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Zeros of partial zeta functions off the critical line

By

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Abstract

We extend the joint universality theorem for Artin L-functions $L(s, \chi_j, K/\mathbb{Q})$ from the previously known strip $1 - \frac{1}{2k} < \operatorname{Re} s < 1$ for $k = \# G(K/\mathbb{Q})$ to the maximal strip $\frac{1}{2} < \operatorname{Re} s < 1$ under an assumption of a weak version of the density hypothesis. Then, we study zeros of partial zeta functions $\zeta(s, \mathcal{A})$ inside the half of the critical strip as an application of universality.

§ 1. Introduction

Let $Q(m,n) = am^2 + bmn + cn^2$ be a positive definite quadratic form with its discriminant $D = b^2 - 4ac < 0$ and $a, b, c \in \mathbb{Z}$. We define the Epstein zeta function attached to Q by

$$E(s, Q) = \sum_{(m,n) \neq (0,0)} \frac{1}{Q(m,n)^s}, \qquad \operatorname{Re} s > 1.$$

Then, it has a meromorphic continuation to $\mathbb{C}$ with one simple pole at $s = 1$, and has the functional equation

$$\Phi(s) := \left(\frac{\sqrt{-D}}{2\pi} \right)^s \Gamma(s) E(s, Q) = \Phi(1-s).$$

Voronin([9] or [4]) studied zeros of $E(s, Q)$ inside the critical strip and proved the following theorem.

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Theorem 1.1 (Voronin). *Suppose that the class number of $\mathbb{Q}(\sqrt{D})$ is $h(D) > 1$. The Epstein zeta function $E(s,Q)$ described above has at least cT zeros on the rectangular region $\sigma_1 < \operatorname{Re} s < \sigma_2$, $0 < \operatorname{Im} s < T$ for each $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$ and some $c > 0$ as $T \to \infty$.*

The main purpose of this paper is extending it to partial zeta functions which are natural algebraic generalizations of Epstein zeta functions.

Let $\mathfrak{f}$ be an integral ideal of the number field K. Then, $J^{\mathfrak{f}}$ denotes the group of all ideals of K which are relatively prime to $\mathfrak{f}$ and $P^{\mathfrak{f}}$ stands for the group of fractional principal ideals (a) such that $a \equiv 1 \mod \mathfrak{f}$ with a totally positive. Choose an element $\mathcal{A}$ of the ray class group $G^{\mathfrak{f}} = J^{\mathfrak{f}}/P^{\mathfrak{f}} \mod \mathfrak{f}$. The partial zeta function $\zeta(s,\mathcal{A})$ attached to $\mathcal{A}$ is defined by

$$\zeta(s,\mathcal{A}) = \sum_{\mathfrak{n}\in\mathcal{A}} \frac{1}{N(\mathfrak{n})^s} \qquad \operatorname{Re} s > 1,$$

where $\mathfrak{n}$ runs through all ideals in O_K and $N(\mathfrak{n})$ denotes the norm of $\mathfrak{n}$. It has a meromorphic continuation to $\mathbb{C}$ with only one simple pole at $s = 1$ and the functional equation

$$\xi(1-s,\mathcal{A}) = \xi(s,\mathcal{A}) := D(\mathfrak{f})^s \Gamma(s)^{r_2} \prod_{m=1}^{r_1} \Gamma\left(\frac{s+a_m}{2}\right) \zeta(s,\mathcal{A}),$$

where r_1 is the number of real places of K, $2r_2$ is the number of complex places of K. The constant $D(\mathfrak{f})$ depends only on $\mathfrak{f}$ and a_m takes the value 0 or 1. For details see Chapter 7 of [8].

The Hecke L-function attached to a ray class character $\psi : G^{\mathfrak{f}} \to S^1$ is defined by

$$L(s,\psi) = \sum_{\mathfrak{n}\in J^{\mathfrak{f}}} \frac{\psi(\mathfrak{n})}{N(\mathfrak{n})^s} = \prod_{\mathfrak{p}\in J^{\mathfrak{f}}} \left(1 - \frac{\psi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \qquad \operatorname{Re} s > 1.$$

Since the functional equation of $\zeta(s,\mathcal{A})$ does not depend on the choice of $\mathcal{A}$, we can deduce the functional equation for the Hecke L-function $L(s,\psi)$ by

$$L(s,\psi) = \sum_{\mathcal{A}\in G^{\mathfrak{f}}} \psi(\mathcal{A})\zeta(s,\mathcal{A}).$$

If we take a representative $\mathfrak{a} \in \mathcal{A}$, then we have

$$\zeta(s,\mathcal{A}) = \frac{1}{h} \sum_{\psi} \bar{\psi}(\mathfrak{a}) L(s,\psi),$$

where ψ runs through all ray class characters ψ defined on $G^{\mathfrak{f}}$, and $h = \#G^{\mathfrak{f}}$.

