline{\Psi}^*) + Sf_m j(\Psi^*_{EX}) &= \mathcal{F}_N, & (37) \\ Fj(\underline{\Phi}^*) + SF_m j(\Phi^*_{EX}) &= \mathcal{F}_F. & (38) \end{aligned}$$

Since $\frac{\Psi_{EX}}{\underline{\Psi}} = \left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}$ and $\frac{\Phi_{EX}}{\underline{\Phi}} = \frac{F_m}{\phi_F F}$, (37) and (38) are the function of $\underline{\Psi}$ and $\underline{\Phi}$, respectively. Because j is decreasing function from infinity to zero on $(0, \infty)$, there exists a unique value of $\underline{\Psi}^*$ and a unique value of $\underline{\Phi}^*$. Accordingly, we obtain the equilibrium values of price indices,

$$\mathcal{P}_F = \left(\frac{\sigma_F F}{\mu R \underline{\Phi}^*}\right)^{\frac{1}{\sigma_F - 1}}$$

and

$$\mathcal{P}_N = \left[\frac{\sigma_N f}{(1-\mu) R (\mathbb{E}\varphi)^{\sigma_N - 1}}\right]^{\frac{1}{\sigma_N - 1}} \frac{1}{\underline{\Psi}^*}.$$

We also have $M_F = pe_F \mathcal{M}_F$, $M_N = pe_N \mathcal{M}_N$, $M_{FX} = pe_{FX} M_F$, and $M_{NX} = pe_{NX} M_N$. Meanwhile, the market-clearing condition for labor is given by

$$\begin{aligned} L &= \mathcal{M}_F \mathcal{F}_F + M_F \frac{\sigma_F - 1}{\sigma_F} \bar{r}_F + M_F F + M_{FX} S F_m \\ &\quad + \mathcal{M}_N \mathcal{F}_N + M_N \frac{\sigma_N - 1}{\sigma_N} \bar{r}_N + M_N f + M_{NX} f_m. \end{aligned}$$

where $\mathbb{E}\varphi = \int_0^\infty \varphi g(\varphi) d\varphi / \int_0^\infty g(\varphi) d\varphi$. Using $\mathcal{F}_N = pe_N \overline{\pi}_N$ and $\mathcal{F}_F = pe_F \overline{\pi}_F$, the market-clearing condition for labor can be written by:

$$L = pe_F \mathcal{M}_F \sigma_F \left(\overline{\pi}_F + F + pe_{FX} S F_m\right) + pe_N \mathcal{M}_N \sigma_N \left(\overline{\pi}_N + f + pe_{FX} f_m\right). \quad (39)$$

Using the definitions of pe_F and pe_N, we can obtain $pe^*_F(\underline{\Phi}^*)$ and $pe^*_N(\underline{\Psi}^*)$, respectively. Accordingly, we obtain $\widetilde{\Phi}(\underline{\Phi}^*)$ and $\widetilde{\Psi}(\underline{\Psi}^*)$, respectively. Therefore, we obtain the following:

$$M^*_{FT} = \mathcal{P}_F^{1-\sigma_F} / \widetilde{\Phi}(\underline{\Phi}^*) \qquad \text{and} \qquad M^*_{NT} = \mathcal{P}_N^{1-\sigma_N} / \widetilde{\Psi}(\underline{\Psi}^*).$$

