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On vector partition functions with negative weights

By

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Abstract

We introduce the notion of weights into the vector partition function and the volume function associated to a sequence of vectors. We prove an explicit formula for the volume function with possibly negative weights. It is a generalization of a formula given by Brion and Vergne in [2].

§1. Introduction

The vector partition function and the volume function associated to a sequence of vectors have been studied from various point of view. For instance, the Kostant partition function for a root system, which is a typical example of vector partition functions, plays an important role in representation theory of Lie groups. We note that a vector partition function (resp. a volume function) counts the number of the lattice points in (resp. measures the volume of) a certain polytope. Hence they are closely related to combinatorics of convex polytopes. We refer to [2], [7], and [1] for known results. For example, in [2] Brion and Vergne gave explicit closed formulas for vector partition functions and volume functions.

In this paper, we introduce the notion of a vector partition function (or volume function) with *weights*, where each weight is an integer. A positive weight merely corresponds to the multiplicity of a vector in the given set of vectors used to define the vector partition function (or volume function). The known results cited above are also available in this positive weight case. On the other hand, the notion of negative weights seems to be new, although it is quite natural when we characterize a vector

partition function (or volume function) by its generating function. Such negative weights appear, for example, in the computation of the dimension of the invariant subspace $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$ in a tensor product representation, where V_{λ_i} is the irreducible representation of a compact Lie group G with highest weight λ_i, if some of λ_i ($i = 1, \ldots, n$) lie on the boundary of the Weyl chamber (see, e.g., [5]).

Our main result Theorem 4.1 gives explicit formulas for the volume functions with possibly negative weights, which generalize some of the Brion-Vergne formulas mentioned above. It is stated as follows.

Theorem 4.1.   *Let $\Delta = (\alpha_1, \ldots, \alpha_N)$, $m = (m_1, \ldots, m_N)$ be as in Definition 3.1 and let $M = m_1 + \cdots + m_N$. Suppose h is in a chamber γ and $y \in \mathbb{R}^N$ is generic. Then we have*

$$\mathcal{V}_\Delta(y, h; m) = \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \left(\frac{e^{-\langle y, v_\sigma(h) \rangle}}{\prod_{k \notin \sigma} (y_k - \sum_{j \in \sigma} c_{jk} y_j)^{m_k}} \right),$$

$$\mathcal{V}_\Delta(h; m) = \frac{1}{(M - d)!} \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma} \frac{\partial_j^{m_j - 1}}{(m_j - 1)!} \left(\frac{\langle y, v_\sigma(h) \rangle^{M-d}}{\prod_{k \notin \sigma} (-y_k + \sum_{j \in \sigma} c_{jk} y_j)^{m_k}} \right),$$

where if $m_i \le 0$, we set $\dfrac{\partial_j^{m_j - 1}}{(m_j - 1)!} = 0$.

See sections 2, 3, and 4 for the details. Applications to geometry and topology of certain spaces will be discussed in another article. We mention that an example in this direction was given in [5].

This paper is organized as follows. In section 2, after giving the definitions of the vector partition function and the volume function associated to a sequence of vectors, we review the Brion-Vergne formulas from [2]. Weights are introduced in section 3. The main theorem above and its example are given in section 4. We prove the theorem in section 5.

§2.   Vector partition function and volume function

In this section, we review some contents of [2].

§2.1.   Definitions

Let E be a real vector space of dimension d and let Λ be a lattice in E. Let $\Delta = (\alpha_1, \ldots, \alpha_N)$ be a sequence of vectors in Λ, all lying in an open half space and spanning E as vector space. For $\lambda \in \Lambda$, we denote by $\mathcal{P}_\Delta(\lambda)$ the number of ways to

express λ as a linear combination of $\alpha_1, \ldots, \alpha_N$ with coefficients in $\mathbb{Z}_{\geq 0}$. Namely, we set

$$\mathcal{P}_\Delta(\lambda) := \sharp\{(x_1, \ldots, x_N) \in (\mathbb{Z}_{\geq 0})^N \,|\, x_1\alpha_1 + \cdots + x_N\alpha_N = \lambda\}.$$

The function $\mathcal{P}_\Delta$ is called the *vector partition function* associated to Δ. Its generating function is given by

$$\sum_\lambda \mathcal{P}_\Delta(\lambda)e^\lambda = \frac{1}{\prod_{i=1}^N (1 - e^{\alpha_i})}$$
$$= (1 + e^{\alpha_1} + e^{2\alpha_1} + \cdots) \cdot \cdots \cdot (1 + e^{\alpha_N} + e^{2\alpha_N} + \cdots),$$

where e^v ($v \in \Lambda$) are elements in the group ring $\mathbb{Z}[\Lambda]$, obeying $e^{v_1}e^{v_2} = e^{v_1 + v_2}$.

