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PROJECTIVE GEOMETRY OF FREUDENTHAL VARIETIES

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0. INTRODUCTION

H. Freudenthal constructed, in a series of his papers (see [10] and its references), the exceptional Lie algebras of type E_8, E_7, E_6 and F_4, with defining various projective varieties. The purpose of our work is to study projective geometry for his varieties of certain type, which are called *varieties of planes* in the symplectic geometry of Freudenthal (see [10, 4.11], [24, 2.3]).

Let $\mathfrak{g}$ be a graded, simple, finite-dimensional Lie algebra over the complex number field $\mathbb{C}$ with grades between -2 and 2, $\dim \mathfrak{g}_2 = 1$ and $\mathfrak{g}_1 \neq 0$, namely a *graded Lie algebra of contact type*: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see §1). We set

$$\mathcal{V} := \{x \in \mathfrak{g}_1 \setminus \{0\} | (\operatorname{ad} x)^2 \mathfrak{g}_{-2} = 0\},$$

and define an algebraic set V in $\mathbb{P}(\mathfrak{g}_1)$ to be the projectivization of $\mathcal{V}$:

$$V := \pi(\mathcal{V}),$$

where $\pi : \mathfrak{g}_1 \setminus \{0\} \to \mathbb{P}(\mathfrak{g}_1)$ is the natural projection. Then we call $V \subseteq \mathbb{P}(\mathfrak{g}_1)$ (with the reduced structure) the *Freudenthal variety* associated to the graded Lie algebra $\mathfrak{g}$ of contact type, which is a natural generalization of Freudenthal's varieties mentioned above: Note that V is not necessarily connected in this general setting. We here consider moreover the projectivization of a closed set $\{x \in \mathfrak{g}_1 | (\operatorname{ad} x)^{k+1} \mathfrak{g}_{-2} = 0\}$, and denote it by V_k: we have

$$\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \mathbb{P},$$

where we set $\mathbb{P} := \mathbb{P}(\mathfrak{g}_1)$ for short. Clearly, V_3 is a quartic hypersurface, V_2 is an intersection of cubics and $V_1 = V$ is an intersection of quadrics, with a few exceptions.

In the literature, several results have been known about the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-space, case-by-case for each exceptional Lie algebra of types E_8, E_7, E_6 and F_4, from the view-point of the invariant theory of prehomogeneous vector spaces (see [13], [15], [20], [23]). By virtue of those results, it can be shown, for example, that the stratification of $\mathbb{P}$ given by the differences of V_k's exactly corresponds to the orbit decomposition of the $\mathfrak{g}_0$-space $\mathfrak{g}_1$ for those exceptional Lie algebras, and also that Freudenthal varieties V associated to the algebras of type E_8, E_7, E_6 and F_4 are respectively projectively equivalent to the 27-dimensional E_7-variety arising from the 56-dimensional irreducible representation, the orthogonal Grassmann variety of isotropic 6-planes in $\mathbb{C}^{12}$ (namely, the 15-dimensional spinor variety), the Grassmann variety of 3-planes in $\mathbb{C}^6$ and the symplectic Grassmann variety of isotropic 3-planes in $\mathbb{C}^6$, with $\dim \mathbb{P} = 55, 31, 19$ and 13, respectively (see Appendix 1): for those homogeneous projective varieties, we refer to [12, §23.3].

In this article we study the Freudenthal varieties V with the filtration $\{V_k\}$ of the ambient space $\mathbb{P}$, from the view-point of projective geometry, not individually but systematically in terms

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of abstract Lie algebras, without depending on the classification of simple Lie algebras as well as on the known results for each case of types E_8, E_7, E_6 and F_4.

Before stating the main result, we note that the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2 \simeq \mathbb{C}$ defines a non-degenerate skew-symmetric form on $\mathfrak{g}_1$, so that this form allows us to identify $\mathfrak{g}_1$ with its dual space, hence $\mathbb{P}$ with its dual space, and $\mathfrak{g}_1$ is even-dimensional. Moreover, the quartic form on $\mathfrak{g}_1$ defining V_3 has a differential which via the symplectic form defines a vector field on $\mathfrak{g}_1$, and this vector field defines a 1-dimensional distribution on $\mathbb{P}$ away from the singular locus of V_3 (see Proposition A1). We denote by L_P the (closure of the) integral curve of this distribution passing through $P \in \mathbb{P} \setminus \operatorname{Sing} V_3$. On the other hand, we have a rational map $\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$ defined by $x \mapsto (\operatorname{ad} x)^3 \mathfrak{g}_{-2}$ with base locus V_2, which turns out to be a Cremona transformation of $\mathbb{P}$: It is deduced that $\gamma^{-1}(V) = V_3 \setminus V_2$, $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$, $\gamma^2 = 1$ on $\mathbb{P} \setminus V_3$, and γ is explicitly given by the partial differentials of q (see Proposition A2). Note that our γ is a special case of the Cremona transformations in [7, Theorem 2.8 (ii)].

