5-2021

Fundamental Aspects of Black Holes

Jacob Fisher Ciafre
Utah State University

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February 11, 2020

Abstract

The literature study here seeks to present the foundations of black hole physics in General Relativity. The report includes a discussion of the Kerr black hole metric, black hole entropy, particle creation, the laws of black hole mechanics, and a bilinear mass formula for the Kerr-Newman black hole solution.

Contents

1 Introduction

Black holes have been at peak public interest in the recent years following the results of the first event horizon telescope [1]. However, crucial fundamental work pertaining to black holes has been ongoing for the last century. In particular, physicists from the 70s produced substantial work for general laws of black holes and relationships of a common solution to Einstein's famous equation [2, 3, 4, 5]. It is widely agreed upon that the most common equilibrium for a black hole is stationary and axisymmetric [5]. The Kerr black hole solution [6] describes such case with zero charge, and the extension to the charged case is Kerr-Newman [7]. This report is organized in the following sections that each represent a publication that help lend to a better understanding of black holes in General Relativity.

Section 2 will focus on Roy Kerr's paper [6]. Kerr finds a solution to Einstein's field equation of a stationary black hole can be described by mass and angular momentum. The solution is asymptotically flat and axisymmetric. The paper shows one method for finding a solution to Einstein's equation. Section 3 examines Bekenstein's literature [3] and shows the relationship between entropy of a black hole and the black holes event horizon area. Section 4 discusses Hawking's work [5] works to approximate the quantum mechanical effects in a curved black hole background. Additionally, a generalized Second Law of Black Holes is presented. Section 5 uses Bardeen's publication [4] to present the four Laws of Black Hole Mechanics. Section 6 outlines ideas primarily from [2]. A bilinear mass formula is found using the fundamental concepts discussed in previous sections. In addition, the Kerr-Newman solution to the Einstein-Maxwell's field equation is examined and practical calculations are shown which find the Hawking temperature, angular velocity, and electric potential.

2 Kerr Black Hole Metric

The Kerr black hole metric is necessary for describing black holes in the context of General Relativity as they are likely to be found in nature. Absent of charge, the description of black holes represented by the Kerr metric are dependent on the irreducible mass and the rotational energy. Beginning with a set of four null vectors consisting of two real and their complex conjugate pair, a line element is constructed.

$$ds^2 = 2tt^* + 2wk \tag{1}$$

The vectors (t, t^*, w, k), otherwise referred to as a null tetrad, forms a complete vector basis over 3+1 dimensional spacetime. This is a formalism for problems in General Relativity constructed by Ezra T. Newman and Sir Roger Penrose, often shortened to NP formalism. Kerr [6] chooses a coordinate system (r, u, ζ) of convenient symmetries, note that ζ is a complex coordinate given by $\zeta = \xi + i\eta$. The complex conjugate is denoted by *.

$$\begin{aligned}
t =& P(r + i\Delta)d\zeta \\
k =& du + 2\,\mathrm{Re}(\Omega d\zeta) \\
w =& dr - 2\,\mathrm{Re}\{[(r - i\Delta)\dot{\Omega} + iD\Delta)d\zeta\} + \{r\dot{P}/P \\
& + \mathrm{Re}\left[P^{-2}D\left(D^*\ln P + \dot{\Omega}^*\right)\right] + \frac{m_1 r - m_2\Delta}{r^2 + \Delta^2}\right\} k
\end{aligned} \tag{2}$$

where

$$\begin{aligned}D =& \partial/\partial\zeta - \Omega\partial/\partial u \\ \Delta =& \operatorname{Im}\left(P^{-2}D^{*}\Omega\right) \\ m =& m_1 + im_2\end{aligned}$$

The dot notation for derivatives is in the direction of u. Kerr then examines the solutions to Einstein's field equation for a variety of cases. One case holds these restrictions on the functions (P, Ω, Δ).

$$\begin{aligned}\dot{\Delta} = \dot{\Omega} = \dot{P} = 0 \\ P \neq 1\end{aligned}$$

Examining the Killing vectors Kerr [6] establishes a metric that is stationary, axially symmetric and asymptotically flat. A useful tool for showing the asymptotic limit of this solution is presented in Section 7.

$$\begin{aligned}ds^2 = \left(r^2 + a^2\cos^2\theta\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) + 2\left(du + a\sin^2\theta d\phi\right) \\ \times\left(dr + a\sin^2\theta d\phi\right) - \left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right) \\ \times\left(du + a\sin^2\theta d\phi\right)^2\end{aligned} \tag{3}$$

Lastly, a transformation to separate the spacelike and timelike coordinates, relative to infinity.

$$(r - ia)e^{i\phi}\sin\theta = x + iy, \quad r\cos\theta = z, \quad u = t + r$$

Yields the final metric in terms of real constant a associated with the black hole rotation and irreducible mass m.