For a positive definite quadratic form $Q(m,n)$ with its discriminant $D < 0$, it is known that

$$E(s,Q) = \omega\zeta(s,\mathcal{A}),$$

where ω is the number of units of $\mathbb{Q}(\sqrt{D})$ and $\mathcal{A}$ is an ideal class corresponding Q. So, we have

$$E(s,Q) = \frac{\omega}{h} \sum_{\psi} \bar{\psi}(\mathfrak{a}) L(s,\psi)$$

and the condition $h(D) > 1$ of Theorem 1.1 means that $h = h(D) = \#G > 1$ for the ideal class group G of $\mathbb{Q}(\sqrt{D})$. In other words, $E(s,Q)$ is a linear combination of at least two Hecke L-functions. Voronin's proof of Theorem 1.1 is based on the joint distribution of those Hecke L-functions in the above equation and Rouché's theorem in complex analysis.

Bauer [1] proved the following theorem concerning zeros of $\zeta(s,\mathcal{A})$ inside the critical strip.

Theorem 1.2 (Bauer)**.** *If T is sufficiently large, then there is a number $c > 0$ such that there are at least cT zeros of $\zeta(s,\mathcal{A})$ in the region with $\frac{1}{2} < \operatorname{Re} s < 1$, $|\operatorname{Im} s| < T$.*

The main ingredient of its proof is the joint universality of Artin L-functions instead of Hecke L-functions. His argument counts the number of zeros in the strip $1 - \frac{1}{2k} < \operatorname{Re} s < 1$ for $k = \#Gal(K/\mathbb{Q})$ because the inequality

$$\int_0^T |f(\sigma + it)|^2 dt \ll T$$

is required for the proof of the joint universality of Artin L-functions and known only for the strip $1 - \frac{1}{2k} < \operatorname{Re} s < 1$ for $k = \#Gal(K/\mathbb{Q})$.

§ 2. Joint distribution of Artin L-functions

Let $K/\mathbb{Q}$ be a normal extension with number field K and G be its Galois group $G(K/\mathbb{Q})$. Let $\rho : G \to GL_m(\mathbb{C})$ be a m-dimensional representation of G in the general linear group $GL_m(\mathbb{C})$. The character $\chi : G \to \mathbb{C}$ of the representation ρ is given by

$$\chi(g) := tr(\rho(g)).$$

The Artin L-function of χ and G is defined by the Euler product

$$L(s,\chi,K/\mathbb{Q}) = \prod_{p:\text{unr.}} L_p(s,\chi), \qquad \operatorname{Re} s > 1,$$

where $L_p(s,\chi) = \det(I - \rho(\sigma_p)p^{-s})^{-1}$ and σ_p denotes one of the conjugate Frobenius automorphisms over p. This definition is independent of the specific representation ρ of the character χ and the chosen conjugate of the Frobenius σ_p.

Brauer's theorem states that every character χ of a finite group G is a $\mathbb{Z}$-linear combination of characters ψ_{l*} induced from characters ψ_l of degree 1 associated to subgroups H_l of G. Thus, for $j \leqslant J$ we have

$$\chi_j = \sum_{l \leqslant l_0} n_{j,l} \psi_l^*,$$

where ψ_l^* are deduced from characters ψ_l of degree 1 associated to subgroups H_l of G and $n_{j,l} \in \mathbb{Z}$. As a consequence, we have

$$L(s, \chi_j, K/\mathbb{Q}) = \prod_{l \leqslant l_0} L(s, \psi_l)^{n_{j,l}} \tag{2.1}$$

and $L(s, \psi_l) = L(s, \psi_l, H_l)$ are Hecke L-functions over number fields contained in K. Note that (2.1) shows that the Artin L-function $L(s, \chi_j, K/\mathbb{Q})$ has a meromorphic continuation to $\mathbb{C}$.