Our main results are summarized as follows (see Theorems A, B, C, D, E, Corollaries A2, B1, B3 and C):

Theorem. *Assume that V is irreducible. Then we have:*

(1) *V is a Legendrian subvariety of $\mathbb{P}$, that is, the projectivization of a Lagrangian subvariety of $\mathfrak{g}_1$, with $\dim V = n - 1$, spans $\mathbb{P}$, and is an orbit of the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_0$, hence smooth, where $\dim \mathfrak{g}_1 = 2n$. In particular, the projective dual V^* of V is equal to the union of tangents to V via the symplectic form.*

(2) *V_2 is the singular locus of V_3, and for any $P \in \mathbb{P} \setminus V_2$, L_P is the line in $\mathbb{P}$ joining P and $\gamma(P)$. Moreover, we have:*
   - (a) *If $P \in \mathbb{P} \setminus V_3$, then L_P is a unique secant line of V passing through P, there is no tangent line to V passing through P, $L_P \cap V$ consists of harmonic conjugates with respect to P and $\gamma(P)$, and $L_P \setminus V \subsetneq \mathbb{P} \setminus V_3$. Moreover, γ preserves L_P, and the automorphism of L_P induced from γ leaves each point in $L_P \cap V$ invariant and permutes P and $\gamma(P)$.*
   - (b) *If $P \in V_3 \setminus V_2$, then there is no secant line of V passing through P, L_P is a unique tangent line to V passing through P, $L_P \cap V = \gamma(P)$, and $L_P \setminus V \subsetneq V_3 \setminus V_2$. Moreover, L_P is contracted by γ to the contact point $\gamma(P)$, and conversely the fibre of γ on $Q \in V$ consists of the points $P \in V_3 \setminus V_2$ such that $Q \in L_P$, or equivalently, P lies on some tangent to V at Q.*

   In particular, V is a variety with one apparent double point, and V_3 is the union of tangents to V.

(3) *For any $P \in V_2 \setminus V$, the family of secants of V passing through P is of dimension at least 1, and all of those secants are isotropic with respect to the symplectic form: In particular, $V_2 \setminus V$ is covered by isotropic secants of V.*

(4) *For any $Q, R \in V$, the secant line joining Q and R is isotropic if and only if the tangents to V at Q and at R are disjoint.*

(5) *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, then there is a twisted cubic curve contained in V to which L_P and L_R are tangent at $\gamma(P)$ and at Q, respectively, where R is a point on some tangent to V at Q away from V_2, determined by P and Q.*

(6) *If $V_2 \neq V$, then V is ruled, that is, covered by lines contained in V.*

(7) *For any $P \in V$, the double projection from P gives a birational map from V onto $\mathbb{P}^{n-1}$, and by the inverse V is written as the closure of the image of a cubic Veronese embedding of a certain affine space $\mathbb{A}^{n-1}$ under some projection to $\mathbb{P}$.*

We show also that *the three conditions, $V = \emptyset$, $V_3 = \mathbb{P}$ and $V_2 = \mathbb{P}$ are equivalent to each* *other* (see Corollary A1), and that *if V is neither empty nor irreducible, then $\mathfrak{g}_1$ decomposes naturally into two irreducible $\mathfrak{g}_0$-submodules of dimension n and V is the (disjoint) union of the projectivizations of those summands* (see Corollary B2).

Finally we should mention that S. Mukai announced a theorem [22, (5.8)] on cubic Veronese varieties without proofs. Our work was originated by looking for proofs of the corresponding statements for Freudenthal varieties (Corollaries A2, B1, C and Theorem D): In fact, we see from his list [22, (5.10)] of cubic Veronese varieties (and the list in Appendix 1) that the notion of our Freudenthal varieties coincides with that of his cubic Veronese varieties. Our result gives a partial explanation for this coincidence (see Theorem D).

This is a joint work with Osami Yasukura. For proofs of the results here, see [19].

1. Preliminaries

For a finite-dimensional, simple Lie algebra $\mathfrak{g}$ of rank ≥ 2, a graded decomposition of contact type is obtained as follows: Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis Δ of the root system R with respect to $\mathfrak{h}$, and fix an order on R defined by Δ. Denote by ρ the highest root of $\mathfrak{g}$, let E_+ and E_- be highest and lowest weight vectors, respectively, and set $H := [E_+, E_-]$. By multiplying suitable scalars, one may assume that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple, that is, those vectors have the following standard relations:

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H.$$

Then, the eigenspace decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} H$ gives $\mathfrak{g}$ a graded decomposition of contact type: In other words, if we set $\mathfrak{g}_\lambda := \{x \in \mathfrak{g} | [H, x] = \lambda x\}$ for $\lambda \in \mathbb{C}$, then it follows that $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\dim \mathfrak{g}_2 = 1$ and $\mathfrak{g}_1 \neq 0$: In fact, $\mathfrak{g}_1 = 0$ if and only if $\mathfrak{g} = \mathfrak{sl}_2$. In terms of root spaces of $\mathfrak{g}$, we have

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \setminus (R_\rho \cup \{\rho\})} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad \mathfrak{g}_{\pm 1} = \bigoplus_{\alpha \in R_\rho} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm\rho} = \mathbb{C}E_\pm,$$

where R^+ is the set of positive roots and $R_\rho := \{\alpha \in R^+ | \rho - \alpha \in R\}$: Indeed, let $\mathfrak{s}_\rho$ be the subalgebra of $\mathfrak{g}$ spanned by E_+, H and E_-, which is isomorphic to $\mathfrak{sl}_2$. Then the irreducible decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module gives the decomposition above (see, for full details, [27]). Conversely, for a graded decomposition $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, taking suitable bases E_+ for $\mathfrak{g}_2$ and E_- for $\mathfrak{g}_{-2}$ with $H := [E_+, E_-]$, one may assume that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple, as before. Then, we see that E_+ and E_- are some highest and lowest weight vectors, respectively, and each $\mathfrak{g}_i$ is recovered as an $(\operatorname{ad} H)$-eigenspace. Therefore, the graded decompositions of contact type are unique up to automorphism of $\mathfrak{g}$, so that the Freudenthal variety V is essentially unique and determined by $\mathfrak{g}$ itself (see Appendix 1).