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3}{r^4 + a^2z^2}(k)^2 \tag{4}$$

where

$$\begin{aligned}\left(r^2 + a^2\right)rk = r^2(xdx + ydy) + ar(xdy - ydx) \\ + \left(r^2 + a^2\right)(zdz + rdt) \\ r^4 - \left(R^2 - a^2\right)r^2 - a^2z^2 = 0, \quad R^2 = x^2 + y^2 + z^2\end{aligned}$$

3 Entropy

In order to establish an entropy relation with the physical quantities of a given black hole the loss of information into its system must be defined. A particle that exists outside of an event horizon of a black hole has some known information associated with it. Minimally, it is known that the particle exists. Upon crossing an event horizon it becomes unclear if that particle exists. The black hole event horizon acts as a unidirectional membrane in this sense for matter. Therefore, some information has been lost in the particle crossing the event horizon of a black hole. Bekenstein [3] finds a generalized Christodoulou's method for inserting a particle of nonzero radius into a black hole, such that the particle minimally increases the area of the event horizon. This draws the link between entropy and area of the event horizon. Given a spherical particle of rest mass μ and proper radius b the area increase ΔA of a Kerr black hole is given by equation (5).

$$(\Delta A)_{\min} = 2\mu b \tag{5}$$

In the limit of a point particle the return is Christodoulou's original method which leaves the irreducible mass unchanged. Although, the physical limitations are dictated by the Compton wavelength. Given a small enough particle to be dictated by quantum effects the lower limit of area increase is such.

$$(\Delta A)_{\text{min}} \geq 2\hbar$$

Using the concept of placing particles inside of a black hole the conclusion is that black hole entropy is some monotonically increasing function of its rationalized area. It is also worth noting that Hawking's theorem [5] agrees with this choice.

$$S_{bh} = f(A)$$

Pick a convenient form for f, and examine the consequences.

$$f(A) = \gamma A$$

γ is a constant. Take two black holes that are placed at a distance such that their gravitational interaction is weak. After merging they settle down to equilibrium there is a single new black hole. The new area of this resultant black hole exceeds the sum of the initial parts area, $(A_{bh})_3 > (A_{bh})_1 + (A_{bh})_2$ this is later discussed as the Second Law of Black Hole Thermodynamics. The Second Law of Black Holes is not violated does not contradict the entropy increase that must occur, since this formation of a new black hole is an irreversible process, if the entropy is proportional by a constant to event horizon area, $S_{\text{bh}} \propto A$. Bekenstein [3] finds that in dimensionless form the entropy of a black hole is thus given in terms of event horizon area A.

$$S_{\text{bh}} = \frac{c^3 A}{4G\hbar} \tag{6}$$

4 Particle Creation

The goal of this section is to approximate the quantum mechanical effects in a curved black hole background [5]. Quantum gravitational effects should be taken into consideration when discussing black hole mechanics. Look no further than the role of quantum mechanics impact on matter fields. Taking a classical spacetime metric that is a solution to Einsteins equation of General Relativity and work with matter fields that obey the wave equation. Dealing with the annihilation and creation operators, a_k and $a_k^\dagger$, in this context is difficult and in the spirit of covariance cannot be split into positive and negative frequency components. It is defined such that the vacuum state cannot annihilate any particles.

$$\boldsymbol{a}_k|0\rangle = 0 \quad \text{for all } k \tag{7}$$

Require that $\{f_k\}$ and $\{\bar{f}_k\}$ together form a complete basis for solutions of the following wave equation integrated over a suitable surface S.

$$\frac{1}{2}i\int_S \left(f_k\bar{f}_{j;a} - \bar{f}_j f_{k;a}\right) d\Sigma^a = \delta_{kj} \tag{8}$$

This does not fix the subspace in which $\{f_k\}$ is chosen to be positive, with reference to the usual Minkowski sign convention. Choose that the positive solutions correspond to flat or asymptotically flat regions. Finally, consider a space described by the solution that contains an initial flat region that transitions to a region of curvature and then to a final flat region. The vacuum state in the first sub-region is not the same as the vacuum state in the third sub-region. If the regions (1,2,3) are denoted by subscripts then the following can be stated [5].

$$(\boldsymbol{a}_k)_1 |0\rangle = 0$$
$$(\boldsymbol{a}_k)_3 |0\rangle \neq 0$$

An interpretation of this is that the gravitational field has caused particle creation. The question should then be, how good is this approximation or in other words, what is the breakdown of this method for regions of high curvature like ones found inside a black hole?