Conjecture 2.1. *Let $L(s, \chi, K/\mathbb{Q})$ be an Artin L-function and write*

$$L(s, \chi, K/\mathbb{Q}) = \prod_{l \leqslant l_0} L(s, \psi_l)^{n_l}$$

for some $n_l \in \mathbb{Z}$. Define $N_\psi(\sigma, T)$ by the number of zeros of Hecke L-function $L(s, \psi)$ on the region $\operatorname{Re} s > \sigma$, $0 < \operatorname{Im} s < T$. Then, there is a constant $c > 0$ such that

$$N_{\psi_l}(\sigma, T) \ll T^{1 + c(\frac{1}{2} - \sigma)} \log T$$

uniformly for $\sigma \geqslant \frac{1}{2}$ and $l \leqslant l_0$.

Now, we are ready to state the joint universality of Artin L-functions.

Theorem 2.2. *Let K be a finite Galois extension of $\mathbb{Q}$ and let $\chi_1, \ldots, \chi_J$ be $\mathbb{C}$-linearly independent characters of the group $G = Gal(K/\mathbb{Q})$. Assume Conjecture 2.1 for $L(s, \chi_j, K/\mathbb{Q})$, $j \leqslant J$. Let $D = D_r(s_0)$ be the closed disc with center s_0 and radius r such that D is contained in the vertical strip $\frac{1}{2} < \operatorname{Re} s < 1$. Suppose that $h_1(s), \ldots, h_J(s)$ are analytic and nonvanishing on $s \in \operatorname{int} D$, and continuous on $s \in D$. Then, for every $\epsilon > 0$ we have*

$$\liminf_{T \to \infty} \frac{1}{T} \left| \left\{ \tau \in [T, 2T] : \max_{j \leqslant J} \max_{s \in D} |L(s + i\tau, \chi_j, K/\mathbb{Q}) - h_j(s)| < \epsilon \right\} \right| > 0.$$

We modify several lemmas from [1] and [6] for the proof of Theorem 2.2.

Lemma 2.3. *Assume Conjecture 2.1 for $L(s, \chi_j, K/\mathbb{Q})$, $j \leqslant J$. Let $\frac{1}{2} < \sigma \leqslant 1$ and $X = T^\kappa$. Then,*

$$\int_T^{2T} \left| \log L(\sigma + it, \chi_j, K/\mathbb{Q}) - \sum_{p \leqslant X} \log L_p(\sigma + it, \chi_j) \right|^2 dt = O\left(T^{1 + c(\frac{1}{2} - \sigma)} \right)$$

for some $c > 0$ and small enough $\kappa > 0$ and all $j \leqslant J$.

Lemma 2 of [6] and (2.1) imply Lemma 2.3.

Lemma 2.4. *Assume Conjecture 2.1 for $L(s, \chi_j, K/\mathbb{Q})$, $j \leqslant J$. Let $D = D_r(s_0) \subset \{s \in \mathbb{C} : \frac{1}{2} < \operatorname{Re} s < 1\}$.*

$$\max_{j \leqslant J} \max_{s \in D} \left| \log L(s + i\tau, \chi_j, K/\mathbb{Q}) - \sum_{p \leqslant X} \log L_p(s + i\tau, \chi_j) \right| \leqslant T^{-c_2}$$

for $\tau \in [T, 2T] \setminus A_T$, $|A_T| \leqslant T^{1-c_1}$.

Lemma 2.4 is a simple consequence of Lemma 4 of [6] and Lemma 2.3.

Define $L_M(s, \chi, \theta) = \prod_{p \in M} f_p(p^{-s} e(-\theta_p))$ and $f_p(t) = \det(I - \rho(\sigma_p)t)^{-1}$. Lemma 2.2 of [1] is on the disc $D_r(1 - \frac{1}{4k})$ with $0 < r < \frac{1}{4k}$, but its proof in fact proves more than written. So, we restate Lemma 2.2 of [1] based on its proof.