Now, we define a symmetric product $\times : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ by the formula:

$$-2a \times b = [b, [a, E_-]] + [a, [b, E_-]],$$

which induces a symmetric map $L : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$ and a ternary product $[,,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ by

$$[a, b, c] = L(a, b)c = [a \times b, c].$$

Note that the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful since $\mathfrak{g}$ is simple (see [27, Lemma 3.2 (1)]): we may assume $\mathfrak{g}_0 \subseteq \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$, so that we identify $L(a, b)$ with $a \times b$. We think of $\mathfrak{g}_1$ as an $\mathfrak{g}_0$-module via the adjoint action: For example, we often write Dx instead of $(\operatorname{ad} D)x$ and $[D, x]$

for $D \in \mathfrak{g}_0$ and $x \in \mathfrak{g}_1$. As the skew-symmetric form $\langle , \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{C}$ and the quartic form on $\mathfrak{g}_1$ defining V_3 mentioned in Introduction, we use the ones determined by

$$2\langle a,b\rangle E_+ = [a,b], \quad 2q(x)E_+ = (\operatorname{ad} x)^4 E_-.$$

Note that the skew-symmetric form $\langle , \rangle$ is *non-degenerate* since $\mathfrak{g}$ is simple (see [27, Lemma 3.2 (2)]).

With the notation above, it follows that

$$\begin{aligned} V = V_1 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | x \times x = 0\}\right), \\ V_2 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | [xxx] = 0\}\right), \\ V_3 &= \pi\left(\{x \in \mathfrak{g}_1 \setminus \{0\} | \langle x, [xxx]\rangle = 0\}\right), \end{aligned}$$

and $q(x) = \langle x, [xxx]\rangle$. Note that $V_0 = \emptyset$ since $[[x, E_-]E_+] = x$ for any $x \in \mathfrak{g}_1$: Indeed, it follows from the Jacobi identity that $[[x, E_-]E_+] = [[x, E_+], E_-] + [x[E_-, E_+]] = [x, -H] = x$ since $[x, E_+] \in \mathfrak{g}_3 = 0$. On the other hand, it follows from Lemma 1 below that $V \neq \mathbb{P}$.

Lemma 1. *Let $\mathfrak{g}_{00}$ be the subalgebra of $\mathfrak{g}_0$ defined by*

$$\mathfrak{g}_{00} := \operatorname{Ker}(\operatorname{ad} E_+ | \mathfrak{g}_0) = \operatorname{Ker}(\operatorname{ad} E_- | \mathfrak{g}_0).$$

Then we have $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}H$, and $\mathfrak{g}_{00}$ is linearly spanned by the elements in $\mathfrak{g}_0$ of the form $a \times b$ with $a, b \in \mathfrak{g}_1$. In particular, $\mathfrak{g}_{00} \neq 0$, and $x \times x \neq 0$ for some $x \in \mathfrak{g}_1$.

Lemma 2 (Asano [3]). *For any $a, b, c \in \mathfrak{g}_1$ and $D \in \mathfrak{g}_{00}$, we have*

(1) $\langle Da, b\rangle + \langle a, Db\rangle = 0$.
(2) $D(a \times b) = Da \times b + a \times Db$.
(3) $D[abc] = [(Da)bc] + [a(Db)c] + [ab(Dc)]$.

If we denote by G_{00} the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_{00}$, then Lemma 2 tells that the symplectic form $\langle , \rangle$, the symmetric product $\times$ and the ternary product $[, ,]$ are equivariant with respect to the action of G_{00}, so that each V_i is stable under the action of G_{00}, that is, a union of some orbits of G_{00}. We should mention that the above proofs of (2) and (3) in Lemma 2 are due to the referee, much simpler than the ones in [3].

Lemma 3 (Asano [3]). *We have $[abc] - [acb] = \langle a, c\rangle b - \langle a, b\rangle c + 2\langle b, c\rangle a$ for any $a, b, c \in \mathfrak{g}_1$.*

2. Basic Results

Proposition 1. *If $x \in \mathcal{V}$, then we have:*

(1) (Asano [2]) $[axx] = 3\langle a, x\rangle x$ *for any $a \in \mathfrak{g}_1$. In particular, if $a \times x = 0$, then $\langle a, x\rangle = 0$.*
(2) $\mathbb{C}x \subseteq \mathfrak{g}_{00}x$.

Proposition 2. *We have $\langle [abc], d\rangle = \langle [cda], b\rangle$ for any $a, b, c, d \in \mathfrak{g}_1$.*

Proposition 3. *If $x \in \mathcal{V}$ and $D, E \in \mathfrak{g}_{00}$, then we have:*

(1) (Asano [2]) $Dx \times x = 0$.
(2) $\langle Dx, x\rangle = 0$.
(3) $\langle Dx, Ex\rangle = 0$.
(4) $[(Dx)(Ex)x] = 0$.

Proposition 4. *For any $a \in \mathfrak{g}_1$, we have:*

(1) $[aaa] \times a = 0$.
(2) $[aa[aaa]] = 3q(a)a$.
(3) $[aaa] \times [aaa] = -3q(a)a \times a$.
(4) $[[aaa][aaa][aaa]] = -9q(a)^2 a$.
(5) $q([aaa]) = 9q(a)^3$.

Proposition 5. *If $b = a + x$ with $a \in \mathfrak{g}_1$ and $x \in \mathcal{V}$, then we have:*

(1) $b \times b = a \times a + 2a \times x$.
(2) $[bbb] = [aaa] + 3[aax] + 6\langle x, a\rangle(a - x)$.
(3) $q(b) = q(a) + 4\langle x, [aaa]\rangle + 12\langle x, a\rangle^2$.

Proposition 6. *For any $a \in \mathfrak{g}_1$, we have:*

(1) $3[aa[aab]] = 8\langle b, [aaa]\rangle a + 8\langle a, b\rangle[aaa] + \langle a, [aaa]\rangle b$ *for any* $b \in \mathfrak{g}_1$.
(2) *If $q(a) \neq 0$, then the linear map $L(a, a)$ has full rank.*

Proposition 7. *For any $a \in \mathfrak{g}_1$ and $x \in \mathcal{V}$, we have*

(1) $[aaa] \times x + 3[aax] \times a + 6\langle x, a\rangle a \times a = 0$.
(2) $3[aax] \times [aax] + 8\langle x, [aaa]\rangle a \times x - 8\langle x, a\rangle[aaa] \times x = 0$. *In particular, if $[aaa] = 0$, then $[aax] \times [aax] = 0$, and moreover, $\mathbb{C}x + \mathbb{C}[aax] \subseteq \mathcal{V} \cup \{0\}$.*

3. A Line Field and a Cremona Transformation

Proposition A1.