More generally, for $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, we define

$$\mathcal{P}_\Delta(y, \lambda) := \sum_{x_1\alpha_1 + \cdots + x_N\alpha_N = \lambda} e^{-(x_1y_1 + \cdots + x_Ny_N)},$$

whose generating function is given by

$$\sum_\lambda \mathcal{P}_\Delta(y, \lambda)e^\lambda = \prod_{i=1}^N \frac{1}{1 - e^{-y_i}e^{\alpha_i}}.$$

By definition, we have $\mathcal{P}_\Delta(0, \lambda) = \mathcal{P}_\Delta(\lambda)$.

For $h \in E$, we define

$$H_\Delta(h) := \{(x_1, \ldots, x_N) \in \mathbb{R}^N \,|\, x_1\alpha_1 + \cdots + x_N\alpha_N = h\},$$
$$X_\Delta(h) := \{(x_1, \ldots, x_N) \in (\mathbb{R}_{\geq 0})^N \,|\, x_1\alpha_1 + \cdots + x_N\alpha_N = h\}.$$

We call $X_\Delta(h)$ the *partition polytope* associated to Δ. Note that it is compact and $\mathcal{P}_\Delta(\lambda)$ is the number of lattice points in it.

Next, we introduce continuous analogues of $\mathcal{P}_\Delta(\lambda)$ and $\mathcal{P}_\Delta(y, \lambda)$. We normalize the Lebesgue measure on E so that the volume of E/Λ is 1, and we consider the standard Lebesgue measure on $\mathbb{R}^N$. They determine the Lebesgue measure ds on $H_\Delta(h)$. For $h \in E$ and $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, let us define

$$\mathcal{V}_\Delta(h) := \text{volume of } X_\Delta(h) = \int_{X_\Delta(h)} ds,$$
$$\mathcal{V}_\Delta(y, h) := \int_{X_\Delta(h)} e^{-(x_1y_1 + \cdots + x_Ny_N)} ds.$$

The function $\mathcal{V}_\Delta$ is called the *volume function* (or *asymptotic partition function*) associated to Δ.

Remark 1.   Under some conditions on Δ and λ, the partition polytope $X_\Delta(\lambda)$ becomes an integral polytope and $\mathcal{P}_\Delta(k \cdot \lambda)$ turns out to be a polynomial of $k \in \mathbb{Z}_{\geq 0}$, which is called the *Ehrhart polynomial*. Then the coefficient of the top term of $\mathcal{P}_\Delta(k \cdot \lambda)$ is equal to $\mathcal{V}_\Delta(\lambda)$. See, e.g., [7].

Moreover, $\mathcal{P}_\Delta(\lambda)$ and $\mathcal{V}_\Delta(\lambda)$ are related to certain invariants, the Riemann-Roch number and the volume, of the toric variety associated to the partition polytope $X_\Delta(\lambda)$.

§ 2.2.   Brion-Vergne formula

In [2], Brion and Vergne gave explicit closed formulas for $\mathcal{P}_\Delta(y, \lambda)$, $\mathcal{V}_\Delta(y, h)$, and $\mathcal{V}_\Delta(h)$. Let us recall those for $\mathcal{V}_\Delta(y, h)$ and $\mathcal{V}_\Delta(h)$. In order to state them, we need some notation.

A subset σ of $\{1, \ldots, N\}$ is called a *basis* of Δ if the sequence $(\alpha_j)_{j \in \sigma}$ is a basis of E. The set of all bases of Δ is denoted by $\mathcal{B}(\Delta)$.

Let $C(\Delta) := \sum_{i=1}^{N} \mathbb{R}_{\geq 0}\alpha_i$ and let $C(\sigma) := \sum_{j \in \sigma} \mathbb{R}_{\geq 0}\alpha_j$ for $\sigma \in \mathcal{B}(\Delta)$. Consider the subdivision of $C(\Delta)$ given by the intersections of the cones $C(\sigma)$ ($\sigma \in \mathcal{B}(\Delta)$). The interior of a maximal cone of this subdivision is called a *chamber*. For a chamber γ, we denote by $\mathcal{B}(\Delta, \gamma)$ the set of all bases $\sigma \in \mathcal{B}(\Delta)$ such that $\gamma \subset C(\sigma)$.

Let $\sigma \in \mathcal{B}(\Delta)$. We define a linear map $v_\sigma : E \to \mathbb{R}^N$ by $v_\sigma(\alpha_j) = w_j$ for $j \in \sigma$, where $(w_1, \ldots, w_N)$ is the standard basis of $\mathbb{R}^N$. We denote by $\mu(\sigma)$ the volume of the parallelepiped

$$\left\{ \sum_{j \in \sigma} t_j \alpha_j \ \middle| \ 0 \leq t_j \leq 1 \quad (j \in \sigma) \right\}.$$

Finally, for $\sigma \in \mathcal{B}(\Delta)$, $j \in \sigma$ and $k \notin \sigma$, we define a real number $c_{jk} = c_{jk}^\sigma$ by $\alpha_k = \sum_{j \in \sigma} c_{jk}\alpha_j$.

