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# Effect of material stiffness variation on shakedown solutions of soils under moving loads

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**ABSTRACT:** Shakedown limits of pavements and railway foundations can be calculated based on shakedown theorems. These values can be used to guide the thickness designs of pavement and railway constructions considering material plastic properties. However, most existing shakedown analyses were carried out by assuming a unique stiffness value for each material. This paper mainly concentrates on the influence of stiffness variation on the shakedown limits of pavements and railway foundations under moving loads. Finite element models as well as a user subroutine UMAT are first developed to obtain the elastic responses of soils considering a linearly increasing stiffness modulus with depth. Then, based on the lower-bound shakedown theorem, shakedown solutions are obtained by searching for the most critical self-equilibrated residual stress field. It is found that for a single-layered structure, the rise of a stiffness changing ratio will give a larger shakedown limit; and the increase is more pronounced when the friction angle is relatively high. For multi-layered pavement and railway systems, neglecting the stiffness variation may overestimate the capacity of the structures.

## INTRODUCTION

Most existing pavement and railway design approaches were developed based on empirical data or elastic theory. However, contribution of material plastic properties to the capacity of pavements or railway foundations was not taken into account. Also, the long-term behavior of pavements and railway foundations subjecting to repeated moving traffic loads was not well considered in most of the existing design approaches. Many filed and laboratory results have demonstrated that when a soil structure is subjected to a moving or cyclic load whose magnitude is larger than the yield limit but smaller than another limit, no further deformation can be observed after some permanent deformation in the first number of load cycles (e.g. Larew and Leonard 1962; Lekarp and Dawson 1998; Werkmeister 2004; Brown et al. 2012 etc.). This phenomenon is known as shakedown and the limit load is termed as shakedown limit. By introducing the shakedown concept, long-term responses of pavements and railway foundations to moving traffic load can be predicted. Shakedown limit, therefore, can be considered as a more rational design load for pavement and railway design against excessive settlement.

Shakedown limits can be determined directly using Melan's (1938) static shakedown theorem (e.g. Sharp and Booker 1984; Raad et al. 1988; Yu and Hossain 1998;

Krabbenhoft at al. 2007; Yu and Wang 2012; Wang and Yu 2013a, 2014) or Koiter's kinematic shakedown theorem (e.g. Collins and Clifffe 1987; Boulbibane and Weichert 1997; Li and Yu 2006). As Melan's static shakedown theorem satisfies internal equilibrium equations and stress boundary conditions, it provides a lower bound to the true shakedown limit; therefore it is also named as lower bound shakedown theorem. Koiter's kinematic shakedown theorem satisfies compatibility condition for plastic strain rate and boundary conditions for velocity and therefore it can be used to predict the upper bound of the true shakedown limit. An advantage of the shakedown approach based on these two fundamental shakedown theorems is that the details of the successive elastic-plastic stress fields are not required. Besides, some shakedown solutions have been verified by using numerical step-by-step analyses (e.g. Wang and Yu, 2013b; Liu et al., 2016).

Pavement and railway systems are layered structures with diverse material properties in those layers. Even within a single type of materials, the material property may also vary at different locations. Typically, the stiffness of soil increases with depth. Stiffness variation with depth has been considered for solving footing problems (e.g. Boswell and Scott 1975; Stark and Booker 1997). However, most of the existing shakedown solutions for pavements and railway foundations were conducted by assuming homogenous materials. In the present study, by assuming a quasi-static response of pavement and railway systems to traffic loads, the effect of material stiffness variation with depth on the shakedown limits will be assessed by using a lower bound shakedown approach.

## PROBLEM DEFINITION

This study considers that a pressure repeatedly moves on the surface of a three-dimensional half-space along one direction (x-direction). The half-space is made of layers of elastic-perfectly plastic materials. The elastic behavior is described by Young's modulus $E$ and Poisson's ratio $v$, while the Young's modulus varies linearly with depth z for one material:

$$E_n(\mathrm{z}) = E_{0n}\left(1 + \mathrm{z}\beta_n\right) \tag{1}$$

where $E_{0n}$ indicates the Young's modulus on the top of $\mathrm{n}^{\mathrm{th}}$ layer; $E_n(\mathrm{z})$ indicates the Young's modulus at depth $z$ of the $\mathrm{n}^{\mathrm{th}}$ layer and $\beta_{\mathrm{n}}$ is defined as stiffness variation ratio. The plastic behaviors of the materials obey the Mohr-Coulomb model.

