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On Moishezon Threefolds Homeomorphic to a Cubic Hypersurface in $\mathbf{P}^4$

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§ 0. Introduction.

The three dimensional projective space $\mathbf{P}^3$ and a smooth quadric hypersurface $\mathbf{Q}^3$ in the four dimensional projective space $\mathbf{P}^4$ have the unique complex structure as Moishezon manifolds ([Ko2, 5.3.5] [N1][N2]).[1] See also [P1][P2] [Ko2, 5.3.13]. The major purpose of the present paper is to report recent progress on the similar problems for Fano 3-folds of index two or more specifically cubic hypersurfaces in $\mathbf{P}^4$.

There are, besides smooth ones, various normal singular cubic hypersurfaces admitting small (smooth) resolutions. *A cubic 3-fold* is by definition a smooth Moishezon 3-fold which is a small resolution

[1] There is a rumor that it has been proved that the six dimensional sphere S^6 has a complex structure, a fortiori, $\mathbf{P}^3$ has an exotic complex structure with nonzero irregularity. It may probably true that any compact complex 3-fold homeomorphic to $\mathbf{P}^3$ (resp. $\mathbf{Q}^3$) is isomorphic to $\mathbf{P}^3$ (resp. $\mathbf{Q}^3$) if the irregularity vanishes.

of a normal cubic hypersurface in $\mathbf{P}^4$. Any cubic 3-fold with $b_2 = 1$ is a simply connected closed 6-manifold with the second integral homology group infinite cyclic, whose first Chern class is divisible by two. Some of cubic 3-folds are shown to have torsion free integral homology groups. Since the second homology group is generated by the dual of a hyperplane section, the cubic form on the second homology group is the same as that of a smooth cubic hypersurface. Therefore if any integral homology group of it is torsion free, (though we do not know whether this is true for arbitrary cubic 3-folds) then the topology of a cubic 3-fold with $b_2 = 1$ is by [W] uniquely determined by its third Betti number b_3. By an inequality which will be proved in (4.2) we see that b_3 is an even integer with $0 \leq b_3 \leq 10$. An arbitrary even integer between 0 and 10 is realized as the third Betti number of some cubic 3-fold with $b_2 = 1$, where a cubic 3-fold with $b_3 = 10$ is a smooth cubic hypersurface. We prove

Theorem 0.1. *Let X be a Moishezon 3-fold with c_1^3 positive, $b_2 = 1$ and $2 \leq b_3 \leq 10$. Then X is homeomorphic to a cubic 3-fold if and only if it is isomorphic to either a cubic 3-fold ($2 \leq b_3 \leq 10$) or a certain blowing down of a small resolution of a blowing-up of $\mathbf{Q}^3$ ($2 \leq b_3 \leq 4$). In particular, any Moishezon 3-fold with c_1^3 positive which is homeomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$ is isomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$.*

Theorem 0.2. *Let X be a Moishezon 3-fold with $b_2 = 1$, $b_3 = 0$. Then X is homeomorphic to a cubic 3-fold if and only if it is* *isomorphic to either a cubic 3-fold or a certain blowing down of a small resolution of a blowing-up of* $\mathbf{Q}^3$.

The blowing down of a blowing-up of a smooth quadric hypersurface $\mathbf{Q}^3$ mentioned above, which we refer to as *a fake cubic 3-fold*, has $b_2 = 1$ and $b_3 \leq 4$. Since any fake cubic 3-fold is simply connected and has torsion free integral homology groups isomorphic to those of one of cubic 3-folds, it is diffeomorphic to some cubic 3-fold by the same reason as before. However it seems that no fake cubic 3-folds are global deformations of cubic 3-folds.

Here we would like to remark that for quadric hypersurfaces in $\mathbf{P}^4$ with Hessian rank four it seems very hard to give their characterization similar to the above because $b_2 \geq 2$. In fact, any normal quadric hypersurface in $\mathbf{P}^4$ with Hessian rank four has a small resolution, which is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, and has infinitely many distinct complex structures as $\mathbf{P}^2$-bundles over $\mathbf{P}^1$.

Our proof of (part of) (0.1) roughly goes as follows. Let X be a Moishezon 3-fold with c_1^3 positive which is homeomorphic to a smooth cubic hypersurface. Then $b_2 = 1$ and $b_3 = 10$, while the canonical line bundle of X is divisible by two. Let L be a (positive) generator of $H^2(X, \mathrm{Z})$ with $L^3 = 3$. Since c_1^3 is positive by the assumption, we have $K_X = -2L$ and $h^0(X, L) = 5$. If the base locus $B := \mathrm{Bs}\,|L|$ is empty, then the associated rational map ρ_L is a birational morphism of X onto a possibly singular normal cubic hypersurface in $\mathbf{P}^4$ with at worst isolated singularities. Thus X is a cubic 3-fold. Then it follows from the inequality proved in section seven that X is isomorphic to a smooth cubic hypersurface.