(1) *The quartic form q on $\mathfrak{g}_1$ has a differential at $a \in \mathfrak{g}_1$ as follows:*

$$dq(a) : t_a\mathfrak{g}_1 \to \mathbb{C}; b \mapsto 4\langle b, [aaa]\rangle,$$

where $t_a\mathfrak{g}_1$ is the Zariski tangent space to $\mathfrak{g}_1$ at a, naturally identified with $\mathfrak{g}_1$.

(2) *In particular, the singular locus of V_3 is equal to V_2.*
(3) *The vector field on $\mathfrak{g}_1$ corresponding to dq via the symplectic form $\langle , \rangle$ induces a 1-dimensional distribution $\mathcal{D}$ on $\mathbb{P}$ away from* $\operatorname{Sing} V_3 = V_2$, *which is given by*

$$\mathcal{D} : \pi(a) \mapsto (\mathbb{C}a + \mathbb{C}[aaa])/\mathbb{C}a,$$

where $\pi(a) \in \mathbb{P} \setminus V_2$ and we naturally identify the Zariski tangent space $t_{\pi a}\mathbb{P}$ with the quotient space $\mathfrak{g}_1/\mathbb{C}a$.

Proposition A2. *Let*

$$\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$$

be a rational map induced from the cubic, $a \mapsto [aaa]$. Then we have:

(1) $\gamma^{-1}(V) = V_3 \setminus V_2$.
(2) $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$.
(3) $\gamma^2 = 1$ *on* $\mathbb{P} \setminus V_3$, *hence γ gives an automorphism of $\mathbb{P} \setminus V_3$.*
(4) *γ is explicitly given by the partial differentials of q.*

In particular, γ is a Cremona transformation of $\mathbb{P}(\mathfrak{g}_1)$ with order 2 if $V_2 \neq \mathbb{P}$.

A *secant line* of V is by definition a line in $\mathbb{P}$ which passes through at least two distinct points of V and is not contained in V. We note that for a line L in $\mathbb{P}$ if the scheme-theoretic intersection $L \cap V$ has length more than 2, then $L \subseteq V$: Indeed, V is an intersection of quadric hypersurfaces.

Theorem A. *Let L_P be the closure of the integral curve of $\mathcal{D}$ through $P \in \mathbb{P} \setminus V_2$, where $\mathcal{D}$ is the 1-dimensional distribution on $\mathbb{P} \setminus V_2$ induced from the quartic form q. Then we have:*

(1) *For any $P \in \mathbb{P} \setminus V_2$, L_P is the line in $\mathbb{P}$ joining P and $\gamma(P)$.*

(2) *If $P \in \mathbb{P} \setminus V_3$, then we have:*
   - (a) *L_P is a secant line of V, and $L_P \cap V$ consists of harmonic conjugates with respect to P and $\gamma(P)$.*
   - (b) *$L_P \setminus V \subseteq \mathbb{P} \setminus V_3$.*
   - (c) *L_P is a unique secant line of V passing through P.*
   - (d) *There is no tangent line to V passing through P.*
   - (e) *$\gamma(L_P \setminus V) = L_P \setminus V$, and the automorphism of L_P induced from γ leaves each point in $L_P \cap V$ invariant and permutes P and $\gamma(P)$.*

(3) *If $P \in V_3 \setminus V_2$, then we have:*
   - (a) *L_P is a tangent line to V, and $L_P \cap V = \{\gamma(P)\}$.*
   - (b) *$L_P \setminus V \subseteq V_3 \setminus V_2$.*
   - (c) *There is no secant line of V passing through P.*
   - (d) *L_P is a unique tangent line to V passing through P.*
   - (e) *$\gamma(L_P \setminus V) = \gamma(P)$, and $\gamma^{-1}(Q) = \{P \in V_3 \setminus V_2 | Q \in L_P\} = T_Q V \setminus V_2$ for any $Q \in V$, where $T_Q V$ is the embedded tangent space to V at Q.*

Corollary A1. *The three conditions, $V = \emptyset$, $V_3 = \mathbb{P}$ and $V_2 = \mathbb{P}$ are equivalent to each other.*

Remark A. It can'be shown that $V = \emptyset$ if and only if the Lie algebra $\mathfrak{g}$ is of type C (see Appendix): In fact, using a theorem of Asano [30, 1.6.Theorem], [4], one can show that if $q \equiv 0$, then $\mathfrak{g} \simeq \mathfrak{sp}_{2n+2}$, where $\dim \mathfrak{g}_1 = 2n$; The converse is checked by an explicit computation.

Recall that a projective variety $V \subseteq \mathbb{P}$ is called a *variety with one apparent double point* if for a general point $P \in \mathbb{P}$ there exists a unique secant line of V passing through P (see [25, IX]).