This classical geometry-quantum treatment of matter should be valid apart from the first $10^{-43}\,s$ of the universe. This analysis is done in the following. Take an observer in a localized region, a inertial coordinate system to that observer can be constructed. Now let the radius of that coordinate system be determined by an upper bound on the curvature of spacetime. Using this setup and the field equation (8) two cases occur. In high frequency modes, in relation to the curvature, the indeterminacy between annihilation operators is exponentially small. However, low frequency modes create and uncertainty in $a_i^\dagger a_i$. Luckily, this will only occur in curvatures hidden by an event horizon.

Hawking [5] also states that black holes emit particles at a steady rate consistent with a body of temperature $\frac{\kappa}{2\pi}$. First discussed by Beckenstein, the idea of a Generalized Second Law of black holes is expanded upon. The particles created outside of the event horizon along with other matter has an entropy S and the entropy of the black hole has entropy $\frac{A}{4}$. The sum $S + \frac{A}{4}$ never decreases.

5 Laws of Black Hole Mechanics

A black hole solution that is stationary axisymmetric asymptotically flat has a timelike, near infinity, and rotational Killing vectors labeled here as K^a and $\tilde{K}^a$ respectively [4]. These Killing vectors obey the relations,

$$\begin{aligned} K_{a;b} &= K_{[a;b]} \\ \tilde{K}_{a;b} &= \tilde{K}_{[a;b]} \\ K_{a;b}\tilde{K}^b &= \tilde{K}_{a;b}K^b \\ K_b^{a;b} &= -R_b^a K^b \\ \tilde{K}_b^{a;b} &= -R_b^a \tilde{K}^b \end{aligned} \tag{9}$$

The standard notation is that covariant derivatives are expressed by semicolons and square brackets imply antisymmetrization. The mass can be found by integrating the time translational Killing vector relation over a 2 surface that is spacelike, asymptotically flat, and tangent to the rotational Killing vector.

$$-4\pi M = \int_{\partial S} K^{a;b} d\Sigma_{ab} = -\int_S R_b^a K^b d\Sigma_a \tag{10}$$

The mass is expressed in integral terms of the mass outside the event horizon and mass of the black hole respectively.

$$M = \int_S \left(2T_a^b - T\delta_a^b\right) K^a d\Sigma_b + \frac{1}{4\pi}\int_{\partial B} K^{a;b} d\Sigma_{ab} \tag{11}$$

where

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \tag{12}$$

Similarly, the rotational Killing vector relation can be integrated to find the angular momentum. In terms of outside and black hole rotational energy contributions.

$$J = -\int_S T_b^a \tilde{K}^b d\Sigma_a - \frac{1}{8\pi}\int_{\partial B} \tilde{K}^{a;b} d\Sigma_{ab} \tag{13}$$

Define a null vector $l^a = dx^a/dt$ such that it is tangent to the generators of the horizon. The time coordinate as usual is the distance along its corresponding Killing vector K^a. In terms of Killing vectors and angular velocity of the black hole the defined vector is expressed as such.

$$l^a = K^a + \Omega \tilde{K}^a \tag{14}$$

The mass expression from before now can be represented.

$$M = \int_S \left(2T_a^b - T\delta_a^b\right) K^a d\Sigma_b + 2\Omega J + \frac{1}{4\pi}\int_{\partial B} l^{a;b} d\Sigma_{ab} \tag{15}$$

where

$$J = -\frac{1}{8\pi}\int_{\delta B} \tilde{K}^{a;b} d\Sigma_{ab} \tag{16}$$

The last term is of equation (15) is the surface tension of the black hole and area of the event horizon. Here is an organized mass integral form for a stationary, axisymmetric asymptotically flat black hole with some outside contribution T_{ab}.

$$M = \int_S \left(2T_a^b - T\delta_a^b\right) K^a d\Sigma_b + 2\Omega J + \frac{\kappa}{4\pi} A \tag{17}$$

Zeroth Law

A stationary black hole has a surface tension κ, defined on the event horizon of surface ∂B, which is constant over the entire region ∂B. The surface temperature is related to the Hawking temperature of the black hole in the following [4].

$$T = \frac{\kappa}{2\pi} \tag{18}$$

First Law

Bardeen [4] also finds the change in mass for any neighboring solutions that are stationary and asymmetric.