Next we assume in general that X is a Moishezon 3-fold with $L^3 = 3$, $K_X = -2L$, $h^0(X, L) = 5$ and that the base locus $B :=$ Bs $|L|$ of $|L|$ is nonempty. Two distinct general members D and D' of $|L|$ have no irreducible components in common so that the complete intersection $\ell = D \cap D'$ is pure one dimensional, which turns out to be a cycle of two smooth rational curves. Thereby B turns out to be a single reduced smooth rational curve. Thus the base locus of ρ_L can be eliminated by blowing up X only once with B center so that we have a morphism from the blowing up $\hat{X}$ of X onto $\mathbf{Q}^3$. By studying the morphism we prove that $\hat{X}$ is a small resolution of a blowing-up of $\mathbf{Q}^3$, and that X is therefore a fake cubic 3-fold. However, no fake cubic 3-fold is homeomorphic to a smooth cubic hypersurface because their third Betti numbers are different. This proves (0.1) when X is homeomorphic to a smooth cubic hypersurface.

In the last we would like to mention that the positivity of b_3 is the obstacle for removing the assumption $c_1^3 > 0$ from (0.1). Since $b_3 \geq 2$ in these cases, X may have nontrivial holomorphic three forms so that $K_X = 0$ or $K_X = 2L$ can happen. We were unable to exclude these possibilities without the assumption on c_1^3. However we need no extra assumption in (0.2) by the vanishing of b_3. It is an interesting question whether there exists a Calabi-Yau 3-fold homeomorphic to some cubic 3-fold with $b_2 = 1$, hence having Euler number -6 or ± 4, or ± 2.

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§ 1. Cubic 3-folds

(1.1) Let W be an irreducible cubic hypersurface in $\mathbf{P}^4$. If W is smooth, then $X := W$ is a Fano 3-fold of index two with Pic $X \simeq \mathbf{Z}$ whose integral homology groups are all torsion free and $b_2 = 1$, $b_3 = 10$. Now we consider also normal cubic hypersurfaces in $\mathbf{P}^4$, which admit small (smooth) resolutions. We call those smooth 3-folds *cubic threefolds*. The purpose of this section is to show that besides smooth cubic hypersurfaces there are smooth cubic 3-folds X with Pic $X \simeq \mathbf{Z}L$, $L^3 = 3$, $K_X = -2L$ and $h^0(X, L) = 5$ whose integral homology groups are all torsion free. By [W] it is easy to determine (classify) the topology of those 3-folds.

Let W be a singular normal cubic hypersurface in $\mathbf{P}^4$, p_0 a singular point of W. Taking homogeneous coordinates $x_0, \cdots, x_4$ such that $p_0 = [0,0,0,0,1]$, we write the defining equation F of W as

$$F = f_2(x)x_4 + f_3(x)$$

where $f_k(x)$ is a homogeneous polynomial of degree k in $x_0, \cdots x_3$.

Lemma 1.2. *f_2 is not identically zero if W has a small resolution.*

Lemma 1.3. *Assume* $f_2(x) = x_0x_1 - x_2x_3$. *Let* Δ *be a curve on* $\mathbf{P}^3$ *defined by* $f_2 = f_3 = 0$. *If* W *has at worst isolated singularities, then* Δ *is reduced. If moreover* W *has a small resolution* $smr : X \to W$, *then any local irreducible component of* Δ *is smooth and* $b_2(X) = b_2(\Delta)$. *The small resolution* X *is simply connected.*

Proof. Let $G(x) = f_2(x)$ and $H(x) = f_3(x)$. Assume $x_k \neq 0$ for some $0 \leq k \leq 3$. Hence we have $G_k \neq 0$ for some k. Therefore

$$F_k := \partial F/\partial x_k = 0 \text{ for any } k$$

$$\Updownarrow$$

$$G = H = 0, \quad x_4 G_k + H_k = 0 \text{ for any } k$$

$$\Updownarrow$$

$$x_4 + H_k/G_k = 0 \text{ for some } k, \quad G = H = 0,$$
$$\operatorname{rank} \begin{pmatrix} G_0 & G_1 & G_2 & G_3 \\ H_0 & H_1 & H_2 & H_3 \end{pmatrix} \leq 1$$

where $G_k := \partial G/\partial x_k$ and $H_k := \partial H/\partial x_k$. Since $G_k \neq 0$ for some k, x_4 at the singular point of W is uniquely determined by $x_4 + H_k/G_k = 0$. It follows that the singularities of W except at p_0 are in one to one correspondence with the singularities of Δ. If Δ is nonreduced along an irreducible component, then Δ is *singular* along the component so that W has nonisolated singularities. Therefore if W has at worst isolated singularities, then Δ is reduced.

We assume that Δ is reduced. Let $smr : X \to W$ be a small resolution. The point p_0 is an ordinary double point of W. On the other hand, any singularity of X except p_0 corresponds to a singularity of Δ as observed above. The singularity of X except p_0 is defined by an equation of the form $xy = f(z, w)$ where $f(z, w) = 0$ is a local equation of Δ at the corresponding singular point of Δ. Hence by (1.2), any local irreducible component of Δ is smooth.