Remark 2.   For $h \in \gamma$, the set of vertices of the partition polytope $X_\Delta(h)$ coincides with $\{v_\sigma(h) \,|\, \sigma \in \mathcal{B}(\Delta, \gamma)\}$. See [2, 3.1].

Now, their formulas are stated as follows.

Theorem 2.1 ([2]).   *Let γ be a chamber and let $h \in \gamma$. Suppose $y \in \mathbb{R}^N$ is generic. Then we have*

$$(2.1) \qquad \mathcal{V}_\Delta(y, h) = \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \frac{e^{-\langle y, v_\sigma(h) \rangle}}{\mu(\sigma) \prod_{k \notin \sigma}(y_k - \sum_{j \in \sigma} c_{jk}y_j)},$$

$$(2.2) \qquad \mathcal{V}_\Delta(h) = \frac{1}{(N-d)!} \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \frac{\langle y, v_\sigma(h) \rangle^{N-d}}{\mu(\sigma) \prod_{k \notin \sigma}(-y_k + \sum_{j \in \sigma} c_{jk}y_j)}.$$

Remark 3.

- The assumption that $y \in \mathbb{R}^N$ is generic means that $y_k - \sum_{j \in \sigma} c_{jk} y_j \neq 0$ for any $\sigma \in \mathcal{B}(\Delta, \gamma)$ and $k \notin \sigma$.

- In fact, both formulas hold for h in the closure $\bar{\gamma}$ of γ.

- The right hand side of (2.2) does not depend on y.

§ 3. Weights

§ 3.1. Positive weights

In the argument above, some of $\alpha_1, \ldots, \alpha_N$ may coincide. In order to treat such a case more definitely, we proceed as follows. Let $m = (m_1, m_2 \ldots, m_N)$ be a sequence of positive integers. Each $m_1, \ldots, m_N$ is called a *weight* or a *multiplicity*. Consider the sequence $\Delta_m = (\alpha_1, \ldots, \alpha_1, \ldots, \alpha_N, \ldots, \alpha_N)$, where each α_i is repeated m_i times, and set

$$
\begin{aligned}
\mathcal{P}_\Delta(\lambda; m) :&= \mathcal{P}_{\Delta_m}(\lambda) \\
&= \sharp \left\{ (x_1^{(1)}, \ldots, x_{m_1}^{(1)}, \ldots, x_1^{(N)}, \ldots, x_{m_N}^{(N)}) \in (\mathbb{Z}_{\geq 0})^M \; \middle| \; \right. \\
&\qquad \left. \left(x_1^{(1)} + \cdots + x_{m_1}^{(1)}\right) \alpha_1 + \cdots + \left(x_1^{(N)} + \cdots + x_{m_N}^{(N)}\right) \alpha_N = \lambda \right\}
\end{aligned}
$$

for $\lambda \in \Lambda$. Here $M = m_1 + \cdots + m_N$. Then the generating function of $\mathcal{P}_\Delta(\lambda; m)$ is given by

$$
(3.1) \qquad \sum_\lambda \mathcal{P}_\Delta(\lambda; m) e^\lambda = \frac{1}{\prod_{i=1}^N (1 - e^{\alpha_i})^{m_i}}.
$$

If we set

$$
\Sigma(x) := \{(x_1, \ldots, x_m) \in (\mathbb{R}_{\geq 0})^m \mid x_1 + \cdots + x_m = x\}
$$

for a fixed $x \in \mathbb{R}_{\geq 0}$, then we have

$$
\int_{\Sigma(x)} dx_1 \cdots dx_m = \frac{x^{m-1}}{(m-1)!}.
$$

Hence the volume function associated to Δ and m should be defined by

$$
(3.2) \qquad \mathcal{V}_\Delta(h; m) := \int_{X_\Delta(h)} \frac{x_1^{m_1-1}}{(m_1-1)!} \cdots \frac{x_N^{m_N-1}}{(m_N-1)!} ds
$$

for $h \in E$.

Remark 4.   This is a special case of Gel'fand-Kapranov-Zelevinsky (GKZ, for short) hypergeometric integral. See [6] for an approach to investigate $\mathcal{V}_\Delta(h; m)$ via the GKZ theory.