Lemma 2.5. *Let $\chi_1, \ldots, \chi_J$ be $\mathbb{C}$-linearly independent characters of the Galois group $G = Gal(K/\mathbb{Q})$, where K is a finite normal algebraic extension of $\mathbb{Q}$. Let $D = D_r(s_0) \subset \{s \in \mathbb{C} : \frac{1}{2} < \operatorname{Re} s < 1\}$. Suppose that $h_1(s), \ldots, h_J(s)$ are analytic and nonvanishing on $s \in \operatorname{int} D$, and continuous on $s \in D$. Then for every pair $\epsilon > 0$ and $y > 0$, there exists a finite set of primes M containing all primes smaller than y and a vector $\theta \in \mathbb{R}^{\mathbb{P}}$ such that*

$$\max_{j \leqslant J} \max_{s \in D} |L_M(s, \chi_j, \theta) - h_j(s)| < \epsilon.$$

Lemma 2.6. *Let $f(s)$ be an analytic function on a region containing $|s| \leqslant R$ and $\alpha > 0$. Then, we have*

$$|f(0)|^\alpha \leqslant \frac{1}{\pi R^2} \int\int_{|s| \leqslant R} |f(s)|^\alpha d\sigma dt.$$

This is a property of subharmonic function $|f(s)|^\alpha$ and its proof can be found in Lemma 3 of [6].

Lemma 2.7.

$$\int_0^T \left| \sum_{n \leqslant N} a_n n^{-it} \right|^2 dt = \sum_{n \leqslant N} |a_n|^2 (T + O(n)).$$

Lemma 2.7 is well-known and we may refer [7] for its proof. Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. By Lemma 2.4 and 2.5, it is enough to show that

$$\max_{j \leqslant J} \max_{s \in D} |\log L_M(s, \chi_j, \theta) - \sum_{p \leqslant X} \log L_p(s + i\tau, \chi_j)| < \epsilon$$

for a positive proportion of $\tau \in [T, 2T] \setminus A_T$, where $X = T^\kappa$ and M is a finite set of primes containing all primes smaller than $y > 0$.

Define the sets $C(\delta, M, T)$ and $C(\delta, M)$ by

$$C(\delta, M, T) = \{\tau \in [T, 2T] : ||\theta_p - \frac{\tau}{2\pi} \log p|| < \frac{\delta}{2} \text{ for all } p \in M\}$$

and

$$C(\delta, M) = \{(\vartheta_p) \in \Omega : ||\vartheta_p - \theta_p|| < \frac{\delta}{2} \text{ for all } p \in M\}$$

where $\theta = (\theta_p) \in \mathbb{R}^{\mathbb{P}}$ is as given in Lemma 2.5 and $||x|| = \min\{|x - n| : n \in \mathbb{Z}\}$. We also use the short expression $C(\delta, X) = C(\delta, \{p \leqslant X\})$ for a real number $X > 0$. By uniform continuity, there exists $\delta > 0$ such that for $\tau \in C(\delta, M, T)$

$$\max_{j \leqslant J} \max_{s \in D} \left| \log L_M(s, \chi_j, \theta) - \sum_{p \in M} \log L_p(s + i\tau, \chi_j) \right| < \epsilon \tag{2.2}$$

and we have

$$|C(\delta, M, T)| \sim |C(\delta, M)| T = \delta^{|M|} T, \qquad T \to \infty$$

by Kronecker's theorem.

Let $D' = D_R(s_0)$ be the disc containing $D = D_r(s_0)$ with $R > r$ and contained in the strip $\frac{1}{2} < \operatorname{Re} s < 1$. Let $\sigma_0 = \min\{\operatorname{Re} s : s \in D'\}$. Take $Y > \max\{y\delta^{|M|(1-2\sigma_0)^{-1}}, \max\{p \in M\}\}$ and let P be the largest prime $\leqslant Y$. We write $Y \setminus M$ for the set $\{p \leqslant Y : p \notin M\}$. By Kronecker's theorem, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_{C(\delta, M, T)} \left| \sum_{p \in M} \log L_p(s + i\tau, \chi_j) - \sum_{p \leqslant Y} \log L_p(s + i\tau, \chi_j) \right|^2 d\tau$$
$$= \int_{C(\delta, M)} \left| \sum_{p \in Y \setminus M} \log L_p(s, \chi_j, \vartheta) \right|^2 d\vartheta_2 \cdots d\vartheta_P \leqslant$$
$$\leqslant \delta^{|M|} \int_0^1 \cdots \int_0^1 \left| \sum_{p \in Y \setminus M} \log L_p(s, \chi_j, \vartheta) \right|^2 \prod_{p \in Y \setminus M} d\vartheta_p = \delta^{|M|} \sum_{p \in Y \setminus M} \sum_{m=1}^{\infty} \frac{|a_{p,j,m}|^2}{p^{2m\sigma}},$$