Corollary A2. *If $V \neq \emptyset$, then V is a variety with one apparent double point. In particular, V is non-degenerate in $\mathbb{P}$.*

4. The Homogeneity

Theorem B. *Let G_{00} be the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_{00}$, where $\mathfrak{g}_{00}$ is the subalgebra of $\mathfrak{g}_0$ defined by $\mathfrak{g}_{00} := \mathrm{Ker}(\mathrm{ad}\, E_\pm | \mathfrak{g}_0)$. Then we have:*

(1) *G_{00} acts transitively on each of irreducible components of $\mathcal{V}$. In particular, we have $t_x \mathcal{V} = \mathfrak{g}_{00} x$ for any $x \in \mathcal{V}$, where $t_x \mathcal{V}$ is the Zariski tangent space to $\mathcal{V}$ at x.*

(2) *$\mathfrak{g}_{00} x = (\mathfrak{g}_{00} x)^\perp$ with $2 \dim \mathfrak{g}_{00} x = \dim \mathfrak{g}_1$ for any $x \in \mathcal{V}$, and $\mathfrak{g}_1 = \mathfrak{g}_{00} x \oplus \mathfrak{g}_{00} y$ for any $x, y \in \mathcal{V}$ with $\langle x, y \rangle \neq 0$.*

Recall that the *tangent variety* of V, denoted by $\mathrm{Tan}\, V$, is the union of embedded tangent spaces to V, and the *projective dual* of V, denoted by V^*, is the set of hyperplanes tangent to V (see, for example, [11, §3]).

Corollary B1. *Assume that $V \neq \emptyset$. Then we have:*

(1) *G_{00} acts transitively on each of irreducible components of V, and V is smooth, equidimensional of dimension $n-1$, where $\dim \mathfrak{g}_1 = 2n$.*

(2) *Denote by L^* the set of hyperplanes containing a linear subspace $L \subseteq \mathbb{P}$. Then we have $(T_Q V)^* = T_Q V$ for any $Q \in V$, hence*

$$\mathrm{Tan}\, V = V^*,$$

where we identify $\mathbb{P}$ with its dual space $\mathbb{P}^\vee := \mathbb{P}(\mathfrak{g}_1^*)$ via the symplectic form $\langle , \rangle$.

Corollary B2. *If V is neither empty nor irreducible, then there are irreducible $\mathfrak{g}_{00}$-modules $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of dimension n such that $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, and we have*

$$V = \mathbb{P}(\mathfrak{s}_1) \sqcup \mathbb{P}(\mathfrak{s}_2),$$

where $\dim \mathfrak{g}_1 = 2n$.

Remark B1. It is known that V is irreducible unless $\mathfrak{g}$ is of type A or C (see Appendix): In fact, if $\mathfrak{g} = \mathfrak{so}_m$, then V is a Segre embedding of $\mathbb{P}^1 \times Q$ in $\mathbb{P}^{2m-9}$, where Q is a quadric hypersurface in $\mathbb{P}^{m-5}$; if $\mathfrak{g}$ is of type G_2, then V is a cubic Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$; for other exceptional Lie algebras $\mathfrak{g}$, see Introduction. Conversely, it follows from a direct computation that we are in the case above if $\mathfrak{g} = \mathfrak{sl}_{n+2}$ with $n \geq 1$.

Corollary B3. *If $V \neq \emptyset$ and $V_2 \neq V$, then V is ruled, that is, covered by lines contained in V.*

Remark B2. It can be shown that $V = V_2$ if and only if $\mathfrak{g}$ is of type G_2.

5. Isotropic Secants

Proposition C. *For $P = \pi(u) \in \mathbb{P}$, let $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ be a rational map induced from $L(u,u)$ with base locus $B_P = \mathbb{P}(\operatorname{Ker} L(u,u))$. If V is irreducible and $P \in V_2 \setminus V$, then $\dim \Phi_P(V \setminus B_P) \geq 1$, hence $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ and $\operatorname{codim} B_P \geq 2$.*

Remark C1. The irreducibility condition for V is essential in Proposition C: In fact, there is an example of u satisfying the assumption above such that $\operatorname{rk} L(u,u) = 1$ in case of $\mathfrak{g} = \mathfrak{sl}_m$, where V is not irreducible (see Remark B3).

Remark C2. It follows easily from Proposition 6 that $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \notin V_2$, and $\operatorname{codim} \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \in V_3$, though we do not use these facts in this article.

Recall that the *secant locus* Σ_P as well as the *tangent locus* Θ_P of V with respect to a given point $P \in \mathbb{P}$ are defined by

$$\Sigma_P^\circ := \{Q \in V | \exists R \in V \setminus \{Q\}, P \in Q * R\}, \quad \Sigma_P := \overline{\Sigma_P^\circ},$$
$$\Theta_P := \{Q \in V | P \in T_Q V\},$$

where we denote by $Q * R$ the line in $\mathbb{P}$ joining Q and R, and by $T_Q V$ the embedded tangent space to V at Q in $\mathbb{P}$ (see, for example, [11]).

Theorem C. *Assume that V is irreducible. Then we have:*

(1) *For any $x, y \in \mathcal{V}$, $\langle x, y \rangle = 0$ if and only if $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y \neq 0$. In particular, a secant line joining $Q, R \in V$ is isotropic with respect to the symplectic form if and only if $T_Q V \cap T_R V \neq \emptyset$.*

(2) *$V_2 \setminus V$ is covered by isotropic secants of V. More precisely, for any $u \in \mathfrak{g}_1$, we have that $[uuu] = 0$ and $u \times u \neq 0$ if and only if $u = x + y$ for some $x, y \in \mathcal{V}$ such that $\langle x, y \rangle = 0$ and $x \times y \neq 0$.*

(3) *If $P \in V_2 \setminus V$, then*

$$\Phi_P(V \setminus B_P) \subseteq \Sigma_P, \quad \Phi_P(V \cap P^\perp \setminus B_P) \subseteq \Theta_P,$$

where $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ is the rational map induced from $L(u,u)$ with base locus $B_P = \mathbb{P}(\operatorname{Ker} L(u,u))$ and $P^\perp = \mathbb{P}(u^\perp)$ with $P = \pi(u)$.

(4) *We have $\dim \Sigma_P \geq 1$ for any $P \in V_2 \setminus V$.*

Remark C3. The irreducibility condition for V is essential in (1) above: In fact, it is easily seen that the conclusion does not hold in case of $\mathfrak{g} = \mathfrak{sl}_m$.