Since p_0 is an ordinary double point of W, $C := smr^{-1}(p_0)$ is a smooth rational curve. Let $\phi_C : \hat{X} \to X$ be the blowing-up of X with C center. Let $f : W \to \mathbf{P}^3$ be a rational map defined by $f([x_0, \cdots, x_4]) = [x_0, \cdots, x_3]$, which is an isomorphism over $\mathbf{P}^3 \setminus Q$ where Q is a smooth quadric surface defined by $f_2 := x_0x_1 - x_2x_3 = 0$ in $\mathbf{P}^3$. Then the induced morphism $\pi := f \cdot smr \cdot \phi_C : \hat{X} \to \mathbf{P}^3$ is birational, which induces an isomorphism of the exceptional set $E := (smr \cdot \phi_C)^{-1}(p_0)$ onto Q ($\subset \mathbf{P}^3$).

Let $S := \pi^{-1}(\Delta)$. Then $\Delta = \pi(S)$, and $smr \cdot \phi_C(S)$ is a cone over Δ in $\mathbf{P}^4$. The divisor S consists of $b_2(\Delta)$ irreducible components. It is clear that $\hat{X} \setminus S \cup E \simeq X \setminus \phi_C(S) \cup C \simeq \mathbf{P}^3 \setminus Q$ and $S \cap E \simeq \Delta$.

Now we compute $b_2(X)$. Let $T := S \cup E$. Since $\mathbf{P}^3 \setminus Q \simeq \hat{X} \setminus T$, we have $\pi_1(\hat{X} \setminus T) \simeq H_1(\hat{X} \setminus T) \simeq \mathbf{Z}/2\mathbf{Z}$. It follows easily from $\pi_1(\hat{X} \setminus T) \simeq \mathbf{Z}/2\mathbf{Z}$ that $\hat{X}$ is simply connected. Similarly $H_2(\hat{X} \setminus T) \simeq H_2(\mathbf{P}^3 \setminus Q) = 0$. By the realtive homology exact sequence for the pair $(\hat{X}, \hat{X} \setminus T)$, we have $H_2(\hat{X}, \hat{X} \setminus T) \simeq H^4(T) \simeq H^4(S) \oplus H^4(E)$, so that $b_2(X) = b_2(\hat{X}) - 1 = \text{rank } H_2(\hat{X}, \hat{X} \setminus T) - 1 = b_4(T) - 1 = b_4(S) = b_2(\Delta)$. This completes the proof. q.e.d.

Lemma 1.4. *Assume $f_2(x) = x_0x_1 - x_2x_3$. If W admits a small* *resolution* $smr : X \to W$ *with* $b_2(X) = 1$, *then the integral homology groups of* X *are torsion free and we have*

$$b_q = 1 \ (q : \text{even}), b_1 = b_5 = 0, b_3 = 8 - 2r$$

where $r = \frac{1}{2}(\deg \omega_\Delta - \deg \omega_{\tilde{\Delta}})$ *for the normalization* $\tilde{\Delta}$ *of* Δ.

Now we recall a theorem of C.T.C. Wall [W, Theorem 5].

Theorem 1.5. (Wall) *Diffeomorphism classes of oriented closed 6-manifolds with torsion free integral homology groups and* $w_2 = 0$ *corresponds bijectively to isomorphism classes of systems of invariants :*

free abelian groups $H := H^2(M)$, $G := H^3(M)$,

a symmetric trilinear form (cup product) $\mu : H \times H \times H \to \mathbf{Z}$, *a homomorphism (Pontrjagin class)* $p_1 : H \to \mathbf{Z}$ *subject to*

$$\mu(x,x,y) = \mu(x,y,y) \mod 2, \quad p_1(x) = 4\mu(x,x,x) \mod 24.$$

In our case p_1 is given by $p_1 = c_1^2 - 2c_2$. It is easy to check that the conditions on p_1 and μ in (1.5) are automatically true in our case. Thus we see

Lemma 1.6. *Let* X *and* Y *be cubic 3-folds with* $b_2 = 1$. *Assume that the integral homology groups of* X *and* Y *are torsion free. Then the following are equivalent.*

(1.6.1) X *and* Y *are homeomorphic.*

(1.6.2) X *and* Y *are diffeomorphic.*

(1.6.3) $b_3(X) = b_3(Y)$.

Proof. It is easy to show that any cubic 3-fold is simply connected. Therefore the assertion of (1.6) follows from (1.5). q.e.d.

Remark 1.7. By computations we see that there are some cubic 3-folds with torsion free integral homology groups also in the case where Hessian rank $f_2 \leq 3$. Therefore in view of (1.6) some cubic threefolds can be diffeomorphic to each other even when they arise from quadratic polynomials f_2 with different Hessian ranks. We know so far no cubic 3-folds whose integral homology groups have torsions.