where $\log L_p(s, \chi_j) = \sum_{m=1}^{\infty} a_{p,j,m} p^{-ms}$. Since $|a_{p,j,m}| \leqslant [K : \mathbb{Q}]$ for all p, j, m, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_{C(\delta, M, T)} \left| \sum_{p \in M} \log L_p(s + i\tau, \chi_j) - \sum_{p \leqslant Y} \log L_p(s + i\tau, \chi_j) \right|^2 d\tau \leqslant c_3 \delta^{|M|} y^{1-2\sigma}$$

for some $c_3 > 0$. Thus, we have
(2.3)
$$\int_{C(\delta,M,T)} \int\int_{D'} \left| \sum_{p \in M} \log L_p(s+i\tau,\chi_j) - \sum_{p \leqslant Y} \log L_p(s+i\tau,\chi_j) \right|^2 d\sigma dt d\tau \leqslant c_4 \delta^{|M|} y^{1-2\sigma_0} T$$

for some $c_4 > 0$ and sufficiently large T.

By lemma 2.7, we have

$$\begin{aligned} \int_T^{2T} \left| \sum_{Y<p\leqslant X} \log L_p(s+i\tau,\chi_j) \right|^2 d\tau &\leqslant 2 \int_T^{2T} \left| \sum_{Y<p\leqslant X} \frac{\chi_j(\sigma_p)}{p^{s+i\tau}} \right|^2 d\tau + O(TY^{2-4\sigma}) \\ &\leqslant 2 \sum_{Y<p\leqslant X} \frac{d^2}{p^{2\sigma}} (T+O(p)) + O(TY^{2-4\sigma}) \\ &\leqslant c_5 T Y^{1-2\sigma} \end{aligned}$$

for some $c_5 > 0$ and all $s \in D'$ and as a consequence
(2.4)
$$\int_{C(\delta,M,T)} \int\int_{D'} \left| \sum_{Y<p\leqslant X} \log L_p(s+i\tau,\chi_j) \right|^2 d\sigma dt d\tau \leqslant c_6 T Y^{1-2\sigma_0} < c_6 \delta^{|M|} y^{1-2\sigma_0} T.$$

Then (2.3) and (2.4) yield

$$\int_{C(\delta,M,T)} \int\int_{D'} \left| \sum_{p \in X\setminus M} \log L_p(s+i\tau,\chi_j) \right|^2 d\sigma dt d\tau \leqslant c_7 \delta^{|M|} y^{1-2\sigma_0} T,$$

where $X \setminus M$ denotes the set $\{p \leqslant X : p \notin M\}$ and $c_7 > 0$ is some constant. From the simple inequality $\max_{n\leqslant N} |\alpha_n| \leqslant \sum_{n\leqslant N} |\alpha_n|$, we have

$$\int_{C(\delta,M,T)} \max_{j\leqslant J} \int\int_{D'} \left| \sum_{p \in X\setminus M} \log L_p(s+i\tau,\chi_j) \right|^2 d\sigma dt d\tau \leqslant c_7 J \delta^{|M|} y^{1-2\sigma_0} T.$$

As a consequence, we have

$$\left| \left\{ \tau \in C(\delta,M,T) : \max_{j\leqslant J} \int\int_{D'} \left| \sum_{p \in X\setminus M} \log L_p(s+i\tau,\chi_j) \right|^2 d\sigma dt \leqslant y^{\frac{1}{2}-\sigma_0} \right\} \right| > \frac{1}{2} \delta^{|M|} T$$

by taking y satisfying $c_7 J y^{\frac{1}{2}-\sigma_0} < \frac{1}{2}$. By Lemma 2.6, we have

$$\left| \left\{ \tau \in C(\delta,M,T) : \max_{j\leqslant J} \max_{s \in D} \left| \sum_{p \in X\setminus M} \log L_p(s+i\tau,\chi_j) \right| \leqslant \frac{1}{\sqrt{\pi}(R-r)} y^{\frac{1}{4}-\frac{1}{2}\sigma_0} \right\} \right| > \frac{1}{2} \delta^{|M|} T.$$