As before, for $y \in \mathbb{R}^N$, variants of $\mathcal{P}_\Delta(\lambda; m)$ and $\mathcal{V}_\Delta(h; m)$ are defined by

$$(3.3) \qquad \sum_\lambda \mathcal{P}_\Delta(y, \lambda; m) e^\lambda = \frac{1}{\prod_{i=1}^N (1 - e^{-y_i} e^{\alpha_i})^{m_i}},$$

$$(3.4) \qquad \mathcal{V}_\Delta(y, h; m) := \int_{X_\Delta(h)} \frac{x_1^{m_1 - 1}}{(m_1 - 1)!} \cdots \frac{x_N^{m_N - 1}}{(m_N - 1)!} e^{-\langle y, x \rangle} ds.$$

§ 3.2.   Negative weights

Even if some of $m_1, \ldots, m_N$ are nonpositive, (3.1) and (3.3) still define $\mathcal{P}_\Delta(\lambda; m)$ and $\mathcal{P}_\Delta(y, \lambda; m)$. (When $m_i = 0$, we can omit α_i from Δ beforehand.) Moreover, (3.2) and (3.4) still make sense, where for $m = 0, -1, -2, \ldots$, $\dfrac{x^{m-1}}{(m-1)!}$ is understood to be

$$\left. \frac{x_+^\nu}{\Gamma(\nu + 1)} \right|_{\nu = m - 1} = \delta^{(|m|)}(x),$$

the $|m|$-th derivative of the delta function with support at $x = 0$. (See, e.g., [3, 4].) We note that, in the situation above, a product of derivatives of delta functions, its restriction to $H_\Delta(h)$, and the integrals (3.2) and (3.4) over X_Δ are well-defined by virtue of the theory of Sato hyperfunction ([4]).

In conclusion, we have introduced the following definition.

Definition 3.1.   Let $\Delta = (\alpha_1, \ldots, \alpha_N)$ be a sequence of vectors in Λ, lying in an open half space of E and spanning E as vector space, and let $m = (m_1, \ldots, m_N) \in \mathbb{Z}^N$. For $\lambda \in \Lambda$, $h \in E$, and $y \in \mathbb{R}^N$, we define $\mathcal{P}_\Delta(\lambda; m)$, $\mathcal{V}_\Delta(h; m)$, $\mathcal{P}_\Delta(y, \lambda; m)$, and $\mathcal{V}_\Delta(y, h; m)$ respectively by (3.1), (3.2), (3.3), and (3.4). We call $\mathcal{P}_\Delta(\lambda; m)$ (resp. $\mathcal{V}_\Delta(h; m)$) the *vector partition function with weights* (resp. the *volume function with weights*).

§ 4.   Result

Let $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$. For $i = 1, \ldots, N$, we denote $\partial_i = \dfrac{\partial}{\partial y_i}$. Our main result is the following, which generalizes the Brion-Vergne formulas (2.1) and (2.2).

Theorem 4.1.   *Let $\Delta = (\alpha_1, \ldots, \alpha_N)$, $m = (m_1, \ldots, m_N)$ be as in Definition 3.1 and let $M = m_1 + \cdots + m_N$. Suppose h is in a chamber γ and $y \in \mathbb{R}^N$ is generic.*

Then we have

(4.1)

$$\mathcal{V}_\Delta(y, h; m) = \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \left(\frac{e^{-\langle y, v_\sigma(h) \rangle}}{\prod_{k \notin \sigma} (y_k - \sum_{j \in \sigma} c_{jk} y_j)^{m_k}} \right),$$

(4.2)

$$\mathcal{V}_\Delta(h; m) = \frac{1}{(M - d)!} \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma} \frac{\partial_j^{m_j - 1}}{(m_j - 1)!} \left(\frac{\langle y, v_\sigma(h) \rangle^{M-d}}{\prod_{k \notin \sigma} (-y_k + \sum_{j \in \sigma} c_{jk} y_j)^{m_k}} \right),$$

where if $m_i \leq 0$, we set $\dfrac{\partial_j^{m_j - 1}}{(m_j - 1)!} = 0$.

A proof is given in the next section. Let us consider a simple example.

Example 4.2 (Volume function for A_2).   Let $\Delta = (\alpha_1, \alpha_2, \alpha_3)$ be the positive root system of type A_2, where $\alpha_3 = \alpha_1 + \alpha_2$. There are two chambers $\gamma_1 = \{p_1 \alpha_1 + p_2 \alpha_2 \,|\, p_1 > p_2 > 0\}$ and $\gamma_2 = \{p_1 \alpha_1 + p_2 \alpha_2 \,|\, p_2 > p_1 > 0\}$ in $C(\Delta)$. For simplicity, suppose $h = p_1 \alpha_1 + p_2 \alpha_2$ is in γ_1. Then for $m = (m_1, m_2, m_3) \in \mathbb{Z}$

$$\mathcal{V}_{A_2}(h; m) = \int_0^{p_2} \frac{(p_1 - t)^{m_1 - 1}}{(m_1 - 1)!} \frac{(p_2 - t)^{m_2 - 1}}{(m_2 - 1)!} \frac{t^{m_3 - 1}}{(m_3 - 1)!} \, dt.$$