Example 1.8. Let $f_2 = x_0x_1 - x_2x_3$. Let S be a surface in $\mathbf{P}^3$ defined by $f_2 = 0$. Then S has two rulings f and g. Let $\Delta :=$ $\{f_2 = f_3 = 0\}$ be a reduced curve of 6 components, the union of 3 fibers of f and 3 fibers of g. Then Δ has 9 double points, whence the cubic hypersurface $W : f_2x_4 + f_3 = 0$ has 10 ordinary double points. In this case W has a small resolution X with $b_2(X) = 6$ and $b_3(X) = 0$ by (1.3) and (4.1). This gives the maximum of the number of ordinary double points on cubic hypersurfaces. This is shown as follows. We choose one of the ordinary double points on a given cubic hypersurface W as $p : [0, 0, 0, 0, 1]$. Then we can choose the equation defining W and the projective coordinates x_i as in (1.1) so that $f_2(x) = x_0x_1 - x_2x_3$. Hence by the proof of (1.3) any ordinary double point of W corresponds to that of a curve Δ, which is a member of $|3e_1 + 3e_2|$. Therefore it is easy to see that the above example gives the maximum.

§ 2. Fake cubic 3-folds and compactifications of $\mathbf{C}^3$

(2.1) In this section we construct some Moishezon threefolds with $K_X = -2L$ whose rational image by ρ_L is $\mathbf{Q}^3$. This example has been mentioned in [Ko2, 5.3.14].

Let Δ be an irreducible Gorenstein curve of $\mathbf{Q}^3$ ($\subset \mathbf{P}^4$) with $\deg \Delta = 5$ and $\deg \omega_\Delta = 2$ whose any local irreducible components are smooth. We assume that Δ lies on a smooth hyperplane section of $\mathbf{Q}^3$, which we denote by Q ($\simeq \mathbf{F}_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$). Let e_1 and e_2 be fibers of two natural rulings of Q. Then we may assume that Δ belongs to $|2e_1 + 3e_2|$.

Let $\hat{f} : \hat{X} \to \mathbf{Q}^3$ be a small resolution of a blowing-up of $\mathbf{Q}^3$ with Δ center, $\hat{\Delta} := \hat{f}^{-1}(\Delta)$ and $\hat{Q}$ the proper transform of Q. It is easy to see that $\hat{Q} \simeq Q$ and $N_{\hat{Q}/\hat{X}} \simeq -e_1 - 2e_2$ even if Δ is singular. Therefore we have a contraction morphism $\phi : \hat{X} \to X$ such that $B := \phi(\hat{Q}) \simeq \mathbf{P}^1$. Let $\hat{L} := \hat{f}^*(O_{\mathbf{Q}^3}(1))$ and $L := \phi_*(\hat{L} + \hat{Q})$. We easily check that

$$\mathrm{Pic}\, X \simeq \mathbf{Z}L, \quad L^3 = 3, \quad \kappa(L) = 3, \quad K_X = -2L.$$

Let $\Lambda := \phi_* \hat{f}^* |O_{\mathbf{Q}^3}(1)|$. Let H be a general smooth hyperplane section of $\mathbf{Q}^3$, $\hat{D} := \hat{f}^{-1}(H)$ and $D := \phi(\hat{D})$. Then $\hat{D}$ is the blowing-up of H with center $H \cap \Delta$, which consists of 6 distinct points. Hence $\hat{D}$ is smooth, while so is D. We easily check that $B = \mathrm{Bs}\, \Lambda \simeq \mathbf{P}^1$.

Moreover for a pair of general members D and D' of Λ, we see that $\ell := D \cap D' = B + C$ is a cycle of smooth rational curves. Moreover

$$LC = 4, \quad LB = -1, \quad \Lambda = |L| \simeq |\hat{L}| \simeq |O_{\mathbf{Q}^3}(1)|.$$

We denote X by $X(\Delta, \mathbf{Q}^3)$ and call it a fake cubic 3-fold, which is uniquely determined by $(\Delta, O_\Delta(1))$. This construction gives no Moishezon 3-folds with $K_X = -2L$ and $b_2(X) = 1$ when Δ lies on a singular quadric surface by (3.2).

It is easy to see the following

Lemma 2.2. *Let Δ be an irreducible Gorenstein curve on a smooth quadric surface Q ($\subset \mathbf{P}^3 \subset \mathbf{P}^4$) with* deg $\Delta = 5$ *and* deg $\omega_\Delta = 2$, *any of whose local irreducible components are smooth. Let* $r = \frac{1}{2}(\deg \omega_\Delta - \deg \omega_{\tilde{\Delta}})$ *for the normalization $\tilde{\Delta}$ of Δ, $X := X(\Delta, \mathbf{Q}^3)$. Then $0 \le r \le 2$ and any homology group $H_q(X, \mathbf{Z})$ of X is torsion free, $b_q = 1$ $(q = 0, 2, 4, 6)$, $b_1 = b_5 = 0$, while $b_3 = 4 - 2r$ (≤ 4) and the Euler number of X is given by $e(X) = 2r (\le 4)$.*

Corollary 2.3. *In the notation in (2.2), $X(\Delta, \mathbf{Q}^3)$ is diffeomorphic to a cubic 3-fold with the same integral homology groups.*

Proof. This follows from Lemmas 1.5-1.6 and 2.2. q.e.d.