The present study will investigate the shakedown limit for two cases considering the effect of stiffness variation. The first case assumes a contact pressure moves on a pavement. The second one considers a train load-induced pressure travels on a railway foundation.

## LOWER BOUND SHAKEDOWN ANALYSIS

Melan's static shakedown theorem states that an elastic-perfectly plastic structure under cyclic or variable loads will shakedown if a self-equilibrated residual stress field exists such that its superposition with a load induced elastic stress field does not exceed the yield criterion anywhere in the structure. It means three components are essential for the calculation of the shakedown limit, which are elastic stress field, residual stress field and yield criterion. In this study, the Mohr-Coulomb criterion is assumed for all materials. For the elastic stress field in the three-dimensional layered system, numerical calculation is required to obtain solutions. In this study, finite element analyses using the commercial software ABAQUS are conducted. The stress-strain relation (Eq. 2) considering the change of Young's Modulus with depth are programmed into a UMAT subroutine and integrated with the software.

$$\begin{bmatrix}\delta\varepsilon_{xx}\\ \delta\varepsilon_{yy}\\ \delta\varepsilon_{zz}\\ \delta\varepsilon_{xy}\\ \delta\varepsilon_{xz}\\ \delta\varepsilon_{zz}\end{bmatrix}=\begin{bmatrix}1/(E_0+E_0\beta z) & -\upsilon/(E_0+E_0\beta z) & -\upsilon/(E_0+E_0\beta z) & & & \\ -\upsilon/(E_0+E_0\beta z) & 1/(E_0+E_0\beta z) & -\upsilon/(E_0+E_0\beta z) & & & \\ -\upsilon/(E_0+E_0\beta z) & -\upsilon/(E_0+E_0\beta z) & 1/(E_0+E_0\beta z) & & & \\ & & & (1+\nu)/(E_0+E_0\beta z) & & \\ & & & & (1+\nu)/(E_0+E_0\beta z) & \\ & & & & & (1+\nu)/(E_0+E_0\beta z)\end{bmatrix}\begin{bmatrix}\delta\sigma_{xx}\\ \delta\sigma_{yy}\\ \delta\sigma_{zz}\\ \delta\sigma_{xy}\\ \delta\sigma_{xz}\\ \delta\sigma_{yz}\end{bmatrix} \quad (2)$$

Two models are established for the pavement case and the railway case respectively. In both cases, only half of the models are simulated in which x-axis represents the travel direction and the plane y = 0 is the symmetric plane. All elements are chosen as C3D20R which stands for Continuum, three-dimensional, 20-noded, reduced-integrated element. The pavement model consists of an asphalt layer, a subbase layer and subsoil, as shown in Figure 1, where $h_n$ respects the thickness of the $n^{th}$ layer. The wheel-pavement contact load ($P$) distributes within a circle of radius $a$ with a maximum pressure $p_{max} = 3P/2\pi a^2$, which can be formulated as:

$$p=\frac{p_{\max}}{a}(a^2-x^2-y^2)^{1/2} \quad (3)$$

The railway model is shown in Figure 2 considering the supporting structure for a typical Rheda 2000 single track. Four axle loads belonging to two adjacent bogies on two carriages are considered. The equivalent reaction modulus ($k_{sub}$) for this system is estimated based on Vesic (1961)'s method and therefore the vertical stress distribution on the top of this structure can be determined based on the beam on elastic foundation theory using the determined $k_{sub}$ value. In the transverse direction (y), the pressure is assumed to be uniformly distributed over y = 0 to y = 1.7m, because a concrete base with a half-width of 1.7m is considered to be located on the top. The pressure distribution used in this study is shown in Figure 2 where $p_{max}$ indicates the maximum pressure. The load distributions are applied on the models using another user subroutine DLOAD.

In order to obtain the residual stress field, the critical residual stress fields of Yu and Wang (2012) will be used, as formulated in Eq. 4. These critical residual stresses are obtained by making sure the total stress state of one point just touches the Mohr-Coulomb yield surface at each depth z=j while fulfilling the self-equilibrium and boundary conditions.

$$\begin{aligned} \sigma^{r}_{xx\text{-min}} &= \min_{z=j}(-M_i + \sqrt{-N_i}) \\ \sigma^{r}_{xx\text{-max}} &= \max_{z=j}(-M_i - \sqrt{-N_i}) \end{aligned} \tag{4}$$

with

$$\begin{aligned} M &= \lambda\sigma^{e}_{xx} - \lambda\sigma^{e}_{zz} + 2\tan\phi_n(c_n - \lambda\sigma^{e}_{zz}\tan\phi_n), \\ N &= 4(1+\tan^2\phi_n)[(\lambda\sigma^{e}_{xz})^2 - (c_n - \lambda\sigma^{e}_{zz}\tan\phi_n)^2]. \end{aligned}$$

in which $i$ represents a general point at depth z=j; $\lambda$ is a dimensionless load factor; $\phi$ is material friction angle, c is material cohesion; $\sigma_{ij}^{e}$ is elastic stress field induced by a unit $p_{max}$; the subscript $n$ ($n$ = 1, 2, 3…) means the $n^{th}$ layer. Tension positive notation is applied throughout this paper.