Direct calculation shows

$$\mathcal{V}_{A_2}(h; m) =$$
$$\sum_{i=0}^{m_1 - 1} \frac{(m_2 + i - 1)!}{(m_2 - 1)!(m_1 - i - 1)! i! (m_2 + m_3 + i - 1)!} (p_1 - p_2)^{m_1 - i - 1} p_2^{m_2 + m_3 + i - 1},$$

while our theorem shows

$$(M - 2)! \cdot \mathcal{V}_{A_2}(h; m) =$$
$$\frac{\partial_1^{m_1 - 1}}{(m_1 - 1)!} \frac{\partial_2^{m_2 - 1}}{(m_2 - 1)!} \frac{(p_1 y_1 + p_2 y_2)^{M-2}}{(-y_3 + y_1 + y_2)^{m_3}} + \frac{\partial_1^{m_1 - 1}}{(m_1 - 1)!} \frac{\partial_3^{m_3 - 1}}{(m_3 - 1)!} \frac{((p_1 - p_2) y_1 + p_2 y_3)^{M-2}}{(-y_2 - y_1 + y_3)^{m_2}}$$

with $M = m_1 + m_2 + m_3$. It might be natural to ask if there is any relation between our formula and the GKZ hypergeometric theory.

§5.   **Proof of Theorem 4.1**

The formula (4.2) follows from (4.1) in the same way with the proof of (2.2) in [2, 3.3]. Hence let us prove (4.1).

Step 1. If all the m_j $(j = 1, \ldots, N)$ are positive, the proof is quite easy. In fact, by applying $\prod_{j=1}^{N} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!}$ to the Brion-Vergne formula ((2.1) in Theorem 2.1)

$$\int_{X_\Delta(h)} e^{-\langle y, x \rangle} ds = \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \frac{e^{-\langle y, v_\sigma(h) \rangle}}{\mu(\sigma) \prod_{k \notin \sigma} (y_k - \sum_{j \in \sigma} c_{jk} y_j)^{m_k}},$$

we have

$$\int_{X_\Delta(h)} \frac{x_1^{m_1 - 1}}{(m_1 - 1)!} \cdots \frac{x_N^{m_N - 1}}{(m_N - 1)!} e^{-\langle y, x \rangle} ds$$

$$= \sum_{\sigma \in \mathcal{B}(\Delta, \gamma)} \prod_{j \in \sigma} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \left(\frac{e^{-\langle y, v_\sigma(h) \rangle}}{\mu(\sigma) \prod_{k \notin \sigma} (y_k - \sum_{j \in \sigma} c_{jk} y_j)^{m_k}} \right),$$

as required. Here note that if $k \notin \sigma$, then $\langle y, v_\sigma(h) \rangle$ does not contain the variable y_k and hence

$$\frac{(-\partial_k)^{m_k - 1}}{(m_k - 1)!} \left(\frac{e^{-\langle y, v_\sigma(h) \rangle}}{y_k - \sum_{j \in \sigma} c_{jk} y_j} \right) = \frac{e^{-\langle y, v_\sigma(h) \rangle}}{(y_k - \sum_{j \in \sigma} c_{jk} y_j)^{m_k}}.$$

Step 2. In order to treat general case, we proceed by induction on $N = \sharp \Delta$.

First, let $N = 1$. If $m_1 > 0$, (4.1) holds by Step 1. If $m_1 \leq 0$, then $\mathcal{V}(y, h; m) = 0$ since $X_\Delta(h)$ and $\{x_1 = 0\}$, the support of $\delta^{(|m_1|)}(x_1)$, are disjoint. On the other hand, the right hand side of (4.1) is also 0 by $\frac{(-\partial_1)^{m_1 - 1}}{(m_1 - 1)!} = 0$.

Next, let us suppose that our formula for $\mathcal{V}_\Delta(y, h; m)$ holds for all Δ and m with $\sharp \Delta \leq N - 1$, whether or not all of the m_j are positive.

Taking Step 1 into account, we consider the case where some of m_j are nonpositive. Without loss of generality, we may assume that $m_N \leq 0$. Then since

$$\frac{x_N^{m_N - 1}}{(m_N - 1)!} = \delta^{(|m_N|)}(x_N),$$

we have

$$(5.1) \qquad \mathcal{V}_\Delta(y, h; m) = \int_{X_\Delta(h)} \left(-\frac{\partial}{\partial x_N} \right)^{|m_N|} \left(\prod_{j=1}^{N-1} \frac{x_j^{m_j - 1}}{(m_j - 1)!} e^{-\langle y, x \rangle} \right) \Bigg|_{x_N = 0} ds.$$

Let us fix a $\sigma_0 \in \mathcal{B}(\Delta, \gamma)$ such that $N \notin \sigma_0$. (If such a σ_0 does not exist, (4.1) obviously holds. In fact, it is easy to see that $X_\Delta(h) \cap \{x_N = 0\} = \emptyset$ in this case. Hence we have $\mathcal{V}_\Delta(y, h; m) = 0$. On the other hand, the right hand side of (4.1) is also 0 by