Corollary C. *If V is irreducible, then $V_3 = \operatorname{Tan} V$.*

6. Double Projections

Proposition D. *For any $x, y \in \mathcal{V}$, let $\Psi_{xy} : \mathfrak{g}_1 \to \mathfrak{g}_1$ be a linear map defined by*

$$\Psi_{xy}(a) := [axy] + \langle a, x\rangle y.$$

(1) *If $\langle x, y\rangle \neq 0$, then $\operatorname{Ker}\Psi_{xy} = \mathfrak{g}_{00}x$ and $\Psi_{xy}(\mathfrak{g}_1) = \mathfrak{g}_{00}y$. In particular, a rational map $\Psi_{PQ} : \mathbb{P} \dashrightarrow \mathbb{P}$ induced from Ψ_{xy} is a double projection from P with image T_QV, that is, a projection with center T_PV onto T_QV, hence defines a morphism*

$$\Psi_{PQ} : \mathbb{P} \setminus T_PV \to T_QV,$$

where T_PV is the embedded tangent space to V at P with $P = \pi(x)$ and $Q = \pi(y)$.

(2) *Moreover for any $R \in V$, the four points R, $[PQR]$, $\Psi_{PR}(Q)$ and $\Psi_{QR}(P)$ are collinear, and $[PQR]$ is the harmonic conjugate of R with respect to $\Psi_{PR}(Q)$ and $\Psi_{QR}(P)$, where we set $[PRR] := \pi([xyz])$ with $R = \pi(z)$. In particular, this holds for general $P, Q, R \in V$ and gives a geometric meaning of our ternary product.*

Remark D1. In terms of the Lie bracket, we have $\Psi_{ab}(c) = [b[a[c, E_-]]]$ by ($\heartsuit$) in the proof of Lemma 1.

Theorem D. *For any $P, Q \in V$, if the secant line joining P and Q is not isotropic, that is, $T_PV \cap T_QV = \emptyset$, then we have:*

(1) $V \setminus P^\perp = (\Psi_{PQ}|_{V \setminus T_PV})^{-1}(T_QV \setminus P^\perp)$.

(2) *The double projection Ψ_{PQ} gives an isomorphism $V \setminus P^\perp \to T_QV \setminus P^\perp$. In fact, a rational map $\Gamma_{QP} : T_QV \dashrightarrow V$ induced from a map $\Gamma_{yx} : \mathfrak{g}_{00}y \to \mathcal{V} \cup \{0\}$ defined by*

$$\Gamma_{yx}(t) := \langle x, [ttt]\rangle x + 3\langle x, t\rangle [ttx] + 12\langle x, t\rangle^2 t$$

gives the inverse of $\Psi_{PQ}|_{V \setminus P^\perp}$, where $P = \pi(x)$ and $Q = \pi(y)$.

(3) *The base locus of Γ_{QP} is $T_QV \cap P^\perp \cap V_2$.*

In particular, if V is irreducible, then Ψ_{PQ} gives a birational map from V to T_QV, and V is the closure of the image of a composition of a cubic Veronese embedding of the affine space $T_QV \setminus P^\perp$ with some projection to $\mathbb{P}$.

Remark D2. The morphism $\Psi_{PQ} : V \setminus T_PV \to T_QV$ is not necessarily surjective: In fact, if $\mathfrak{g}$ is of type G_2, then for any $P \in V$, $P^\perp$ is the osculating plane to the twisted cubic $V \subseteq \mathbb{P}^3$ at P, $V \cap P^\perp = \{P\}$, and $\Psi_{PQ}(V \setminus T_PV) = T_QV \setminus P^\perp$ for any $Q \in V$ with $P \neq Q$.

Remark D3. We have proved in the above that $\Psi_{xy} : \mathcal{V} \setminus x^\perp \to \mathfrak{g}_{00}y \setminus x^\perp$ is an isomorphism.

Remark D4. We here give another expression of the inverse map of the double projection Ψ_{PQ}. We first note that there is an isomorphism of affine spaces,

$$\iota : \mathfrak{g}_{00}y \cap x^\perp \to T_QV \setminus P^\perp$$

defined by $\iota(a) := \pi(a + y)$. Indeed, the inverse is given by $\iota^{-1}(\pi(t)) := \frac{\langle x,y\rangle}{\langle x,t\rangle}t - y$ for $\pi(t) \in T_QV \setminus P^\perp$, where $T_QV = \mathbb{P}(\mathfrak{g}_{00}y)$ and $P^\perp = \mathbb{P}(x^\perp)$. Now let $\rho : \mathfrak{g}_{00}y \cap x^\perp \to V$ be the composition of ι with the rational map $\Gamma_{QP} : T_QV \dashrightarrow V$ in Theorem D (2). Then ρ is the inverse of Ψ_{PQ} via ι, and it follows from part (1) and (4) of Proposition 3 that

$$\rho(a) = \pi\left(\frac{\langle x, [aaa]\rangle}{12\langle x, y\rangle^2}x + \frac{1}{4\langle x, y\rangle}[aax] + a + y\right).$$

In particular, the Freudenthal variety V is equal to the closure of the image of the affine space $\mathfrak{g}_{00}y \cap x^\perp$ under the cubic Veronese embedding ρ.