By taking a real number y satisfying $\frac{1}{\sqrt{\pi}(R-r)} y^{\frac{1}{4}-\frac{1}{2}\sigma_0} < \epsilon$, we have

$$|E_T| > \frac{1}{2}\delta^{|M|}T,$$

where

$$E_T = \left\{ \tau \in C(\delta, M, T) : \max_{j \leqslant J} \max_{s \in D} \left| \sum_{p \in M} \log L_p(s+i\tau, \chi_j) - \sum_{p \leqslant X} \log L_p(s+i\tau, \chi_j) \right| \leqslant \epsilon \right\}.$$

Therefore, there exists a $\delta > 0$ and a finite set M such that

$$\max_{j \leqslant J} \max_{s \in D} |\log L(s+i\tau, \chi_j, K/\mathbb{Q}) - \log h_j(s)| < 4\epsilon$$

for $\tau \in E_T \setminus A_T$ with $|E_T \setminus A_T| \geqslant |E_T| - |A_T| \geqslant \frac{1}{2}\delta^{|M|}T + o(T)$. □

§ 3. Zeros of the partial zeta functions

We extends Theorem 1.1 to partial zeta functions subject to Conjecture 2.1 as an application of joint universality of Artin L-functions. Let K be a number field and $G^{\mathfrak{f}}$ be its ray class group. By class field theory, there is a unique Abelian extension L of K with $G^{\mathfrak{f}} \simeq G(L/K)$. Thus, every Abelian Artin L-function is a Hecke L-function, and vice versa. There is a unique minimal normal extension N of $\mathbb{Q}$ containing L.

Theorem 3.1. *Assume Conjecture 2.1 for $L(s, \chi, N/\mathbb{Q})$ for all characters χ defined on $G(N/\mathbb{Q})$, where the field N is as described above. Suppose that $\#G^{\mathfrak{f}} > 1$ and $\mathcal{A} \in G^{\mathfrak{f}}$. Then, the number of zeros of the partial zeta function $\zeta(s, \mathcal{A})$ on the rectangular region $\sigma_1 < \operatorname{Re} s < \sigma_2$, $0 < \operatorname{Im} s < T$ is bigger than*

$$\gg T$$

for any fixed $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$.

Proof. Suppose that $\chi \neq 1$ is an irreducible character of $G(L/K) \simeq G^{\mathfrak{f}}$. By Frobenius reciprocity, we know that

$$(\chi^*, 1)_{G(N/\mathbb{Q})} = (\chi, 1_{|G(N/K)})_{G(N/K)} = (\chi, 1)_{G(N/K)} = 0.$$

If we denote the irreducible characters of $G(N/\mathbb{Q})$ by $\phi_1 := 1, \phi_2, \ldots, \phi_k$, then for every non-trivial character of $G(L/K)$ we have

$$\chi^* = \sum_{j=2}^{k} m_j \phi_j, \qquad m_j \in \mathbb{Z}_{\geqslant 0}.$$

For the induced character 1^* of the trivial character 1 defined on $G(L/K)$, we get

$$(1^*, 1)_{G(N/\mathbb{Q})} = (1, 1_{|G(N/K)})_{G(N/K)} = (1, 1)_{G(N/K)} = 1.$$

Therefore, we have

$$1^* = \phi_1 + \sum_{j=2}^{k} n_j \phi_j, \qquad n_j \in \mathbb{Z}_{\geqslant 0}.$$

So we get

$$L(s, 1) = L(s, 1^*, N/\mathbb{Q}) = L(s, \phi_1, N/\mathbb{Q}) \prod_{j=2}^{k} L(s, \phi_j, N/\mathbb{Q})^{n_j} \tag{3.1}$$

and for the non-trivial Abelian characters χ of $G(L/K)$

$$L(s, \chi) = \prod_{j=2}^{k} L(s, \phi_j, N/\mathbb{Q})^{m_j}. \tag{3.2}$$

Since the irreducible characters ϕ_j are linearly independent, we apply Theorem 2.2 to $L(s, \chi_j, N/\mathbb{Q})$ with $1 \leqslant j \leqslant k$. For any $\epsilon > 0$, there exists a set $A_\epsilon \subset [T, 2T]$ with