Remark 2.4. It is possible to construct similar examples when the curve Δ is on a singular hyperplane section. We also call this a fake cubic 3-fold.

By choosing $\Delta \in |2e_1 + (6-m)e_2|$ with mild singularities, we can also produce examples of Moishezon 3-folds X with Pic $X = \mathbf{Z}L$, $K_X = -2L$ and $L^3 = m$.

To be more precise, let Q be a smooth hyperpplane section of $\mathbf{Q}^3$, and $\Delta \in |2e_1 + (6-m)e_2|_Q$ $(-\infty < m \leq 3)$ an irreducible curve on Q whose any local irreducible components are smooth. Let $\hat{f} : \hat{X} \to \mathbf{Q}^3$ be a small resolution of the blowing up $B_\Delta(\mathbf{Q}^3)$ of $\mathbf{Q}^3$ with Δ center, $\hat{\Delta} := \hat{f}^{-1}(\Delta)$, and $\hat{B}$ the proper transform of Q. Then we see $N_{\hat{B}/\hat{X}} \simeq O_{\hat{B}}(-e_1 - (5-m)e_2)$ so that we have a Moishezon 3-fold X and a contraction morphism $\phi : \hat{X} \to X$ such that $B := \phi(\hat{B}) \simeq \mathbf{P}^1$ and $\hat{X} \setminus \hat{B} \simeq X \setminus B$. Letting $\hat{L} = \hat{f}^*(O_{\mathbf{Q}^3}(1))$ and $L := \phi_*(L)$, we have

$$\text{Pix } X = \mathbf{Z}L, \quad K_X = -2L, \quad L^3 = m \ (\leq 3),$$
$$|L| \simeq |\hat{L}| \simeq |O_{\mathbf{Q}^3}(1)|, \quad \text{Bs } |L| = B.$$

We denote X by $X(\Delta, \mathbf{Q}^3)$. We also note that if $m \geq 6$, then there is no irreducible Δ. If $m = 4$ or 5, then Bs $|L|$ is empty and X is either a small resolution of a complete intersection of two quadric hypersurfaces in $\mathbf{P}^5$ $(m = 4)$ or a 3-fold hyperplane section of $\text{Gr}(2,5) \subset \mathbf{P}^9$ $(m = 5)$. This is shown directly or by using a theorem of Kollár (3.1).

It is possible to construct similar examples when the curve Δ is on a singular hyperplane section. These yield compactifications of $\mathbf{C}^3$ by an irreducible divisor.

§ 3. Moishezon 3-folds with $b_2 = 1$ and $K_X = -2L$

First we recall a theorem of Kollár.

Theorem 3.1. (Kollár [Ko2, 5.3.12]) *Let X be a Moishezon 3-fold X with $K_X = -2L$, $s := h^0(X, L) \geq 4$ and $b_2 = 1$. Let ρ_L be the rational map associated with $|L|$. Then (X, ρ_L) is one of the following;*

(3.1.1) $s = 4$, ρ_L is a birational rational map of X onto $\mathbf{P}^3$,

(3.1.2) $s = 4$, X is a double cover of $\mathbf{P}^3$ ramified along a quartic by the morphism ρ_L,

(3.1.3) $s = 5$, ρ_L is a birational rational map of X onto $\mathbf{Q}^3$,

(3.1.4) $s = 5$, ρ_L is a birational morphism of X onto a cubic hypersurface in $\mathbf{P}^4$,

(3.1.5) $s = 6$, ρ_L is a birational morphism of X onto a complete intersection of a pair of quadric hypersurfaces in $\mathbf{P}^5$,

(3.1.6) $s = 7$, ρ_L is an isomorphism of X onto a 3-fold hyperplane section of the Grassmanian $\mathrm{Gr}(2, 5) \subset \mathbf{P}^9$, which we call V_5.

We shall nearly settle (3.1.3) by proving (3.2) and (3.3).

Theorem 3.2. *Let X be a Moishezon 3-fold with $b_2(X) = 1$, $L^3 = 3$, $K_X = -2L$ and $h^0(X, L) = 5$. If $\mathrm{Bs}\,|L|$ is nonempty, then X is a fake cubic 3-fold. In particular, X is homeomorphic to no smooth cubic hypersurface.*

Theorem 3.3. *Let X be a Moishezon 3-fold with $b_2(X) = 1$, $L^3 = m$, $K_X = -2L$ and $h^0(X, L) = 5$. If $\mathrm{Bs}\,|L|$ is empty, then $m = 3$ and*

X is a cubic 3-fold. If Bs $|L|$ *is nonempty, then* $-\infty < m \leq 3$ *and X is isomorphic to* $X(\Delta, \mathbf{Q}^3)$ *resp. the blowing up-and-down or the blowing down of a* $\mathbf{Q}$*-factorialization of the blowing up* $B_\Delta(\mathbf{Q}^3)$ *of* $\mathbf{Q}^3$ *with center a curve* Δ *of* $\mathbf{Q}^3$ $(\subset \mathbf{P}^4)$ *where* Δ *is an irreducible reduced curve contained in a smooth (resp. singular) hyperplane section of* $\mathbf{Q}^3$ *with* $\deg \Delta = 8 - m$ *and* $\deg \omega_\Delta = 8 - 2m$.