7. Twisted Cubic Curves

Proposition E. *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, then we have:*

(1) *$Q \in L_{\Phi_P(Q)}$ and $\Phi_P^3(Q) = \gamma(P) \in L_P = L_{\Phi_P^2(Q)}$ with $\Phi_P(Q), \Phi_P^2(Q) \in V_3 \setminus V_2$.*

(2) *$L_P \cap L_{\Phi_P(Q)} = \emptyset$, hence $Q, \Phi_P(Q), \Phi_P^2(Q)$ and $\Phi_P^3(Q)$ are linearly independent in $\mathbb{P}$.*

Theorem E. *For any $P \in V_3 \setminus V_2$ and $Q \in V$ such that the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, that is, $T_Q V \cap T_{\gamma(P)} V = \emptyset$, let $\mathbb{P}_{PQ}$ be the linear subspace of dimension 3 in $\mathbb{P}$ spanned by Q, $\Phi_P(Q)$, $\Phi_P^2(Q)$ (or equivalently P) and $\Phi_P^3(Q) = \gamma(P)$, that is, spanned by L_P and $L_{\Phi_P(Q)}$, the unique tangent lines to V passing through P and $\Phi_P(Q)$. Then we have:*

(1) *The intersection $V \cap \mathbb{P}_{PQ}$ is a twisted cubic curve in $\mathbb{P}_{PQ} \simeq \mathbb{P}^3$ given explicitly by the image of L_P under the cubic map $\Gamma_{\gamma(P)Q}$:*

$$V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P).$$

(2) *The twisted cubic curve in $\mathbb{P}_{PQ}$ above has the following properties:*
   (a) *L_P and $L_{\Phi_P(Q)}$ are respectively the tangent lines at $\gamma(P)$ and at Q, and*
   (b) *$\gamma(P)^\perp \cap \mathbb{P}_{PQ}$ and $Q^\perp \cap \mathbb{P}_{PQ}$ are respectively the osculating planes at $\gamma(P)$ and at Q, which are spanned by L_P and $\Phi_P(Q)$ and by $L_{\Phi_P(Q)}$ and $\Phi_P^2(Q)$, respectively.*

Remark E1. The morphism $\Gamma_{\gamma(P)Q} : L_P \to \mathbb{P}_{PQ}$ is given by

$$(\lambda : \mu) \mapsto (2\lambda^3 : 6\lambda^2\mu : 9\lambda\mu^2 : 9\mu^3)$$

in terms of homogeneous coordinate with respect to the basis $\{D^2x, D^3x\}$ for L_P and $\{x,\ Dx,\ D^2x,\ D^3x\}$ for $\mathbb{P}_{PQ}$.

Remark E2. Set $E := L(Dx, Dx)$, $F := [D, E]$ with $D := L(t,t)$ as in the above, and denote by $\mathfrak{g}_{00PQ}$ the subalgebra of $\mathfrak{g}_{00}$ generated by D, E and F. Then it follows that

$$[F, D] = \frac{4}{3}\langle D^3x, x\rangle D, \quad [F, E] = -\frac{4}{3}\langle D^3x, x\rangle E,$$

so that $\mathfrak{g}_{00PQ}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2$. If we denote by $\mathfrak{g}_{1PQ}$ the subspace of $\mathfrak{g}_1$ spanned by x, Dx, D^2x and D^3x, then we see that $\mathfrak{g}_{1PQ}$ is an irreducible $\mathfrak{g}_{00PQ}$-module of dimension 4 with

$$F(D^kx) = (2k-3)\frac{2}{3}\langle D^3x, x\rangle D^kx,$$

and the twisted cubic curve $V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P)$ is a unique closed orbit in $\mathbb{P}_{PQ} = \mathbb{P}(\mathfrak{g}_{1PQ})$ under the natural action of the group of inner automorphisms of $\mathfrak{g}_{00}$ with Lie algebra $\mathfrak{g}_{00PQ}$.

Thus, for any $P \in V_3 \setminus V_2$ and $Q \in V$ with $T_{\gamma(P)}V \cap T_Q V = \emptyset$, a subalgebra $\mathfrak{g}_{00PQ}$ of $\mathfrak{g}_{00}$ isomorphic to $\mathfrak{sl}_2$ and an irreducible $\mathfrak{g}_{00PQ}$-submodule $\mathfrak{g}_{1PQ}$ of $\mathfrak{g}_1$ with dimension 4 are associated to P and Q. If $\mathfrak{g}$ is of type G_2, then $\mathfrak{g}_{00PQ}$ and $\mathfrak{g}_{1PQ}$ are respectively equal to $\mathfrak{g}_{00}$ and $\mathfrak{g}_1$ themselves.

Appendix 1. A Classification of Freudenthal Varieties

We here give a classification of Freudenthal varieties V in terms of the root data of $\mathfrak{g}$. It would be interesting to compare V with the adjoint variety associated to $\mathfrak{g}$ since those varieties are closely related to each other: In fact, for a simple graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, denote by V the Freudenthal variety associated to $\mathfrak{g}$, as before, and denote by X the orbit of the inner automorphism group of $\mathfrak{g}$ through $\pi(E_+)$ in $\mathbb{P}(\mathfrak{g})$, which is the minimal closed orbit in $\mathbb{P}(\mathfrak{g})$, called the *adjoint variety* associated to $\mathfrak{g}$ (see [16]). Then, according to [17, Theorem B], we have $V = X \cap \mathbb{P}(\mathfrak{g}_1)$.