$$\liminf_{T\to\infty} \frac{1}{T} |A_\epsilon| > 0 \tag{3.3}$$

such that

$$\left| L(s + i\tau, \phi_1, N/\mathbb{Q}) - (s - s_0 - \sum_{\chi \neq 1} \bar{\chi}(\mathfrak{a})) \right| < \epsilon$$

and

$$|L(s + i\tau, \phi_j, N/\mathbb{Q}) - 1| < \epsilon$$

for any $2 \leqslant j \leqslant k$, $s \in D_{s_0}(r) \subset \{z \in \mathbb{C} : \sigma_1 < \operatorname{Re} z < \sigma_2\}$ and $\tau \in A_\epsilon$ and for an integral ideal $\mathfrak{a} \in \mathcal{A}$. By (3.1) and (3.2), we have

$$\left| L(s + i\tau, 1) - (s - s_0 - \sum_{\chi \neq 1} \bar{\chi}(\mathfrak{a})) \right| < \epsilon$$

and for $\chi \neq 1$ we find

$$|L(s + i\tau, \chi) - 1| < \epsilon$$

for all $s \in D_{s_0}(r)$ and $\tau \in A_\epsilon$.

Note that

$$\zeta(s, \mathcal{A}) = \frac{1}{h} \sum_{\chi} \bar{\chi}(\mathfrak{a}) L(s, \chi) \tag{3.4}$$

for some $\mathfrak{a} \in \mathcal{A}$ and $h = \#G^{\mathfrak{f}}$. We have

$$\begin{aligned}
&\left|\zeta(s+i\tau,\mathcal{A}) - \frac{s-s_0}{h}\right| \\
&\quad \leqslant \frac{1}{h}\left(\sum_{\chi\neq 1}|\bar{\chi}(\mathfrak{a})L(s+i\tau,\chi)-\bar{\chi}(\mathfrak{a})| + \left|L(s+i\tau,1)-(s-s_0-\sum_{\chi\neq 1}\bar{\chi}(\mathfrak{a}))\right|\right) < \epsilon
\end{aligned}$$

for all $s \in D_{s_0}(r)$ and all $\tau \in A_\epsilon$. Suppose that $\epsilon < \frac{r}{h}$, then

$$\left|\zeta(s+i\tau,\mathcal{A}) - \frac{s-s_0}{h}\right| < \left|\frac{s-s_0}{h}\right|$$

on the circle $|s-s_0| = r$. Inside the disc $|s-s_0| < r$, there is exactly one zero of $\zeta(s+i\tau,\mathcal{A})$ by Rouché Theorem for each $\tau \in A_\epsilon$. By (3.3), we complete the proof of Theorem 3.1. □

Let $N_{\zeta(s,\mathcal{A})}(\sigma_1,\sigma_2;T)$ be the number of zeros of $\zeta(s,\mathcal{A})$ on the rectangular region $\sigma_1 < \operatorname{Re} s < \sigma_2$, $0 < \operatorname{Im} s < T$. Theorem 3.1 gives a lower bound for $N_{\zeta(s,\mathcal{A})}(\sigma_1,\sigma_2;T)$ on the assumption of Conjecture 1. What can we say about an upper bound for $N_{\zeta(s,\mathcal{A})}(\sigma_1,\sigma_2;T)$? The following theorem gives an answer.

Theorem 3.2. *Let K be a number field and $G^{\mathfrak{f}}$ be its ray class group. Assume Conjecture 1 for all Hecke L-functions $L(s,\chi)$ with $\chi : G^{\mathfrak{f}} \to S^1$. Suppose that $\#G^{\mathfrak{f}} > 1$ and $\mathcal{A} \in G^{\mathfrak{f}}$. Then, the number of zeros of the partial zeta function $\zeta(s,\mathcal{A})$ on the rectangular region $\operatorname{Re} s > \sigma_0$, $0 < \operatorname{Im} s < T$ is less than*

$$\ll T$$

for any fixed $\sigma_0 > \frac{1}{2}$.

Proof. By Littlewood's lemma, we have

$$2\pi\int_\sigma^\infty N(u,T)du = \int_0^T \log|\zeta(\sigma+it,\mathcal{A})|dt + O(\log T),$$

where $N(u,T)$ denotes the number of zeros of $\zeta(s,\mathcal{A})$ on the region $\operatorname{Re} s > u$, $0 < \operatorname{Im} s < T$. Thus, it is enough to show that the integral on the right is less than $\ll T$.