See (2.4) (resp. (3.5)) for the detail when Δ is contained in a smooth (resp. singular) hyperplane section of $\mathbf{Q}^3$.

Proof of (3.3). If Bs $|L|$ is empty, then $m = 3$ and X is a cubic 3-fold by (3.1) or [Ko2, 5.3.12]. Now we assume that $B :=$ Bs $|L|$ is nonempty. Then we see $B \simeq \mathbf{P}^1$. Moreover we have $h^0(O_B(L)) = 0$, whence $LB \leq -1$. This proves that $m = L^3 = LB + LC = LB + 4 \leq 3$. Let $\hat{X}$ be the blowing-up of X with B center, $\hat{B}$ the total transform of B. The we have a morphism $f : \hat{X} \to \mathbf{Q}^3$. Let $\hat{L} := f^*(O_{\mathbf{Q}^3}(1)$.

There are two cases $\hat{B} \simeq \mathbf{F}_0$ or $\hat{B} \simeq \mathbf{F}_2$. In each case we have

Lemma 3.4. *If* $\hat{B} \simeq \mathbf{F}_0$, *then* $\hat{X}$ *is a small resolution of* $B_\Delta(\mathbf{Q}^3)$. *The curve* Δ *is an irreducible reduced curve contained in a smooth hyperplane section of* $\mathbf{Q}^3$ *with* $\deg \Delta = 8 - m$ *and* $\deg \omega_\Delta = 8 - 2m$, *any of whose local irreducible components is smooth.*

Lemma 3.5. *Assume* $\hat{B} \simeq \mathbf{F}_2$. *Let* e_∞ *be a smooth rational curve on* $\mathbf{F}_2$ *with* $e_\infty^2 = -2$, I_{e_∞} *the defining ideal of* e_∞ *in* $\hat{X}$, $I := I_{e_\infty}^{2-k} + I_{\hat{B}}$ $(m = 2k, 2k+1)$ *and* $\phi_I : B_I(\hat{X}) \to \hat{X}$ *the blowing-up of* $\hat{X}$ *with* *the ideal I center. Then we have a natural surjective morphism* $g^{\mathrm{norm}} : B_I(\hat{X}) \to B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$.

(3.5.1) If $m = 2k$, then $m \leq 1$, and g^{norm} contracts $E := \phi_I^{-1}(e_\infty) \simeq$ $\mathbf{F}_0$ into a smooth rational curve. There exists a contraction morphism $cont_E : B_I(\hat{X}) \to \mathrm{Cont}_E\, B_I(\hat{X})$ of $B_I(\hat{X})$ onto a 3-dimensional normal Moishezon space $\mathrm{Cont}_E\, B_I(\hat{X})$ and a surjective morphism $h^{\mathrm{norm}} : \mathrm{Cont}_E\, B_I(\hat{X}) \to B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$ such that $g^{\mathrm{norm}} = h^{\mathrm{norm}} \cdot cont_E$. We have $cont_E(E) \simeq \mathbf{P}^1$ and $\mathrm{Cont}_E\, B_I(\hat{X})$ is a $\mathbf{Q}$-factorialization of $B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$ by h^{norm}.

(3.5.2) If $m = 2k+1$, then $m \leq 1$ and $B_I(\hat{X})$ is a normal $\mathbf{Q}$-factorialization of $B_{\Delta}^{\mathrm{norm}}(\mathbf{Q}^3)$ by g^{norm}.

§ 4. V_5 and cubic 3-folds

Theorem 4.1. (Kollár) *Let X be a Moishezon 3-fold homeomorphic to a V_5. Then it is isomorphic to a V_5.*

Lemma 4.2. *Let W be a normal cubic hypersurface with isolated singularities which admits a small resolution $f : X \to W$. Let H be a general hyperplane section of W, and $L := f^*(H)$. Let $s :=$the number of singular points of W, and $e(X)$ the Euler number of X. Then X is a Moishezon 3-fold with $K_X = -2L$, $h^q(X, O_X) = 0$ $(q \geq 1)$. Moreover there exists a coherent sheaf $\mathcal{G}$ on X with supp $\mathcal{G}$ contained in the exceptional set of f such that $\frac{1}{2}e(X) + 3 = h^0(X, \mathcal{G}) \geq s$.*