ADJOINT VARIETIES AND FREUDENTHAL VARIETIES

$\mathfrak{g}$	$X \subseteq \mathbb{P}(\mathfrak{g})$	$\mathfrak{g}_{00}$	$V \subseteq \mathbb{P}(\mathfrak{g}_1)$
$\mathfrak{sl}_m$	$(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}) \cap (1) \subseteq \mathbb{P}^{m^2-2}$	$\mathfrak{gl}_1 \oplus \mathfrak{sl}_{m-2}$	$\mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} \subseteq \mathbb{P}^{2m-5}$
$\mathfrak{so}_m$	$\mathbb{G}_{\text{orthog.}}(2, m) \subseteq \mathbb{P}^{\binom{m}{2}-1}$	$\mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4}$	$\mathbb{P}^1 \times Q^{m-6} \subseteq \mathbb{P}^{2m-9}$
$\mathfrak{sp}_{2m}$	$v_2\mathbb{P}^{2m-1} \subseteq \mathbb{P}^{\binom{2m+1}{2}-1}$	$\mathfrak{sp}_{2m-2}$	$\emptyset \subseteq \mathbb{P}^{2m-3}$
$\mathfrak{e}_6$	$E_6(\omega_2)^{21} \subseteq \mathbb{P}^{77}$	$\mathfrak{sl}_6$	$\mathbb{G}(3,6) \subseteq \mathbb{P}^{19}$
$\mathfrak{e}_7$	$E_7(\omega_1)^{33} \subseteq \mathbb{P}^{132}$	$\mathfrak{so}_{12}$	$S_5 = \mathbb{G}_{\text{orthog.}}(6, 12) \subseteq \mathbb{P}^{2^5-1}$
$\mathfrak{e}_8$	$E_8(\omega_8)^{57} \subseteq \mathbb{P}^{247}$	$\mathfrak{e}_7$	$E_7(\omega_6) \subseteq \mathbb{P}^{55}$
$\mathfrak{f}_4$	$F_4(\omega_1)^{15} \subseteq \mathbb{P}^{51}$	$\mathfrak{sp}_6$	$\mathbb{G}_{\text{sympl.}}(3,6) \subseteq \mathbb{P}^{13}$
$\mathfrak{g}_2$	$G_2(\omega_2)^5 \subseteq \mathbb{P}^{13}$	$\mathfrak{sl}_2$	$v_3\mathbb{P}^1 \subseteq \mathbb{P}^3$

Notation: We denote by $\cap(1)$ cutting by a general hyperplane, and by v_d the Veronese embedding of degree d. We denote by $\mathbb{G}(r, m)$ a Grassmann variety of r-planes in $\mathbb{C}^m$, and denote by $\mathbb{G}_{\text{orthog.}}(r, m)$ and by $\mathbb{G}_{\text{symp.}}(r, m)$ respectively an orthogonal and a symplectic Grassmann varieties of isotropic r-planes in $\mathbb{C}^m$. A simple exceptional Lie algebra of Dynkin type G is denoted by the lowercase of G in the German character, as in [12], a simple algebraic group of type G is denoted by just G, and for a dominant integral weight ω of G, the minimal closed orbit of G in $\mathbb{P}(V_\omega)$ is denoted by $G(\omega)$, where V_ω is the irreducible representation space of G with highest weight ω: For example, $\mathfrak{g}_2$ in the list is the simple Lie algebra of type G_2, and $G_2(\omega_2)$ is the minimal closed orbit of an algebraic group of type G_2 in $\mathbb{P}(V_{\omega_2})$, where ω_2 is the second fundamental dominant weight with the standard notation of Bourbaki [6].

APPENDIX 2. THE FILTRATION OF THE AMBIENT SPACE

- $\mathfrak{e}_{6,7,8}, \mathfrak{f}_4$
  - $\mathbb{P}$
  - |
  - $V_3 = \operatorname{Tan} V$
  - |
  - $V_2 = \operatorname{Sing} V_3$
  - |
  - V
  - |
  - $\emptyset$

- $\mathfrak{g}_2$
  - $\mathbb{P}^3$
  - |
  - $V_3 = \operatorname{Tan} V$
  - |
  - $(V_2)_{\text{red}} = \operatorname{Sing} V_3 = V = v_3\mathbb{P}^1$
  - |
  - $\emptyset$

- $\mathfrak{sl}_{m \geq 3}$

  ∘ $\mathbb{P}^{2m-5}$
  |
  ∘ $(V_3)_{\text{red}} = V_2 = Z(\sum x_i y_i)$
  |
  ∘ $V = \mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} = Z(x_0, \ldots, x_{m-3}) \sqcup Z(y_0, \ldots, y_{m-3})$
  |
  ∘ $\emptyset$

- $\mathfrak{sp}_{2m}$

  ∘ $\mathbb{P}^{2m-3} = V_3 = V_2$
  |
  ∘ $V = \emptyset$

- $\mathfrak{so}_8$

  $\mathfrak{g}_{00} = \mathfrak{so}_4 \oplus \mathfrak{sl}_2 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$
  $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^4 \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = M_{2,2,2}$

  ∘ $\mathbb{P}^7$
  |
  ∘ $V_3 = \operatorname{Tan} V = Z(\text{hyper-determinant for } M_{2,2,2})$
  / | \
  $(\mathbb{P}^3 \times \mathbb{P}^1$ ∘ ∘ ∘ $\mathbb{P}^1 \times \mathbb{P}^3) \cdots\cdots V_2 = \operatorname{Sing} V_3$
  \ | /
  ∘ $V = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
  |
  ∘ $\emptyset$

- $\mathfrak{so}_{m \geq 9}$

  $\mathfrak{g}_{00} = \mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4}$
  $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^{m-4} = M_{m-4,2}$
  $V_{20} = \pi(\{X \in M_{m-4,2} | \operatorname{rk} X = 2, {}^{\tau}XX = 0\})$
  $= \{[a \otimes c + b \otimes d] | [a] * [b] = \mathbb{P}^1, [c] * [d] \subseteq Q\}$
  : dim $2m - 12$

  $V_{11} = \pi(\{X \in M_{m-4,2} | \operatorname{rk} X = 1, \operatorname{rk}{}^{\tau}XX = 1\})$
  $= \mathbb{P}^1 \times \mathbb{P}^{m-5}$

  ∘ $\mathbb{P}^{2m-9}$
  |
  ∘ $V_3 = \operatorname{Tan} V$
  / \
  $(V_{20}$ ∘ ∘ $V_{11}) \cdots V_2 = \operatorname{Sing} V_3$
  \ /
  ∘ $V = \mathbb{P}^1 \times Q^{m-6}$
  |
  ∘ $\emptyset$

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