Combining zero cut-off profit conditions for online and offline firms with $M^*_F = pe^*_F(\underline{\Phi}^*)\mathcal{M}_F$ and $M^*_N = pe^*_N(\underline{\Psi}^*)\mathcal{M}_N$, we obtain (27). Combining (39) and (27) yields $\mathcal{M}^*_F$ and $\mathcal{M}^*_N$

in equilibrium as follows:

$$\mathcal{M}_N^* = \frac{L}{\sigma_N p e_N^*(\underline{\Psi}^*)\left[\frac{(1-\mu)f}{\mu F}\frac{\underline{\Phi}^*\widetilde{\Psi}(\underline{\Psi}^*)}{\widetilde{\Phi}(\underline{\Phi}^*)(\underline{\Psi}^*)^{\sigma_N-1}(\mathbb{E}\varphi)^{\sigma_N-1}}\left(\bar{\pi}_F + F + pe_{FX}SF_m\right) + \bar{\pi}_N + f + pe_{FX}f_m\right]},$$

and

$$\mathcal{M}_F^* = \frac{L}{\sigma_F p e_F^*(\underline{\Phi}^*)\left[\frac{\mu F}{(1-\mu)f}\frac{\widetilde{\Phi}(\underline{\Phi}^*)(\underline{\Psi}^*)^{\sigma_N-1}(\mathbb{E}\varphi)^{\sigma_N-1}}{\underline{\Phi}^*\widetilde{\Psi}(\underline{\Psi}^*)}\left(\bar{\pi}_N + f + pe_{FX}f_m\right) + \bar{\pi}_F + F + pe_{FX}SF_m\right]}.$$

The indirect utility is the same as (28).

7.3 Increase of F_m and f_m

We investigate the effects of an increase in F_m and f_m in the following. Using (36) and (38),

$$\partial\Phi_{EX} = \frac{F_m}{\phi_F F}\partial\underline{\Phi} + \frac{\underline{\Phi}}{\phi_F F}\partial F_m$$

and

$$Fj'(\underline{\Phi}^*)\partial\underline{\Phi}^* + SF_m j'(\Phi_{EX}^*)\partial\Phi_{EX}^* + Sj(\Phi_{EX}^*)\partial F_m = 0$$

yields

$$\frac{\partial\underline{\Phi}}{\partial F_m} = -\frac{S\left[F_m j'(\Phi_{EX}^*)\frac{\underline{\Phi}}{\phi_F F} + j(\Phi_{EX}^*)\right]}{Fj'(\underline{\Phi}^*) + SF_m j'(\Phi_{EX}^*)\frac{F_m}{\phi_F F}}.$$

Since $\frac{j_F'(\Phi_{EX}^*)\Phi_{EX}^*}{j_F(\Phi_{EX}^*)} = -\left(1 + \frac{1}{k_F(\Phi_{EX}^*)}\right)$, we obtain

$$\frac{\partial\underline{\Phi}}{\partial F_m} = \frac{Spe_F}{Fj'(\underline{\Phi}^*) + SF_m j'(\Phi_{EX}^*)\frac{F_m}{\phi_F F}} < 0.$$

Thus, we obtain

$$\frac{\partial\Phi_{EX}^*}{\partial F_m} = \frac{-Fj'(\underline{\Phi}^*)\frac{\partial\underline{\Phi}^*}{\partial F_m} - Sj(\Phi_{EX}^*)}{SF_m j'(\Phi_{EX}^*)} > 0.$$

Using (33) and (37),

$$\frac{\Psi_{EX}}{\underline{\Psi}} = \left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}$$

and

$$fj(\underline{\Psi}^*) + Sf_m j(\Psi^*_{EX}) = \mathcal{F}_N$$

yield

$$\partial\Psi_{EX} = \left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}} \partial\underline{\Psi} + \frac{1}{\sigma_N - 1}\Psi_{EX}\frac{\phi_N S f}{f_m}\partial f_m$$

and

$$fj'(\underline{\Psi}^*)\partial\underline{\Psi}^* + Sf_m j'(\Psi^*_{EX})\partial\Psi^*_{EX} + Sj(\Psi^*_{EX})\partial f_m = 0.$$