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega \delta J + \int \Omega \delta dJ + \int \bar{\mu} \delta dN + \int \bar{\theta} \delta dS \tag{19}$$

The integral terms included describe a perfect fluid behaving a circular flow to a central black hole. The surface tension κ is related by a constant to so-called Hawking temperature of the black hole shown in equation (18) and the area is related to entropy seen in equation (6). Therefore, the first term of equation (19) is often written as follows.

$$\frac{\kappa}{8\pi}\delta A = T\delta S \tag{20}$$

Second Law

The change in area of an event horizon over any length of time is positive.

$$\delta A \geqq 0 \tag{21}$$

Moreover, should two black holes merge, the resultant event horizon area A_3 of the new object is larger than the sum of the preexisting parts $A_1 + A_2$.

$$A_3 > A_1 + A_2$$

This law is stronger than the thermodynamic counterpart. Entropy can be transferred from one system to another given the total entropy does not increase. However, event horizon area is non-transferable in this tone, but that each individual change in area is positive.

Third Law

The surface tension κ can not be reduced to zero by finite operations. The contribution of each particle added in attempt to reduce κ goes to zero as the limit of surface tension approaches zero. Additionally, the violation of this law lends mechanisms for creating a naked singularity.

6 Bilinear Mass Formula

In the framework of the Kerr-Newman black hole solution that solves Einstein-Maxwell equations, a convenient relation of the mass in terms of the physical invariants can be constructed. In Boyer-Lindquist coordinates the line element follows.

$$\begin{aligned} ds^2 &= -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 \\ &+ \left(dt - a\sin^2\theta d\phi\right)^2 \frac{\Delta}{\rho^2} \\ &- \left(\left(r^2 + a^2\right)d\phi - adt\right)^2 \frac{\sin^2\theta}{\rho^2} \end{aligned} \tag{22}$$

where
$\rho^2 = r^2 + a^2\cos^2\theta$
$\Delta = r^2 + a^2 - 2Mr + Q^2$
$a = \frac{J}{M}$
True to no-hair theorem, which is discussed later in this section, the black hole is described by mass M, rotational Kerr parameter a, and electric charge Q. The radial equation $\Delta = 0$ gives the radii of the inner and outer event horizon respectively.

$$r_- = M - \sqrt{M^2 - a^2 - Q^2} \tag{23}$$
$$r_+ = M + \sqrt{M^2 - a^2 - Q^2} \tag{24}$$

with the restriction $0 \leq a^2 + Q^2 \leq M^2$. Following the prescription for finding the area of the outer event horizon $r = r_+$ at some time $t = \textit{constant}$ then $dt, dr, \Delta \to 0$. The metric at the event horizon is then,

$$dA^2 = g_{(\theta,\theta)}d\theta^2 + g_{(\phi,\phi)}d\phi^2 \tag{25}$$

where

$$\begin{aligned} g_{(\theta,\theta)} &= -\rho^2 \\ g_{(\phi,\phi)} &= -\frac{\sin^2\theta}{\rho^2}\left(r^2+a^2\right)^2 \end{aligned}$$

Thus the outer event horizon can be constructed.

$$\begin{aligned} A &= \int_0^{\pi}\int_0^{2\pi} d\theta d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} \\ &= 4\pi\left(a^2+r_+^2\right) \\ &= 4\pi\left(2M^2-Q^2+2M\sqrt{M^2-\left(\frac{J}{M}\right)^2-Q^2}\right) \end{aligned} \tag{26}$$

Equation (26) can be inverted to find the irreducible mass in terms of event horizon area A, angular momentum J, and charge Q.

$$M = \sqrt{\frac{A}{16\pi}+\frac{4\pi J^2}{A}+\frac{Q^2}{2}+\frac{\pi Q^4}{A}} \tag{27}$$

Smarr [2] uses first law of black holes, equation (19), to express the differential mass. The differential mass formula is an exact differential and thus can be integrated over any convenient path belonging to (A,J,Q) space. The subscripts denote parameters fixed along integration path.