By substituting the elastic stress fields and either of the critical residual stress fields into the Mohr-Coulomb yield criterion $f(\sigma) \leq 0$, the present shakedown problem is presented as a mathematical optimisation problem:

$$\begin{aligned} &\max \lambda \\ &\text{s.t.}\begin{cases} f(\sigma^{r}_{xx}(\lambda\sigma^{e}), \lambda\sigma^{e}) \leq 0, \\ \sigma^{r}_{xx}(\lambda\sigma^{e}) = \sigma^{r}_{xx\text{-max}} \text{ or } \sigma^{r}_{xx\text{-min}}. \end{cases} \end{aligned} \tag{5}$$

The mathematical optimization process is programmed using MATLAB. For each layer, one maximum admissible $\lambda$ could be found, marked as $\lambda^{n}_{sd}$, and therefore $\lambda^{n}_{sd}p_{max}$ is the shakedown limit of the $n^{th}$ layer. The minimum value among all them $\lambda_{sd}p_{max}$ is then recorded as the shakedown limit of the whole structure.

## RESULTS AND DISCUSSIONS

### Model Verification

A single-layered problem is first investigated by giving identical material properties ($E_0$=100MPa, $v$=0.3) to all layers. Zero stiffness variation ratios are applied to all layers and the results are compared with the analytical solutions of Wang and Yu (2013a). It is found the maximum difference is 3.6%.

Figure 3 also demonstrates the effect of the stiffness variation ratio on the shakedown limit of the single-layered structure. It indicates that the shakedown limit increases with the stiffness variation ratio and the change is more pronounced when the friction angle is high.

**Pavement Case**

Typical material properties are shown in Table 1 for a pavement structure considering a temperature of 35°C. If all the materials are with constant stiffness, the shakedown limit of the layered structure is 445kPa and failure will initiate in the second layer. Figure 4 demonstrates the effect of the stiffness variation ratio ($\beta$) on the shakedown limits. For all the cases considered here, the second layer is always the most critical layer and shakedown limits of the structure are always smaller than 445kPa. Therefore, neglecting the stiffness variation property may overestimate the pavement shakedown limit. Moreover, because the increase of $\beta_1$ and $\beta_3$ leads to a larger shakedown limit of the second layer, it can be expected that the shakedown limit of the second layer could exceed 445kPa when $\beta_1$ or $\beta_3$ is large enough. However, a reduction in the shakedown limit can be observed along with the rise of $\beta_2$.

**Railway Case**

Concerning a typical three-layered track system whose material properties are given in Table 2, the second layer will fail prior to the other two layers for the constant stiffness case. The shakedown limit of the layered structure is 16kPa. Figure 5 illustrates the effect of the stiffness variation ratio on the shakedown limits. It can be seen that the second layer is still the weakest layer. In addition, it is found that the increasing stiffness variation ratio in one layer will result in a smaller shakedown limit in that layer but larger shakedown limits in the other two layers. Comparing with the constant stiffness case, changes of the stiffness variation ratios can lead to either larger or smaller shakedown limit. For instant, larger shakedown limits can be observed when $\beta_3$ is larger than 2.16 (Figure 5a) or $\beta_2$ is smaller than 0.667 (Figure 5c). Besides, it can be expected that distress may first occur in the first layer instead of the second layer when $\beta_1$ is large enough.

## CONCLUSIONS

(1) The lower bound shakedown approach has been adopted to solve pavement and railway problems considering a linearly increasing stiffness modulus with depth.. Finite element models and a user subroutine UMAT have been developed to obtain the elastic responses of soils. The results are well validated.

(2) For the single-layered problems, the rise of the stiffness variation ratio leads to a larger shakedown limit. More obvious changes can be observed when the friction angle is high.

(3) For the typical layered pavement system, the second layer is the most critical layer. The shakedown limit of the layered system increases with the rise of $\beta_1$ and $\beta_3$ as well as the decrease of $\beta_2$. Neglecting the stiffness variation property may overestimate the pavement shakedown limit.