Thus, we obtain

$$fj'(\underline{\Psi}^*)\partial\underline{\Psi}^* + Sf_m j'(\Psi^*_{EX})\left[\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}} \partial\underline{\Psi} + \frac{1}{\sigma_N - 1}\Psi_{EX}\frac{\phi_N S f}{f_m}\partial f_m\right] + Sj(\Psi^*_{EX})\partial f_m = 0,$$

which is equivalent to

$$\frac{\partial\underline{\Psi}}{\partial f_m} = \frac{-S\left[j'(\Psi^*_{EX})\frac{1}{\sigma_N - 1}\Psi_{EX}\phi_N S f/f_m + j(\Psi^*_{EX})\right]}{fj'(\underline{\Psi}^*) + Sf_m j'(\Psi^*_{EX})\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}}.$$

Note that $\frac{j'_N(\Psi^*_{EX})\Psi^*_{EX}}{j_N(\Psi^*_{EX})} = -(\sigma_N - 1)\left(1 + \frac{1}{k(\Psi^*_{EX})}\right)$. Thus, we have

$$\frac{\partial\underline{\Psi}}{\partial f_m} = \frac{-Sj(\Psi^*_{EX})\left[-\left(1 + \frac{1}{k(\Psi^*_{EX})}\right)\phi_N S f/f_m + 1\right]}{fj'(\underline{\Psi}^*) + Sf_m j'(\Psi^*_{EX})\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}}.$$

Since $\frac{\Psi_{EX}}{\underline{\Psi}} = \left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}} < 1$, we obtain

$$\frac{\partial\underline{\Psi}}{\partial f_m} < 0.$$

Furthermore, we obtain

$$\frac{\partial\Psi^*_{EX}}{\partial f_m} = \frac{-fj'(\underline{\Psi}^*)\frac{\partial\underline{\Psi}^*}{\partial f_m} - Sj(\Psi^*_{EX})}{Sf_m j'(\Psi^*_{EX})} > 0.$$

The results obtained in this subsection are the same as the results in Melitz (2003).

7.4 Decrease in the transport costs

We investigate the impact of an increase in transport costs in the following. Using (36) and (38),

$$\partial\Phi_{EX} = \frac{F_m}{\phi_F F}\partial\underline{\Phi} - \frac{F_m}{\phi_F^2 F}\underline{\Phi}\partial\phi_F$$

and

$$Fj'(\underline{\Phi}^*)\partial\underline{\Phi} + SF_m j'(\Phi_{EX}^*)\partial\Phi_{EX} = 0 \Leftrightarrow \frac{\partial\Phi_{EX}}{\partial\phi_F} = -\frac{Fj'(\underline{\Phi}^*)\frac{F_m}{\phi_F^2 F}\underline{\Phi}}{Fj'(\underline{\Phi}^*) + SF_m j'(\Phi_{EX}^*)\frac{F_m}{\phi_F F}}$$

yields

$$\frac{\partial\underline{\Phi}}{\partial\phi_F} = \frac{SF_m j'(\Phi_{EX}^*)\frac{F_m}{\phi_F^2 F}}{Fj'(\underline{\Phi}^*) + SF_m j'(\Phi_{EX}^*)\frac{F_m}{\phi_F F}} > 0$$

and

$$\frac{\partial\Phi_{EX}}{\partial\phi_F} = -\frac{F_m}{\phi_F F}\frac{Fj'(\underline{\Phi}^*)}{Fj'(\underline{\Phi}^*) + SF_m j'(\Phi_{EX}^*)\frac{F_m}{\phi_F F}}\frac{1}{\phi_F} < 0.$$

In other words, as transport costs decrease, $\underline{\Phi}$ increases and Φ_{EX} decreases as shown in Melitz (2003).