We are going to use a simple inequality. First,

$$\left|\frac{1}{J}\sum_{j\leqslant J} z_j\right| \leqslant \frac{1}{J}\sum_{j\leqslant J}|z_j| \leqslant \max_{j\leqslant J}|z_j|.$$

Take logarithms on both sides, then

$$\log\left|\frac{1}{J}\sum_{j\leqslant J} z_j\right| \leqslant \max_{j\leqslant J}\log|z_j|. \tag{3.5}$$

By (3.4) and (3.5), we have

$$\int_0^T \log|\zeta(\sigma+it,\mathcal{A})|dt \ll \int_0^T \max_{\chi}\log|L(\sigma+it,\chi)|dt.$$

Apply Lemma 2 of [6], then we have

$$\begin{aligned}\int_0^T \max_{\chi}\log|L(\sigma+it,\chi)|dt &\leqslant \int_0^T \max_{\chi}\left|\log|L(\sigma+it,\chi)|dt - \operatorname{Re}\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|dt\\ &\quad+\int_0^T \max_{\chi}\left|\operatorname{Re}\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|dt\\ &\leqslant \sum_{\chi}\int_0^T \left|\log|L(\sigma+it,\chi)|dt - \operatorname{Re}\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|dt\\ &\quad+\sum_{\chi}\int_0^T \left|\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|dt,\end{aligned}$$

where $X = T^{\kappa}$, $0<\kappa<\frac{1}{2}$, $a(p,\chi)=\sum_{N\mathfrak{p}=p,\mathfrak{p}|p}\chi(\mathfrak{p})$, $|a(p,\chi)|\leqslant [K:\mathbb{Q}]$. By Cauchy's inequality and Lemma 2 of [6], we have

$$\begin{aligned}&\int_0^T \left|\log|L(\sigma+it,\chi)|dt - \operatorname{Re}\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|dt\\ &\qquad\ll \sqrt{T}\left(\int_0^T \left|\log|L(\sigma+it,\chi)|dt - \operatorname{Re}\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|^2 dt\right)^{\frac{1}{2}} \ll T,\end{aligned}$$

and by Lemma 2.7

$$\begin{aligned}\int_0^T \left|\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|dt &\ll \sqrt{T}\left(\int_0^T \left|\sum_{p\leqslant X}\frac{a(p,\chi)}{p^{\sigma+it}}\right|^2 dt\right)^{\frac{1}{2}}\\ &= \sqrt{T}\left(\sum_{p\leqslant X}\frac{|a(p,\chi)|^2}{p^{2\sigma}}(T+O(p))\right)^{\frac{1}{2}} \ll T.\end{aligned}$$

Thus, the proof is complete. $\square$

§ 4. Concluding remarks

The author [5] improved Theorem 1.1 by obtaining asymptotic formula $cT + o(T)$ for the number of zeros of Epstein zeta function $E(s, Q)$ on the rectangular region $\frac{1}{2} < \sigma_1 < \operatorname{Re} s < \sigma_2$, $0 < \operatorname{Im} s < T$ with the constant c has an integral formula $c = \int_{\sigma_1}^{\sigma_2} \mu(\sigma) d\sigma$ for some density function $\mu(\sigma)$. The main ingredient of the proof is the method given by Borchsenius and Jessen [2]. Based on Theorems 3.1 and 3.2, we expect the following statement.

Conjecture 4.1. *Let K be a number field and let $G^{\mathfrak{f}}$ be its ray class group. Let $\mathcal{A} \in G^{\mathfrak{f}}$. Then, the number of zeros of partial zeta function $\zeta(s, \mathcal{A})$ on the region $\frac{1}{2} < \sigma_1 < \operatorname{Re} s < \sigma_2$, $0 < \operatorname{Im} s < T$ is*

$$= cT + o(T),$$

where $c = \int_{\sigma_1}^{\sigma_2} \mu(\sigma) d\sigma$ for some density function $\mu(\sigma)$ depending on $\mathcal{A}$.

The simplest case $K = \mathbb{Q}$ is considered and completed by the author with Haseo Ki [3].

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