$\dfrac{(-\partial_N)^{m_N-1}}{(m_N-1)!} = 0$.) Let us set $h = \sum_{j\in\sigma_0} p_j\alpha_j$. We take $(x_j)_{j\notin\sigma_0}$ as a coordinate of the hyperplane $H_\Delta(h)$. Recall that $\alpha_k = \sum_{j\in\sigma_0} c^{\sigma_0}_{jk}\alpha_j$ for $k\notin\sigma_0$. Hence we see that

$$x_j = p_j - \sum_{k\notin\sigma_0} c^{\sigma_0}_{jk}x_k \qquad (j\in\sigma_0)$$

and

$$\langle y, x\rangle = \sum_{j\in\sigma_0} y_j p_j + \sum_{k\notin\sigma_0}\left(y_k - \sum_{j\in\sigma_0} c^{\sigma_0}_{jk}y_j\right)x_k.$$

It follows that

$$\frac{\partial x_j}{\partial x_N} = \begin{cases} -c^{\sigma_0}_{jN} & (j\in\sigma_0) \\ 0 & (j\notin\sigma_0). \end{cases}$$

and

$$\frac{\partial}{\partial x_N}e^{-\langle y,x\rangle} = -\left(y_N - \sum_{j\in\sigma_0} c^{\sigma_0}_{jN}y_j\right)e^{-\langle y,x\rangle}.$$

Hence the integrand in (5.1) is computed as follows.

$$\left(-\frac{\partial}{\partial x_N}\right)^{|m_N|}\left(\prod_{j=1}^{N-1}\frac{x_j^{m_j-1}}{(m_j-1)!}e^{-\langle y,x\rangle}\right)\Bigg|_{x_N=0}$$

$$= \sum_{u_j,u_e}\binom{|m_N|}{u_j,u_e}\prod_{j=1}^{N-1}\left(-\frac{\partial}{\partial x_N}\right)^{u_j}\frac{x_j^{m_j-1}}{(m_j-1)!}\left(-\frac{\partial}{\partial x_N}\right)^{u_e}e^{-\langle y,x\rangle}\Bigg|_{x_N=0}$$

$$= \sum_{u_j,u_e}\binom{|m_N|}{u_j,u_e}\prod_{j\in\sigma_0^c}\frac{x_j^{m_j-1}}{(m_j-1)!}\prod_{j\in\sigma_0}\frac{(c^{\sigma_0}_{jN})^{u_j}x_j^{m_j-u_j-1}}{(m_j-u_j-1)!}\left(y_N - \sum_{j\in\sigma_0} c^{\sigma_0}_{jN}y_j\right)^{u_e}e^{-\langle y',x'\rangle},$$

where the sum is taken over all $u_j\in\mathbb{Z}_{\geq0}$ $(j\in\sigma_0)$ and $u_e\in\mathbb{Z}_{\geq0}$ such that $\sum_{j\in\sigma_0} u_j + u_e = |m_N|$. and we set

$$\binom{|m_N|}{u_j,u_e} := \frac{|m_N|!}{(\prod_{j\in\sigma_0} u_j!)u_e!}.$$

Note that $y' = (y_1,\ldots,y_{N-1})$, $x' = (x_1,\ldots,x_{N-1})$, and σ_0^c is the complement of σ_0 in $\{1,\ldots,N-1\}$.

Therefore, we see that $\mathcal{V}_\Delta(y,h;m)$ becomes a linear combination of integrals over

$X_{\Delta'}(h)$, where $\Delta' := \{\alpha_1, \ldots, \alpha_{N-1}\}$, as follows.

$$(5.2) \quad \mathcal{V}_\Delta(y, h; m) = \sum_{u_j, u_e} \binom{|m_N|}{u_j, u_e} \prod_{j \in \sigma_0} (c_{jN}^{\sigma_0})^{u_j} \left(y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j \right)^{u_e}$$

$$\int_{X_{\Delta'}(h)} \prod_{j \in \sigma_0^c} \frac{x_j^{m_j - 1}}{(m_j - 1)!} \prod_{j \in \sigma_0} \frac{x_j^{m_j - u_j - 1}}{(m_j - u_j - 1)!} e^{-\langle y', x' \rangle} ds',$$

where ds' is the Lebesgue measure on

$$H_{\Delta'}(h) = \{(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \mid x_1 \alpha_1 + \cdots + x_{N-1} \alpha_{N-1} = h\}.$$

Step 3. Let γ' be the chamber in $C(\Delta')$ that contains h. By the assumption of our induction, the integral

$$\int_{X_{\Delta'}(h)} \prod_{j \in \sigma_0^c} \frac{x_j^{m_j - 1}}{(m_j - 1)!} \prod_{j \in \sigma_0} \frac{x_j^{m_j - u_j - 1}}{(m_j - u_j - 1)!} e^{-\langle y', x' \rangle} ds'$$

in (5.2) is computed as follows.