Using (33) and (37),

$$\partial\Psi_{EX} = -\frac{1}{\sigma_N - 1}\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}\frac{1}{\phi_N}\underline{\Psi}\partial\phi_N + \left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}\partial\underline{\Psi}$$

and

$$fj'(\underline{\Psi}^*)\partial\underline{\Psi} + Sf_m j'(\Psi_{EX}^*)\partial\Psi_{EX} = 0$$

yields

$$\frac{\partial\Psi_{EX}}{\partial\phi_N} = -fj'(\underline{\Psi}^*)\frac{\frac{1}{\sigma_N - 1}\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}\frac{1}{\phi_N}\underline{\Psi}}{fj'(\underline{\Psi}^*) + Sf_m j'(\Psi_{EX}^*)\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}} < 0$$

and

$$\frac{\partial\Psi_{EX}}{\partial\phi_N} = -fj'(\underline{\Psi}^*)\frac{\frac{1}{\sigma_N - 1}\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}\frac{1}{\phi_N}\underline{\Psi}}{fj'(\underline{\Psi}^*) + Sf_m j'(\Psi_{EX}^*)\left(\frac{f_m}{\phi_N S f}\right)^{\frac{1}{\sigma_N - 1}}} < 0.$$

In other words, as transport costs decrease, $\underline{\Psi}$ increases and Ψ_{EX} decreases as shown in Melitz (2003).

7.5 Increase in the number of trading partners

We investigate the impact of an increase in S in the following. Using (36) and (38),

$$\partial \Phi_{EX} = \frac{F_m}{\phi_F F} \partial \underline{\Phi}$$

and

$$F j'(\underline{\Phi}^*) \partial \underline{\Phi} + S F_m j'(\Phi^*_{EX}) \partial \Phi_{EX} + F_m j(\Phi^*_{EX}) \partial S = 0$$

yields

$$\frac{\partial \underline{\Phi}}{\partial S} = \frac{-F_m j(\Phi^*_{EX}) \underline{\Phi}}{F j'(\underline{\Phi}^*) \underline{\Phi} + S F_m j'(\Phi^*_{EX}) \Phi_{EX}} > 0.$$

Thus, we also have $\partial \Phi_{EX} / \partial S > 0$, which is the same as shown in Melitz (2003). The increase in the number of trading partners provides the least productive firms to exit because of the increase in the average profit level.

Using (33) and (37), we have

$$\partial \Psi_{EX} = \left(\frac{f_m}{\phi_N S f} \right)^{\frac{1}{\sigma_N - 1}} \partial \underline{\Psi} - \frac{1}{\sigma_N - 1} \left(\frac{f_m}{\phi_N S f} \right)^{\frac{1}{\sigma_N - 1}} \underline{\Psi} \frac{1}{S} \partial S$$

and

$$f j'(\underline{\Psi}^*) \partial \underline{\Psi} + S f_m j'(\Psi^*_{EX}) \partial \Psi_{EX} + f_m j(\Psi^*_{EX}) \partial S = 0,$$

which yields

$$\frac{\partial \underline{\Psi}}{\partial S} = \frac{f_m j'(\Psi^*_{EX}) \frac{1}{\sigma_N - 1} \Psi_{EX} \underline{\Psi} - f_m j(\Psi^*_{EX}) \underline{\Psi}}{f j'(\underline{\Psi}^*) \underline{\Psi} + S f_m j'(\Psi^*_{EX}) \Psi_{EX}} > 0. \tag{40}$$

This result shows the same sign as shown in Melitz (2003), however, the size of the impact is larger than that of Melitz (2003). More precisely, if we assume that exporting online firms incur fixed costs for each remote area, the first term in the numerator of (37) disappears. That is, with an increase in S, $\underline{\Psi}$ increases more with the cost-saving of exporting online firms by the utilization of online market than without cost-saving.