Proof. Since the singularities of W are rational by [E], $h^q(O_X) =$

$h^q(O_W) = 0$ for $q \geq 1$ and $\chi(O_X) = 1$. Let T_X be the holomorphic tangent bundle of X. Then we have exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O_X & \longrightarrow & O_X(L)^{\oplus 5} & \longrightarrow & f^*T_{\mathbf{P}^4} & \longrightarrow & 0 \\
0 & \longrightarrow & T_X & \longrightarrow & f^*T_{\mathbf{P}^4} & \longrightarrow & O_X(3L) & & \\
0 & \longrightarrow & T_X & \longrightarrow & f^*T_{\mathbf{P}^4} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & O_X(3L) & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}$$

where $\mathcal{F}$ and $\mathcal{G}$ are the cokernels of the sequences, while the second sequence is the normal sequence over the smooth locus of W. Note that $h^q(X, mL) = h^q(W, mH) = 0$ for any $q \geq 1$ and $m \geq 0$ because $f_*(mL) \simeq mH$ and $R^q f_*(mL) = 0$ for $q \geq 1$. From the above sequences we infer

$$h^q(X, f^*T_{\mathbf{P}^4}) = 0 \ (q \geq 1), \quad h^3(X, T_X) = h^0(X, \Omega^1_X(-2L)) = 0$$
$$h^q(X, \mathcal{G}) = h^{q+1}(X, \mathcal{F}) = h^{q+2}(X, T_X) = 0 \ (q \geq 1)$$

It follows that

$$\begin{aligned}
h^0(X, \mathcal{G}) &= \chi(X, \mathcal{G}) \\
&= \chi(X, 3L) - \chi(X, f^*T_{\mathbf{P}^4}) + \chi(X, T_X) \\
&= \chi(X, 3L) - 5\chi(X, L) + \chi(X, O_X) + \chi(X, T_X) \\
&= 34 - 25 + 1 + (12c_1^3 - 19c_1c_2 + 12c_3)/24 \\
&= e(X)/2 + 3
\end{aligned}$$

where $c_1c_2 = 24\chi(X, O_X) = 24$ and $c_1 := c_1(X) = f^*c_1(W) = 2c_1(L)$ because the resolution $f : X \to W$ is small. Let $F := F(x_0, \cdots, x_4)$ be the defining equation of W, I the Jacobian ideal of F (generated by $\partial F/\partial x_i$). Let J be the ideal of O_X generated by $f^*(\partial F/\partial x_i)$. Then by the definition of the above sequences we have $\mathcal{G} \simeq (O_X/J) \otimes O_X(3L)$. Since the singularities of W are isolated, $(O_X/J) \otimes O_X(3L) \simeq O_X/J$. Therefore $h^0(X, \mathcal{G}) \geq s$, where s :=the number of singular points of W. This completes the proof of (4.2).

q.e.d.

Proposition 4.3. *Let W be a normal cubic hypersurface in $\mathbf{P}^4$ which admits a small resolution $f : X \to W$, and H a general hyperplane section of W. Then $H_1(X, \mathbf{Z}) = 0$, and the inclusion homomorphism $i_* : H_2(H, \mathbf{Z}) \to IH_2(W, \mathbf{Z}) \simeq H_2(X, \mathbf{Z})$ is surjective where $IH_2(W, \mathbf{Z})$ is the second intersection homology of W for any perversity. In particular, $b_2(X) \leq 7$ and $-6 \leq e(X) \leq 16$.*

Proof. Let W be a normal cubic hypersurface which admits a small resolution X. By [GM, 6.2], $IH_2(W, \mathbf{Z}) \simeq H_2(X, \mathbf{Z})$ by the smoothness of X. The first assertion of (4.2) follows from [GM, 7.1]. Similarly $b_2(X) \leq b_2(H) \leq 7$. Because H is a smooth cubic surface so that it is $\mathbf{P}^2$ blown-up at 6 points. We also see $e(X) = 2 + 2b_2 - b_3 \leq 16$, we have $e(X) \geq -6$ by (4.2). q.e.d.

Corollary 4.4. *Let W be a normal cubic hypersurface in $\mathbf{P}^4$ with at worst ordinary double points. Then the number of ordinary double points on W is at most 10.*

Proof. We have $s \leq 10$ by (1.9), though we have only $s \leq 11$ by (4.2) and (4.3). q.e.d.

Theorem 4.5. *Let X be a Moishezon 3-fold homeomorphic to a cubic 3-fold with $b_2 = 1$ and $6 \leq b_3 \leq 10$. If c_1^3 is positive or if X has no holomorphic 3-forms, then X is isomorphic to a cubic 3-fold. In particular if X is homeomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$ and if $c_1^3 > 0$, then X is isomorphic to a smooth cubic hypersurface in $\mathbf{P}^4$.*