$$dM = TdA + \Omega dJ + \Phi dQ \tag{28}$$

$$\left(\frac{\partial M}{\partial A}\right)_{J,Q} = T,\ \left(\frac{\partial M}{\partial J}\right)_{A,Q} = \Omega,\ \left(\frac{\partial M}{\partial Q}\right)_{A,J} = \Phi$$

Using the mass from (26) the Hawking temperature T, angular velocity Ω, and electric potential Φ are found respectively for the Kerr-Newman black hole.

$$\begin{aligned} T &= \frac{1}{32\pi M}-\frac{2\pi J^2}{MA^2}-\frac{\pi Q^4}{2MA^2} \\ \Omega &= \frac{4\pi J}{MA} \\ \Phi &= \frac{Q}{2M}+\frac{2\pi Q^3}{MA} \end{aligned} \tag{29}$$

The no hair theorem [8] refers to the concept that a black hole can be described by three quantities, the irreducible mass, rotational energy, and charge. The final bilinear form for a black hole in an asymptotically flat black hole background that relates the event horizon area quantities important in the no hair theorem:

$$M = 2TA + 2\Omega J + \Phi Q \tag{30}$$

7 Appendix: Mathematica and EDCRTCode

Wolfram Mathematica and other symbolic software have become vital to the progression of studying systems in General Relativity a useful package created by Sotirios Bonanos will be discussed. The overall goal of the package is to calculate Riemannian Geometry tenors and provide tensor operations within the Mathematica framework. Wolfram Mathematica supplies a free download in their achieve and it is open source in the sense that the underlying Mathematica code can be viewed freely. Below is a useful table for remembering the short hand.

Quick Reference Guide		
Description	Retrievable	Definition
Metric	gdd	$g_{\beta v}$
Christoffel Symbols	GUdd	$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]$
Riemann Tensor	RUddd	$R^{\alpha}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\beta\mu}$
Ricci Tensor	Rdd	$R_{\beta v} = R^{\alpha}_{\beta\alpha v}$
Ricci Scalar	R	$R = g^{\beta v} R_{\beta v}$
Einstein Tensor	EUd	$G^{\mu}_{v} = R^{\mu}_{v} - \frac{1}{2} g^{\mu}_{v} R$

The following example illustrates the power of this package and Mathematica using only five lines of input, highlighted in yellow. The additional text is the output.

```
1) << EDCRGTCcode.m
2) coord = {t, r, θ, ϕ};
```

3) Metric =

$$\begin{pmatrix} \frac{-a^2 \mathrm{Sin}[\theta]^2 + \Delta[r]}{\rho[r,\theta]^2} & 0 & 0 & \frac{a \mathrm{Sin}[\theta]^2 (a^2 + r^2 - \Delta[r])}{\rho[r,\theta]^2} \\ 0 & -\frac{\rho[r,\theta]^2}{\Delta[r]} & 0 & 0 \\ 0 & 0 & -\rho[r,\theta]^2 & 0 \\ \frac{a \mathrm{Sin}[\theta]^2 (a^2 + r^2 - \Delta[r])}{\rho[r,\theta]^2} & 0 & 0 & \frac{\mathrm{Sin}[\theta]^2 \left(-(a^2+r^2)^2 + a^2 \mathrm{Sin}[\theta]^2 \Delta[r]\right)}{\rho[r,\theta]^2} \end{pmatrix};$$

```
4) RGtensors[Metric, coord]
```

SetDelayed: Tag Classify in Classify[x_] is Protected.

gdd =

$$\begin{pmatrix} \frac{-a^2 \mathrm{Sin}[\theta]^2 + \Delta[r]}{\rho[r,\theta]^2} & 0 & 0 & \frac{a \mathrm{Sin}[\theta]^2 (a^2 + r^2 - \Delta[r])}{\rho[r,\theta]^2} \\ 0 & -\frac{\rho[r,\theta]^2}{\Delta[r]} & 0 & 0 \\ 0 & 0 & -\rho[r,\theta]^2 & 0 \\ \frac{a \mathrm{Sin}[\theta]^2 (a^2 + r^2 - \Delta[r])}{\rho[r,\theta]^2} & 0 & 0 & \frac{\mathrm{Sin}[\theta]^2 \left(-(a^2+r^2)^2 + a^2 \mathrm{Sin}[\theta]^2 \Delta[r]\right)}{\rho[r,\theta]^2} \end{pmatrix}$$