Then, we obtain

$$\partial \Psi_{EX} = \Psi^*_{EX} \left[\frac{-f_m j(\Psi^*_{EX}) - \frac{1}{\sigma_N - 1} \underline{\Psi}^* \frac{1}{S} f j'(\underline{\Psi}^*)}{f j'(\underline{\Psi}^*) \underline{\Psi}^* + S f_m j'(\Psi^*_{EX}) \Psi^*_{EX}} \right] \partial S.$$

Therefore, we have

$$\frac{\partial \Psi_{EX}}{\partial S} \gtreqless 0 \Leftrightarrow f_m j(\Psi^*_{EX}) + \frac{1}{\sigma_N - 1} \underline{\Psi} \frac{1}{S} f j'(\underline{\Psi}^*) \gtreqless 0. \tag{41}$$

The second term in (41) appears due to the new setting on the fixed costs of online firms, which weakens the increase in Ψ_{EX} with S or leads to a decrease in Ψ_{EX} with S. This result differs from that of Melitz (2003).

Because $\underline{\Psi}^*$ and $\underline{\Phi}^*$ increase with S, we find that indirect utility increases with S. Because $\underline{\Psi}^*$ increases more with cost-saving than without cost-saving, we obtain the following proposition.

Proposition 4. The indirect utility with cost-saving on exporting is higher than indirect utility without cost-saving in the online market.

This result is obtained due to the tough competition in the online market.

7.6 Increase in the lowest quality of online goods

Setting the lowest quality of online goods as $\underline{\varphi}$, we examine the impact of an increase in the lowest quality of online goods on the following:

$$\mathbb{E}\varphi(\underline{\varphi}) = \frac{\int_{\underline{\varphi}}^{\infty} \varphi g(\varphi) d\varphi}{\int_{\underline{\varphi}}^{\infty} g(\varphi) d\varphi}.$$

Note that we set $\underline{\varphi} = 0$ above. The increase in $\underline{\varphi}$ may result from the regulation by the provider of the platform or the mechanism to clarify the reputation of online goods. We have:

$$\frac{\partial \mathbb{E}\varphi}{\partial \underline{\varphi}} = \frac{-\underline{\varphi} g(\underline{\varphi})}{\int_{\underline{\varphi}}^{\infty} g(\varphi) d\varphi} + \frac{g(\underline{\varphi}) \int_{\underline{\varphi}}^{\infty} \varphi g(\varphi) d\varphi}{\left[\int_{\underline{\varphi}}^{\infty} g(\varphi) d\varphi\right]^2} = \frac{g(\underline{\varphi})}{\int_{\underline{\varphi}}^{\infty} g(\varphi) d\varphi} \left[-\underline{\varphi} + \mathbb{E}\varphi(\underline{\varphi})\right] > 0.$$

In other words, an increase in $\underline{\varphi}$ provides a higher expected value of φ. The derivation process of $\underline{\Psi}$ shows that the increase in $\underline{\varphi}$ does not change $\underline{\Psi}$. As a result, the increase in $\underline{\varphi}$ decreases $\mathcal{P}_N$, which implies an increase in the indirect utility.

Proposition 5. The means to increase the lowest quality of online goods improve the indirect utility.

8 Conclusion

Our analysis shows how online and offline firms behave under a clear distinction between productivity and quality. These channels may affect online and offline firms differently. In particular, the entry costs, the fixed costs or the consumption of online and offline firms work via two channels in the opposite direction such that the indirect utility under the coexistence of online and offline firms is higher than that under the existence of only offline firms.

With regard to covid-19, we focus on the increase in the consumption share from online market as shown in Watanabe and Omori (2021). Since it is easy to compare prices in the online market with those in the offline market, the elasticities of substitution in the online market may be larger than that in the offline market. Thus, we find that, as the consumption share of varieties produced by online firms increases, a smaller initial investment cost of online firms is needed or a larger initial investment cost of offline firms is allowed to ensure that the indirect utility under the coexistence of online and offline firms is higher than that under the existence of only offline firms. Furthermore, this paper analyzes the cost-saving of exporting online firms, which shows a clear difference between online and offline markets. We show that this property of the online market improves welfare. Furthermore, we show that regulation by the supplier of the platform or the mechanism to clarify the reputation of online goods provides higher welfare.

As a future direction of research, an analysis of the online market platform, which is not included in this paper, remains to be done.

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