Proof. Let X be a Moishezon 3-fold homeomorphic to a smooth cubic hypersurface. Then $b_1 = 0$, $b_2 = 1$ and $b_3 = 10$. Hence $h^1(X, O_X) = h^2(X, O_X) = 0$ and $h^3(X, O_X) \leq b_3/2 = 5$ so that Pic $X \simeq H^2(X, \mathbf{Z}) \simeq \mathbf{Z}$. Let L be a generator of Pic X with $L^3 = 3$. Then $K_X = -(2q+2)L$ for some integer q because $c_1 \mod 2$ is a topological invariant. Then as in [M] we have

$$-4 \leq \chi(X, O_X) = \chi(Z, O_Z(q)) = (q+1)(q^2+2q+2)/2,$$

where Z is a smooth cubic hypersurface in $\mathbf{P}^4$ and $O_Z(1)$ is the hyperplane bundle. Hence $(q, h^3(X, O_X)) = (0,0),\ (-1,1),\ (-2,2)$. When q is negative, we have $c_1^3 = 3(2q+2)^3 \leq 0$. By either of the assumptions we have $q = 0$, $h^3(X, O_X) = 0$ and $K_X = -2L$. Then by [Ko2, 5.3.12] $h^0(X, L) = 5$. By (3.2), $B :=$ Bs $|L|$ is empty. Therefore the half anti-canonical map ρ_L of X is defined everywhere and X is a cubic 3-fold because $b_2 = 1$.

Let W be the cubic hypersurface which is the image of X by ρ_L. Then the singularities of W are isolated, hence normal. Since X is a small resolution of W, the singularities of W are terminal so that they are rational [E]. Hence by (4.2), we have $5 - b_3/2 = e(X)/2 + 3 \geq s$, where $s :=$ the number of isolated singular points on W. It follows from $b_3 = 10$ that W is smooth and $X \simeq W$.

Next we consider the case where X is homeomorphic to a cubic 3-fold with $6 \leq b_3 \leq 8$. We see in the same manner as above that $K_X = -2L$ and $h^0(X, L) = 5$ for the generator L of Pic X with $L^3 = 3$ by [Ko2]. Let $B :=$ Bs $|L|$. If B is empty, then X is a cubic 3-fold by $b_2 = 1$, while if B is nonempty, then $X \simeq X(\Delta, \mathbf{Q}^3)$ by (4.2), which contradicts $b_3(X(\Delta, \mathbf{Q}^3)) \leq 4$ by (2.2). q.e.d.

Theorem 4.6. *Let X be a Moishezon 3-fold homeomorphic to a cubic 3-fold with $b_2 = 1$, then X is isomorphic to either a cubic 3-fold or a fake cubic 3-fold if one of the following conditions is satisfied.*

(4.6.1) $2 \leq b_3 \leq 4$, *either c_1^3 is positive or* $h^{3,0}(X) = 0$,

(4.6.2) $b_3 = 0$.

BIBLIOGRAPHY

[B] E.Brieskorn, *Die Auflösung der rationalen Singularitäten holomorphen Abbildungen*, Math. Ann. **178** (1968), 255-270.

[E] R.Elkik, Rationalité des singularités canoniques, Invent. Math. **64** (1981), 1-6.

[F] A.Fujiki, On the blowing down of analytic spaces, Publ. R. Inst. Math. Sci. Kyoto Univ. **10** (1975), 473-507.

[GM] M.Goresky and R.MacPherson, *Intersection homology* II, Invent. Math. **71** (1983), 77-129.

[Ka] Y.Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. Math. **127** (1988), 93-163.

[Ko1] J.Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15-36.

[Ko2] J.Kollár, *Flips, flops, minimal models etc.*, Surveys in Diff. Geom. **1** (1991).

[M] J.Morrow, *A survey of some results on complex Kähler manifolds*, Global Analysis, Univ. Tokyo Press and Princeton University Press (1969), 315-324.

[N1] I.Nakamura, *Moishezon threefolds homeomorphic to* $\mathbf{P}^3$, Jour. Math. Soc. Japan **39** (1987), 521-535.

[N2] I.Nakamura, *Threefolds homeomorphic to a hyperquadric in* $\mathbf{P}^4$, Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, Kinokuniya, Tokyo, Japan (1987), 379-404.

[N3] I.Nakamura, *On Moishezon manifolds homeomorphic to* $\mathbf{P}^n_{\mathbf{C}}$, Jour. Math. Soc. Japan **44** (1992), 667-692.

[N4] I.Nakamura, *Moishezon fourifolds homeomorphic to* $\mathbf{Q}^4_{\mathbf{C}}$, Osaka J. Math to appear.

[P1] T.Peternell, *A rigidity theorem for* $\mathbf{P}_3(\mathbf{C})$, Manuscripta Math. **50** (1985), 397-428.

[P2] T.Peternell, *Algebraic structures on certain 3-folds*, Math. Ann. **274** (1986), 133-156.

[R] M.Reid, *Minimal models of canonical 3-folds*, Advanced Studies in Pure Mathematics, Kinokuniya, Tokyo, Japan, vol. 1 (1983), 131-180.

[W] C.T.C.Wall, *Classification problems in differential topology* V, *On certain 6-mani-folds*, Invent. Math. **1**, (1966), 355-374.

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