$$(5.3) \quad \sum_{\sigma \in B(\Delta', \gamma')} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma \cap \sigma_0^c} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \prod_{j \in \sigma \cap \sigma_0} \frac{(-\partial_j)^{m_j - u_j - 1}}{(m_j - u_j - 1)!}$$

$$\left(\frac{e^{-\langle y', v_\sigma(h) \rangle}}{\prod_{k \in \sigma^c \cap \sigma_0^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{m_k} \prod_{k \in \sigma^c \cap \sigma_0} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{m_k - u_k}} \right)$$

$$= \sum_{\sigma \in B(\Delta', \gamma')} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma \cap \sigma_0^c} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \prod_{j \in \sigma \cap \sigma_0} \frac{(-\partial_j)^{m_j - u_j - 1}}{(m_j - u_j - 1)!}$$

$$\left(\frac{e^{-\langle y', v_\sigma(h) \rangle} \prod_{k \in \sigma^c \cap \sigma_0} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{u_k}}{\prod_{k \in \sigma^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{m_k}} \right),$$

where σ^c is the complement of σ in $\{1, \ldots, N-1\}$.

Let $\sigma \in \mathcal{B}(\Delta', \gamma')$. Then the factor $\prod_{j \in \sigma_0} (c_{jN}^{\sigma_0})^{u_j} \left(y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j \right)^{u_e}$ in (5.2) is

rewritten as follows.

(5.4)

$$\prod_{j \in \sigma_0} (c_{jN}^{\sigma_0})^{u_j} \left(y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j \right)^{u_e}$$

$$= \prod_{j \in \sigma \cap \sigma_0} (c_{jN}^{\sigma_0})^{u_j} \left(y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j \right)^{u_e} \prod_{k \in \sigma^c \cap \sigma_0} (c_{kN}^{\sigma_0})^{u_k}$$

$$= \frac{(\prod_{j \in \sigma \cap \sigma_0} u_j!) u_e!}{(u_e + u_{\sigma \cap \sigma_0})!} \prod_{j \in \sigma \cap \sigma_0} \frac{(-\partial_j)^{u_j}}{u_j!} \left(y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j \right)^{u_e + u_{\sigma \cap \sigma_0}} \prod_{k \in \sigma^c \cap \sigma_0} (c_{kN}^{\sigma_0})^{u_k},$$

where $u_{\sigma \cap \sigma_0} = \sum_{j \in \sigma \cap \sigma_0} u_j$.

Step 4. It follows from (5.2), (5.3), and (5.4) that

$$(5.5) \qquad \mathcal{V}_\Delta(y, h; m) = \sum_{u_j, u_e} \sum_{\sigma \in B(\Delta', \gamma')} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma \cap \sigma_0^c} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} F(u_j, u_e, \sigma),$$

where

$$F(u_j, u_e, \sigma)$$

$$= \frac{|m_N|!}{(\prod_{k \in \sigma^c \cap \sigma_0} u_k!)(u_e + u_{\sigma \cap \sigma_0})!} \prod_{j \in \sigma_0 \cap \sigma} \frac{(-\partial_j)^{u_j}}{u_j!} \left(y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j \right)^{u_e + u_{\sigma \cap \sigma_0}}$$

$$\prod_{j \in \sigma \cap \sigma_0} \frac{(-\partial_j)^{m_j - u_j - 1}}{(m_j - u_j - 1)!} \left(\frac{e^{-\langle y', v_\sigma(h) \rangle} \prod_{k \in \sigma^c \cap \sigma_0} \left(c_{kN}^{\sigma_0} \left(y_k - \sum_{j \in \sigma} c_{jk}^{\sigma} y_j \right) \right)^{u_k}}{\prod_{k \in \sigma^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^{\sigma} y_j \right)^{m_k}} \right).$$

Furthermore, $\displaystyle\sum_{u_j, u_e} F(u_j, u_e, \sigma)$ is equal to

$$\sum_{u_k} \binom{|m_N|}{u_k, |m_N| - u_{\sigma^c \cap \sigma_0}} \prod_{j \in \sigma \cap \sigma_0} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \left(\left(y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j \right)^{|m_N| - u_{\sigma^c \cap \sigma_0}} \right.$$

$$\times \left. \frac{e^{-\langle y', v_\sigma(h) \rangle} \prod_{k \in \sigma^c \cap \sigma_0} \left(c_{kN}^{\sigma_0} \left(y_k - \sum_{j \in \sigma} c_{jk}^{\sigma} y_j \right) \right)^{u_k}}{\prod_{k \in \sigma^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^{\sigma} y_j \right)^{m_k}} \right),$$

the sum over all $u_k \in \mathbb{Z}_{\geq 0}$ $(k \in \sigma^c \cap \sigma_0)$ such that $u_{\sigma^c \cap \sigma_0} = |m_N|$, where we set