(4) For the typical railway system, the second layer is also the most critical layer. Increasing stiffness variation ratio in one layer results in a smaller shakedown limit in that layer, but larger shakedown limits in the other two layers. Comparing with the constant stiffness cases, the consideration of material stiffness variation lead to either larger or smaller shakedown limit of the railway foundation.

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## TABLES

**Table. 1 Material properties for a pavement**

| | $E_{0n}$ (MPa) | $\nu_n$ | $h_n$ (m) | $c_n$ (kPa) | $\phi_n$ (°) |
|---|---|---|---|---|---|
| Asphaltic layer | 690 | 0.3 | 0.4 | 1 | 45 |
| Subbase | 75 | 0.3 | 0.3 | 2 | 35 |
| Subsoil | 15 | 0.4 | ∞ | 10 | 0 |

**Table. 2 Material properties for a railway foundation**

| | $E_{0n}$ (MPa) | $\nu_n$ | $h_n$ (m) | $c_n$ (kPa) | $\phi_n$ (°) |
|---|---|---|---|---|---|
| Anti-frozen layer | 200 | 0.3 | 0.4 | 300 | 30 |
| Subgrade bed | 150 | 0.3 | 2.3 | 2 | 40 |
| Subsoil | 60 | 0.4 | ∞ | 10 | 0 |

