Unconditionally Stable Space-Time Finite Element Method for the Shallow Water Equations

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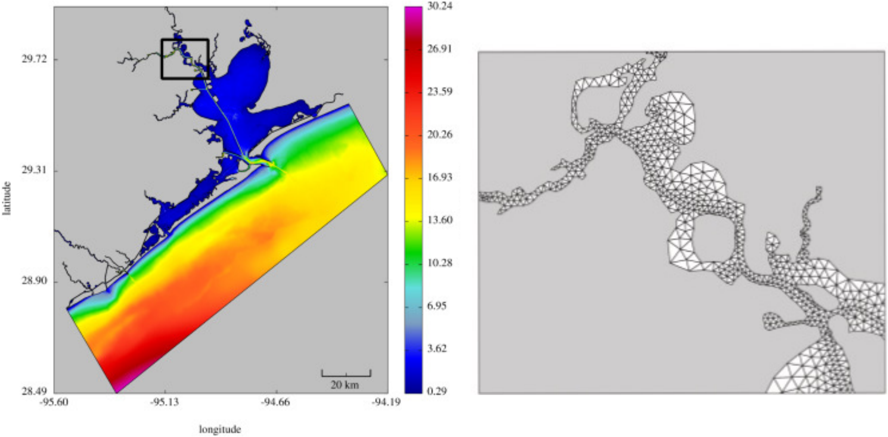
The Coastal Domain



Galveston Bay bathymetry and example mesh[1]

[1] S.R. Brus et al. "High-order discontinuous Galerkin methods for coastal hydrodynamics applications". In: *Computer Methods in Applied Mechanics and Engineering* 355 (2019), pp. 860 –899. ISSN: 0045-7825.

Mathematical Model - The 2D Shallow Water Equations

Find $(\zeta, \mathbf{u})$ such that:

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{\nabla} \cdot (\zeta \mathbf{u}) = 0, \text{ in } \Omega \times (0, T]$$
$$\frac{\partial \mathbf{u}}{\partial t} + \underbrace{\mathbf{u} \cdot (\boldsymbol{\nabla} \mathbf{u})}_{\textit{convective}} + \underbrace{\tau_b \mathbf{u}}_{\textit{friction}} + g \boldsymbol{\nabla} \zeta - \underbrace{\nu \Delta \mathbf{u}}_{\textit{viscous}} = \mathbf{f}, \text{ in } \Omega \times (0, T] \quad (1)$$

Corresponding Space-Time Weak Formulation:

Find $(\zeta, \mathbf{u}, \boldsymbol{\sigma}) \in U(\Omega_T)$ such that:

$$B((\zeta, \mathbf{u}, \boldsymbol{\sigma}), (v, \mathbf{w}, \mathbf{p})) = F(v, \mathbf{w}, \mathbf{p}), \quad \forall (v, \mathbf{w}, \mathbf{p}) \in V(\mathcal{P}_h) \quad (2)$$

- Test space $V(\mathcal{P}_h)$ is constructed using the philosophy of the discontinuous Petrov-Galerkin method[2]

[2] Leszek Demkowicz and Jay Gopalakrishnan. "A class of discontinuous Petrov–Galerkin methods. II. Optimal test functions". In: *Numerical Methods for Partial Differential Equations* 27.1 (2011), pp. 70–105.

Remarks

- In the corresponding finite element discretization we use classical *continuous* basis functions for the trial space, whereas the test space consist of *optimal* discontinuous functions computed on-the-fly locally on each element that guarantee discrete stability.
- The discontinuous Petrov-Galerkin philosophy leads to a linear system of equations that is symmetric and positive definite.
- Element wise Riesz representers of the approximation error yield error indicators that are employed to drive adaptive mesh refinements in both space and time.
- The discontinuity of the test functions and Riesz representers makes the method well suited for the parallel environment in TACC computers.

▶ As a verification, consider a purely convective 2D stationary problem with $\Omega = (0,1)^2$, $\zeta_{ex} = \cos(x - y \cdot x)$ $u^x_{ex} = u^y_{ex} = \cos^2(\pi x + y)\sin(\pi x^3 + y \cdot x)\sin(\pi x + y)$.

Adaptive Mesh Refinement