$u_{\sigma^c \cap \sigma_0} = \sum\limits_{k \in \sigma^c \cap \sigma_0} u_k$. By the following lemma, this implies that

$$(5.6) \qquad \sum_{u_j, u_e} F(u_j, u_e, \sigma)$$

$$= \prod_{j \in \sigma \cap \sigma_0} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \left(\frac{e^{-\langle y', v_\sigma(h) \rangle} \left(y_N - \sum_{j \in \sigma} c_{jN}^\sigma y_j \right)^{|m_N|}}{\prod_{k \in \sigma^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{m_k}} \right).$$

Lemma 5.1.

$$y_N - \sum_{j \in \sigma_0} c_{jN}^{\sigma_0} y_j + \sum_{k \in \sigma^c \cap \sigma_0} \left(c_{kN}^{\sigma_0} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right) \right) = y_N - \sum_{j \in \sigma} c_{jN}^\sigma y_j.$$

Proof. We see that

$$\alpha_N = \sum_{k \in \sigma_0} c_{kN}^{\sigma_0} \alpha_k = \sum_{k \in \sigma^c \cap \sigma_0} c_{kN}^{\sigma_0} \alpha_k + \sum_{k \in \sigma \cap \sigma_0} c_{kN}^{\sigma_0} \alpha_k$$

$$= \sum_{k \in \sigma^c \cap \sigma_0} c_{kN}^{\sigma_0} \sum_{j \in \sigma} c_{jk}^\sigma \alpha_j + \sum_{k \in \sigma \cap \sigma_0} c_{kN}^{\sigma_0} \alpha_k$$

$$= \sum_{j \in \sigma \cap \sigma_0^c} \sum_{k \in \sigma^c \cap \sigma_0} c_{kN}^{\sigma_0} c_{jk}^\sigma \alpha_j + \sum_{j \in \sigma \cap \sigma_0} \left(\sum_{k \in \sigma^c \cap \sigma_0} c_{kN}^{\sigma_0} c_{jk}^\sigma + c_{jN}^{\sigma_0} \right) \alpha_j.$$

Hence we have

$$c_{jN}^\sigma = \begin{cases} \sum_{k \in \sigma^c \cap \sigma_0} c_{kN}^{\sigma_0} c_{jk}^\sigma & (j \in \sigma \cap \sigma_0^c) \\ c_{jN}^{\sigma_0} + \sum_{k \in \sigma^c \cap \sigma_0} c_{kN}^{\sigma_0} c_{jk}^\sigma & (j \in \sigma \cap \sigma_0), \end{cases}$$

which implies the lemma. $\qquad \qquad \square$

Step 5. From (5.5) and (5.6) we have

$$\mathcal{V}_\Delta(y, h; m)$$

$$= \sum_{\sigma \in B(\Delta', \gamma')} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma \cap \sigma_0^c} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \prod_{j \in \sigma \cap \sigma_0} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!}$$

$$\left(\frac{e^{-\langle y', v_\sigma(h) \rangle} \left(y_N - \sum_{j \in \sigma} c_{jN}^\sigma y_j \right)^{|m_N|}}{\prod_{k \in \sigma^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{m_k}} \right)$$

$$= \sum_{\sigma \in B(\Delta', \gamma')} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma} \frac{(-\partial_j)^{m_j - 1}}{(m_j - 1)!} \left(\frac{e^{-\langle y', v_\sigma(h) \rangle} \left(y_N - \sum_{j \in \sigma} c_{jN}^\sigma y_j \right)^{|m_N|}}{\prod_{k \in \sigma^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{m_k}} \right).$$

Since $\dfrac{(-\partial_j)^{m_j-1}}{(m_j-1)!} = 0$ if $m_j \leq 0$, the sum above is actually taken over $\sigma \in \mathcal{B}(\Delta', \gamma')$ such that $m_j > 0$ for $\forall j \in \sigma$. Moreover, since h belongs to both γ' and γ, we see that

$$\gamma' \subset C(\sigma) \iff \gamma \subset C(\sigma).$$

for $\sigma \in \mathcal{B}(\Delta')$. Therefore, we obtain

$$\mathcal{V}_\Delta(y, h; m) = \sum_{\sigma \in B(\Delta, \gamma)\, \text{s.t.}\, N \notin \sigma} \frac{1}{\mu(\sigma)} \prod_{j \in \sigma} \frac{(-\partial_j)^{m_j-1}}{(m_j-1)!} \left(\frac{e^{-\langle y, v_\sigma(h)\rangle}}{\prod_{k \in \sigma^c} \left(y_k - \sum_{j \in \sigma} c_{jk}^\sigma y_j \right)^{m_k}} \right),$$

where this time σ^c is the complement of σ in $\{1, \ldots, N\}$. Since $\dfrac{(-\partial_j)^{m_N-1}}{(m_N-1)!} = 0$, the formula above is nothing but (4.1). This completes the proof of Theorem 4.1.

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