## FIGURES

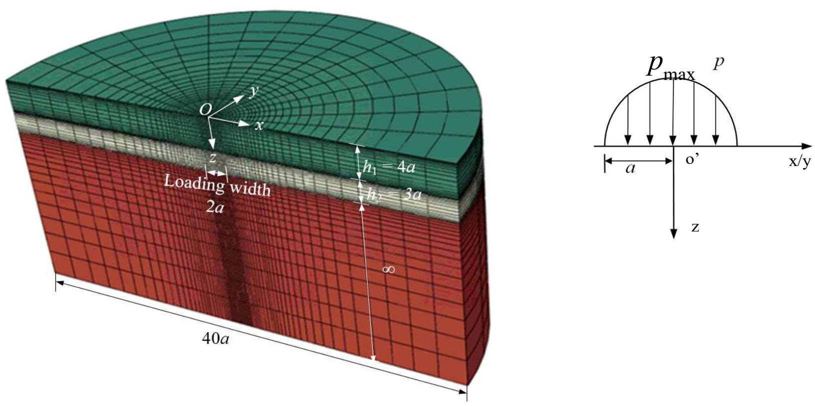


**FIG. 1. Load distribution and finite element model for a pavement**

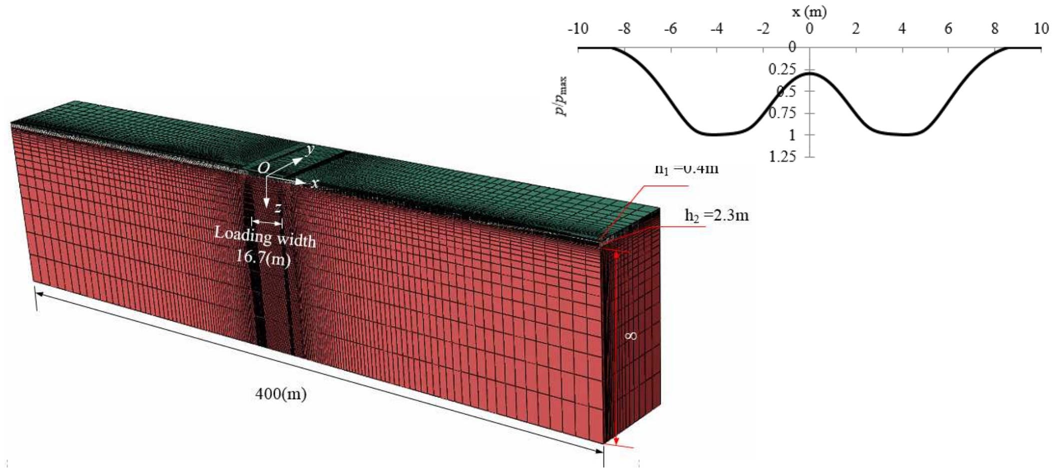


**FIG. 2. Load distribution and finite element model for a railway foundation**

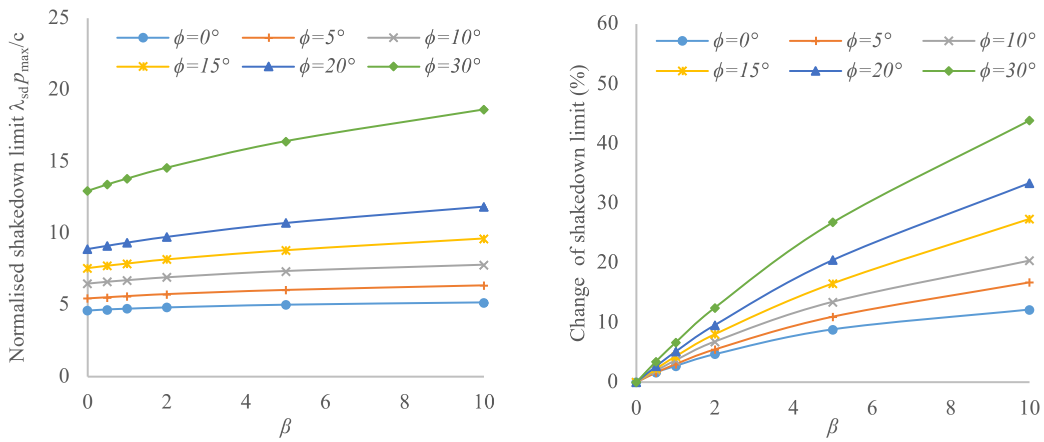


**FIG. 3. Effect of stiffness variation ratio on the shakedown limits of a single-layered soil structure**

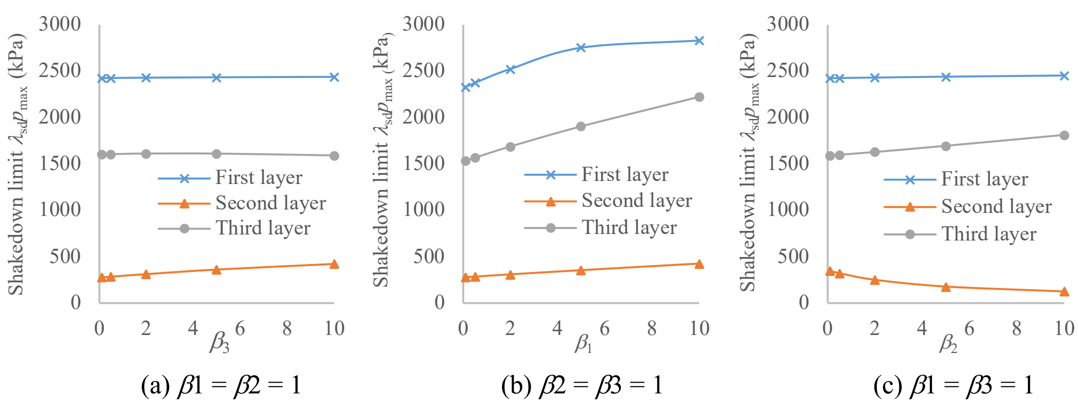


**FIG. 4. Effect of stiffness variation ratio on shakedown limits of a pavement**

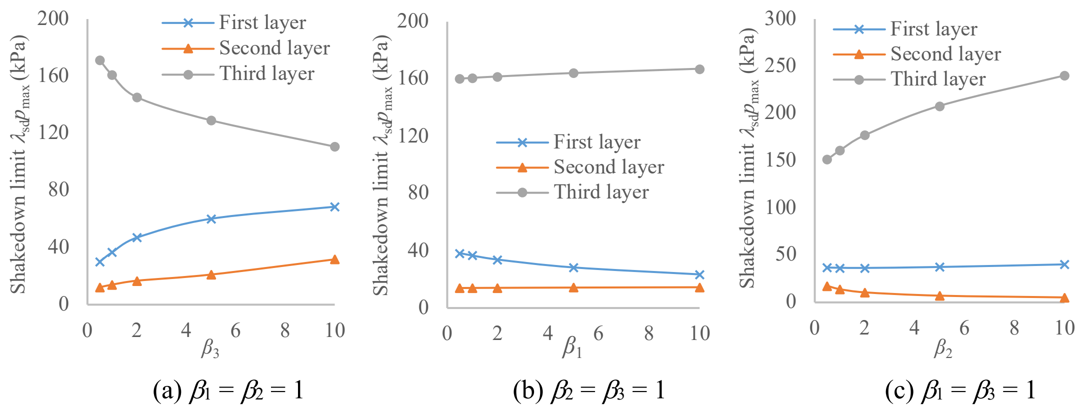


**FIG. 5. Effect of stiffness variation ratio on shakedown limits of a railway foundation**