LineElement =

$$\frac{2 a\, d[t]\, d[\phi]\, \mathrm{Sin}[\theta]^2 (a^2 + r^2 - \Delta[r])}{\rho[r,\theta]^2} - \frac{d[t]^2 (a^2 \mathrm{Sin}[\theta]^2 - \Delta[r])}{\rho[r,\theta]^2} + \frac{d[\phi]^2 \mathrm{Sin}[\theta]^2 (-a^4 - 2a^2 r^2 - r^4 + a^2 \mathrm{Sin}[\theta]^2 \Delta[r])}{\rho[r,\theta]^2} - d[\theta]^2 \rho[r,\theta]^2 - \frac{d[r]^2 \rho[r,\theta]^2}{\Delta[r]}$$

gUU =

$$\begin{pmatrix} \frac{(a^4 + 2a^2 r^2 + r^4 - a^2 \mathrm{Sin}[\theta]^2 \Delta[r])\, \rho[r,\theta]^2}{(-a^2 - r^2 + a^2 \mathrm{Sin}[\theta]^2)^2 \Delta[r]} & 0 & 0 & \frac{a (a^2 + r^2 - \Delta[r])\, \rho[r,\theta]^2}{(-a^2 - r^2 + a^2 \mathrm{Sin}[\theta]^2)^2 \Delta[r]} \\ 0 & -\frac{\Delta[r]}{\rho[r,\theta]^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho[r,\theta]^2} & 0 \\ \frac{a (a^2 + r^2 - \Delta[r])\, \rho[r,\theta]^2}{(-a^2 - r^2 + a^2 \mathrm{Sin}[\theta]^2)^2 \Delta[r]} & 0 & 0 & -\frac{\mathrm{Csc}[\theta]^4 (-a^2 + \mathrm{Csc}[\theta]^2 \Delta[r])\, \rho[r,\theta]^2}{(-a^2 + a^2 \mathrm{Csc}[\theta]^2 + r^2 \mathrm{Csc}[\theta]^2)^2 \Delta[r]} \end{pmatrix}$$

```
gUU computed in 0.031 sec
Gamma computed in 0.047 sec
Riemann(dddd) computed in 0.375 sec
Riemann(Uddd) computed in 1.485 sec
Ricci computed in 0.718 sec
Weyl computed in 1.75 sec
Einstein computed in 0.719 sec
All tasks completed in 5.125 seconds
```

```
5) GUdd[[1, 3, 4]]
```

Out[4]= $\frac{a^3 \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta]^3 (a^2 + r^2 - \Delta[r])}{(-a^2 - r^2 + a^2 \mathrm{Sin}[\theta]^2)^2}$

The steps chosen are explained as follows.

1. Import the package
2. Choose a set of coordinates, in this case Boyer-Lindquist
3. Write the metric in matrix notation, here the stationary axially symmetric rotating non-charged solution is chosen. This is the same Kerr solution presented in section 1 but with a more convenient set of coordinate system. Example: the upper left corner is metric component g_{tt} and bottom right is $g_{\phi\phi}$
4. Pass the metric and coordinate choices are passed to a function that calculates geometry tensors some of which were discussed earlier
5. Recall the Christoffel Symbol $\Gamma^{t}_{\theta\phi}$

In general, steps 1-4 provide a powerful tool in analyzing a given spacetime and checking if a solution solves Einsteins equation. The package is also able to recognize when a spacetime metric is flat space. Taking the solution presented in equation (3) at the asymptotic limit such that $m, a \to 0$, metric resembles the following.

```
<< EDCRGTCcode.m
coord = {r, θ, ϕ, u};
```

$$\text{Metric} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\,\text{Sin}[\theta]^2 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix};$$

```
RGtensors[Metric, coord]
SetDelayed: Tag Classify in Classify[x_] is Protected.
```

$$\text{gdd} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\,\text{Sin}[\theta]^2 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{LineElement} = 2\,d[r]\,d[u] - d[u]^2 + r^2\,d[\theta]^2 + r^2\,d[\phi]^2\,\text{Sin}[\theta]^2$$

$$\text{gUU} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{\text{Csc}[\theta]^2}{r^2} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
gUU computed in 0. sec
Gamma computed in 0. sec
Riemann(dddd) computed in 0. sec
Flat Space!
Aborted after 0.015625 seconds
```

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